GROUP REPRESENTATIONS IN INDEFINITE METRIC SPACES

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Let V be a n -dimensional complex vector space with scalar product $(,)$ and let η be a Hermitian non-singular linear operator on V . The indefinite metric of V is given by $\langle \phi, \psi \rangle = (\phi, \eta \psi)$. Let G be a finite group of symmetry transformations of the indefinite metric space V . According to a generalisation of Wigners theorem [1,2] one then comes to deal with a n -dimensional projective linear-antilinear (PLA) representation of G ; each $g \in G$ is represented by an operator $D(g)$ on V which is either η -unitary (ηU), or η -antiunitary (ηAU), or η -pseudounitary (ηPU), or η -pseudoantiunitary (ηPAU). In terms of matrices, taken with respect to an orthonormal basis of V we then have, instead of unitarity,

$$D^\dagger(g) \eta D(g) = (-)_g \eta^g. \quad (1)$$

Here η is now a Hermitian non-singular matrix, $(-)_g$ is equal to -1 if g is represented by a ηPU or a ηPAU operator and equal to $+1$ otherwise and the superscript g denotes complex conjugation if and only if g is represented by a ηAU or a ηPAU operator. Let G_0 be the normal subgroup of G consisting of those $g \in G$ which are represented by ηU operators and let a, b and c be elements of G (if any exist) which are represented by ηAU , ηPU and ηPAU operators respectively. Then aG_0 , bG_0 and cG_0 denote the cosets of G with respect to G_0 which are represented by ηAU , ηPU and ηPAU operators respectively. We will only consider here the case that G_0 has index 1 or 2. This leaves the following 4 possibilities: $G = G_0$ (case I), $G = G_0 + aG_0$ (case II), $G = G_0 + bG_0$ (case III) and $G = G_0 + cG_0$ (case IV). For any non-singular $n \times n$ -matrix A an equivalence transformation of D is given by $D'(g) = A^{-1} D(g) A^g$ and $\eta' = A^\dagger \eta A$. D is said to be decomposable if there exists an equivalence transformation such that

$$D'(g) = \begin{pmatrix} D'_1(g) & 0 \\ 0 & D'_2(g) \end{pmatrix} \quad \forall g \in G; \quad \eta' = \begin{pmatrix} \eta'_1 & 0 \\ 0 & \eta'_2 \end{pmatrix} \quad (2)$$

A PLA representation D satisfying (1) is decomposable into a direct sum of undecomposable PLA representations. If D is irreducible it is undecomposable but if D is undecomposable it is not necessarily irreducible. We have investigated which are the

undecomposable PLA representations, with the following results.

In case I and II the undecomposable PLA representations are just the irreducible PLA representations.

In the cases III and IV we distinguish between two types of irreducible PLA representations D : those for which $D \upharpoonright G_0$ is irreducible (type A) and those for which $D \upharpoonright G_0$ is reducible (type B).

Consider case III. If D is an irreducible PLA representation of type B then the irreducible PLA representation D' , defined by $D'(g) = (-)^g D(g)$ is not equivalent with D . Then D and D' are said to be related. The undecomposable PLA representations of G are the irreducible PLA representations of type A and the direct sums of two related irreducible PLA representations of type B. It follows that in each PLA representation of case III related irreducible PLA representations have equal multiplicity.

In case IV the undecomposable PLA representations of G are the irreducible PLA representations of type B and the direct sums of two irreducible PLA representations of type A. It follows that in each PLA representation of case IV the irreducible PLA representations of type A have even multiplicity. More details and full proofs will be published in [3].

- [1] L. Bracci, G. Morchio and F. Strocchi : Commun. Math. Phys. 41, 289-299 (1975).
- [2] P.M. van den Broek : "Symmetry transformations in indefinite metric spaces", Memorandum nr. 428, Department of Applied Mathematics, Twente University of Technology, The Netherlands (1983).
- [3] P.M. van den Broek : "Group representations in indefinite metric spaces", Journ. Math. Phys. (to be published).