Paths, Cycles and Related Partitioning Problems in Graphs

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PATHS, CYCLES
AND RELATED PARTITIONING PROBLEMS
IN GRAPHS

DISSERTATION

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by

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born on the 3rd of December 1974
in Guangdong, China
This dissertation has been approved by the promotor:
Prof.dr.ir. H.J. Broersma
Preface

This thesis is based on joint work of the author with several different collaborators during the last five years. It is composed of a short introductory chapter, followed by five technical chapters. These five chapters are all based on associated research papers that are in different stages of submission, refereeing, acceptance or publication, and that are listed below together with several other joint publications of the author.

The underlying research papers as well as the corresponding chapters in this thesis are the result of continuous part-time research efforts of the author, in collaboration with other researchers, on paths and cycles in graphs.

The first chapter contains a brief introduction, with some background and motivation for the research in this field, and with some remarks on earlier work that inspired the author to contribute to this field. The details of the author’s contributions are presented in the subsequent five chapters, that are all self-contained. The second chapter deals with the extremal digraphs one has to exclude when relaxing a classical degree condition for the existence of Hamiltonian cycles in digraphs. The third and fourth chapter deal with sufficient conditions for path extendability and cycle extendability in digraphs, respectively. In the fifth chapter, in the context of path and cycle properties, we study the number of 2-paths in oriented graphs and tournaments. We also present some applications to demonstrate the relevance of conditions in terms of the number of 2-paths. In the sixth chapter, we study monochromatic clique and multicolored cycle partitioning problems in edge-colored graphs, from the perspective of computational complexity and algorithmic solutions.
The thesis has been written as a collection of independent papers. To guarantee the independence and readability of the chapters, we chose to maintain the structure of journal papers, with a short introduction and background, and the definitions of concepts and terminology at the beginning of each chapter. The author apologizes for any repetition.

Papers underlying this thesis


[3] Hamiltonicity, pancyclicity and cycle extendability in bipartite tournaments, preprint. (with X. Zhang, and D. Lou) (Chapter 4)


Some other recent joint publications by the author:


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Chapter 1

Introduction

The main theme of this thesis is centered around paths and cycles in graphs, while there is also one chapter devoted to partitioning problems that do not only involve cycles but also cliques. The graphs under consideration are sometimes undirected, mostly directed, and occasionally random or edge-colored.

Intuitively stated, a graph is a mathematical concept consisting of a set of elements, together with a set of (ordered or unordered) pairs of its elements, where the latter set of pairs is a subset of the set of all (ordered or unordered) possible pairs of its elements. This means that graphs can model a variety of practical situations in which one is faced with a set of objects and relationships between (some of) the pairs of these objects. If these relationships are symmetric, they are modeled by unordered pairs; otherwise, they are modeled by ordered pairs. In this thesis, we adopt the usual terminology, and refer to the elements of a graph as its vertices, and to the pairs that are contained in the subset of pairs of its elements as its edges (for unordered pairs) or its arcs (for ordered pairs). Given a graph that models a practical situation, or given a graph just as a mathematical object, one might be interested in structural properties of the graph, or in conditions that guarantee such properties. As an example, in many applications it is important to know whether the graph that models the situation, is connected, meaning that there exist paths between every pair of its vertices. Here, with a (directed) path we mean a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ and edges (arcs) $v_i v_{i+1}$ (we usually omit the brackets and comma in the pairs that correspond to edges or arcs) between
all successive vertices in the sequence, between the two vertices $v_1$ and $v_k$ of the pair. So, we require connecting paths between all the pairs of vertices, not only between the pairs that are joined by an edge (arc). Similarly, a (directed) cycle is a sequence of distinct vertices $v_1, v_2, \ldots, v_k$ and edges (arcs) $v_i v_{i+1}$ between all successive vertices in the sequence, with an additional edge (arc) $v_k v_1$.

Paths and cycles are among the most commonly studied structures in graphs, and have attracted many researchers for a relatively long time. Tracing back to 1857, Sir William Rowan Hamilton invented the Icosian Game, which involves constructing a cycle containing every vertex of the graph that represents a dodecahedron. Started in such a form as a recreational game, path and cycle problems became a hot area of research since the middle of the last century, and now form an important branch of graph theory, encompassing many challenging problems, deep results and important applications.

Named after Sir William Rowan Hamilton, a Hamiltonian path (cycle) of a graph $G$ is a path (cycle) in $G$ that contains every vertex of $G$. Correspondingly, a graph with a Hamiltonian path (cycle) is called traceable (Hamiltonian). Problems on Hamiltonian paths and cycles have many practical application, for instance in interconnection network design, in VLSI design and in DNA analysis. However, the problem of deciding whether a given graph admits a Hamiltonian path (or Hamiltonian cycle) is generally NP-complete, which means that an efficient algorithm to solve this decision problem is not likely to exist. Similarly, there exists no elegant, easy to apply set of conditions that characterize in which cases a graph is Hamiltonian. This was, and still is, one of the main motivations for the vast amount of research on Hamiltonian paths and Hamiltonian cycles in graphs, trying to tackle various related problems from a computational, algorithmic or theoretical perspective.

In theoretical research, a lot of graph parameters and properties, involving the vertex and edge degrees, the cardinalities of neighborhood sets, the number of edges, and the independence number and vertex connectivity, have been associated with traceability and Hamiltonicity. We will not introduce the necessary terminology and definitions for the above notions here, but postpone this until it is essential to understand the details in the upcoming chapters. Most of the corresponding results appear in the form of sufficient conditions for the existence of Hamiltonian cycles in certain graph classes. One of the
main results in this thesis improves a classical degree sum condition due to Woodall ([106]) for Hamiltonicity in digraphs.

It is a common approach within graph theory to strengthen or extend existing results on the existence of certain structures, by trying to strengthen the conclusion or relax the conditions for such structures. There are several ways to generalize or extend the notion of a Hamiltonian cycle or path. One can ask for the number of Hamiltonian cycles, instead of just the existence of a Hamiltonian cycle, that is implied by a certain condition. This leads to enumeration problems, that are not a subject of this thesis. One can further require that these Hamiltonian cycles are edge-disjoint, leading to problems involving Hamiltonian cycle packings or decompositions, that are not the subject of this thesis either. In a slightly different direction, one can work on cycle factors of a graph. A cycle factor of a graph is a set of disjoint cycles, the union of which covers all the vertices of the graph. Thus, a Hamiltonian cycle is a special cycle factor consisting of only one cycle. Some sufficient conditions for Hamiltonicity, or their strengthenings, imply cycle factors with additional constraints on the number or the length of the cycles.

The main idea behind most of the results in this thesis, is to establish conditions that guarantee the existence of cycles with many different lengths in a graph. Let $n$ denote the number of vertices of the graphs we consider, unless otherwise specified. Then, a graph containing cycles of every length from 3 to $n$ is called pancyclic. In 1971 and 1973, Bondy defined the concept of pancyclicity in some pioneering work in this direction (See [23] and [24]). There, he also raised the following meta-conjecture that has become well-known within the graph theory community.

**Meta-conjecture.** Almost any nontrivial condition on a graph which implies that the graph is Hamiltonian also implies that the graph is pancyclic (possibly with some exceptional graphs that can be easily characterized).

In many cases, sufficient conditions for Hamiltonicity imply that the graphs are dense (contain relatively many edges), and therefore the meta-conjecture of Bondy is likely to hold. This meta-conjecture motivated countless subsequent contributions of results on pancyclicity and other related concepts, such as panconnectedness. A graph is called panconnected, if there is a path of length $k$ from $u$ to $v$ for every two distinct vertices $u$ and $v$, and for every $k$ with
$3 \leq k \leq n - 1$. Hence, panconnectedness is the counterpart of pancyclicity for the existence of paths instead of cycles with many different lengths.

Later, around 1990, Hendry went further, by defining cycle extendability and path extendability ([56], [57], [58], [59] and [60]). He observed that many proofs for Hamiltonicity are based on a proof by contradiction or contraposition. In such proofs, one assumes that the graph is not Hamiltonian, starts by considering a maximal non-Hamiltonian cycle $C$, and tries to find a longer cycle $C'$, hence a contradiction, or a violation of the condition in the hypothesis of the result. As Hendry observed, in many cases this $C'$ consists of all the vertices of $C$, and one additional vertex. This led to the notion of cycle extendability: Hendry defined the cycle $C$ to be extendable, whenever such a cycle $C'$ exists. The definition of path extendability is similar. A non-Hamiltonian path $P$ of length at least one is called extendable if there exists another path $P'$ with the same end vertices of $P$, whose vertex set consists of the vertex set of $P$ and one additional vertex. A graph $G$ is called cycle (path) extendable, if every non-Hamiltonian cycle (path of length at least 1) of $G$ is extendable. If $G$ is cycle extendable and has a cycle $C$ of length 3, we can start from $C$, repeatedly apply the operation of extending a cycle, until we get a Hamiltonian cycle. Then, we have cycles of every length from 3 to $n$. In this sense, cycle extendability is stronger property than pancyclicity, and similarly, path extendability is stronger than panconnectedness.

It is then natural to ask whether Bondy’s meta-conjecture also holds for cycle extendability and path extendability, which is one of the themes of this thesis. An interesting observation in the work of Hendry and our work in this thesis reveals that, with respect to cycle extendability and path extendability, the validity of Bondy’s meta-conjecture in undirected graphs and digraphs is different. As an example, if we consider a Dirac-type degree condition for Hamiltonicity in undirected graphs, involving a lower bound on the degree of every vertex in the graph, we will find that this condition, sometimes with a slight strengthening, often implies pancyclicity, cycle extendability and even path extendability. Moreover, usually their counterparts in digraphs also imply pancyclicity. However, to guarantee cycle extendability and path extendability in digraphs, one needs to raise the lower bound on the vertex degrees significantly.

Another interesting problem in this context is, to answer the question in
which classes of graphs Bondy’s meta-conjecture always holds, that is, in which classes Hamiltonicity and pancyclicity become equivalent. One can find several examples in the literature. In his well-known work on Hamiltonicity in squares of graphs, Fleischner ([42]) proved that in for squares of graphs, Hamiltonicity and pancyclicity are equivalent. As another example, in tournaments, strongness has been shown to be equivalent to Hamiltonicity ([28]) and pancyclicity ([90]). As before, in our studies we want to go one step further, and try to include cycle extendability into this equivalence relation. Next, we consider another example. A chordal graph is a graph in which every cycle of length at least 3 has a chord, i.e., an edge joining two vertices of the cycle that are nonadjacent on the cycle. It is easy to deduce that a chordal graph with a Hamiltonian cycle is pancyclic. Hendry raised the problem whether a Hamiltonian chordal graph is cycle extendable. This problem can be understood as a query on the equivalence between Hamiltonicity and cycle extendability in chordal graphs. For several subclasses of chordal graphs, this equivalent relation has been verified ([33], [3] and [2]). However, the equivalence is not valid in general chordal graphs, as has been shown in [76] recently. For classes of digraphs, Hendry ([57]) has shown that in tournaments Hamiltonicity and cycle extendability are equivalent. One of our contributions is an analogous result that verifies the equivalence of these properties in bipartite tournaments.

Next to theoretical problems and results on the existence of various path and cycle structures, a lot of closely related computational and algorithmic problems can be raised. A natural and important one for applications, is the problem of determining a cycle factor of a graph, in contrast to just proving sufficiency results for its existence. As defined above, a cycle factor of a graph $G$ is a collection of mutually disjoint cycles of $G$ such that each vertex of $G$ is contained in exactly one of these cycles. We consider this cycle factor problem in edge-colored graphs, that belongs to the very broad and well-studied area of graph partitioning problems. These partitioning problems of edge-colored graphs have many applications, e.g., in graph-based data mining. In the latter field, the vertices of a graph denote the entities in a system, usually a large network, and the edges between the vertices represent relationships between the entities, with different colors denoting different kinds of relationships. According to different requirements in real-world applications or theoretical considerations, different structures are studied, e.g.,
cliques, cycles, paths and trees, with respect to these partitioning problems. In the final chapter of this thesis, we prove computational complexity and algorithmic results for what are called the multicolored cycle problem and the monochromatic clique problem.

For definitions of the relevant concepts, we refer to the subsequent chapters. The rest of the thesis is organized as follows. In Chapter 2, we improve a classical result involving a degree sum condition for the existence of Hamiltonian cycles in digraphs due to Woodall. In particular, we characterize all the exceptional (non-Hamiltonian) digraphs that satisfy a slightly relaxed degree sum condition, i.e., we characterize all the extremal digraphs for the original condition. Due to a relationship between cycles in digraphs and matching alternating cycles in bipartite graphs, the result of Woodall turns out to be equivalent to a result of Las Vergnas. The latter result deals with the existence of matching alternating Hamiltonian cycles in balanced bipartite graphs. We actually improve the result due to Las Vergnas in Chapter 2.

In Chapter 3, we establish a number of extremal and degree conditions for path extendability in general digraphs. Moreover, we prove that every path of length at least two in a regular tournament is extendable, with some exceptions that we have characterized.

In Chapter 4, we investigate the equivalence between Hamiltonicity and cycle extendability in bipartite tournaments. Our results show that in bipartite tournaments, Hamiltonicity is equivalent to even cycle extendability, and also equivalent to even pancyclicity, even vertex-pancyclicity, and fully even cycle extendability, with one exceptional class that we have characterized.

In Chapter 5, as a relevant condition for path and cycle properties, we estimate the number of 2-paths between every two vertices in random oriented graphs and random tournaments. We also demonstrate some applications of the 2-path conditions, by using them to derive some path and cycle properties in tournaments.

In Chapter 6, we investigate the inapproximability and complexity of the problems of finding the minimum number of monochromatic cliques and multicolored cycles that, respectively, partition the vertex set $V(G)$ of an edge-colored graph $G$, where $G$ avoids some forbidden induced subgraphs.
Chapter 2

Directed Hamiltonian cycles in digraphs and matching alternating Hamiltonian cycles in bipartite graphs

Hamiltonian problems, and their many variations, have been studied extensively for more than half a century. We refer the interested readers to the surveys of Gould ([48] and [49]), Kawarabayashi ([67]) and Broersma ([26]) to trace the developments in this field. Recently, several approximate solutions, based on probabilistic methods, to many traditional Hamiltonian problems and conjectures in digraphs have appeared ([69], [68], [35] and [74]), which are surveyed by Kühn and Osthus ([73]).

Hamiltonicity and related properties are also important in practical applications. For example, in network design, the existence of Hamiltonian cycles in the underlying topology of an interconnection network provides advantages for the routing algorithm to make use of a ring structure, while the existence of a Hamiltonian decomposition allows the load to be equally distributed, making the network more robust ([20]).

There exist many well-known degree and degree sum conditions for guaranteeing Hamiltonicity. Often, the lower bounds in such conditions are best
possible. However, one could still try to reduce the bounds and aim to characterize all exceptional non-Hamiltonian graphs, that is, the extremal graphs for the conditions. This type of research often leads to the discovery of interesting topological structures. In this chapter, we apply this approach to Woodall's condition for the existence of directed Hamiltonian cycles in digraphs.

2.1 Terminology, notations and preliminary results

In this chapter, we consider finite, simple and connected graphs, and finite and simple digraphs. For the terminology not defined in this paper, the reader is referred to [25] and [12].

Let \( G \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). We denote by \( \nu \) or \( |G| \) the cardinality of \( V(G) \), also called the order of \( G \). For \( u \in V(G) \), we denote by \( d(u) \) the degree of \( u \), and by \( N(u) \) or \( N_G(u) \) the set of neighbors of \( u \) in \( G \), so \( d(u) = |N(u)| \). For a subgraph \( H \) of \( G \) and a vertex \( u \in V(G) \setminus V(H) \), we denote by \( N_H(u) \) the set of neighbors of \( u \) in \( H \). For any two disjoint vertex sets \( X, Y \) of \( G \), we denote by \( e(X,Y) \) the number of edges of \( G \) from \( X \) to \( Y \). For \( u, v \in V(G) \), we denote by \( d(u, v) \) the distance between \( u \) and \( v \), that is, the length of a shortest path connecting \( u \) and \( v \). By \( uv + (uv−) \) we mean that the vertices \( u \) and \( v \) are adjacent (nonadjacent). If a vertex \( u \) sends (no) edges to \( X \), where \( X \) is a subgraph or a vertex subset of \( G \), we write \( u \rightarrow X \) (\( u \twoheadrightarrow X \)). By \( nK_2 \), we denote a graph consisting of a disjoint union of \( n \) independent edges.

Let \( D \) be a digraph with vertex set \( V(D) \) and arc set \( A(D) \). \( u, v \) and \( w \) distinct vertices of \( D \). We denote by \( |D| \) the cardinality of \( V(D) \), and by \( d^+(u) \) and \( d^−(u) \) the out-degree and in-degree of \( u \), respectively. The degree of \( u \) is the sum of its out-degree and in-degree. The minimum out-degree and in-degree of the vertices in \( D \), is denoted by \( δ^+(D) \) and \( δ^−(D) \), respectively. We let \( δ_0(D) = \min\{δ^+(D), δ^−(D)\} \). Let \( (u,v) \) denote an arc from \( u \) to \( v \). If \( (u,v) \in A(D) \) or \( (v,u) \in A(D) \), we say that \( u \) and \( v \) are adjacent. If \( (w,u) \in A(D) \) and \( (w,v) \in A(D) \), then we say that the pair \( \{u,v\} \) is dominated, while if \( (u,w) \in A(D) \) and \( (v,w) \in A(D) \), we say that the pair \( \{u,v\} \) is dominating. The complete digraph on \( n \geq 1 \) vertices, denoted by \( \overrightarrow{K}_n \), is obtained from the complete graph \( K_n \) by replacing every edge \( xy \) with
two arcs \((x, y)\) and \((y, x)\). Without causing ambiguity, we use \(I_n\) to denote a graph or a digraph consisting of \(n\) independent vertices, so without any edges or arcs. A \textit{transitive tournament} is an orientation of a complete graph for which the vertices can be numbered in such a way that \((i, j)\) is an edge if and only if \(i < j\).

Let \(C = u_0u_1 \ldots u_{m-1}u_0\) be a cycle in a graph \(G\). Throughout this chapter, the subscript of \(u_i\) is reduced modulo \(m\). We always orient \(C\) such that \(u_{i+1}\) is the successor of \(u_i\). For \(0 \leq i, j \leq m-1\), the path \(u_iu_{i+1}\ldots u_j\) is denoted by \(u_iC^+u_j\), while the path \(u_iu_{i-1}\ldots u_j\) is denoted by \(u_iC^-u_j\). For a path \(P = v_0v_1 \ldots v_{p-1}\) and \(0 \leq i, j \leq p-1\), the segment of \(P\) from \(v_i\) to \(v_j\) is denoted by \(v_iPv_j\).

A \textit{matching} \(M\) of \(G\) is a subset of \(E(G)\) in which no two elements are adjacent. If every \(v \in V(G)\) is covered by an edge in \(M\), then \(M\) is said to be a \textit{perfect matching} of \(G\). For a matching \(M\), an \(M\)-\textit{alternating path} (\(M\)-\textit{alternating cycle}) is a path (cycle) of which the edges appear alternatingly in \(M\) and \(E(G) \setminus M\). We call an edge in \(M\) or an \(M\)-alternating path starting and ending with edges in \(M\) a \textit{closed} \(M\)-\textit{alternating path}, while an edge in \(E(G) \setminus M\) or an \(M\)-alternating path starting and ending with edges in \(E(G) \setminus M\) is called an \textit{open} \(M\)-\textit{alternating path}.

The following results of Dirac and Ore for the existence of Hamiltonian cycles in graphs are basic and well-known.

**Theorem 2.1.** (Dirac, 1952 [40]) If \(G\) is a simple graph with \(|G| \geq 3\) and every vertex of \(G\) has degree at least \(|G|/2\), then \(G\) has a Hamiltonian cycle.

**Theorem 2.2.** (Ore, 1960 [95]) Let \(G\) be a simple graph. If for every distinct nonadjacent vertices \(u, v\) of \(G\), we have \(d(u) + d(v) \geq |G|\), then \(G\) has a Hamiltonian cycle.

Below are some of their digraph analogues.

**Theorem 2.3.** (Ghouila-Houri, 1960 [46]) Let \(D\) be a strong digraph. If the degree of every vertex of \(D\) is at least \(|D|\), then \(D\) has a directed Hamiltonian cycle.

**Theorem 2.4.** ([12], Corollary 6.4.3) If \(D\) is a digraph with \(\delta^0(D) \geq |D|/2\), then \(D\) has a directed Hamiltonian cycle.
Chapter 2

Theorem 2.5. (Woodall, 1972 [106]) Let $D$ be a digraph. If for every vertex pair $u$ and $v$, where there is no arc from $u$ to $v$, we have $d^+(u) + d^-(v) \geq |D|$, then $D$ has a directed Hamiltonian cycle.

It is not hard to verify that the bounds in the above theorems are tight. Nash-Williams [93] raised the problem of describing all the extremal digraphs for Theorem 2.3, that is, all digraphs with minimum degree at least $|D| - 1$ that do not have a directed Hamiltonian cycle. As a partial solution to this problem, Thomassen proved a structural theorem on the extremal graphs.

Theorem 2.6. (Thomassen, 1981 [102]) Let $D$ be a strong non-Hamiltonian digraph, with minimum degree $|D| - 1$. Let $C$ be a longest directed cycle in $D$. Then any two vertices of $D - C$ are adjacent, every vertex of $D - C$ has degree $|D| - 1$ (in $D$), and every component of $D - C$ is complete. Furthermore, if $D$ is strongly 2-connected, then $C$ can be chosen such that $D - C$ is a transitive tournament.

Darbinyan characterized the digraphs of even order that are extremal for both Theorem 2.3 and Theorem 2.4. We present his result here without a description of the extremal graphs.

Theorem 2.7. (Darbinyan, 1986 [37]) Let $D$ be a digraph of even order such that the degree of every vertex of $D$ is at least $|D| - 1$ and $\delta^0(D) \geq |D|/2 - 1$. Then either $D$ is Hamiltonian or $D$ belongs to a non-empty finite family of non-Hamiltonian digraphs.

We study the extremal graphs of Theorem 2.5 in this chapter. In contrast to Theorem 2.6 and Theorem 2.7, we obtain a complete characterization of all the extremal graphs.

For other results on degree sum conditions for the existence of Hamiltonian cycles in digraphs see [13], [14], [16], [37], [38], [83], [88], [110], [111], and a good summary is given in Chapter 6 of [12].

Another interesting aspect of directed Hamiltonian cycle problems is their connection with the problem of matching alternating Hamiltonian cycles in bipartite graphs. Given a bipartite graph $G$ with a perfect matching $M$, if we orient the edges of $G$ towards the same part, then contracting all edges in $M$, we get a digraph $D$. An $M$-alternating Hamiltonian cycle of $G$ corresponds
to a directed Hamiltonian cycle of $D$, and vice versa. Hence, Theorem 2.5 is equivalent to the following theorem.

**Theorem 2.8.** (Las Vergnas, 1972 [77]) Let $G = (B,W)$ be a balanced bipartite graph of order $\nu$. If for any $b \in B$ and $w \in W$, where $b$ and $w$ are nonadjacent, we have $d(w) + d(b) \geq \nu/2 + 2$, then for every perfect matching $M$ of $G$, there is an $M$-alternating Hamiltonian cycle.

Hence, we also determine the extremal graphs for the result of Las Vergnas in this chapter.

Theorem 2.8 is an instance of the problem of cycles containing matchings, which studies the conditions that enforce certain matchings to be contained in certain cycles. Related work can be found in [8], [9], [19], [52], [62], [66], [98] and [105]. In particular, Berman proved the following.

**Theorem 2.9.** (Berman, 1983 [19]) Let $G$ be a graph on $\nu \geq 3$ vertices. If for any pair of independent vertices $x, y \in V(G)$, we have $d(x) + d(y) \geq \nu + 1$, then every matching lies in a cycle.

Similarly to the aforementioned work, Jackson and Wormald determined all the extremal graphs of a generalized version of Berman’s result. We present their result without a description of the extremal graphs.

**Theorem 2.10.** (Jackson and Wormald, 1990 [62]) Let $G$ be a graph on $\nu$ vertices, and let $M$ be a matching of $G$ such that (1) $d(x) + d(y) \geq \nu$ for all pairs of independent vertices $x, y$ that are incident with $M$. Then $M$ is contained in a cycle of $G$ unless equality holds in (1) and several exceptional cases happen.

We will state our main results and present their proofs in the following sections.

### 2.2 Main results

Let $m$, $n \geq 1$ be integers. Let $\mathcal{D}_1$ be the set of all digraphs obtained by identifying one vertex of $\overrightarrow{K}_{n+1}$ with one vertex of $\overrightarrow{K}_{m+1}$. Let $\mathcal{D}_2$ be an arbitrary digraph on $n$ vertices, and take a copy of $I_{n+1}$. Let $\mathcal{D}_2$ be the set of all digraphs obtained by adding arcs of two directions between every vertex
of $I_{n+1}$ and every vertex of $D_2$. Let $D_3$ be as shown in Figure 2.1, and take a copy of $\overrightarrow{K}_n$. Let $D_3$ be the set of all graphs constructed by adding arcs of two directions between $v_i$, $i = 0, 1$, and every vertex of $\overrightarrow{K}_n$, and possibly, adding any of the arcs $(v_0, v_1)$ and $(v_1, v_0)$, or both. Finally, let $D_4$ be the digraph showed in Figure 2.2. Our main result is as follows.

**Theorem 2.11.** Let $D$ be a digraph. If, for every vertex pair $u$ and $v$, where there is no arc from $u$ to $v$, we have $d^+(u) + d^-(v) \geq |D| - 1$, then $D$ has a directed Hamiltonian cycle, unless $D \in D_1$, $D_2$ or $D_3$, or $D = D_4$.

Let $G_1$ be the class of graphs $G$ constructed by identifying an edge of one $K_{m+1,m+1}$ and one $K_{n+1,n+1}$, and $\mathcal{M}_1$ be the set of all perfect matchings of $G$ containing the identified edge. Let $G_2$ be the class of graphs $G$, constructed by taking a copy of $(n+1)K_2$ with bipartition $(B, W)$, and an arbitrary bipartite graph $G_2$ with bipartition $(B_1, W_1)$, where $|B_1| = |W_1| = n$, which has at least one perfect matching, then connecting every vertex in $B$ to every vertex in $W_1$, and every vertex in $W$ to every vertex in $B_1$. Furthermore, let $\mathcal{M}_2$ be the set of all perfect matchings of $G$, containing all the edges in $(n + 1)K_2$ (shown thick in Figure 2.3). Let $G_3$ be as shown in Figure 2.4, and $\mathcal{G}_3$ the set of the graphs $G$ constructed by taking one copy of $K_{n,n}$ with bipartition $(B, W)$, and connecting every vertex in $B$ to $w_0$ and $w_1$, every vertex in $W$ to $b_0$ and $b_1$, and possibly, adding any of the edges $w_0b_1$, $w_1b_0$, or both. Let $\mathcal{M}_3$ be the set of perfect matchings of $G$, containing the thick edges in $G_3$. Finally, we let graph $G_4$ be the graph in Figure 2.5, and $M_4$ the perfect matching of it, consisting of the thick edges. We obtain the following version of our main theorem.

**Theorem 2.12.** Let $G = (W, B)$ be a bipartite graph with a perfect matching $M$. If, for every vertex pair $w \in W$ and $b \in B$ with $wb-$, we have $d(w)+d(b) \geq \nu/2 + 1$, then $G$ has an $M$-alternating Hamiltonian cycle, unless one of the following holds.

1. $G \in G_1$, and $M \in \mathcal{M}_1$.
2. $G \in G_2$, and $M \in \mathcal{M}_2$.
3. $G \in G_3$, and $M \in \mathcal{M}_3$.

Since the two results are equivalent, we only prove Theorem 2.12 in the
next section. Before that, let us say a few words on the non-existence of $M$-alternating Hamiltonian cycles in the four exceptional cases. In Case (1), an $M$-alternating cycle of $G$ must contain the identified edge, whose endvertices form a vertex cut of $G$, so $G$ does not have an $M$-alternating Hamiltonian cycle. In Case (2), if there is an $M$-alternating Hamiltonian cycle $C$ of $G$, then the edges on $C$ that belong to $M$ must be in $(n+1)K_2$ and $G_2$ alternatingly, but there is one more such edge in $(n+1)K_2$, a contradiction. In Case (3), we can not have an $M$-alternating Hamiltonian cycle containing both $e_0$ and $e_1$. Finally in Case (4), the non-existence of any $M$-alternating Hamiltonian cycle can be verified directly.

### 2.3 Proof of Theorem 2.12

Let $G = (W, B)$ be a bipartite graph satisfying the condition of the theorem, and let $M$ be a perfect matching of $G$. Suppose that $G$ does not have an
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Figure 2.3: Exceptional graph family: $G_2$

Figure 2.4: Exceptional graph family: $G_3$

Figure 2.5: Exceptional graph $G_4$
$M$-alternating Hamiltonian cycle. We prove the theorem by characterizing $G$.

The following two lemmas will be used in our proof.

**Lemma 2.13.** Let $G = (W, B)$ be a bipartite graph with a perfect matching $M$. Let $C = u_0u_1 \ldots u_{2m-1}u_0$ be a longest $M$-alternating cycle in $G$, where $u_{2i} \in W$, $u_{2i+1} \in B$, and $u_{2i}u_{2i+1} \in M$, $0 \leq i \leq m - 1$. Let $b \in B$, $w \in W$ be the ending vertices of a closed $M$-alternating path $P$ in $G - C$. Then, for every $0 \leq i \leq m - 1$, either $u_{2i}b-$ or $u_{2i-1}w-$. Furthermore, if $b \rightarrow C$ and $w \rightarrow C$, then $|N_C(b)| + |N_C(w)| \leq m - |P|/2 + 1$.

**Proof.** If there exists $0 \leq k \leq m - 1$, such that $u_{2k}b+$ and $u_{2k-1}w+$, then $u_{2k}\overline{C}^+u_{2k-1}wp_{b_{2k}}$ is an $M$-alternating cycle longer than $C$, a contradiction. Thus, for $0 \leq i \leq m - 1$, either $u_{2i}b-$ or $u_{2i-1}w-$.

If $b \rightarrow C$ and $w \rightarrow C$, let $u_{2r} \in N_C(b)$ and $u_{2s-1} \in N_C(w)$ be such that $P' = u_{2s}C^+u_{2r-1}$ is the shortest. Then, there is no neighbor of $w$ and $b$ on $P'$. Since $C$ is longest, we have $|P'| \geq |P|$. So $|N_C(w)| + |N_C(b)| \leq 2 + (|C| - |P'| - 2)/2 = m - |P'|/2 + 1 \leq m - |P|/2 + 1$. 

**Lemma 2.14.** Let $G$ be a bipartite graph with a perfect matching $M$. Let $C = u_0u_1 \ldots u_{2m-1}u_0$ be a longest $M$-alternating cycle in $G$, where $u_{2i}u_{2i+1} \in M$, $0 \leq i \leq m - 1$. Let $C_1$ be an $M$-alternating cycle in $G - C$. For any vertex set $\{u_{2i-1}, u_{2i}\}$, $0 \leq i \leq m - 1$, either $u_{2i-1} \rightarrow C_1$ or $u_{2i} \rightarrow C_1$.

**Proof.** Suppose there exists $0 \leq k \leq m - 1$ such that $u_{2k-1} \rightarrow C_1$ and $u_{2k} \rightarrow C_1$. Let $b \in N_{C_1}(u_{2k})$ and $w \in N_{C_1}(u_{2k-1})$. We can always find a closed $M$-alternating path, $P$, as a subsegment of $C_1$, connecting $b$ and $w$. Then $u_{2k}\overline{C}^+u_{2k-1}wp_{b_{2k}}$ is an $M$-alternating cycle longer than $C$, contradicting our condition. 

In our proof, some important intermediate results are stated and proved as claims.

**Claim 2.1.** There is an $M$-alternating cycle in $G$ whose length is at least $\nu/2 + 1$.

**Proof.** Let $P = u_0u_1 \ldots u_{2p-1}$ be a longest closed $M$-alternating path in $G$. Then, all neighbors of $u_0$ and $u_{2p-1}$ in $G$ should be on $P$. 


If \( u_0u_{2p-1}+ \), then we obtain a cycle \( C = u_0u_1 \ldots u_{2p-1}u_0 \). Since \( P \) is the longest, \( e(V(C), V(G - C)) = 0 \). However, \( G \) is connected, so \( C \) must be an \( M \)-alternating Hamiltonian cycle and the claim holds.

If \( u_0u_{2p-1}- \), by our condition, \( d(u_0) + d(u_{2p-1}) \geq \nu/2 + 1 \). Without lost of generality, assume that \( d(u_0) \geq d(u_{2p-1}) \) and let \( u_{2i-1} \) be the neighbor of \( u_0 \) with the maximum \( i, 1 \leq i \leq p \). Then, \( i \geq (\nu/2 + 1)/2 \) and \( u_0Pu_{2i-1}u_0 \) is an \( M \)-alternating cycle with length at least \( 2i \geq \nu/2 + 1 \). This proves our claim.

Now let \( C = u_0u_1 \ldots u_{2m-1}u_0 \) be a longest \( M \)-alternating cycle in \( G \), where \( u_{2i} \in W, u_{2i-1} \in B \) and \( u_{2i}u_{2i+1} \in M \). Let \( G_1 = G - C \). Denote the neighborhood and degree of \( v \in V(G_1) \) in \( G_1 \) by \( N_1(v) \) and \( d_1(v) \). By Claim 2.1, \( |G_1| \leq \nu/2 - 1 \).

Let \( P_1 = v_0v_1 \ldots v_{2p_1-1} \) be a longest closed \( M \)-alternating path in \( G_1 \), where \( v_{2i} \in W \) and \( v_{2i+1} \in B \), \( 0 \leq i \leq p_1 - 1 \). Then \( N_1(v_0), N_1(v_{2p_1-1}) \subseteq V(P_1) \), and \( d_1(v_0), d_1(v_{2p_1-1}) \leq p_1 \). Firstly, we prove that \( v_0 \rightarrow C \) and \( v_{2p_1-1} \rightarrow C \).

If \( v_0 \rightarrow C \) and \( v_{2p_1-1} \rightarrow C \), then \( d(v_0) + d(v_{2p_1-1}) \leq 2p_1 \leq |G_1| \leq \nu/2 - 1 \). By the condition of our theorem, \( v_0v_{2p_1-1}+ \), and we get a cycle \( C_1 = v_0v_1 \ldots v_{2p_1-1}v_0 \) in \( G_1 \). By Lemma 2.14, for any two vertices \( u_{2i-1} \) and \( u_{2i} \) on \( C \), at least one of them, say \( u_{2i} \rightarrow C_1 \). Then \( d(u_{2i}) \leq \nu/2 - p_1 \). But then \( d(u_{2i}) + d(v_{2p_1-1}) \leq \nu/2 \), contradicting the condition of the theorem.

Now suppose only one of \( v_0 \) and \( v_{2p_1-1} \), say \( v_0 \rightarrow C \). Let a neighbor of \( v_0 \) on \( C \) be \( u_{2j-1} \). By Lemma 2.13, \( u_{2j} \) sends no edge to \( P_1 \), so \( d(u_{2j}) \leq \nu/2 - p_1 \), and \( d(u_{2j}) + d(v_{2p_1-1}) \leq \nu/2 \), again contradicting the condition of the theorem.

Therefore \( v_0 \rightarrow C \) and \( v_{2p_1-1} \rightarrow C \).

By Lemma 2.13, \( |N_C(v_0)| + |N_C(v_{2p_1-1})| \leq m - p_1 + 1 \). Therefore,

\[
\begin{align*}
d(v_0) + d(v_{2p_1-1}) & \leq 2p_1 + (m - p_1 + 1) \\
& = m + p_1 + 1 \\
& \leq m + |G_1|/2 + 1 \\
& = \nu/2 + 1. 
\end{align*}
\]

If \( v_0v_{2p_1-1}- \), then by our condition, \( d(v_0) + d(v_{2p_1-1}) \geq \nu/2 + 1 \) and hence
equalities in (2.1) hold. But then we must have $v_0v_{2p_1-1}+$, a contradiction. So $v_0v_{2p_1-1}+$, and we get a cycle $C_1 = v_0v_1 \ldots v_{2p_1-1}v_0$.

If $G_1 - C_1$ is nonempty, then there exists an edge $wb \in M \cap E(G_1 - C_1)$, where $w \in W$ and $b \in B$. By the choice of $P_1$, $e(V(C_1), V(G_1 - C_1)) = 0$. By our condition, $d(w) + d(b) + d(v_0) + d(v_{2p_1-1}) \geq 2(v/2 + 1) = v + 2$. However, by Lemma 2.13, $|N_C(w)| + |N_C(b)| \leq m$, and hence $d(w) + d(b) \leq |G_1| - 2p_1 + m$, while $d(v_0) + d(v_{2p_1-1}) \leq m + p_1 + 1$ by (2.1), therefore $d(w) + d(b) + d(v_0) + d(v_{2p_1-1}) \leq v + 2p_1 + 2 + 1 = v + p_1 + 1 < v + 1$, a contradiction. Hence, $G_1 - C_1$ must be empty. Then $|G_1| = 2p_1$ and $C_1$ is an $M$-alternating Hamiltonian cycle of $G_1$.

We claim that every vertex of $G_1$ sends some edges to $C$. Let $v$ be any vertex in $G_1$. Since $G_1$ has an $M$-alternating Hamiltonian cycle $C_1$, we can choose a closed $M$-alternating Hamiltonian path $P_1$ of $G_1$ starting from $v$. By the above discussion, $v$ sends some edges to $C$.

For a longest $M$-alternating cycle $C$ in $G$, we call the graph $G_1 = G - C$ a critical graph (with respect to $C$) and a closed $M$-alternating Hamiltonian path of $G_1$, $P_1 = v_0v_1 \ldots v_{2p_1-1}$, where $v_{2i} \in W$ and $v_{2i+1} \in B$, a critical path, or a critical edge if $|P_1| = 2$. For a critical path $P_1$, we can always find $u_{2s-1} \in N_C(v_0)$ and $u_{2r} \in N_C(v_{2p_1-1})$, such that $P_2 = u_{2s}C^+u_{2r-1}$ is the shortest. We let $R = u_{2s}C^+u_{2s-1}$.

By Lemma 2.14, $u_{2s} \to G_1$ and $u_{2r-1} \to G_1$. Further, for any edge $u_{2i-1}u_{2i}$ on $R$, we must have $e(\{u_{2i-1}, u_{2i}\}, \{u_{2s}, u_{2r-1}\}) \leq 1$, or we get an $M$-alternating Hamiltonian cycle

\[ u_{2r}C^+u_{2i-1}u_{2s}C^+u_{2r-1}u_{2i}C^+u_{2s-1}v_0P_1v_{2p_1-1}u_{2r}. \]

Hence,

\[ d(u_{2s}) + d(u_{2r-1}) \leq |P_2| + 2 + (|R| - 2)/2 = |P_2| + |R|/2 + 1. \tag{2.2} \]

Moreover,

\[ d(v_0) + d(v_{2p_1-1}) \leq 2p_1 + 2 + (|R| - 2)/2 = 2p_1 + |R|/2 + 1. \tag{2.3} \]
So,

\[ d(u_{2s}) + d(u_{2r-1}) + d(v_0) + d(v_{2p_1-1}) \leq 2p_1 + |P_2| + |R| + 2 = \nu + 2(2.4) \]

However \( v_0u_{2r-1} - \) and \( v_{2p_1-1}u_{2s} - \), and by our condition,

\[ d(u_{2s}) + d(u_{2r-1}) + d(v_0) + d(v_{2p_1-1}) \geq 2(\nu/2 + 1) = \nu + 2. \tag{2.5} \]

So all equalities in (2.2), (2.3), (2.4) and (2.5) must hold. To get equality in (2.3), \( v_0 \) (respectively \( v_{2p_1-1} \)) must be adjacent to all vertices in \( V(G_1) \cap B \) (respectively \( V(G_1) \cap W \)), and for every edge \( u_{2i-1}u_{2i} \) on \( R \), we have \( e(\{u_{2i-1}, u_{2i}\}, \{v_0, v_{2p_1-1}\}) = 1 \). Thus, for a critical path \( P_1 = v_0v_1 \ldots v_{2p_1-1} \), we find two closed \( M \)-alternating paths \( R \) and \( P_2 \) as segments of \( C \), such that \( V(C) = V(R) \cup V(P_2) \), where the ending vertices of \( R \) are adjacent to \( v_0 \) and \( v_{2p_1-1} \), respectively, and for every edge \( u_{2i-1}u_{2i} \notin M \) on \( R \), we have \( e(\{u_{2i-1}, u_{2i}\}, \{v_0, v_{2p_1-1}\}) = 1 \), while \( e(V(P_2), \{v_0, v_{2p_1-1}\}) = 0 \). We call \( P_2 \) the opposite path, and \( R \) the central path for \( P_1 \).

Furthermore, to get equality in (2.2), \( u_{2s} \) (respectively \( u_{2r-1} \)) must be adjacent to all vertices in \( V(P_2) \cap B \) (respectively \( V(P_2) \cap W \)). In particular \( u_{2s}u_{2r-1} - \).

**Claim 2.2.** A critical graph \( G_1 \) is complete bipartite.

**Proof.** Since \( C_1 \) is an \( M \)-alternating Hamiltonian cycle of \( G_1 \), for any vertex \( v \in V(G_1) \), \( P_1 \) can be chosen so that it is starting from \( v \). By the equality of (2.3), \( v \) sends edges to every vertex in the opposite part of \( G_1 \). \( \square \)

Let \( G_2 = G[V(P_2)] \). We call \( G_2 \) the opposite graph. We choose \( C, G_1 \) and \( P_1 \) so that the opposite path \( P_2 \) is the shortest.

**Claim 2.3.** \( e(V(G_1), V(G_2)) = 0 \), and \( u_{2s-1} \) (respectively \( u_{2r} \)) is adjacent to every vertex in \( V(G_1) \cap W \) (respectively \( V(G_1) \cap B \)).

**Proof.** If \( |G_1| = 2 \) the conclusion holds. We assume that \( |G_1| \geq 4 \).

For any closed \( M \)-alternating Hamiltonian path \( P'_1 \) of \( G_1 \) with ending vertices \( w \in W \) and \( b \in B \), we can find an opposite path \( P'_2 \) and a central path \( R' \)
for $P'_1$. Since $P_2$ is chosen as the shortest, $|P'_2| \geq |P_2|$ and $|R'| \leq |R|$. Similar to (2.3) we have

$$d(w) + d(b) \leq 2p_1 + |R'|/2 + 1 \leq 2p_1 + |R|/2 + 1.$$  

(2.6)

Together with (2.2), we have

$$d(u_{2s}) + d(u_{2r-1}) + d(w) + d(b) \leq \nu + 2.$$  

(2.7)

Since $u_{2r}$ and $u_{2s-1}$ send edges to $G_1$, which has an $M$-alternating Hamiltonian cycle, by Lemma 2.14, $u_{2r-1} \mapsto G_1$ and $u_{2s} \mapsto G_1$, and hence $wu_{2r-1}u_{2s}$ and $bu_{2s-1}u_{2s}$. By the condition given,

$$d(u_{2s}) + d(u_{2r-1}) + d(w) + d(b) \geq 2(\nu/2 + 1) = \nu + 2.$$  

(2.8)

Hence all equalities in (2.6), (2.7) and (2.8) must hold. Therefore $|R| = |R'|$, $|P'_2| = |P_2|$, $d(w) = d(v_0) = \nu/2 + 1 - d(u_{2r-1})$ and $d(b) = d(v_{2p_1-1}) = \nu/2 + 1 - d(u_{2s})$. In other words, all opposite paths (respectively all central paths) have the same length. Since any vertex in $G_1$ can be an ending vertex of an $M$-alternating Hamiltonian path, all vertices in $V(G_1) \cap W$ have the same degree $\nu/2 + 1 - d(u_{2r-1})$, and all vertices in $V(G_1) \cap B$ have the same degree $\nu/2 + 1 - d(u_{2s})$.

Let $b \neq v_{2p_1-1}$ be a vertex in $V(G_1) \cap B$, and assume that $b$ has a neighbor $u_{2r'}$ on $P_2$. Since $G_1$ is complete bipartite we can always find a closed $M$-alternating path $P''_1$ connecting $v_0$ and $b$ in $G_1$. (Note that $P''_1$ need not be Hamiltonian. If $b = v_1$, $P''_1$ can only be the edge $v_0v_1$.) Let $P''_2 = u_{2s}C^+u_{2r'-1}$ and $R'' = u_{2r'}C^+u_{2s-1}$. For any vertex pair $\{u_{2i-1}, u_{2i}\}$ on the path $R''$, we have $e(\{u_{2i-1}, u_{2i}\}, \{u_{2s}, u_{2r'-1}\}) \leq 1$, or we get an $M$-alternating cycle

$$u_{2r'}C^+u_{2i-1}u_{2s}C^+u_{2r'-1}u_{2i}C^+u_{2s-1}v_0P''_1bu_{2r'},$$

which is longer than $C$. Therefore,

$$d(u_{2s}) + d(u_{2r'-1}) \leq |P''_2| + 2 + (|R''| - 2)/2 = |P''_2| + |R''|/2 + 1 < |P_2| + |R|/2 + 1.$$  

By $d(v_0) + d(b) = d(v_0) + d(v_{2p_1-1}) = 2p_1 + |R|/2 + 1$, we have $d(u_{2s}) + d(u_{2r'-1}) + d(v_0) + d(b) < (|P_2| + |R|/2 + 1) + (2p_1 + |R|/2 + 1) = \nu + 2$. However, since $u_{2r}b$ and $u_{2r'-1}v_0$, by our condition, $d(u_{2s}) + d(u_{2r'-1}) + d(v_0) + d(b) \geq \nu + 2$, a contradiction. Hence $b$, and similarly any $w \in V(G_1) \cap W$, must not have any neighbor on $P_2$. That is, $e(V(G_1), V(G_2)) = 0.$
For any closed $M$-alternating Hamiltonian path $P'_1$ of $G_1$ with ending vertices $w \in W$ and $b \in B$, let $P'_2$ be an opposite path of it. Since $w$ and $b$ send no edges to $P_2$, $P_2$ must be part of $P'_2$. However, all opposite paths have the same length, so $|P'_2| = |P_2|$, and therefore $P'_2 = P_2$. Then, $wu_{2s-1}^+$ and $bu_{2r}^+$. Since any vertex in $G_1$ can be an ending vertex of a closed $M$-alternating Hamiltonian path of $G_1$, this completes the proof of the second part of the claim.

Claim 2.4. $G_2$ is complete bipartite, and $u_{2s-1}$ (respectively $u_{2r}$) is adjacent to every vertex in $V(G_2) \cap W$ (respectively $V(G_2) \cap B$).

Proof. By the above discussions, $u_{2s}u_{2r-1}^+$ and thus we have a cycle $C_2 = u_{2s}C^+u_{2r-1}u_{2s}$. Since $e(V(G_1), V(G_2)) = 0$, for every edge $u_{2j-1}u_{2j}$ on $P_2$, where $s + 1 \leq j \leq r - 1$, we can replace $u_{2j-1}$ with $u_{2s}$ and $u_{2s}$ with $u_{2j}$ in (2.2), (2.4) and (2.5), and all equalities must hold. So, $u_{2j-1}$ (respectively $u_{2j}$) must be adjacent to all vertices in $V(P_2) \cap W$ (respectively $V(P_2) \cap B$), $u_{2j-1}u_{2r}$ and $u_{2j}u_{2s-1}^+$, therefore the claim holds.

For convenience we change some notations henceforth. We let $|G_2| = 2p_2$ and the vertices of $G_2$ be $v_0', v_1', \ldots, v_{2p_2-1}'$, where $v_{2j}'v_{2j+1}' \in M$, for $0 \leq j \leq p_2 - 1$, and we let $R = u_0u_1 \ldots u_{2r-1}$.

Now we discuss the situations case by case, with respect to the length of $R$ and the distribution of edges between $R$ and $G_i$, $i = 1, 2$.

Case 1. $|R| = 2$.

Then $R = u_0u_1$. By Claim 2.3 and Claim 2.4, for any $0 \leq i \leq p_1 - 1$ and $0 \leq j \leq p_2 - 1$, $u_0v_{2i+1}^+$, $u_0v_{2j+1}'$, $u_1v_{2i}^+$ and $u_1v_{2j}'$. Therefore $G \in \mathcal{G}_1$ and $M \in \mathcal{M}_1$.

Case 2. $|R| \geq 4$.

Claim 2.5. For $j = 1, 2$, and every edge $u_{2i-1}u_{2i}$, $1 \leq i \leq r - 1$, exactly one of $u_{2i-1} \rightarrow G_j$ and $u_{2i} \rightarrow G_j$ holds. Furthermore, if $u_{2i-1} \rightarrow G_j$ (respectively $u_{2i} \rightarrow G_j$), it is adjacent to all vertices in $V(G_j) \cap W$ (respectively $V(G_j) \cap B$).

Proof. Firstly, we prove that for $j = 1, 2$ and every edge $u_{2i-1}u_{2i}$, $1 \leq i \leq r - 1$, $u_{2i-1} \rightarrow G_j$ or $u_{2i} \rightarrow G_j$. By Lemma 2.14, the conclusion holds for $G_1$. Now we prove it for $G_2$. Suppose to the contrary that there exists $1 \leq l \leq r - 1$ such that $u_{2l-1} \rightarrow G_2$ and $u_{2l} \rightarrow G_2$, and let $v_{2l}' \in N_{G_2}(u_{2l-1})$ and $v_{2l+1}' \in N_{G_2}(u_{2l})$. If $|G_2| = 2$ or $t \neq s$, we can find a closed $M$-alternating Hamiltonian
path $Q$ of $G_2$ connecting $v'_2$, and $v'_{2r-1}$, and hence we have an $M$-alternating Hamiltonian cycle

$$u_0Ru_{2l-1}v'_2Qv'_{2l-1}u_{2l}Ru_{2r-1}v_0P_1v_{2p_1-1}u_0$$

of $G$, contradicting our assumption. If $|G_2| \geq 4$ and $t = s$, let $P'_2$ be a closed $M$-alternating Hamiltonian path of $G_2 - \{v'_2, v'_{2+1}\}$. Then $P'_2$ is an opposite path for $P_1$, with the central path $u_0Ru_{2l-1}v'_2v'_{2l+1}u_{2l}Ru_{2r-1}$, which is shorter than $P_2$, contradicting our choice of $P_2$. Hence $u_{2l-1} \Rightarrow G_2$ or $u_{2i} \Rightarrow G_2$, for $1 \leq i \leq r - 1$.

Arbitrarily choose $0 \leq l \leq p_1 - 1$ and $0 \leq k \leq p_2 - 1$. We have $d(v_{2l}) + d(v_{2l+1}) \leq 2p_1 + 2 + (|R| - 2)/2 = 2p_1 + r + 1$ and similarly $d(v'_{2k}) + d(v'_{2k+1}) \leq 2p_2 + r + 1$. So

$$d(v_{2l}) + d(v_{2l+1}) + d(v'_{2k}) + d(v'_{2k+1}) \leq 2p_1 + 2p_2 + 2r + 2 = \nu + 2. \quad (2.9)$$

However, $v_{2l}v'_{2k+1}$ and $v_{2l+1}v'_{2k}$, and by the condition of the theorem,

$$d(v_{2l}) + d(v'_{2k+1}) + d(v_{2l+1}) + d(v'_{2k}) \geq 2(\nu/2 + 1) = \nu + 2, \quad (2.10)$$

and all equalities must hold. To obtain equalities, for $j = 1, 2$, and every edge $u_{2l-1}u_{2l}$, $1 \leq i \leq r - 1$, exactly one of $u_{2i-1} \Rightarrow G_j$ and $u_{2i} \Rightarrow G_j$ must hold. Furthermore, since $l$ and $k$ are arbitrarily chosen, this proves that if $u_{2l-1} \rightarrow G_j$ (respectively $u_{2i} \rightarrow G_j$), it is adjacent to all vertices in $V(G_j) \cap W$ (respectively $V(G_j) \cap B$).

Let $1 \leq i \leq r - 1$. We define $E_1$ ($E'_i$) to be the set of edges $u_{2i-1}u_{2i}$, where $u_{2i-1}v_{2j}$, for every $0 \leq j \leq p_1 - 1$ ($u_{2i-1}v'_{2k}$, for every $0 \leq k \leq p_2 - 1$), and $E_2$ ($E'_2$) to be the set of edges $u_{2i-1}u_{2i}$, where $u_{2i}v_{2j+1}$, for every $0 \leq j \leq p_1 - 1$ ($u_{2i}v'_{2k+1}$, for every $0 \leq k \leq p_2 - 1$).

By Claim 2.5, for every $1 \leq i \leq r - 1$, $u_{2i-1}u_{2i} \in E_1 \cap E'_1$, $E_1 \cap E'_2$, $E_2 \cap E'_1$ or $E_2 \cap E'_2$. Accordingly, we say that $u_{2i-1}u_{2i}$ is an edge of type I, II, III or IV for $G_1$, $G_2$ and $R$. Let the number of edges $u_{2i-1}u_{2i}$ belonging to $E_1 \cap E'_1$, $E_1 \cap E'_2$, $E_2 \cap E'_1$ and $E_2 \cap E'_2$ be $t_{11}$, $t_{12}$, $t_{21}$ and $t_{22}$, respectively. We have $d(v_0) = t_{11} + t_{12} + p_1 + 1$, $d(v_1) = t_{22} + t_{21} + p_1 + 1$, $d(v'_0) = t_{11} + t_{21} + p_2 + 1$ and $d(v'_1) = t_{22} + t_{12} + p_2 + 1$.

Since equalities hold in (2.9) and (2.10), we have $d(v_{2l}) + d(v'_{2k+1}) = d(v_{2l+1}) + d(v'_{2k}) = \nu/2 + 1$ for any $0 \leq l \leq p_1 - 1$ and $0 \leq k \leq p_2 - 1$. 
Hence,

\[
t_{11} + t_{22} + 2t_{12} + p_1 + p_2 + 2 = d(v_0) + d(v'_1)
= \nu/2 + 1
= d(v_1) + d(v'_0)
= t_{11} + t_{22} + 2t_{21} + p_1 + p_2 + 2. \quad (2.11)
\]

So \( t_{12} = t_{21} \).

We let \( t_1 = t_{11}, t_2 = t_{22} \) and \( t_0 = t_{12} = t_{21} \). Then \( t_1 + t_2 + 2t_0 = r - 1 \).

We summarize some structural results in the form of observations.

**Observation 2.1.** If there exists \( 1 \leq j < i \leq r - 1 \), such that \( u_{2i-1}u_{2i} \in E_1 (E'_j) \) and \( u_{2j-1}u_{2j} \in E'_2 (E_2) \), then \( u_{2j-1}u_{2i} \).

**Proof.** If \( u_{2j-1}u_{2i+} \), we obtain an \( M \)-alternating Hamiltonian cycle

\[
(0)Ru_{2j-1}u_{2i}R_{u_{2i-1}v_0'}P_2v_{2p_2-1}u_{2j}Ru_{2i-1}v_0P_1v_{2p_1-1}u_0,
\]
contradicting our assumption.

**Observation 2.2.** If there exists \( 1 \leq i \leq r - 2 \), such that \( u_{2i-1}u_{2i} \in E_1 \) and \( u_{2i+1}u_{2i+2} \in E_2 \), then \( u_{2i}u_{2i+1} \) is a critical edge, \( |G_1| = |G_2| \) and exactly one of \( u_{2i+1}v_0' \) and \( u_{2i+1}u_0+ \) \( u_{2i+1}v_0' \) and \( u_{2i+1}u_0+ \) holds.

If there exists \( 1 \leq i \leq r - 2 \), such that \( u_{2i-1}u_{2i} \in E_1' \) and \( u_{2i+1}u_{2i+2} \in E_2' \), then \( u_{2i}u_{2i+1} \) is a critical edge, \( |G_1| \) and exactly one of \( u_{2i+1}v_0+ \) \( u_{2i+1}v_0' \) and \( u_{2i+1}u_0+ \) holds.

**Proof.** Suppose there exists \( 1 \leq i \leq r - 2 \), such that \( u_{2i-1}u_{2i} \in E_1 \) and \( u_{2i+1}u_{2i+2} \in E_2 \). Then \( u_{2i}u_{2i+1} \) is a critical edge with respect to the \( M \)-alternating cycle

\[
u_0Ru_{2i-1}v_0P_1v_{2p_1-1}u_{2i+2}Ru_{2i-1}v'_0P_2v_{2p_2-1}u_0,
\]
where \( P_1 \) is an opposite path. Since \( G_1 \) is critical, \( |G_1| = 2 \). Since \( |P_1| = 2 \), and \( P_2 \) is the shortest opposite path, \( |G_2| = 2 \). Since \( u_0v'_1 \) \( (u_{2r-1}v_0') \) are on a central path for the critical edge \( u_{2i}u_{2i+1} \) and the opposite path \( v_0v_1 \), exactly one of \( u_{2i+1}u_0+ \) \( u_{2i+1}v_0' \) and \( u_{2i+1}u_0+ \) holds.
Now suppose there exists $1 \leq i \leq r - 2$, such that $u_{2i-1}u_{2i} \in E'_1$ and $u_{2i+1}u_{2i+2} \in E'_2$. Then $u_{2i}u_{2i+1}$ is a critical edge with respect to the $M$-alternating cycle

$$u_0Ru_{2i-1}v'_0P_2v'_2P_2'v_{2p_2-1}u_{2i+2}Ru_{2i-1}v_0P_1v_{2p_1-1}u_0,$$

where $P_2'$ is an opposite path. Since $G_1$ is critical, $|G_1| = 2$. Since $u_0v_1$ ($u_{2r-1}v_0$) are on a central path for the critical edge $u_{2i}u_{2i+1}$ and the opposite path $P_2$, exactly one of $u_{2i+1}v_0$ and $u_2v_1$ ($u_{2i+1}v_0$ and $u_{2i}v_{2r-1}$) holds.

**Observation 2.3.** If there exists $1 \leq i < k < j \leq r - 1$, such that $u_{2i-1}u_{2i} \in E_1 (E'_1)$, $u_{2j-1}u_{2j} \in E_2 (E'_2)$, $u_{2k-1}u_{2k} \in E'_1 (E_2')$ and $u_{2k-1}v_0 +$, then $u_{2i}u_{2j-1}$.

**Proof.** Then for $1 \leq i < k < j \leq r - 1$, such that $u_{2i-1}u_{2i} \in E_1 (E'_1)$, $u_{2j-1}u_{2j} \in E_2 (E'_2)$, $u_{2k-1}u_{2k} \in E'_1 (E_2')$ and $u_{2k-1}v_0 +$, then $u_{2i}u_{2j-1}$.

By symmetry, the claim holds in the other situation.

By symmetry, the claim holds in the other situation.

**Claim 2.6.** $|G_1| = 2$.

**Proof.** Suppose $|G_1| \geq 4$. By Observation 2.2, there does not exist $1 \leq i \leq r - 1$, such that $u_{2i-1}u_{2i} \in E_1 (E'_1)$ and $u_{2i+1}u_{2i+2} \in E_2 (E'_2)$. Therefore, there can not exist $i < j$, such that $u_{2i-1}u_{2i} \in E_1 (E'_1)$ and $u_{2j-1}u_{2j} \in E_2 (E'_2)$. In other words, there exists an integer $0 \leq k_1 \leq r - 1$ ($0 \leq k_2 \leq r - 1$), such that for all $i \leq k_1 (j \leq k_2)$, $u_{2i-1}u_{2i} \in E_2 (u_{2j-1}u_{2j} \in E'_2)$ and for all $i > k_1 (j > k_2)$, $u_{2i-1}u_{2i} \in E_1 (u_{2j-1}u_{2j} \in E'_1)$. It is easily seen that $t_0 = 0$ and $k_1 = k_2$. We let $k = k_1 = k_2$.

Suppose that $t_1$, $t_2 \neq 0$, or equivalently, $1 \leq k \leq r - 2$. Consider the vertices $u_{2k-1}$ and $u_{2k+2}$. By Observation 2.1, for all $j \geq k + 1$, $u_{2k-1}u_{2j-1}$, and for all $j \leq k$, $u_{2k+2}u_{2j-1}$. Particularly, $u_{2k-1}u_{2k+2}$. But then we have $d(u_{2k-1}) \leq k + 1$, $d(u_{2k+2}) \leq r - k$ and $d(u_{2k-1}) + d(u_{2k+2}) \leq r + 1 < \nu/2 + 1$, contradicting our condition.

Suppose one of $t_1$ and $t_2$, say $t_1 = 0$. Then for $1 \leq i \leq r - 1$, $d(u_{2i-1}) \leq r$. Moreover $d(v_0) = p_1 + 1$, so $d(u_{2i-1}) + d(v_0) \leq r + p_1 + 1 < \nu/2 + 1$, but $v_0u_{2r-1}$, a contradiction.

So we must have $|G_1| = 2$. 

\[ \square \]
Claim 2.7. Either \( t_0 = 0 \), or \( t_1 = t_2 = 0 \).

**Proof.** Suppose that \( t_0 > 0 \), and one of \( t_1 \) and \( t_2 \) is greater than 0. Without lost of generality, we may assume that \( t_1 \geq t_2 \), and so \( t_1 > 0 \).

Let \( u_{2i-1} u_{2i} \in E_1 \cap E'_1 \), \( 1 \leq i \leq r-1 \), be such that \( i \) is the maximum. Then by our condition, \( d(u_{2i}) + d(v_1) \geq \nu/2 + 1 \). Hence, \( d(u_{2i}) \geq \nu/2 + 1 - d(v_1) = \nu/2 + 1 - (t_2 + t_0 + 2) = t_1 + t_0 + \nu/2 - r \). By Observation 2.1, \( u_{2i} \) can not be adjacent to any \( u_{2j-1} \), where \( u_{2j-1} u_{2j} \in E_2 \cup E'_2 \) and \( j < i \). Hence \( u_{2i} \) sends at least \( t_0 + \nu/2 - r - (t_1 + 1) = t_0 + \nu/2 - r - 1 \) edges to \( \{u_{2r-1}\} \cup \{u_{2j-1} : u_{2j-1} u_{2j} \in E_2 \cup E'_2, j > i+1\} \). Since \( t_0 > 0 \) and \( \nu/2 - r \geq 2 \), \( u_{2i} \to \{u_{2j-1} : u_{2j-1} u_{2j} \in E_2 \cup E'_2, j > i + 1\} \), so there exists at least one \( u_{2j-1} u_{2j} \) such that \( j > i + 1 \) and \( u_{2j-1} u_{2j} \in E_2 \cup E'_2 \).

By our choice of \( u_{2i-1} u_{2i}, u_{2i+1} u_{2i+2} \in E_2 \cup E'_2 \). If \( u_{2i+1} u_{2i+2} \in E_2 \), then by Observation 2.2, \( u_{2i+1} u_{2i+1} \) is a critical edge, and exactly one of \( u_{2i} v_1 + 1 \) and \( u_{2i+1} u_0 + 1 \) holds. By \( u_{2i-1} u_{2i} \in E'_1 \) we have \( u_{2i} v_1 + 1 \), therefore \( u_{2i+1} u_0 + 1 \). If \( u_{2i+1} u_{2i+2} \in E'_2 \), then again by Observation 2.2, \( u_{2i+1} u_{2i+1} \) is a critical edge, and exactly one of \( u_{2i} v_1 + 1 \) and \( u_{2i+1} u_0 + 1 \) holds. By \( u_{2i-1} u_{2i} \in E_1 \) we have \( u_{2i} v_1 + 1 \), hence \( u_{2i+1} u_0 + 1 \).

Now we discuss different situations of \( u_{2i+1} u_{2i+2} \).

- If \( u_{2i+1} u_{2i+2} \in E_2 \cap E'_2 \), let \( j > i + 1 \) be such that \( u_{2i} u_{2j-1} + 1, u_{2j-1} u_{2j} \in E_2 \cup E'_2 \). By Observation 2.3, \( u_{2i+2} u_{2j-1} + 1 \), a contradiction.

- If \( u_{2i+1} u_{2i+2} \in E_1 \cap E'_2 \) or \( E_2 \cap E'_1 \), without lost of generality, we may assume that \( u_{2i+1} u_{2i+2} \in E_1 \cap E'_2 \) since \( u_{2i+1} u_{2i+1} \) is a critical edge and \( u_{2i+1} u_0 + 1 \), by Observation 2.2, we have \( u_{2i} u_{2j-1} + 1 \). For \( j > i + 1 \), where \( u_{2j-1} u_{2j} \in E_2 \), by Observation 2.3, \( u_{2j-1} u_{2j} \). Therefore \( u_{2i} \) sends at least \( t_0 + \nu/2 - 1 - r \geq t_0 + 1 \) edges to \( \{u_{2j-1} : u_{2j-1} u_{2j} \in E_1 \cap E'_2, j > i + 1\} \). However, the number of such \( u_{2j-1} \) is at most \( t_0 \), a contradiction.

**Case 2.1.** \( t_0 = 0 \).

Without lost of generality, we may assume that \( t_1 > 0 \), and let \( u_{2i-1} u_{2i} \in E_1 \cap E'_1 \).

- If there exists \( u_{2j-1} u_{2j}, j < i \), such that \( u_{2j-1} u_{2j} \in E_2 \cap E'_2 \), then \( u_{2j-1} u_{2i} \) by Observation 2.1.

- If there exists \( u_{2j-1} u_{2j}, j > i + 1 \), such that \( u_{2j-1} u_{2j} \in E_2 \cap E'_2 \), then there exists \( i \leq k \leq j - 1 \), such that \( u_{2k-1} u_{2k} \in E_1 \cap E'_1 \) and \( u_{2k+1} u_{2k+2} \in E_2 \cap E'_2 \). By Observation 2.2, \( u_{2k} u_{2k+1} \) is a critical edge, and since \( u_{2k+1} u_0 \) and
We have $u_{2k}v_1$, and we have $u_{2k}u_{2r-1}^+$ and $u_{2k+1}u_0^+$. By Observation 2.3, $u_{2i}u_{2j-1}^-$. Hence, for all $u_{2j-1}u_{2j} \in E_2 \cap E_2'$, $j \neq i+1$, $u_{2i}u_{2j-1}^-$. So, $d(u_{2i}) \leq t_1 + 2$. But then

$$\nu/2 + 1 \leq d(u_{2i}) + d(v_1) \leq t_1 + t_2 + 4 = (\nu - 2p_2 - 4)/2 + 4 = \nu/2 - p_2 + 2.$$ 

Since $p_2 \geq 1$, all equalities must hold, hence $p_2 = 1$ and $2r - 1 = \nu - 5$. Furthermore, to get $d(u_{2i}) = t_1 + 2$, we must have the following.

(a) $u_{2i+1}u_2, \nu \neq u_{2i-7}u_{\nu-6}$.
(b) $u_{2i}u_{2j-1}^-$, for all $u_{2j-1}u_{2j} \in E_1 \cap E_1'$.
(c) $u_2, u_{\nu-5}^+$.

By (a), $t_2 \geq 0$, and similarly, for any $u_{2i-1}u_{2i} \in E_2 \cap E_2'$, we can prove the following.

(d) $u_{2i-3}u_{2i-2} \in E_1 \cap E_1'$, hence $u_{2i-1}u_{2i} \neq u_1u_2$.
(e) $u_{2i-1}u_{2j}^+$, for all $u_{2j-1}u_{2j} \in E_2 \cap E_2'$.
(f) $u_{2i-1}u_0^+$.

So, the edges $u_{2i-1}u_{2i}$, $1 \leq i \leq \nu/2 - 3$, belong to $E_1 \cap E_1'$ and $E_2 \cap E_2'$ alternately. Moreover, $u_1u_2 \in E_1 \cap E_1'$ and $u_{\nu-7}u_{\nu-6} \in E_2 \cap E_2'$, $u_{4j+1}u_{4j+2}u_{E_1} \subseteq E_1 \cap E_1'$ and $u_{4j+1}u_{4j+2}u_{E_1} \subseteq E_2 \cap E_2'$, for $0 \leq j \leq n-2$. The vertex set $\{u_{4j+1}, u_{4j+2} : 0 \leq j \leq n-2\} \cup \{v_0, v_0', u_{4n-3}\}$, as well as $\{u_{4j+3}, u_{4j+4} : 0 \leq j \leq n-2\} \cup \{v_1, v_1', u_0\}$, induce complete bipartite subgraphs, respectively.

Let $B_1 = \{u_{4j+1} : 0 \leq j \leq n-1\}$, $W = \{u_{4j+2} : 0 \leq j \leq n-2\} \cup \{v_0, v_0'\}$, $B = \{u_{4j+3} : 0 \leq j \leq n-2\} \cup \{v_1, v_1'\}$ and $W_1 = \{u_{4j} : 0 \leq j \leq n-1\}$. By the above discussion, there can be no more edge between $B$ and $W$. But we can add edges between $B_1$ and $W_1$ freely, to obtain all graphs $G \in G_2$, with $M \in M_2$.

**Case 2.2.** $t_1 = t_2 = 0$. Since $t_1 + t_2 + 2t_0 = r - 1$, we have $r = 2t_0 + 1$ and $r$ must be odd.

If there exists $1 \leq i \leq r - 2$, such that $u_{2i-1}u_{2i} \in E_1 \cap E_1'$ and $u_{2i+1}u_{2i+2} \in E_2 \cap E_2'$, $(u_{2i-1}u_{2i} \in E_2 \cap E_2'$ and $u_{2i+1}u_{2i+2} \in E_1 \cap E_1')$, we say that an A-change (B-change) occurs at $u_{2i-1}$. If there exist $i$ and $j$, such that $2 \leq
i + 1 < j ≤ r − 2, and there is an A-change (B-change) occurring at \( u_{2i-1} \) and a B-change (A-change) occurring at \( u_{2j-1} \), we say that a change couple occurs at \((u_{2i-1}, u_{2j-1})\).

**Case 2.2.1.** \(|G_2| \geq 4\).

There can not be any A-change, or by Observation 2.2, \(|G_1| = |G_2| = 2\). To avoid any A-change, for \(1 \leq i \leq (r - 1)/2\), \(u_{2i-1}u_{2i} \in E_2 \cap E'_1\) and for \((r + 1)/2 \leq i \leq r - 1\), \(u_{2i-1}u_{2i} \in E_1 \cap E'_2\).

Suppose that \(r = 3\). It is not hard to see that \(u_0u_3-\) and \(u_2u_5-\), while each of \(u_0u_5\) and \(u_1u_4\) can exist or not. Hence we obtain all the graphs in the class \( \mathcal{G}_3 \), except those with \(n = 1\).

If \(r \geq 5\), then \(u_{r-1}u_r\) becomes a critical edge, with the central path \(u_{r+1}Ru_{r-1}v_0v_1u_0Ru_{r-2}\) and the opposite graph \(G_2\) (Figure 2.6). Consider the edge \(v_1u_0\) and \(u_1u_2\). We have \(v_1u_{r-1}+, u_0 \rightarrow G_2, u_1 \rightarrow G_2\), and by Claim 2.7, \(u_2u_{r+}\). But then an A-change occurs at \(v_1\), a contradiction.

![Figure 2.6: Contradiction in Case 2.2.1](image)

**Case 2.2.2.** \(|G_2| = 2\).

Then \(\nu = 4n + 6\), for some \(n \geq 1\). For \(n = 1\), it is not hard to verify that \(G \in \mathcal{G}_3, M \in \mathcal{M}_3\), and we obtain all graphs in \(\mathcal{G}_3\) together with Case 2.2.1. For \(n = 2\), it can be checked that \(G = G_4, M = M_4\). Henceforth, we assume that \(n \geq 3\), and then \(r = 2n + 1 \geq 7\).

We call \(G_1\) and \(G_2\) a critical edge pair with central path \(R\). Since we have discussed all other cases, we may assume that for every critical edge pair and the central path, every edge of the central path that is not in \(M\) is of type II.
or III.

Let there be a change couple occurring at \((u_{2i-1}, u_{2j-1})\). Without lost of generality, suppose that an A-change occurs at \(u_{2i-1}\) and a B-change occurs at \(u_{2j-1}\). Then \(u_{2i}u_{2i+1}\) and \(u_{2j}u_{2j+1}\) are critical edges. Since \(u_{2i}u_{2i+1}\) and \(v_1v_0\) is a critical edge pair, with the central path \(u_{2i+2}Ru_{2r-1}v_0'v_1'v_0Ru_{2i-1}\), by our assumption, \(u_{2j-1}u_{2j}\) and \(u_{2j-1}u_{2j+2}\) are of type II or III. By \(u_{2j}v_1+1\) and \(u_{2j+1}v_0+\), we have \(u_{2j-1}u_{2j+1}\) and \(u_{2j}u_{2j+1}\). Similarly, we have \(u_{2i-1}u_{2j}\) and \(u_{2i+1}u_{2j+1}\). However, we get an \(M\)-alternating Hamiltonian cycle

\[u_0Ru_{2i-1}u_{2i}u_{2i+1}Ru_{2j-1}u_{2j}u_{2j+1}v_0v_1Ru_{2r-1}v_0v_1u_0\]

then, a contradiction. Therefore, there cannot be any change couple.

By symmetry, we may assume that \(u_1u_2 \in E_1 \cap E'_2\), and let \(r_0 > 0, r_1 > r_0\) and \(r_2 \geq r_1\) be such that \(u_1u_2, \ldots, u_{2r_0-2}u_{2r_0} \in E_1 \cap E'_2, u_{2r_0+1}u_{2r_0+2}, \ldots, u_{2r_1-1}u_{2r_1} \in E_2 \cap E'_1, u_{2r_1+1}u_{2r_1+2}, \ldots, u_{2r_2-1}u_{2r_2} \in E_1 \cap E'_2\) and if \(u_{2r_2+1}u_{2r_2+2} \in E_2 \cap E'_1\).

If \(r_1 - r_0 \geq 2\) and \(r_2 - r_1 \geq 1\), then a change couple occurs at \((u_{2r_0-1}, u_{2r_1-1})\), a contradiction. Hence, \(r_1 - r_0 = 1\) or \(r_2 = r_1\).

If \(r_1 - r_0 = 1\), then \(r_2 > r_1\), and the edge \(u_{2r_2+1}u_{2r_2+2}\) exists. If \(r_2 - r_1 \geq 2\), a change couple occurs at \((u_{2r_1-1}, u_{2r_2-1})\), a contradiction. Therefore \(r_2 = r_1 + 1\). Moreover, if any B-change occurs at \(u_{2j-1}\) where \(j \geq r_2 + 1\), we obtain a change couple \((u_{2r_1-1}, u_{2j-1})\), again leading to a contradiction. Hence, we must have \(u_{2r_2+1}u_{2r_2+2}, \ldots, u_{2r_3}u_{2r_3-2} \in E_2 \cap E'_1\), and then \(r_0 = (r - 3)/2, r_1 = (r - 1)/2\) and \(r_2 = (r + 1)/2\).

Then we can see that \(u_{r+1}u_{r+2}\) and \(v_1v_0\) is a critical edge pair, with the central path \(u_{r+3}Ru_{2r-1}v_0'v_1'u_0Ru_r\). Again we may assume that the edges of the central path not in \(M\) are of type II or III. Consider the edges \(u_{r-4}u_{r-3}\) and \(u_{r-2}u_{r-1}\). Since \(u_{r-4}v_0+\) and \(u_{r-1}v_1+\), we must have \(u_{r-3}u_{r+1}\) and \(u_{r-2}u_{r+1}\). Since \(r \geq 7, 2r - 3 > r + 3\). Consider the edges \(u_{2r-3}u_{2r-2}\). Since \(v_1u_{2r-2}+\), we must have \(u_{2r-3}u_{r+1}+\). But then we find a change couple occurring at \((u_{2r-3}, u_{r-4})\), a contradiction (Figure 2.7).

Suppose \(r_2 = r_1\). Then we have \(u_1u_2, \ldots, u_{2r-2}u_{r-1} \in E_1 \cap E'_2\) and \(u_{r-1}u_{r+1}, u_{2r-3}u_{2r-2} \in E_2 \cap E'_1\). Then, \(u_{r-1}u_r\) and \(v_0v_1\) is a critical edge pair, with the central path \(u_{r+1}Ru_{2r-1}v_0'v_1'u_0Ru_{r-2}\). For the edges \(u_{2i-1}u_{2i}\)
2.4 Concluding Remarks

Most of the degree sum conditions for Hamiltonian problems deal with independent vertex sets. In our work, we tried to strengthen the condition of our main theorem, by replacing “for every vertex pair \( u \) and \( v \), where there is no arc from \( u \) to \( v \)” with “for every vertex pair \( u \) and \( v \)”. Naturally, if the former condition guarantees Hamiltonicity without exceptions, then such a strengthening brings nothing. But in the case where there are exceptions, we do find some differences. Let \( D' \) be a subset of \( D_1 \), in which \( n = m \). Let \( D'_4 \) be a subset of \( D_3 \), where \( n = 1 \). We obtain the following result.

**Theorem 2.15.** Let \( D \) be a digraph. If for every vertex pair \( u \) and \( v \), we have \( d^+(u) + d^-(v) \geq |D| - 1 \), then \( D \) has a directed Hamiltonian cycle, unless \( D \in D'_1, D_2 \) or \( D'_3 \), or \( D = D_4 \).

As a corollary, we can improve an Ore-type condition as well. Given an undirected graph \( G \), if we replace every edge \( uv \in E(G) \) with two arcs \( uv \) and \( vu \), we obtain a digraph \( D \). Applying Theorem 2.11 on \( D \), we obtain the following result.

Let \( n, m \geq 1 \), and \( G_5 \) be the set of graphs obtained by identifying one vertex of a complete graph \( K_{m+1} \) and one vertex of a complete graph \( K_{n+1} \),
where \( n, m \geq 1 \). Let \( G_6 \) be the set of all graphs obtained by joining every vertex of a graph \( I_{n+1} \) to every vertex of an arbitrary graph on \( n \) vertices.

**Corollary 2.16.** Let \( G \) be a graph. If for every distinct nonadjacent vertex pair \( u \) and \( v \), we have \( d(u) + d(v) \geq |G| - 1 \), then \( G \) has a Hamiltonian cycle, unless \( G \in G_5 \), or \( G \in G_6 \).

A slightly different result can be found in [78]. The result presented there involves only one exceptional class, but it considers only 2-connected graphs.

**Theorem 2.17.** (Li, Li and Feng, 2007) Let \( G \) be a 2-connected graph with \( |G| \geq 3 \). If \( d(u) + d(v) \geq |G| - 1 \) for every pair of vertices \( u \) and \( v \) with \( d(u,v) = 2 \), then \( G \) has a Hamiltonian cycle, unless \( |G| \) is odd and \( G \in G_6 \).

Stimulated by the above results, we conjecture that the lower bound on the degree sum in the following result can be reduced by 1, with some exceptional cases that can be characterized in a similar way as in the presented results of this chapter.

**Theorem 2.18.** (Bang-Jensen, Gutin and Li, 1996 [14]) Let \( D \) be a strong digraph such that for every pair of dominating non-adjacent and every pair of dominated non-adjacent vertices \( \{u,v\} \), we have \( \min\{d^+(u) + d^-(v), d^-(u) + d^+(v)\} \geq |D| \). Then \( D \) has a directed Hamiltonian cycle.
Chapter 3

Extremal and degree conditions for path extendability in digraphs

3.1 Introduction, terminology and notation

Cycle and path properties of graphs and digraphs have been studied extensively since the early 1950s, and have been a popular field of research within graph theory and computational complexity ever since. An important milestone in this field is the meta-conjecture of Bondy ([23]), the idea of which is that sufficient conditions for the existence of a Hamiltonian cycle are often sufficient conditions for a graph to be pancyclic, that is, to contain cycles of every length from 3 up to the number of vertices in the graph. It became an important motivation for the succeeding research on various concepts involving cycles and paths of many lengths in undirected graphs and digraphs.

With the objective to pursue the theme further, Hendry introduced the concepts of path and cycle extendability, first in his doctoral thesis ([56], 1985) and later in a series of papers around 1990 ([57], [58] and [60]).

In this chapter, we will focus on results related to cycle and path extendability in digraphs. Therefore we give the definitions of these extendability properties in digraphs only. Let $D$ be a digraph and let $C$ be a non-Hamiltonian cycle of $D$. $C$ is called extendable, if there exists another cycle
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C′ in D such that V(C) ⊆ V(C′), and |V(C′)| = |V(C)| + 1. Let P be a non-Hamiltonian path of D with at least two vertices. P is called extendable, if there exists another path P′ in D with the same initial vertex and terminal vertex of P, such that V(P) ⊆ V(P′), and |V(P′)| = |V(P)| + 1. A digraph D is called cycle (respectively path) extendable if D has a cycle (respectively a path with at least two vertices), and every non-Hamiltonian cycle (respectively non-Hamiltonian path with at least two vertices) of D is extendable. We can see from the definition of path extendability that a path with only one vertex is not extendable. However, we would like to exclude this trivial case in our discussion and define a non-extendable path to be a non-Hamiltonian path with at least two vertices that is not extendable.

In the original works of Hendry, different kinds of results on path and cycle extendability are studied, such as results involving degree conditions, extremal conditions, forbidden induced subgraphs and stability. Since then, there has been a lot of subsequent work on extendability properties. In particular, the question of Hendry asking whether every Hamiltonian chordal graph is cycle extendable ([56]) has caught a lot of attention. It remained open for nearly three decades. Although several supporting results appeared (Abueida et al. [2], Abueida and Sritharan [3], and Chen et al. [33]), showing that the answer is yes when restricted to some special classes of graphs, the general problem was answered negatively recently by counterexamples constructed by Lafond and Seamone ([76], 2013). For digraphs, related research has been focused mainly on cycle extendability in tournaments and in-tournaments (Tewes and Volkmann [100], and Meierling [87]). We refer to the respective publications for the relevant definitions that we have not specified here.

Whereas cycle extendability in undirected graphs and digraphs, as well as path extendability in undirected graphs have been well studied in the above-mentioned works, there seem to be very few results on path extendability in digraphs so far. However, the following easy observation shows that, unlike many other path and cycle properties, path extendability is a stronger property than cycle extendability.

**Proposition 3.1.** Every path extendable digraph is cycle extendable.

**Proof.** Let C be a non-Hamiltonian cycle in a path extendable digraph D, and let a be an arc of C. Then the path P = C − a is extendable to a path P′,
hence $C' = P' \cup a$ is a cycle extending $C$.  

Along with Bondy’s meta-conjecture, it is natural and interesting to further generalize the sufficient conditions for the many other cycle and path properties to path extendability.

In this chapter we make the first attempt by proving several path extendability counterparts of known results for cycle extendability in digraphs. We focus mainly on degree conditions and extremal conditions for path extendability in digraphs, and on path extendability in tournaments. It turns out that these new results are more complex than their cycle extendability counterparts, particularly for the result in tournaments.

Proposition 3.1 also explains why we will see that our path extendability results require stronger conditions than their known counterparts on cycle extendability, and why the exceptional digraphs for path extendability in regular tournaments have more complicated structures than those for cycle extendability.

We finish this introductory section with some terminology and notation.

In this chapter, most of the terms and notation are adopted from [12]. We use $D$ to denote a digraph and $T$ to denote a tournament. The vertex set and arc set of $D$ are denoted by $V(D)$ and $A(D)$, respectively. We often use $n$ to denote $|V(D)|$. Let $F$ be a digraph different from $D$. $F \subseteq D$ means that $F$ is isomorphic to a subdigraph of $D$. Let $G$ be an undirected graph. $\overrightarrow{G}$ stands for the complete biorientation of $G$, which is a digraph obtained from $G$ by replacing every edge $\{u, v\}$ with two arcs $uv$ and $vu$. The undirected graph on $n$ vertices with an edge between every two distinct vertices is called a complete graph, and denoted by $K_n$. And $\overrightarrow{K_n}$, the complete biorientation of $K_n$, is called a complete digraph. Let $U$ be a subset of $V(D)$. We use $D(U)$ (or simply $\langle U \rangle$ in cases when no confusion can occur) to denote the subdigraph induced by $U$. Let $X$ and $Y$ be two disjoint vertex subsets of $V(D)$, or two disjoint subdigraphs of $D$. Then $X \to Y$ means that every vertex in $X$ dominates every vertex in $Y$, i.e., there is an arc (directed) from every vertex of $X$ to every vertex of $Y$. We use $d^+(X,Y)$ to denote the number of arcs from $X$ to $Y$, and let $d(X,Y) = d^+(X,Y) + d^+(Y,X)$. When $X$ or
Y contains just one vertex, say $X = \{u\}$ or $Y = \{v\}$, we write $u$ instead of $X$ or $v$ instead of $Y$ in $X \rightarrow Y$, $d^+(X,Y)$ and $d(X,Y)$ for convenience. Let $F$ be a subdigraph of $D$ and $v \in V(F)$. We call $N^-_F(v) = \{u \in F : u \rightarrow v\}$ and $N^+_F(v) = \{u \in F : v \rightarrow u\}$ the in-neighborhood and out-neighborhood of $v$ in $F$, and call $N_F(v) = N^+_F(v) \cup N^-_F(v)$ the neighborhood of $v$ in $F$. The in-degree, out-degree and total degree of $v$ in $F$ is defined as $d^-_F(v) = |N^-_F(v)|$, $d^+_F(v) = |N^+_F(v)|$ and $d_F(v) = d^-_F(v) + d^+_F(v)$, respectively. When $F = D$ we usually omit the subscript $F$. The minimum in-degree, minimum out-degree and minimum total degree of the vertices in $D$ is denoted by $\delta^-(D)$, $\delta^+(D)$ and $\delta(D)$, respectively.

### 3.2 Main results

Throughout the rest of the chapter, when we consider cycle extendability or path extendability, we always refer to these properties in digraphs. As we saw earlier and from our results below we will see that path extendability is closely related to cycle extendability. We will cite several results on cycle extendability from [57] and [91] here, and state the counterparts on path extendability that we are going to prove in the next section.

The following two theorems involve extremal conditions on the number of arcs that guarantee cycle extendability, for digraphs and strong digraphs, respectively.

**Theorem 3.1.** ([57]) Let $D$ be a digraph of order $n \geq 2$ with at least $(n-1)^2$ arcs. Then $D$ is cycle extendable unless either $n = 3$ and $D = \overrightarrow{P}_3$ or $D$ is obtained from $\overrightarrow{K}_{n-1}$ by adding a new vertex $v$ and arcs from $v$ to all vertices of $\overrightarrow{K}_{n-1}$ or from all vertices of $\overrightarrow{K}_{n-1}$ to $v$.

**Theorem 3.2.** ([57]) Let $D$ be a strong digraph of order $n \geq 3$ with more than $n^2 - 3n + 4$ arcs. Then $D$ is cycle extendable.

We will prove the following counterpart of Theorem 3.1.

To increase the readability of this chapter, we decided to postpone the definitions of the exceptional classes in the statements of our new results to the next subsection. This holds for the below theorem, as well as for Theorems 3.6, 3.7 and 3.11.
**Theorem 3.3.** Let $D$ be a digraph of order $n \geq 3$ with at least $(n-1)^2 + 1$ arcs. Then $D$ is path extendable unless $D \in \mathcal{D}_3(n)$, $D \in \mathcal{D}_4(n)$, $D = D_{5.1}$, or $D = D_{5.2}$.

As we will see, all the exceptional digraphs in Theorem 3.3 are strong. Therefore, unlike the case for cycle extendability as expressed by Theorem 3.1 and Theorem 3.2, for strong digraphs we cannot reduce the lower bound on the number of arcs that guarantees path extendability in Theorem 3.3.

Next, we turn to the following two known results on degree conditions for cycle extendability. We need some definitions first.

For a given integer $t > 1$, let $D_0(t) = \{D_0(t,s) : 0 \leq s \leq t\}$, where $D_0(t,s)$ is the digraph defined as follows. The vertices are partitioned into three (possibly empty) sets $V_1$, $V_2$, and $V_3$ of orders $t$, $s$, and $t-s$, respectively, with arcs between all pairs of vertices, except that there are no arcs from $V_1$ to $V_2$, and no arcs from $V_3$ to $V_1$.

For $t \geq 3$, let $D_1(t)$ be the digraph defined as follows. The vertices are partitioned into three sets $V_1$, $V_2$ and $V_3$, each of order $t$, with arcs between all pairs of vertices, except that there are no arcs from $V_1$ to $V_2$, and no arcs from $V_3$ to $V_1$. For $t \geq 3$, let $D_2(t)$ be the digraph obtained from $D_1(t)$ by deleting all arcs from $V_2$ to $V_3$.

**Theorem 3.4.** ([57]) Let $D$ be a digraph of order $n \geq 2$ satisfying $\delta(D) \geq 3n/2 - 2$. Then $D$ is cycle extendable unless either $n$ is even and $D \in D_0(n/2)$ or $n = 4$ and $D = \overrightarrow{C}_4$.

**Theorem 3.5.** ([57]) Let $D$ be a digraph of order $n \geq 7$ with $\delta^+(D), \delta^-(D) \geq 2n/3 - 1$. Then $D$ is cycle extendable unless $n = 3t$ and $D_2(t) \subseteq D \subseteq D_1(t)$.

We will prove the following counterparts of Theorems 3.4 and 3.5, respectively.

**Theorem 3.6.** Let $D$ be a digraph of order $n \geq 3$ satisfying $\delta(D) \geq 3n/2 - 1$. Then $D$ is path extendable unless $D \in \mathcal{D}_6(n/2)$ with $n$ even, $D = D_{7.1}$ or $D = D_{7.2}$.

**Theorem 3.7.** Let $D$ be a digraph of order $n \geq 8$ with $\delta^+(D), \delta^-(D) \geq (2n-2)/3$. Then $D$ is path extendable unless $n = 3t+1$, $D_9(t) \subseteq D \subseteq D_8(t)$, and $d^+_F(u) \geq 1$ for $u \in \{v_t\} \cup V_2$ and $d^-_F(w) \geq 1$ for $w \in V_3 \cup \{v_0\}$, where $F = D\langle\{v_0, v_t\} \cup V_2 \cup V_3\rangle$. 

\[\]
The following known results show that strong and regular tournaments are very likely to be cycle extendable, in some sense. We call the digraph with only one vertex a trivial digraph. Thus, a nontrivial tournament is one that contains at least two vertices. If the nontrivial tournament is strong or regular, then it must contain at least three vertices.

**Theorem 3.8.** ([91]) A strong tournament is not cycle extendable if and only if its vertex set can be partitioned into three nonempty sets $W$, $X$, and $Y$ such that $\langle W \rangle$ is a nontrivial strong tournament, $W \rightarrow X$, and $Y \rightarrow W$.

**Theorem 3.9.** ([57]) A regular tournament is not cycle extendable if and only if its vertex set can be partitioned into three nonempty sets $W$, $X$, and $Y$ such that $\langle W \rangle$ is a nontrivial regular tournament, $W \rightarrow X$, $Y \rightarrow W$, and $|X| = |Y|$.

While strong tournaments are cycle extendable with only one exceptional class, as shown in Theorem 3.8, path extendability is not a general property even for regular tournaments. For example, the two smallest regular tournaments with 3 vertices and 5 vertices are not path extendable. And among the three regular tournaments with 7 vertices (see Figure 3.1), only (b) is path extendable\(^1\). These examples suggest that perhaps only small tournaments or short paths are not extendable. So a natural question is whether one can always extend long(er) paths.

In fact, the following known theorem seems to indicate that we might restrict our consideration to longer paths. The length of a path (or cycle) in

\(^{1}\)In Appendix A, we prove the path (non-)extendability of the regular tournaments on 7 vertices.
Path extendability in digraphs

$D$ is defined to be the number of arcs of it.

**Theorem 3.10.** (Alspach et al. [6], see also Thomassen [101]) Let $uv$ be an arc of a regular tournament $T$ with $n \geq 7$ vertices. Then there exist paths from $u$ to $v$ of length $k$ in $T$, for $3 \leq k \leq n - 1$.

Before we present our result on path extendability in regular tournaments, we introduce a few more terms.

A path $P$ of length $k$ is called a $k$-path. Let $D$ be a digraph. $D$ is called $k$-path extendable if there exists a non-Hamiltonian path of length $k$ in $D$, and every path in $D$ of length $k$ is extendable. $D$ is called $\{k+\}$-path extendable if there exists a path of length at least $k$ in $D$, where $0 < k \leq n - 1$, and every non-Hamiltonian path in $D$ of length at least $k$ is extendable.

We will prove that every regular tournament on at least 7 vertices is $\{2+\}$-path extendable unless it belongs to two classes of exceptional digraphs or is isomorphic to the tournament in Figure 3.1 (a), which is denoted by $T_2(7)$ according to our definition of exceptional digraphs in the next section.

**Theorem 3.11.** A regular tournament $T$ with at least 7 vertices is $\{2+\}$-path extendable, unless $T \in T_3$, $T \in T_4$ or $T = T_2(7)$.

In the next section, we present the proofs of the new results we listed above.

### 3.3 Proofs of our main results

We start this section with the following lemma that is a useful tool in some of the proofs of the main results.

**Lemma 3.12.** Let $P = v_0v_1 \ldots v_{p-1}$ be a non-extendable path in digraph $D$, $F = D - V(P)$, and $u \in V(F)$. Then

(a) for $0 \leq i \leq p - 2$, at most one of $v_i \rightarrow u$ and $u \rightarrow v_{i+1}$ holds,

(b) $d(u, P) \leq p + 1$, and if equality holds, then $u \rightarrow v_0$ and $v_{p-1} \rightarrow u$.

**Proof.** For any $u \in V(F)$ and any $0 \leq i \leq p - 2$, if $v_i \rightarrow u$ and $u \rightarrow v_{i+1}$, then the path $v_0 \ldots v_i u v_{i+1} \ldots v_{p-1}$ extends $P$, a contradiction. Therefore (a) holds.
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Figure 3.2: \( D_{5.1} \) and \( D_{5.2} \)

Each arc between \( u \) and \( P \) belongs to the union of the following \( p+1 \) sets: \( \{uv_0\}, \{v_0u, uv_1\}, \ldots, \{v_{p-2}u, uv_{p-1}\} \) and \( \{v_{p-1}u\} \). By (a) we can have at most one arc from each set. Therefore \( d(u, P) \leq p + 1 \). And if equality holds, then we have the arcs \( uv_0 \) and \( v_{p-1}u \).

\[ \square \]

### 3.3.1 Proof of Theorem 3.3

Before we prove Theorem 3.3, we firstly define the exceptional classes \(^2\) in the statement of the theorem.

Let \( n \geq 3 \) be an integer, and \( \mathcal{D}_3(n) = \{D_3(n, s) : 0 \leq s \leq n - 2\} \), where \( D_3(n, s) \) is the digraph of order \( n \), whose vertices can be partitioned into three sets \( V_0 = \{v_0, v_1\}, V_1 \) and \( V_2 \), where \( |V_1| = s \), \( \langle V_0 \rangle = K_2 \), \( \langle V_1 \cup V_2 \rangle = \overrightarrow{K_{n-2}} \), \( V_1 \cup V_2 \rightarrow v_0, v_1 \rightarrow V_1 \cup V_2 \), \( v_0 \rightarrow V_1 \) and \( V_2 \rightarrow v_1 \).

Let \( n \geq 3 \), and \( \mathcal{D}_4(n) = \{D_{4.1}(n), D_{4.2}(n)\} \), where \( D_{4.1}(n) \) (\( D_{4.2}(n) \)) is the digraph of order \( n \) obtained from \( \overrightarrow{K_{n-1}} \) by adding a vertex \( u \), which is adjacent to (adjacent from) all vertices of \( \overrightarrow{K_{n-1}} \), together with one arc from \( \overrightarrow{K_{n-1}} \) to \( u \) (\( u \) to \( \overrightarrow{K_{n-1}} \)).

Let \( D_{5.1} \) and \( D_{5.2} \) be the digraphs in Figure 3.2, where a line with both arrow ends stands for two arcs of both directions between two vertices.

We repeat the statement of Theorem 3.3 for convenience.

**Theorem 3.3.** Let \( D \) be a digraph of order \( n \geq 3 \) with at least \( (n - 1)^2 + 1 \) arcs. Then \( D \) is path extendable unless \( D \in \mathcal{D}_3(n), D \in \mathcal{D}_4(n), D = D_{5.1}, \) or \( D = D_{5.2} \).

**Proof.** Suppose that \( D \) is not path extendable, and let \( P = v_0v_1 \ldots v_{p-1} \) be a

\(^2\)We will prove that all digraphs in the exceptional classes of our results are not path extendable in Appendix A.
non-extendable path of $D$. Then, by (b) of Lemma 3.12,
\[1+(n-1)^2 \leq |A(D)| \leq 2\left(\frac{p}{2}\right) + (n-p)(p+1) + 2\left(\frac{n-p}{2}\right) = n^2 - pn - p + p^2. \quad (3.1)\]
Therefore,
\[(p-2)(p-n+1) \geq 0.\]
However, $2 \leq p \leq n-1$. So, equality must hold and either $p = 2$ or $p = n-1$. Furthermore, equality in (3.1) implies that $\langle V(P) \rangle$ and $F = D - V(P)$ are complete, and for every $u \in V(F)$,
\[d(u, P) = p + 1. \quad (3.2)\]
By (b) of Lemma 3.12, $u \rightarrow v_0$ and $v_{p-1} \rightarrow u$.

Suppose $p = 2$. Let $P = v_0v_1$, and $V(P) = V_0$. For any $u \in V(F)$, $d(u, P) = 3$. Since $u \rightarrow v_0$ and $v_1 \rightarrow u$, it follows that exactly one of $v_0 \rightarrow u$ and $u \rightarrow v_1$ holds.

Let $V_1$ be the set of vertices $u \in F$ such that $v_0 \rightarrow u$, and $V_2$ be the set of vertices $u \in F$ such that $u \rightarrow v_1$. Then $D \in D_3(n)$.

Suppose that $p = n-1$ and $p \geq 3$. Then $F$ contains just one vertex $u$. Further suppose that $p \geq 4$. Besides $uv_0$ and $v_{p-1}u$, the possible arcs between $u$ and $F$ belong to one of the following two sets:
\[
\{v_iu : 0 \leq i \leq p-2\},
\]
and
\[
\{uv_j : 1 \leq j \leq p-1\}.
\]
However, since $\langle V(P) \rangle$ is complete, if we have arcs from both sets, it is easy to verify that $P$ is extendable, a contradiction. Therefore, we have arcs from only one set. And by (3.2), we must have all arcs from this set. Therefore, we can see that $D \in D_4(n)$.

If $p = 3$, it is not hard to verify that $D \in D_4(3)$, $D = D_{5,1}$ or $D = D_{5,2}$. □

### 3.3.2 Proof of Theorem 3.6

Let $n = 2t$, $t \geq 2$ be an even integer. Let $D_6(t)$ be the class of digraphs $D$ defined as follows. Let $F$ be any digraph with $n_1 = n - 2$ vertices and $\delta(F) \geq 3n_1/2 - 1$, whose vertex set is partitioned into $V_0$ and $V_1$ with $|V_0| = n - 2$...

If $p = 3$, it is not hard to verify that $D \in D_4(3)$, $D = D_{5,1}$ or $D = D_{5,2}$. □
Figure 3.3: $D_{7,1}$ and $D_{7,2}$

$|V_1| = n_1/2$. Let $D$ be obtained from $F$ by adding a $K_2$ with vertex set \{v_0, v_1\}, and all arcs from $F$ to $v_0$, all arcs from $v_1$ to $F$, all arcs from $v_0$ to $V_0$ and all arcs from $V_1$ to $v_1$.

Let $D_{7,1}$ and $D_{7,2}$ be the digraphs in Figure 3.3.

For convenience, we repeat the statement of the theorem.

**Theorem 3.6.** Let $D$ be a digraph of order $n \geq 3$ satisfying $\delta(D) \geq 3n/2 - 1$. Then $D$ is path extendable unless $D \in D_6(n/2)$ with $n$ even, $D = D_{7,1}$ or $D = D_{7,2}$.

**Proof.** Let $D$ be a digraph satisfying the condition of the theorem but having a non-extendable path $P = v_0v_1 \ldots v_{p-1}$. Let $F = D - V(P)$.

Let $u \in V(F)$. Then by the hypotheses of the theorem and (b) of Lemma 3.12,

$$3n/2 - 1 \leq d(u) \leq p + 1 + 2(n - p - 1) = 2n - p - 1.$$  \hfill (3.3)

So

$$p \leq n/2. \hfill (3.4)$$

Considering the degree sum of all $v_i$, $0 \leq i \leq p - 1$, we have

$$(3n/2 - 1)p \leq \sum_{i=0}^{p-1} d(v_i) = 2|A(V(P))| + d(P, F) \leq 2 \cdot 2 \cdot \binom{p}{2} + (p+1)(n-p). \hfill (3.5)$$

Hence, $(2p - n)(p - 2) \geq 0$.

However $p \geq 2$, and together with (3.4), we have $(2p - n)(p - 2) \leq 0$. So

$$(2p - n)(p - 2) = 0.$$
Therefore either $p = n/2$, or $p = 2$. Furthermore, (3.5) holds with equalities. Note that equalities in (3.5) imply that $\langle V(P) \rangle$ is complete, that $d(v_i) = 3n/2 - 1$ for all $i \in \{0, 1, \ldots, p-1\}$, and that $d(u, P) = p + 1$ for all $u \in V(F)$. In particular, $n$ is even.

If $p = 2$, then $\langle V(P) \rangle = \overrightarrow{K_2}$, $d(v_0) = d(v_1) = 3n/2 - 1$, and for every $u \in V(F)$,

$$d(u, P) = 3.$$  (3.6)

By (3.6), (b) of Lemma 3.12, and equalities in (3.5), for every $u \in V(F)$, $u \rightarrow v_0$ and $v_1 \rightarrow u$. Therefore,

$$3n/2 - 1 = d(v_0) = 2 + (n - 2) + d^+(v_0, F) = n + d^+(v_0, F),$$

and

$$3n/2 - 1 = d(v_1) = 2 + (n - 2) + d^+(F, v_1) = n + d^+(F, v_1).$$

So,

$$d^+(v_0, F) = d^+(F, v_1) = n/2 - 1.$$  (3.7)

Let $n_1 = |V(F)| = n - 2$. The degree of every $u \in V(F)$ satisfies $d_F(u) = d_D(u) - d(u, P) = d_D(u) - 3 \geq 3n/2 - 1 - 3 = 3n_1/2 - 1$. Let $N^+_F(v_0) = V_0$ and $N^-_F(v_1) = V_1$. Then $V_0$ and $V_1$ are disjoint, for if there exists a vertex $w \in V_0 \cap V_1$, then $v_0 w v_1$ extends $P = v_0 v_1$. By (3.7), $|V_0| = |V_1| = n/2 - 1 = n_1/2$. We see that $D \in \mathcal{D}_6(n/2)$.

Now suppose $p = n/2$ and $p \geq 3$. Then, (3.3) holds with equalities. Equalities in (3.3) and (3.5) imply that every vertex in $D$ is of degree $3n/2 - 1$, $F$ and $\langle V(P) \rangle$ are complete digraphs on $n/2$ vertices, and $d(u, P) = p + 1 = n/2 + 1$ for every $u \in V(F)$.

By (b) of Lemma 3.12, for every $u \in V(F)$, $u \rightarrow v_0$ and $v_{p-1} \rightarrow u$. Hence, $d^+(F, v_0) = d^+(v_{p-1}, F) = |V(F)| = n/2$, and

$$d^+(v_0, F) = d(v_0) - d^+(F, v_0) + 2(\langle V(P) \rangle - 1)$$

$$= 3n/2 - 1 - n/2 - 2(n/2 - 1)$$

$$= 1,$$  (3.8)

and similarly,

$$d^+(F, v_{p-1}) = 1.$$  (3.9)
Suppose that \( p \geq 4 \). By (a) and equality in (b) of Lemma 3.12, exactly one of \( v_i \to u \) and \( u \to v_{i+1} \) holds for \( 0 \leq i \leq p-2 \), and every \( u \in V(F) \). Let \( i = 0 \). By (3.8), there exists only one vertex \( u_0 \in V(F) \) such that \( v_0 \to u_0 \) holds. Therefore, for all \( u \in V(F) \setminus \{ u_0 \}, u \to v_1 \). So there are \( p-1 \) vertices in \( F \) dominating \( v_1 \). Similarly, if we let \( i = p-2 \), by (3.9), we can deduce that \( p-1 \) vertices in \( F \) are dominated by \( v_{p-2} \). By \( n = 2p \geq 8, |V(F)| = n/2 < 2(p-1) = n - 2 \). So, there must exist a vertex \( u_0 \in F \) such that \( u_0 \to v_1 \) and \( v_{p-2} \to u_0 \). But then the path \( v_0 v_2 \ldots v_{p-2} u_1 v_1 v_{p-1} \) extends \( P \), a contradiction. Therefore \( p = 3 \).

By (3.8) and (3.9), \( u_0 \) has an out-neighbor \( u_0 \in V(F) \) and \( v_2 \) has an in-neighbor \( u_1 \in V(F) \). If \( u_0 \neq u_1 \), we can conclude that \( D = D_{7,1} \). If \( u_0 = u_1 \), we can conclude that \( D = D_{7,2} \).

### 3.3.3 Proof of Theorem 3.7

The proof of Hendry’s result on in-degree and out-degree conditions (Theorem 3.5) for cycle extendability is the longest and hardest part in [57]. Here, we introduce a new technique, in the form of a contraction operation, to transfer path non-extendability into cycle non-extendability. This enables us to use Theorem 3.5 in our proof of Theorem 3.7.

Let \( D \) be a digraph, and let \( u \) and \( v \) be two vertices of \( D \). Delete all in-arcs of \( u \) and all out-arcs of \( v \). Then contract \( u \) and \( v \) into one vertex, obtaining a digraph \( D' \). Denote the resulting digraph as \( D' = D/\{u^-, v^+\} \). We have the following lemma.

**Lemma 3.13.** Let \( P = v_0 v_1 \ldots v_{p-1} \) be a path in a digraph \( D \). Let \( D' = D/\{v_0^-, v_{p-1}^+\} \), and denote the vertex that \( v_0 \) and \( v_{p-1} \) are contracted into as \( v_0' \). Let \( C = v_0' v_1 \ldots v_{p-2} v_0' \). Then, \( P \) is an extendable path of \( D \) if and only if \( C \) is an extendable cycle of \( D' \).

**Proof.** Suppose that \( P \) is extendable to \( P' = v_0 u_1 \ldots u_{p-1} v_{p-1} \) in \( D \), where \( \{v_i : 1 \leq i \leq p-2\} \subseteq \{u_j : 1 \leq j \leq p-1\} \). Then \( C' = v_0' u_1 \ldots u_{p-1} v_0' \) is a cycle in \( D' \) that extends \( C \).

Suppose that in \( D' \), \( C \) is extendable to a cycle \( C' = v_0' u_1 \ldots u_{p-1} v_0' \), where \( \{v_i : 1 \leq i \leq p-2\} \subseteq \{u_j : 1 \leq j \leq p-1\} \). Then, \( P' = v_0 u_1 \ldots u_{p-1} v_{p-1} \) is a path in \( D \) that extends \( P \). \( \square \)

Let \( t \geq 3 \) be an integer. Let \( D_8(t) \) be the digraph with vertex set \( \{v_0, v_t\} \cup \ldots \cup \{v_{t-1}, v_0\} \) and edge set \( \{(v_i, v_{i+1}) : 0 \leq i \leq t-1\} \). The digraph \( D_8(t) \) is extendable by the following minimum extendability result.
Let $d = F(2 \langle V \rangle)$. A non-extendable path in $D$ is not cycle extendable. Therefore, $d(2 \langle V \rangle) = 3$. The arc set of $D_8(t)$ contains all possible arcs between all pairs of vertices, except that there is no arc from $V_1 \cup \{v_0\}$ to $V_2$, and no arc from $V_3$ to $V_1 \cup \{v_1\}$. Let $D_9(t)$ be obtained from $D_8(t)$ by deleting all arcs from $V_2 \cup \{v_1\}$ to $V_3 \cup \{v_0\}$.

Next we are going to prove Theorem 3.7. For convenience, we repeat the statement of the theorem.

**Theorem 3.7.** Let $D$ be a digraph of order $n \geq 8$ with $\delta^+(D), \delta^-(D) \geq (2n - 2)/3$. Then $D$ is path extendable unless $n = 3t + 1$, $D_9(t) \subseteq D \subseteq D_8(t)$, and $d_F^+(u) \geq 1$ for $u \in \{v_1\} \cup V_2$ and $d_F^-(w) \geq 1$ for $w \in V_3 \cup \{v_0\}$, where $F = D\langle \{v_0, v_1\} \cup V_2 \cup V_3 \rangle$.

Proof. Suppose that $D$ is not path extendable, and let $P = v_0v_1 \ldots v_{p-1}$ be a non-extendable path in $D$. Let $D' = D/\{v_0', v_{p-1}'\}$ and $v_0'$ be the vertex that $v_0$ and $v_{p-1}$ are contracted to. Then, by Lemma 3.13, $C = v_0'v_1 \ldots v_{p-2}v_0'$ is a non-extendable cycle in $D'$ of length $p - 1$.

Furthermore, the order of $D'$, $n' = n - 1 \geq 7$. The in-degree and out-degree of every vertex in $V(D') \setminus v_0'$ is less than those in $D$ by at most one. Also, $d_{D'}^+(v_0') = d_F^+(v_0)$ and $d_{D'}^-(v_0') = d_F^-(v_{p-1})$, hence $\delta^+(D'), \delta^-(D') \geq (2n - 2)/3 - 1 = 2n'/3 - 1$. So, $D'$ satisfies the condition of Theorem 3.5, but is not cycle extendable. Therefore, $n' = 3t$, and $D_2(t) \subseteq D' \subseteq D_1(t)$.

By the definition of $D_1(t)$ and $D_2(t)$, the vertices of $D_1(t)$ and $D_2(t)$ can be partitioned into three sets $V_1', V_2$ and $V_3$, each of order $t$. According to the proof of Theorem 3.5 in [57], the non-extendable cycle $C$ of $D'$ is a Hamiltonian cycle of $D'(V_1')$. Therefore $t = p - 1$, and $C = v_0'v_1 \ldots v_{t-1}v_0'$. Let $V_1 = V_1' \setminus \{v_0'\} = \{v_1, \ldots, v_{t-1}\}$. $D$ can be obtained form $D'$ by splitting $v_0'$ into $v_0$ and $v_{p-1} = v_t$, where $N^+(v_0) \setminus \{v_t\} = N^+(v_0') = V_1 \cup V_3$ and $N^-(v_t) \setminus \{v_0\} = N^-(v_t') = V_1' \setminus \{v_0'\}$.

![Figure 3.4: The construction of $D$ from $D'$](image)

Figure 3.4: The construction of $D$ from $D'$. $V_1 \cup V_2 \cup V_3$, where $|V_1| = t - 1$, $|V_2| = t$ and $|V_3| = t$. The arc set of $D_8(t)$ contains all possible arcs between all pairs of vertices, except that there is no arc from $V_1 \cup \{v_0\}$ to $V_2$, and no arc from $V_3$ to $V_1 \cup \{v_1\}$. Let $D_9(t)$ be obtained from $D_8(t)$ by deleting all arcs from $V_2 \cup \{v_1\}$ to $V_3 \cup \{v_0\}$.
Chapter 3

Let \( N^-(v'_0) = V_1 \cup V_2 \) (see Figure 3.4 (a)), then adding some arcs to \( v_0 \), some arcs from \( v_t \), and possibly the arc \( v_0v_t \), so that the degree conditions of the theorem are satisfied and \( P = v_0v_1 \ldots v_t \) is non-extendable.

By definition, the arcs from \( V'_1 \) to \( V_2 \) and from \( V_3 \) to \( V'_1 \) are forbidden in \( D' \). Hence, there is no arc from \( \{v_0\} \cup V_1 \) to \( V_2 \), and no arc from \( V_3 \) to \( V_1 \cup \{v_t\} \) in \( D \). To satisfy the degree conditions in \( D \), some other arcs must then exist in \( D \). In the table below, for the degrees listed in column 1 to satisfy the lower bounds of the theorem, we can conclude the dominating relations in column 2 (see Figure 3.4 (b)).

<table>
<thead>
<tr>
<th>Degrees</th>
<th>Dominating relations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Out-degree of ( v_0 ) and in-degree of ( v_t )</td>
<td>( v_0 \to v_t )</td>
</tr>
<tr>
<td>In-degree and out-degree of the vertices in ( V_1 )</td>
<td>( V_1 \to v_0 ) and ( v_t \to V_1 )</td>
</tr>
<tr>
<td>In-degree of the vertices in ( V_2 )</td>
<td>( v_t \to V_2 )</td>
</tr>
<tr>
<td>Out-degree of the vertices in ( V_3 )</td>
<td>( V_3 \to v_0 )</td>
</tr>
</tbody>
</table>

Table 3.1: Degrees whose lower bounds imply some dominating relations

Now we can see that \( D_9(t) \subseteq D \). Moreover, we can add arcs from \( V_2 \cup \{v_t\} \) to \( V_3 \cup \{v_0\} \) freely, without changing the non-extendibility of \( P \) (see Figure 3.4 (c)), hence \( D \subseteq D_8(t) \). Furthermore, to satisfy the out-degree condition on \( V_2 \cup \{v_t\} \), as well as the in-degree condition on \( V_3 \cup \{v_0\} \), we must have \( d_F^+(u) \geq 1 \) for every \( u \in V_2 \cup \{v_t\} \) and \( d_F^-(w) \geq 1 \) for every \( w \in V_3 \cup \{v_0\} \) where \( F = D(\{v_0, v_t\} \cup V_2 \cup V_3) \).

3.3.4 Proof of Theorem 3.11

In order to prove Theorem 3.11 and describe the exceptional digraphs, we need to introduce some additional terminology and notation, and prove several auxiliary results that will cover some of the special cases.

Let \( v \) be a vertex of \( D \), the semi-degrees of \( v \) in \( D \) is defined as the ordered pair \( (d^+(v), d^-(v)) \). An almost regular tournament is a tournament on an even number of vertices, in which the in-degree and out-degree of every vertex differ by 1. In an almost regular tournament on \( n \) vertices, half of the vertices have semi-degrees \((n/2, n/2-1)\), and the other half have semi-degrees \((n/2-1, n/2)\).

Let \( T \) be a regular tournament on \( n = 2m + 1 \geq 7 \) vertices. Then
\[
d^+(v) = d^-(v) = m \text{ for all } v \in V(T).
\]
Let \( P = v_0v_1 \ldots v_{p-1} \) be a non-extendable path of \( T \) with \( p \) vertices. Let \( F = D - V(P) \) and \( q = |F| \). Then \( p+q = 2m+1 \).

Throughout this section we will keep using the above notation, in particular in the statements of the results, except in Lemma 3.15 and in Theorem 3.11.

We firstly consider the case that \( p = 3 \).

Let \( \mathcal{T}_1 \) be the class of regular tournaments whose vertices can be partitioned into three parts \( V = \{v_0, v_1, v_2\}, N_0^+ \) and \( N_0^- \), where \( \langle V \rangle \) is a directed triangle, \( |N_0^+| = |N_0^-| \geq 3, V \rightarrow N_0^+ \) and \( N_0^- \rightarrow V \).

Let \( D_{10} \) be the digraph in Figure 3.5. Let \( \mathcal{T}_2 \) be the class of regular tournaments whose vertices can be partitioned into four parts \( V = \{v_0, v_1, v_2\}, \{u_0, u_1\}, N_0^+ \) and \( N_0^- \), where \( \langle V \cup \{u_0, u_1\} \rangle = D_{10} + u_0u_1 \) or \( D_{10} + u_1u_0 \), \( |N_0^+| = |N_0^-| \geq 1, V \rightarrow N_0^+ \) and \( N_0^- \rightarrow V \). See Figure 3.6.

**Theorem 3.14.** If \( p = 3 \), then either \( m \geq 4 \) and \( T \in \mathcal{T}_1 \), or \( T \in \mathcal{T}_2 \).
By our assumption, \( m \geq 3 \), and therefore \( F \) is not empty. We have
\[
d^+(v_1, F) = d^+(F, v_1) = m - 1, \quad N^+_F(v_1) \subseteq N^+_F(v_2) \quad \text{and} \quad N^-_F(v_1) \subseteq N^-_F(v_0).
\]
Since \( T \) is a tournament, \( N^+_F(v_1) \cup N^-_F(v_1) = F \).

If \( v_2 \to v_0 \), then
\[
N^-(v_0) = N^-_F(v_1) \cup \{v_2\} \quad \text{and} \quad N^+(v_2) = N^+_F(v_1) \cup \{v_0\}.
\]
Since \( T \) is a regular tournament,
\[
v_0 \to N^+_F(v_1) \quad \text{and} \quad N^-_F(v_1) \to v_2.
\]
Let \( N^+_0 = N^+_F(v_1) \) and \( N^-_0 = N^-_F(v_1) \). It is easy to see that \( |N^+_0| = |N^-_0| = m - 1 \geq 2 \). Now let \( x, y \in N^-_0 \) and suppose without loss of generality that \( xy \in A(D) \). Notice that \( d^+(x) \geq 4 \), but
\[
d^-(x) \leq |N^+_0| + |N^-_0 \setminus \{x, y\}| = (m - 1) + (m - 3) = 2m - 4.
\]
Since \( T \) is regular, it follows that \( m \geq 4 \). So \( |N^+_0| = |N^-_0| = m - 1 \geq 3 \), and \( T \in T_1 \).

if \( v_0 \to v_2 \), then all in-neighbors of \( v_0 \) are in \( F \). Thus \( |N^-_F(v_0)| = |N^-(v_0)| = m \). Since \( |N^+_F(v_1)| = m - 1 \) and \( N^-_F(v_1) \subseteq N^-_F(v_0) \), there must be exactly one in-neighbor of \( v_0 \), denoted by \( u_0 \), in \( N^+_F(v_1) = F \setminus N^-_F(v_1) \). Therefore \( u_0 \to v_0 \) and \( v_0 \to N^+_F(v_1) \setminus \{u_0\} \). Similarly, there must be a vertex \( u_1 \in N^-_F(v_1) \), such that \( v_2 \to u_1 \) and \( N^-_F(v_1) \setminus \{u_1\} \to v_2 \). Let \( N^+_0 = N^+_F(v_1) \setminus \{u_0\} \) and \( N^-_0 = N^-_F(v_1) \setminus \{u_1\} \). By \( m \geq 3 \), \( |N^+_0| = |N^-_0| = m - 2 \geq 1 \). We have \( T \in T_2 \).

To show that regular tournaments satisfying the definition of \( T_1 \) and \( T_2 \) exist, we now construct regular tournaments in \( T_1 \) and \( T_2 \), of all possible orders.

By definition, a tournament in \( T_1 \) has at least 9 vertices. We construct a regular tournament \( T_1(n) \) on \( n \) vertices such that \( T_1(n) \in T_1 \), for every odd integer \( n \geq 9 \). In \( T_1(9) \), \( \langle N^+_0 \rangle \) and \( \langle N^-_0 \rangle \) are directed triangles, and \( \langle N^+_0 \rangle \to \langle N^-_0 \rangle \). For \( n \geq 11 \), \( T_1(n) \) is defined as follows. Let \( N^+_0 = W_0^+ \cup \{w^+_1, w^+_2, w^+_3\} \) and \( N^-_0 = W_0^- \cup \{w^-_1, w^-_2, w^-_3\} \), such that \( \langle \{w^+_1, w^+_2, w^+_3\} \rangle \) and \( \langle \{w^-_1, w^-_2, w^-_3\} \rangle \) are directed triangles, and \( \{w^+_1, w^+_2, w^+_3\} \to \{w^-_1, w^-_2, w^-_3\} \). Furthermore, let \( W_0^+ \cup W_0^- \) be an almost regular tournament on \( 2m - 8 \) vertices such that the vertices in \( W_0^+ \) (respectively \( W_0^- \)) have semi-degrees \( (m-4, m-5) \) (respectively...
(m - 5, m - 4)). Finally, let \( \{w_1^+, w_1^-\} \rightarrow W_0^+, W_0^+ \rightarrow \{w_2^+, w_3^+, w_2^-, w_3^-\}, W_0^- \rightarrow \{w_1^+, w_1^-\}, \) and \( \{w_2^+, w_3^+, w_2^-, w_3^-\} \rightarrow W_0^- \).

By definition, a tournament in \( T_2 \) has at least 7 vertices. We construct a tournament \( T_2(n) \in T_2 \) on \( n \) vertices, for every odd integer \( n \geq 7 \). In \( T_2(n), u_0 \rightarrow u_1, (N_0^+ \cup N_0^-) \) is an almost regular tournament, in which the vertices in \( N_0^+ \) (respectively \( N_0^- \)) have semi-degrees \( (m-2, m-3) \) (respectively \( (m-3, m-2) \)), \( N_0^+ \rightarrow \{u_0, u_1\} \) and \( \{u_0, u_1\} \rightarrow N_0^- \). Note that \( T_2(7) \) is the unique tournament in \( T_2 \) with 7 vertices, and isomorphic to the digraph in Figure 3.1 (a). We will keep using \( T_2(7) \) to denote this digraph and refer to it later. Furthermore, if \( n \geq 9 \), we can always find a cycle \( u_0u_1u_2u_3u_0 \), such that \( u_2 \in N_0^- \) and \( u_3 \in N_0^+ \). By reversing the direction of the cycle we have another tournament in \( T_2 \) on \( n \) vertices, in which \( (V \cup \{u_0, u_1\}) = D_{10} + u_1u_0 \).
Now we handle the general case. We start with some more notation and an auxiliary lemma.

Let $P = v_0v_1 \ldots v_{p-1}$ be a path with $p$ vertices in a digraph $D$. Let $0 \leq i < j \leq p - 1$ be two integers, the segment of $P$ from $v_i$ to $v_j$ is denoted by $P[v_i, v_j]$. If $p \geq 4$, then $v_1 \neq v_{p-2}$, and we let $P' = P[v_1, v_{p-2}]$. Let $u$ be a vertex in $D - V(P)$. If $u \to P$ ($P \to u$), we say that $u$ is a dominating (dominated) vertex of $P$. If $u$ dominates some vertices and is dominated by some vertices of $P$, we say that $u$ is a hybrid vertex of $P$. If there exists an integer $i$ with $0 \leq i \leq p - 2$, such that $v_i \to u$ and $u \to v_{i+1}$, then we say that $u$ can be inserted into $P$. Let $T$ be a tournament, let $P = v_0v_1 \ldots v_{p-1}$ be a non-extendable path of $T$, and let $u$ be a hybrid vertex of $D$. Then it is easy to see that there must exist an integer $k$ with $0 \leq k \leq p - 2$, such that $u$ dominates all $v_i$, $0 \leq i \leq k$, and is dominated by all $v_j$, $k + 1 \leq j \leq p - 1$. We say that $u$ switches at $v_k$ on $P$.

We will see that, when $p \geq 4$, the number of hybrid vertices of $P'$ helps to divide our discussion into different cases.

**Lemma 3.15.** Let $P = v_0v_1 \ldots v_{p-1}$ be a non-extendable path of a tournament $T$ with $p \geq 4$, and let $u$ be a hybrid vertex of $P$ which switches at $v_r$ with $1 \leq r \leq p - 3$. Then either $v_{r+1} \to v_i$ for all $0 \leq i \leq r - 1$, or $v_j \to v_r$ for all $r + 2 \leq j \leq p - 1$.

**Proof.** Suppose there exist $i$ and $j$ such that $0 \leq i \leq r - 1$ and $r + 2 \leq j \leq p - 1$, $v_i \to v_{r+1}$ and $v_r \to v_j$. Then the path

$$v_0 \ldots v_i v_{r+1} \ldots v_{j-1} u v_i \ldots v_r v_j \ldots v_{p-1}$$

extends $P$, a contradiction. \hfill \Box

Let $D_{11}$ be the digraph in Figure 3.7. Let $T_3$ be the class of regular tournaments whose vertex set can be partitioned into four parts $V = \{v_0, v_1, v_2, v_3, v_4\}$, $\{u_0, u_1\}$, $N_0^+$ and $N_0^-$, where $V \cup \{u_0, u_1\} = D_{11} + u_0u_1$ or $D_{11} + u_1u_0$, $|N_0^+| = |N_0^-| = 0$ or $|N_0^+| = |N_0^-| \geq 3$, $V \to N_0^+$ and $N_0^- \to V$. See Figure 3.8.

**Theorem 3.16.** If $p \geq 4$, then $P'$ has no or exactly two hybrid vertices. Moreover, if $P'$ has exactly two hybrid vertices, then $T \in T_3$. 
Proof. Suppose that \( P' \) has a hybrid vertex \( u \). Then \( u \) is also a hybrid vertex of \( P \). Let the number of hybrid vertices of \( P \) be \( h \), so \( h \geq 1 \). For every hybrid vertex \( w \) of \( P \), \( w \rightarrow v_0 \) and \( v_{p-1} \rightarrow w \). Therefore \( d^+(v_{p-1}, F) - d^+(v_0, F) = h \).

Since \( p \geq 4 \), \( v_1 \neq v_{p-2} \). We have \( v_{p-2} \rightarrow u \) and \( u \rightarrow v_1 \). And \( u \) is also a hybrid vertex of \( P \). For \( 1 \leq i \leq p - 3 \), if \( v_0 \rightarrow v_{i+1} \) and \( v_i \rightarrow v_{p-1} \), then the path \( v_0 v_{i+1} \ldots v_{p-2} u v_1 \ldots v_{p-1} \) extends \( P \), a contradiction. Therefore, for \( 1 \leq i \leq p - 3 \), at most one of \( v_0 \rightarrow v_{i+1} \) and \( v_i \rightarrow v_{p-1} \) holds. Hence,

\[
d^+_P(v_0) + d^+_P(v_{p-1}) \leq (p - 3) + 1 + 2 = p + 1.
\]

And, since \( T \) is regular,

\[
d^+(v_0, F) + d^+(F, v_{p-1}) = 2m - d^+_P(v_0) - d^+_P(v_{p-1}) \geq 2m - (p + 1) = q - 2.
\]

Since \( T \) is a tournament,

\[
d^+(v_{p-1}, F) + d^+(F, v_{p-1}) = q.
\]

So

\[
h = d^+(v_{p-1}, F) - d^+(v_0, F) \leq 2.
\]

Since every hybrid vertex of \( P' \) is also a hybrid vertex of \( P \), \( P' \) has no more than two hybrid vertices either.

In the following, we let

\[
N^+_P(v_0) = N^+_0, \quad N^-_P(v_{p-1}) = N^-_0, \quad |N^+_0| = n^+_0 \quad \text{and} \quad |N^-_0| = n^-_0.
\]

Since \( P \) is not extendable, we must have \( P \rightarrow N^+_0 \) and \( N^-_0 \rightarrow P \), that is, all vertices in \( N^+_0 \) (\( N^-_0 \)) are dominated (dominating) vertices of \( P \).

Suppose that \( h = 1 \). Then \( u \) is the only hybrid vertex of both \( P \) and \( P' \). Assume that \( u \) switches at \( r \), \( 1 \leq r \leq p - 3 \). Then \( V(F) = N^+_0 \cup \{u\} \cup N^-_0 \) and \( n^+_0 + n^-_0 = q - 1 \). Since \( T \) is regular, \( d^+(F, P) = d^+(P, F) \). Therefore,

\[
n^+_0 p + (p - 1 - r) = n^-_0 p + r + 1,
\]

that is,

\[
(n^+_0 - n^-_0 + 1)p = 2(r + 1). \tag{3.10}
\]

Since \( 1 \leq r \leq p - 3 \),

\[
0 < 2(r + 1) < 2p.
\]
By (3.10), \(2(r + 1)\) is divisible by \(p\), so
\[2(r + 1) = p.\]

Then \(p\) is even,
\[r = p/2 - 1,\quad \text{and } n^+_0 = n^-_0.\]

By Lemma 3.15, either \(v_{r+1} \rightarrow v_i\) for all \(0 \leq i \leq r - 1\), or \(v_j \rightarrow v_r\) for all \(r + 2 \leq j \leq p - 1\). Thus either
\[d^+(v_{r+1}) \geq n^+_0 + 1 + (r - 1 + 1) + 1 = (q - 1)/2 + 1 + (p/2 - 1) + 1 = m + 1,\]
or
\[d^-(v_r) \geq n^-_0 + 1 + (p - 1 - (r + 2) + 1) + 1 = (q - 1)/2 + 1 + (p/2 - 1) + 1 = m + 1,\]
both of which are contradictions because \(d^+(v) = d^-(v) = m\) for all \(v \in V(T)\).

Therefore we can assume that \(h = 2\). Let the two hybrid vertices of \(P\) be \(u_0 = u\) and \(u_1\). Note that although \(u_0\) is also a hybrid vertex of \(P'\), \(u_1\) is not necessarily a hybrid vertex of \(P'\). We assume that \(u_0\) switches at \(v_r\) on \(P\), where \(1 \leq r \leq p - 3\), and \(u_1\) switches at \(v_s\) on \(P\), where \(0 \leq s \leq p - 2\). Then
\[V(F) = N^+_u \cup \{u_0, u_1\} \cup N^+_1,\]
and \(n^+_0 + n^-_0 + 2 = q\). Again, considering the arcs between \(F\) and \(P\), we have
\[n^+_0 p + (p - 1 - r) + (p - 1 - s) = n^-_0 p + (r + 1) + (s + 1),\]
that is,
\[(n^+_0 - n^-_0 + 2)p = 2(r + s + 2).\]

Since \(1 \leq r + s \leq 2p - 5\),
\[0 < 2(r + s + 2) < 4p.\]

However, \(2(r + s + 2)\) is divisible by \(p\), so
\[2(r + s + 2) \in \{p, 2p, 3p\}\]
Note that \(n^+_0 + n^-_0 + 2 = q\). Since \(T\) is regular, \(|V(T)|\) is odd. Therefore it follows from \(|V(T)| = p + q\) that \(p\) and \(q\) are of different parity. We distinguish three cases.

Case 1: \(r + s + 2 = p/2\). In this case, we have that \(q\) is odd and \(p\) is even,
\[n^+_0 = (q - 3)/2,\quad \text{and } n^-_0 = (q - 1)/2;\]
Case 2: \( r + s + 2 = p \). In this case, we have that \( q \) is even and \( p \) is odd, and
\[
n^+_0 = n^-_0 = q/2 - 1; \tag{3.11}
\]

Case 3: \( r + s + 2 = 3p/2 \). In this case, we have that \( q \) is odd and \( p \) is even,
\[
n^+_0 = (q - 1)/2, \text{ and } n^-_0 = (q - 3)/2. \tag{3.12}
\]

We will prove that the hypotheses of Case 1 and Case 3 lead to contradictions. But we need only consider one of them, say Case 3. For, if we reverse all the arcs in \( T \), and replace the index \( i \) with \( p - 1 - i \) for all \( v_i \) with \( 0 \leq i \leq p - 1 \), we change one of these two cases into the other. Furthermore, whether \( T \) is path extendable will not be changed under the operation of reversing all arcs. Before discussing the cases, we state a few general facts.

For every \( 1 \leq i \leq p - 3 \), if \( v_0 \rightarrow v_{i+1} \) and \( v_i \rightarrow v_{p-1} \), then the path
\[
v_0v_{i+1}\ldots v_{p-2}u_0v_1\ldots v_iv_{p-1}
\]
extends \( P \), a contradiction. Therefore at most one of \( v_0 \rightarrow v_{i+1} \) and \( v_i \rightarrow v_{p-1} \) holds. Thus,
\[
d^+(v_0) + d^-(v_{p-1}) \leq n^+_0 + n^-_0 + (p-3) + 4 = (q-2)+(p-3)+4 = p+q-1 = 2m,
\]
and equality must hold. Hence, for all \( 1 \leq i \leq p - 3 \), exactly one of \( v_0 \rightarrow v_{i+1} \) and \( v_i \rightarrow v_{p-1} \) holds. Furthermore, \( v_0 \rightarrow v_{p-1}, v_0 \rightarrow v_1 \), and \( v_{p-2} \rightarrow v_{p-1} \). Since \( T \) is a tournament, for every \( 1 \leq i \leq p - 3 \), \( v_{i+1} \rightarrow v_0 \) and \( v_i \rightarrow v_{p-1} \) or \( v_0 \rightarrow v_{i+1} \) and \( v_{p-1} \rightarrow v_i \). We denote by \( A_0 \) \((A_1)\) the set of arcs \( v_iv_{i+1} \), where \( v_{i+1} \rightarrow v_0 \) and \( v_i \rightarrow v_{p-1} \) \((v_0 \rightarrow v_{i+1} \text{ and } v_{p-1} \rightarrow v_i)\), for \( 1 \leq i \leq p - 3 \).

Note that for every \( 1 \leq i \leq p - 3 \), \( v_iv_{i+1} \) belongs to \( A_0 \) or \( A_1 \). The set of in-neighbors of \( v_{p-1} \) consists of \( N^{-}_0 = N_{F}(v_{p-1}), \{v_0,v_{p-2}\} \) and \( \{v_i : v_iv_{i+1} \in A_0\} \). Therefore,
\[
|A_0| = d^-(v_{p-1}) - n^-_0 - 2 = m - 2 - n^-_0. \tag{3.13}
\]
Similarly, considering the out-neighbors of \( v_0 \), we have
\[
|A_1| = d^+(v_0) - n^+_0 - 2 = m - 2 - n^+_0. \tag{3.14}
\]
Now we consider Case 3. By $r + s + 2 = 3p/2$, $r \leq p - 3$ and $s \leq p - 2$, we have

\[ r \geq p/2 \text{ and } s \geq p/2 + 1. \]

We can always assume that $s \geq r$. For, if $s = p - 2$, then $r = p/2$ and $s \geq r$. And since $s \geq p/2 + 1 \geq 1$, if $s < p - 2$ and $s < r$, then both $u_0$ and $u_1$ are hybrid vertices of $P'$, and we can exchange $u_0$ and $u_1$ without affecting our proof.

By (3.12), (3.13) and (3.14),

\[ |A_0| = m - 2 - n_0^- = m - (q - 3)/2 - 2 = p/2 - 1, \]

and

\[ |A_1| = m - 2 - n_0^+ = m - (q - 1)/2 - 2 = p/2 - 2. \]

By Lemma 3.15, either $v_{r+1} \to v_i$ for all $0 \leq i \leq r - 1$, or $v_j \to v_r$ for all $r + 2 \leq j \leq p - 1$. If the former holds, then

\[
\begin{align*}
    d^+(v_{r+1}) &= d^+_p(v_{r+1}) + d^+(v_{r+1}, F) \\
                 &\geq (r + 1) + (n_0^+ + 1) \\
                 &\geq n_0^+ + r + 2 \\
                 &\geq (q - 1)/2 + p/2 + 2 \\
                 &= m + 2, 
\end{align*}
\]

(a contradiction. Therefore $v_j \to v_r$ for all $r + 2 \leq j \leq p - 1$, which implies that $v_r v_{r+1} \in A_1$. Similarly, $v_j \to v_s$ for all $s + 2 \leq j \leq p - 1$.

We prove that for all $r \leq i \leq p - 3$, $v_r v_{i+1} \in A_1$. We already have $v_r v_{r+1} \in A_1$. Suppose there exists $r + 1 \leq i \leq p - 3$, such that $v_i v_{i+1} \in A_0$. If there further exists $0 \leq j \leq r - 1$ such that $v_j \to v_{i+1}$, then the path

\[ u_0 \cdots v_j v_{i+1} \cdots v_{p-2} u_0 v_{p-1} \]

extends $P$, a contradiction. Hence, for all $0 \leq j \leq r - 1$, $v_{i+1} \to v_j$. But then similar to (3.15), we have

\[
\begin{align*}
    d^+(v_{i+1}) &\geq n_0^+ + r + 2 \geq (q - 1)/2 + p/2 + 2 = m + 2, 
\end{align*}
\]

a contradiction. So, for all $r \leq i \leq p - 3$, $v_i v_{i+1} \in A_1$.

Let $k = \max\{j|2 \leq j \leq r, v_{j-1} v_j \in A_0\}$. Then $v_k v_{k+1} \in A_1$. The path

\[ P_1 = v_0 v_{k+1} \cdots v_{p-2} u_0 v_1 \cdots v_{k-1} v_{p-1} \]
is a path with vertex set $V(P) \cup \{u_0\} \setminus \{v_k\}$. Since $P$ is not extendable, $v_k$ cannot be inserted into $P_1$. By $u_0 \rightarrow v_k$, we must have
\[ \{v_1, v_2, \ldots, v_{k-1}, v_{p-1}\} \rightarrow v_k. \]

Since $s \geq r$ and $u_0 \rightarrow v_k$, we have $u_1 \rightarrow v_k$.

Therefore,
\[ d^-(v_k) \geq n_0^- + 2 + k = (q+1)/2 + k. \]

So,
\[ k \leq d^-(v_k) - (q+1)/2 = m - (q+1)/2 = p/2 - 1. \]

Thus,
\[ |A_1| \geq (p-2) - k \geq p/2 - 1, \]
contradicting $|A_1| = p/2 - 2$. Therefore Case 3, and also Case 1, can not occur.

Now we consider Case 2. Since $r + s = p - 2$ and $r \leq p - 3$, we see that $s \geq 1$. Similarly to the discussion of Case 3, we can assume that $r \leq s$. Since $p$ is odd and $r + s = p - 2$, we have
\[ r \leq (p-3)/2 < (p-1)/2 \leq s. \]

And by $r \geq 1$, we have $s \leq p - 3$. Therefore $u_1$ is also a hybrid vertex of $P'$, so $P'$ has exactly two hybrid vertices. By (3.11), (3.13) and (3.14),
\[ |A_0| = |A_1| = m - 2 - (q/2 - 1) = (p-3)/2. \]

By Lemma 3.15, either $v_{r+1} \rightarrow v_i$ for all $0 \leq i \leq r - 1$, or $v_j \rightarrow v_r$ for all $r + 2 \leq j \leq p - 1$. Suppose that the latter holds. Note that the $N^-_F(v_r) = \{u_0, u_1\} \cup N^-_0$. Together with $r \leq (p-3)/2$, we have
\[ d^-(v_r) \geq 2 + n_0^- + 1 + ((p-1) - (r+2) + 1) \]
\[ = n_0^- + p - r + 1 \]
\[ \geq q/2 - 1 + p - (p-3)/2 + 1 \]
\[ = (p+q)/2 + 3/2 \]
\[ = m + 2, \]
contradicting $d^-(v_r) = m$. Therefore, we have $v_{r+1} \rightarrow v_i$ for all $0 \leq i \leq r - 1$. Furthermore, by $v_{r+1} \rightarrow v_0$, we have that $v_r v_{r+1} \in A_0$. 

Now we prove that for all \( 1 \leq i \leq r \), \( v_i v_{i+1} \in A_0 \). Suppose there exists \( 1 \leq i \leq r - 1 \), such that \( v_i v_{i+1} \in A_1 \). If there further exists \( r + 2 \leq j \leq p - 1 \) such that \( v_i \to v_j \), then the path
\[
v_0 v_{i+1} \ldots v_{j-1} u_0 v_1 \ldots v_i v_j \ldots v_{p-1}
\]
extends \( P \), a contradiction. Hence, for all \( r + 2 \leq j \leq p - 1 \), \( v_j \to v_i \). Note that \( N_F(v_i) = \{u_0, u_1\} \cup N_0^- \). Therefore,
\[
d^+(v_i) \geq 2 + n_0^- + 1 + ((p - 1) - (r + 2) + 1) = n_0^- + p - r + 1 \geq m + 2,
\]
a contradiction, where the final inequality is obtained in (3.16). So we must have \( v_i v_{i+1} \in A_0 \) for all \( 1 \leq i \leq r \).

By similar arguments, we can conclude that for all \( s \leq j \leq p - 3 \), \( v_j v_{j+1} \in A_1 \).

Let \( k = \max\{j \mid r + 1 \leq j \leq s, v_{j-1} v_j \in A_0\} \) (possibly \( r + 1 = k = s \)). Then \( v_k v_{k+1} \in A_1 \). Consider the paths
\[
P_1 = v_0 v_{k+1} \ldots v_{p-2} u_0 v_1 \ldots v_{k-1} v_{p-1}
\]
and
\[
P_2 = u_0 v_{k+1} \ldots v_{p-2} u_1 v_1 \ldots v_{k-1} v_{p-1}.
\]
Since \( P \) is not extendable, \( v_k \) cannot be inserted into \( P_1 \) or \( P_2 \). And since \( v_k \to u_0 \) and \( u_1 \to v_k \), we have \( v_k \to \{v_0, v_{k+1}, \ldots, v_{p-2}\} \) and \( \{v_1, \ldots, v_{k-1}, v_{p-1}\} \to v_k \). Since \( r + 1 \leq k \leq s \), \( v_k \to u_0 \) and \( u_1 \to v_k \). Together with \( v_k \to N_0^+ \) and \( N_0^- \to v_k \), we totally determine the direction of arcs between \( v_k \) and the other vertices. Therefore,
\[
m = d^+(v_k) = n_0^+ + 1 + (p - 2 - k) = q/2 + p - k - 1
\]
and
\[
m = d^-(v_k) = n_0^- + 1 + (k - 1) + 1 = q/2 + k.
\]
Therefore \( k = (p - 1)/2 \). If \( p \geq 7 \), then \( k \geq 3 \), and the path
\[
v_0 v_1 v_k v_{k+1} \ldots v_{p-2} u_0 v_2 \ldots v_{k-1} v_{p-1}
\]
extends \( P \), a contradiction. Therefore \( p = 5 \). It is easy to see that \( r = 1 \), \( s = k = 2 \), \( v_1 v_2 \in A_0 \) and \( v_2 v_3 \in A_1 \). Considering the degrees of \( v_0 \), \( v_1 \), \( v_3 \) and \( v_4 \), we have \( v_1 \to v_3 \) and \( v_0 \to v_4 \). Hence \( \langle V(P) \cup \{u_0, u_1\} \rangle = D_{11} + u_0 u_1 \) or \( D_{11} + u_1 u_0 \). Furthermore, if \( |N_0^+| = |N_0^-| > 0 \), by considering the degrees of the vertices in \( N_0^+ \), we have that \( |N_0^+| = |N_0^-| \geq 3 \). Therefore \( T \in T_3 \). □
Now we construct tournaments in $T_3$ of all possible orders. There is only one regular tournament $D_{11} + u_0u_1$ in $T_3$ with $|N_0^+| = |N_0^-| = 0$, which is isomorphic to the digraph in Figure 3.1 (c). If $|N_0^+| = |N_0^-| \geq 3$, then $n \geq 13$.
For every odd integer $n \geq 13$, we construct a regular tournament $T_3(n)$ on $n$ vertices such that $T_3(n) \in T_3$. Let $N_0^+ = W_0^+ \cup \{w_1^+, w_2^+, w_3^+\}$ and $N_0^- = W_0^- \cup \{w_1^-, w_2^-, w_3^-\}$. In $T_3(n)$, $(W_0^+ \cup W_0^-)$ is either empty or an almost regular tournament in which the semi-degrees of the vertices in $W_0^+$ (respectively $W_0^-$) are $(m - 6, m - 7)$ (respectively $(m - 7, m - 6)$), $(\{w_1^+, w_2^+, w_3^+\})$ and $(\{w_1^-, w_2^-, w_3^-\})$ are directed triangles. The directions of the other arcs follow Table 3.2, in which the vertices or vertex sets in the first column dominate those in the second column. Furthermore, if $n \geq 17$, we can always find a cycle $u_0u_1u_2u_3u_0$ in $T_3(n)$, such that $u_2 \in W_0^+$ and $u_3 \in W_0^-$. By reversing the direction of the cycle we have another regular tournament in $T_3$ of $n$ vertices, in which $u_1 \rightarrow u_0$.

<table>
<thead>
<tr>
<th>Dominating vertices or vertex sets</th>
<th>Dominated vertices or vertex sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_0$</td>
<td>$u_1, {w_1^-, w_2^-, w_3^-}$ and $W_0^+$</td>
</tr>
<tr>
<td>$u_1$</td>
<td>${w_1^+, w_2^+, w_3^+}$ and $W_0^+$</td>
</tr>
<tr>
<td>${w_1^-, w_2^-, w_3^-}$</td>
<td>$u_0, u_1, W_0^-$ and ${w_1^-, w_2^-, w_3^-}$</td>
</tr>
<tr>
<td>${w_1^+, w_2^+, w_3^+}$</td>
<td>$W_0^-$</td>
</tr>
<tr>
<td>$W_0^-$</td>
<td>${w_1^-, w_2^-, w_3^-}$ and ${w_1^+, w_2^+, w_3^+}$</td>
</tr>
<tr>
<td>$W_0^+$</td>
<td>$u_0$ and $u_1$</td>
</tr>
</tbody>
</table>

Table 3.2: The direction of some arcs in $T_3(n)$

We discuss the case that $P'$ has no hybrid vertex in the next theorem. Let $T_4$ be the set of regular tournaments whose vertices can be partitioned into 5 sets $V = \{v_i, 0 \leq i \leq p - 1\}$, $N_0^+, N_1^+, N_0^-$ and $N_1^-$, where $p \geq 3$ is odd, $|N_0^+| = |N_0^-| = n_0$, $|N_1^+| = |N_1^-| = n_1$, $n_0 + n_1 \geq p$, $v_0v_1 \ldots v_{p-1}$ is a path, $V \rightarrow N_0^+, N_0^- \rightarrow V, V\{v_0\} \rightarrow N_1^+, N_1^- \rightarrow v_0, N_1^- \rightarrow V\{v_{p-1}\}$ and $v_{p-1} \rightarrow N_1^-$.  

Note that $T_1$ is a subset of $T_4$ in which $p = 3, N_1^- = N_1^+ = 0$, and $T_2\{T_2(7)\}$ is a subset of $T_4$ in which $p = 3, N_1^+ = \{u_0\}$, and $N_1^- = \{u_1\}$. Therefore, by Theorem 3.14, if $p = 3$, then $T \in T_4$ or $T = T_2(7)$. 

Theorem 3.17. If $p \geq 4$ and $P'$ has no hybrid vertex, then $T \in T_4$.

Proof. Suppose $P'$ has no hybrid vertex and $p \geq 4$. The vertices in $F$ can be partitioned into 4 sets: those dominated by $V(P)$, denoted by $N_0^+$; those dominated by $V(P)$ except $v_0$, denoted by $N_1^+$; those dominating $V(P)$, denoted by $N_0^-$; and those dominating $V(P)$ except $v_{p-1}$, denoted by $N_1^-$. We further let

$$|N_0^+| = n_0^+, \quad |N_1^+| = n_1^+, \quad |N_0^-| = n_0^-, \quad \text{and} \quad |N_1^-| = n_1^-.$$ 

Then

$$n_0^+ + n_1^+ + n_0^- + n_1^- = q. \quad (3.17)$$

Since $T$ is regular, $d^+(P, F) = d^+(F, P)$, that is

$$n_0^+ p + n_1^+ (p - 1) + n_0^- p + n_1^- (p - 1) + n_1^+,$$

which simplifies to

$$(n_0^+ - n_0^-)p = (n_1^- - n_1^+)(p - 2). \quad (3.18)$$

Again by the regularity of $T$, we have

$$d^+(P', F) + d^+(P', \{v_0, v_{p-1}\}) = d^+(F, P') + d^+(\{v_0, v_{p-1}\}, P').$$

Therefore,

$$|(n_0^+ + n_1^+ - n_0^- - n_1^-)(p - 2)| = |d^+(P', F) - d^+(F, P')|$$

$$= |d^+(\{v_0, v_{p-1}\}, P') - d^+(P', \{v_0, v_{p-1}\})|$$

$$\leq 2(p - 2) - 1 - 1$$

$$= 2(p - 3).$$

Since $p \geq 4$ and $n_0^+ + n_1^+ - n_0^- - n_1^-$ is an integer, we have

$$|n_0^+ + n_1^+ - n_0^- - n_1^-| \leq 1.$$ 

Therefore

$$(n_0^+ - n_0^-) - (n_1^- - n_1^+) \in \{-1, 0, 1\}.$$

Together with (3.18) we have three cases.
Case 1: $p$ is even, $q$ is odd,
\[ n_0^+ - n_0^- = p/2 - 1 \text{ and } n_1^- - n_1^+ = p/2; \] (3.19)

Case 2: $p$ is odd, $q$ is even,
\[ n_0^+ = n_0^- \text{ and } n_1^+ = n_1^-; \] (3.20)

Case 3: $p$ is even, $q$ is odd,
\[ n_0^+ - n_0^- = 1 - p/2 \text{ and } n_1^- - n_1^+ = -p/2; \]

If we reverse the direction of all arcs in $T$ and replace the index $i$ by $p - 1 - i$ for all $v_i$ with $0 \leq i \leq p - 1$, we can switch between Case 1 and Case 3. Furthermore, note that $T_4$ is closed under the operation of reversing all arcs in a tournament. Therefore, Case 1 and Case 3 are equivalent and we will only discuss Case 1.

We firstly consider Case 2. Let $n_0 = n_0^+ = n_0^-$ and $n_1 = n_1^+ = n_1^-$. By (3.17),
\[ n_0 + n_1 = q/2. \] (3.21)

By
\[ m = d^-(v_0) \geq 2n_1 + n_0 = n_1 + q/2, \]
we have
\[ n_1 \leq m - q/2 = (p - 1)/2. \] (3.22)

Considering the in-degree of the vertices in subdigraph $(N_0^+ \cup N_1^+)$, we have
\[ \frac{p + q - 1}{2}(n_0 + n_1) = \sum_{v \in (N_0^+ \cup N_1^+)} d^-(v) \geq pm_0 + (p - 1)n_1 + \frac{(n_0 + n_1)(n_0 + n_1 - 1)}{2}. \] (3.23)

Replacing $n_0 + n_1$ by $q/2$, (3.23) simplifies to
\[ q^2 - 2pq + 8n_1 \geq 0. \]
And by (3.22)\[ q^2 - 2pq + 4(p - 1) \geq 0, \]
that is\[ (q - 2p + 2)(q - 2) \geq 0. \tag{3.24} \]
Since $p$ is odd, $p \geq 5$. And since $T$ is regular and $q$ is even, we must have $q \geq 4$ to balance the in-degree and out-degree of the vertices in $N_0^+$ or $N_1^+$. Hence by (3.24)\[ q \geq 2p - 2. \tag{3.25} \]

Suppose $q \geq 2p$. By (3.21) and (3.22),\[ n_0 = q/2 - n_1 \geq p - n_1, \]
and therefore $T \in T_4$.

If $q = 2p - 2$, then all equalities in (3.22) and (3.23) must hold. In particular, $d^-(v_0) = 2n_1 + n_0$. Therefore, $v_0 \to P'$. Similarly, $P' \to v_{p-1}$. Thus $P'$ is a non-extendable path with $p - 2$ vertices. If $|P'| \geq 5$, then taking $P'$ as $P$ we reduce the discussion to the case $q \geq 2p$. If $|P'| = 3$, then it can be proved that $T \in T_1 \subseteq T_4$, and $T$ is uniquely determined up to isomorphism.

Firstly, we have $p = 5$, $q = 8$, $n = 13$, and two non-extendable paths $P$ and $P'$. Note that considering the non-extendable path $P$ we cannot say that $T \in T_4$, because in the definition of $T_4$ we pose the restriction that $n_0 + n_1 \geq p$, while in this case we have $n_0 + n_1 = q/2 = 4 \leq p = 5$. However, if we replace $P$ with $P'$ we can prove that $T$ belongs to $T_1 \subseteq T_4$ and is unique. Note that $V(P') \to N_0^+ \cup N_1^+ \cup \{v_4\}$ and $N_0^- \cup N_1^- \cup \{v_0\} \to V(P')$. By the regularity of $T$, we must have $v_3 \to v_1$, therefore $V(P')$ induce a cycle. Since $|N_0^+ \cup N_1^+ \cup \{v_4\}| = |N_0^- \cup N_1^- \cup \{v_0\}| = 5 \geq 3$, $T \in T_1$.

By (3.22), $n_1 \leq (p - 1)/2 = 2$ and hence $n_0 = q/2 - n_1 \geq 2$. Since $v_0 \to N_0^+ \cup V(P'), 6 = d^+(v_0) \geq 3 + n_0$ and $n_0 \leq 3$. So $n_0 \in \{2, 3\}$. Suppose $n_0 = 3$. Since $\{v_i: 0 \leq i \leq 4\} \to N_0^+$ and every vertex has in-degree 6, the three vertices in $N_0^+$ must induce a cycle, and $N_0^+ = N_0^+ \cup N_1^- \cup N_0^-$. However, for the only vertex $w \in N_1^+$, $N^-(w) \supseteq N_0^+ \cup \{v_1, v_2, v_3, v_4\}$ and so $d^-(w) \geq 7$, a contradiction. So $n_0 = 2$ and hence $n_1 = 2$. By the regularity of $T$, we must have $v_0 \to v_4$. Now let $H = \langle N_0^+ \cup N_0^- \cup N_1^+ \cup N_1^- \rangle$, $N_0^+ = \{u_0^+, u_1^+, v_1, v_2, v_3, v_4\}$, $N_0^- = \{u_0^-, w_0^+, w_1^+\}$, and $N_1^- = \{w_0^-, w_1^-\}$. Furthermore, without loss of generality we may assume $u_0^+ \to u_1^+, u_0^- \to w_1^-, u_0^+ \to w_1^+$ and $w_0^- \to w_1^-$. It is
not hard to calculate that \( d_H^-(v_0^+) = d_H^+(v_0^+) = 1 \) and \( d_H^-(w_0^+) = d_H^+(w_1^+) = 2 \). Together with \( u_0^+ \rightarrow u_1^+ \) and \( w_0^+ \rightarrow w_1^+ \), we deduce that \( u_1^+ \rightarrow \{w_0^+, w_1^+\} \), \( w_1^+ \rightarrow u_0^+ \), \( w_0^+ \rightarrow w_0^+ \), and \( N_0^+ \cup N_1^+ \rightarrow N_0^- \cup N_1^- \). Similarly, we can determine the direction of all arcs between the vertices in \( N_0^- \cup N_1^- \), and so \( T \) is uniquely determined.

We now consider Case 1. By (3.17) and (3.19),
\[
    n_1^- + n_0^+ = (p + q - 1)/2 = m.
\]
However,
\[
    m = d^+(v_{p-1}) \geq n_0^+ + n_1^+ + n_1^- = m + n_1^-.
\]
Therefore
\[
    n_1^+ = 0, \tag{3.26}
\]
and
\[
    \{v_i, 0 \leq i \leq p - 2\} \rightarrow v_{p-1}. \tag{3.27}
\]
By (3.17), (3.19), and (3.26), we have
\[
    n_0^+ = (q - 1)/2, \quad n_1^+ = 0, \quad n_0^- = (q - p + 1)/2 \quad \text{and} \quad n_1^- = p/2. \tag{3.28}
\]

Then the path \( P_1 = P[v_0, v_{p-2}] \) is a non-extendable path of length \( p - 1 \) in \( T \), with \( V(P_1) \rightarrow N_0^+ \cup \{v_{p-1}\} \) and \( N_0^- \cup N_1^- \rightarrow V(P_1) \), where \( |N_0^+ \cup \{v_{p-1}\}| = |N_0^- \cup N_1^-| = (q + 1)/2 \), which can be viewed as a special case of Case 2.

Therefore \( T \in \mathcal{T}_4 \). \( \square \)

Now we construct tournaments in \( \mathcal{T}_4 \) in which \( p, n_0 \) and \( n_1 \) take every possible value, respectively. Specifically, for every odd integer \( p \geq 3, n_1 \leq (p - 1)/2 \) and \( n_0 \geq p - n_1 \), we construct a regular tournament \( T_4(p, n_0, n_1) \in \mathcal{T}_4 \) on \( p + 2(n_0 + n_1) \) vertices. For \( p = 3, n_1 = 0 \) and \( n_0 \geq 3 \), we let \( T_4(3, n_0, 0) = T_1(3 + 2n_0) \). For \( p = 3, n_1 = 1 \) and \( n_0 \geq 2 \), we let \( T_4(3, n_0, 1) = T_2(5 + 2n_0) \).

For \( p \geq 5 \), \( T_4(p, n_0, n_1) \) are constructed as follows. Let \( N_0^+ = W_0^+ \cup W_1^+ \) and \( N_0^- = W_0^- \cup W_1^- \) be partitions of \( N_0^+ \) and \( N_0^- \), such that \( |W_0^+| = |W_0^-| = n_0 + n_1 - p, \) and \( |W_1^+| = |W_1^-| = p - n_1 \). Furthermore, let \( X_0 \cup X_1 \) be a partition of \( W_1^+ \cup N_1^+ \cup W_1^- \cup N_1^- \) such that \( |X_0| = (3p - 1)/2 \) and \( |X_1| = (p + 1)/2 \). The arcs are arranged so that the following statements are satisfied.
(1) If \( n_1 = 0 \), then \( \langle V(P) \rangle \) is a regular tournament on \( p \) vertices.

(2) If \( n_1 \geq 1 \), then \( \langle V(P') \rangle \) is a regular tournament on \( p - 2 \) vertices. Further, \( v_0 \to v_{p-1} \). There is a partition \( V_0 \cup V_1 \) of the vertex set \( \{v_1, \ldots, v_{p-2}\} \) such that \( |V_0| = (p - 3)/2 + n_1, |V_1| = (p - 1)/2 - n_1 \) and \( v_1, v_{p-2} \in V_0 \) (note that by \( p \geq 5 \) and \( n_1 \geq 1 \), \( |V_0| = (p - 3)/2 + n_1 \geq 2 \)), where \( v_0 \to V_0, V_0 \to v_{p-1}, v_{p-1} \to V_1 \) and \( V_1 \to v_0 \).

(3) \( \langle W_1^+ \cup N_1^+ \rangle \) and \( \langle W_1^- \cup N_1^- \rangle \) are two regular tournaments on \( p \) vertices.

(4) \( \langle W_0^+ \cup W_0^- \rangle \) is an almost regular tournament in which the vertices in \( W_0^+ \) have semi-degrees \( (n_0 + n_1 - p, n_0 + n_1 - p - 1) \) and those in \( W_0^- \) have semi-degrees \( (n_0 + n_1 - p - 1, n_0 + n_1 - p) \).

(5) The arcs between \( \langle W_1^+ \cup N_1^+ \rangle \) and \( \langle W_1^- \cup N_1^- \rangle \) are from \( \langle W_1^+ \cup N_1^+ \rangle \) to \( \langle W_1^- \cup N_1^- \rangle \), except that there are \( n_1 \) independent arcs from \( N_1^- \) to \( N_1^+ \).

(6) \( W_0^+ \to X_0, X_0 \to W_0^-, W_0^- \to X_1 \) and \( X_1 \to W_0^+ \).

Theorem 3.11 now follows from the above results. Let \( T \) be a regular tournament that is not \( \{2+\}\)-path extendable. Let \( P \) be a non-extendable path on \( p \geq 3 \) vertices in \( T \). If \( p = 3 \), then by Theorem 3.14 and the former discussion, \( T \in \mathcal{T}_4 \) or \( T = T_2(7) \). If \( p \geq 4 \), then by Theorem 3.16 and Theorem 3.17, \( T \in \mathcal{T}_3 \) or \( T \in \mathcal{T}_4 \). For convenience, we repeat the statement of Theorem 3.11.

**Theorem 3.11.** A regular tournament \( T \) with at least 7 vertices is \( \{2+\}\)-path extendable, unless \( T \in \mathcal{T}_3 \), \( T \in \mathcal{T}_4 \) or \( T = T_2(7) \).

### 3.4 Open problems

We have studied extremal and degree conditions for path extendability in digraphs, as well as path extendability in regular tournaments. We conclude the section with a few problems on cycle extendability and path extendability.

Proposition 3.1 and Lemma 3.13 reveal some interesting connections between cycle extendability and path extendability. Therefore, we raise the following question, which might motivate further research on the relationship between these two properties.
Problem 3.1. Which conditions imply that a cycle extendable digraph is path extendable?

In [109], Zhang proved the following elegant condition on the existence of paths of every length between two end vertices of an arc in a tournament.

A digraph $D$ with $n$ vertices is called arc-pancyclic (respectively arc-antipancyclic) if for each $uv \in A(D)$, there is a path from $v$ to $u$ (respectively from $u$ to $v$) of length $k$ for $k = 2, 3, \ldots, n-1$, denoted by $P_k(u, v)$ (respectively $P'_k(u, v)$). A tournament $T$ is called completely strong path-connected, if $T$ is arc-pancyclic and arc-antipancyclic.

**Theorem 3.18.** ([109]) A tournament $T$ is completely strong path-connected if and only if for every $uv \in A(T)$, the paths $P_2(u, v)$ and $P'_2(u, v)$ exist, and $T$ does not belong to a well-defined exceptional class of tournaments.

For convenience, we say that a digraph $D$ satisfies Zhang’s condition, if for every arc $uv$ in $D$, the paths $P_2(u, v)$ and $P'_2(u, v)$ exist. Obviously, a tournament satisfying Zhang’s condition is strong. Therefore, by Theorem 3.8, it is cycle extendable with some exception, which we will not describe in detail here. Given that path extendability does not generally hold for tournaments, and Zhang’s condition seems to fill a gap of Theorem 3.11, i.e., the non-extendability of arcs, we ask whether it can imply path extendability in tournaments.

**Problem 3.2.** Does Zhang’s condition imply path extendability in tournaments?

In [101], Thomassen introduced the concept of irregularity. The irregularity $i(T)$ of a tournament $T$ is $\max |d^+(u) - d^-(u)|$ over all vertices $u \in V(T)$. A digraph $D$ with $n$ vertices is strongly panconnected if for any integer $k$, $3 \leq k \leq n-1$, and any two vertices $u, v$ of $D$, there is a path of length $k$ from $u$ to $v$ and a path of length $k$ from $v$ to $u$. It is shown that if a tournament $T$ has a sufficiently large number of vertices relative to its irregularity, then $T$ has strong pancyclic and antipancyclic properties.

**Theorem 3.19.** ([101]) Let $T$ be a tournament with $n$ vertices, and $i(T) \leq k$. Let $u, v$ be two vertices of $T$. If $vu$ is an arc of $T$ and $n \geq 5k + 3$, then $vu$ is contained in cycles of all lengths $m$, $4 \leq m \leq n$, and if $uv$ is an arc of $T$ and
\[ n \geq 5k + 9, \text{ then there exist paths from } u \text{ to } v \text{ of all lengths } m, 3 \leq m \leq n - 1. \]

In particular, a tournament with \( n \) vertices and irregularity at most \( (n - 9)/5 \) is strongly panconnected.

A natural question is whether there are similar conditions for cycle extendability and path extendability. Note that strongly panconnectedness implies strongness, and hence cycle extendability, in tournaments. So the last upper bound on irregularity in Theorem 3.19 also implies cycle extendability. This provides an answer to this question. But we do not know whether the bound is best possible.

**Problem 3.3.** Let \( T \) be a tournament with \( n \) vertices, and \( i(T) \leq k \). Does there exist a function \( f_1 \) (respectively \( f_2 \)), such that if \( n \geq f_1(k) \) (respectively \( n \geq f_2(k) \)), then \( T \) is cycle extendable (respectively path extendable)?
Chapter 4

Hamiltonicity, Pancyclicity and Cycle Extendability in Bipartite Tournaments

4.1 Introduction

Throughout this chapter, we consider both finite undirected and finite directed graphs. When it is necessary to distinguish between undirected and directed graphs, we use the shortened terms digraph, dicycle and dipath for directed graph, directed cycle and directed path, respectively. When there is no cause for confusion, the terms graph, cycle and path may refer to both undirected and directed versions.

Cycles in graphs have been studied extensively both on theoretical and algorithmic aspects. Research works on them have formed a field with numerous significant achievements and challenging problems. For example, the central problem of determining whether there is a Hamiltonian cycle in a given graph is \textbf{NP}-complete. Therefore, many research works focus on the existence of Hamiltonian cycles in some special classes of graphs and polynomial algorithms to find them, such as [53], [15] and [70]. On the other hand, sufficient conditions for Hamiltonicity of graphs have taken up a great proportion of the theoretical results. A typical kind of such conditions is that of degree-constrained conditions, including Dirac’s condition on the degrees of single
vertices and Ore’s condition on the degree sum of nonadjacent vertex pairs, and their many generalizations (See, e.g., [46], [92], [106], and Chapter 2 of this thesis). There are many other variations on cycle problems, including cycles in random graphs (See, e.g., [21] and [44]) or in edge-colored graphs (See, e.g., [22] and [1]). Recently, Kühn and Osthus ([73]) surveyed approximate solutions of several long-standing problems and conjectures on dicycles.

A milestone in the research of cycle problems is the meta-conjecture of Bondy that almost any nontrivial condition on graphs which implies that the graph is Hamiltonian also implies that the graph is pancyclic (see [24]). It has led to a large amount of subsequent works on cycles and paths of many lengths in graphs. Particularly, with the objective to pursue the theme further, Hendry introduced the concept of cycle extendability in graphs and studied it from various perspectives ([56], [57] and [60]). We give the related definitions in digraphs here. Let $D$ be a digraph. A dicycle $C$ of $D$ is called extendable if there exists a dicycle $C'$ of $D$ such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 1$. If $D$ contains a dicycle, and every non-Hamiltonian dicycle of $D$ is extendable, we say that $D$ is dicycle extendable. Let $D$ be a bipartite digraph on an even number $n$ of vertices. If $D$ contains dicycles of length $k$ (a dicycle of length $k$ will be denoted by a $k$-dicycle) for every $k \in \{4, 6, \ldots, n\}$ (through every vertex $v$ of $D$), then $D$ is called even (vertex-)pancyclic. A dicycle $C$ of $D$ is called even extendable if there exists a dicycle $C'$ of $D$ such that $V(C) \subset V(C')$ and $|V(C')| = |V(C)| + 2$. If every dicycle of $D$ is even extendable, we say that $D$ is even dicycle extendable. If $D$ is even dicycle extendable and contains a 4-dicycle, then we say that it is fully even dicycle extendable. Following the work of Hendry, many research works on cycle extendability came forth. For example, dicycle extendability has been studied in [100] and [87]. Extendability of cycles can be viewed as a strengthening of Hamiltonicity and pancyclicity.

Along Bondy’s idea, it is natural and interesting to investigate the equivalent relationship between Hamiltonicity and some other stronger properties such as pancyclicity and cycle extendability, in certain graph classes. One famous example is the work of Fleischner on equivalence of Hamiltonicity, pancyclicity, Hamiltonian connectedness and panconnectedness in the square of a graph ([42]). In the study of cycle extendability, one problem of such
kind that has caught a lot of attention is Hendry’s question ([56]), which can be viewed as an inquiry on the equivalence between Hamiltonicity and cycle extendability in chordal graphs. This equivalent relationship has been verified in some graph classes ([3] and [33]) which are subclasses of chordal graphs, but was disproved in [76] recently. However, in digraphs, there is relatively little progress in this direction. In this chapter, we investigate Hamiltonicity and cycle extendability in an important class of digraphs, namely bipartite tournaments. It turns out that, with only one exceptional class of digraps, Hamiltonicity, even pancyclicity, even vertex-pancyclicity, even dicycle extendability and fully even dicycle extendability are all equivalent in bipartite tournaments.

A tournament is a digraph with exactly one arc between every pair of distinct vertices, i.e., it is an orientation of a complete graph. A multipartite tournament is a digraph whose vertices can be partitioned into \( k \geq 2 \) sets such that there is exactly one arc between every two vertices from different sets, and no arc between any pair of vertices from the same set. When \( k = 2 \), a multipartite tournament is called a bipartite tournament. Tournaments and multipartite tournaments are arguably the most studied digraph classes for dicycle problems (See, e.g., the surveys [51] and [11]). Since these graphs are relatively dense, many cycle properties may be expected to hold for these graphs. Camion ([28]) proved that every strong tournament is Hamiltonian. Moon ([90]) further proved that every strong tournament is vertex-pancyclic. Hendry ([57]) restated the result of Moon ([91]) that strong tournaments are dicycle extendable, with some exceptions, as follows.

**Theorem 4.1.** A strong tournament is dicycle extendable, unless its vertex set can be partitioned into three nonempty sets \( W, X, \) and \( Y \) such that \( (W) \) is a nontrivial strong tournament, \( W \) completely dominates \( X \), and \( Y \) completely dominates \( W \), i.e., every vertex of \( W \) (\( Y \)) sends an arc to every vertex of \( X \) (\( W \)).

The subclass of bipartite digraphs that corresponds to tournaments is the class of bipartite tournaments, which attracted quite some interest from the research community. Many important results on bipartite tournaments can be found in the literature. An early result due to Jackson involves the following Ore-type condition.
Theorem 4.2. ([61]) Let $T$ be a strong bipartite tournament such that for every $u, v \in V(T)$ where $uv \not\in A(T)$,
\[d^+(u) + d^-(v) \geq m.\]
Then $T$ contains a dicycle of length at least $2m$.

As a corollary, this result implies that every regular bipartite tournament is Hamiltonian.

We now introduce a class of bipartite tournaments, $T(r, r, r, r)$, that play the role of exceptional digraphs for many theorems in this chapter. The vertex set of $T(r, r, r, r)$ can be partitioned into 4 parts $V_0, V_1, V_2$ and $V_3$, with $|V_i| = r$, for $0 \leq i \leq 3$, where every vertex in $V_i$ dominates every vertex in $V_{i+1}$ (the subscripts are taken modulo 4 here, and also in the sequel).

The following result shows that the property of regular bipartite tournaments with respect to cycles is actually much stronger than in the above corollary.

Theorem 4.3. (Amar and Manoussakis [7]) An $r$-regular bipartite tournament is even arc-pancyclic unless it is isomorphic to $T(r, r, r, r)$.

The following property, which will also be useful in the proof of our main result, is important for the design of effective algorithms to find a Hamiltonian dicycle in a bipartite tournament ([84]).

Theorem 4.4. (Gutin [12], Häggkvist and Manoussakis [53]) A bipartite tournament is Hamiltonian if and only if it is strong and has a dicycle factor $F$.

Here, a dicycle factor is a spanning subgraph consisting of dicycles. It can be seen from Theorem 4.4 that in bipartite tournaments strongness is not enough to guarantee the existence of Hamiltonian cycles. But it turns out that being Hamiltonian implies stronger cycle properties. Beineke and Little proved that in bipartite tournaments Hamiltonicity implies even pancyclicity ([17]). Häggkvist and Manoussakis further proved that Hamiltonicity implies even vertex-pancyclicity as well.

Theorem 4.5. (Häggkvist and Manoussakis [53]) A Hamiltonian bipartite tournament $T$ is even vertex-pancyclic, unless $T = T(r, r, r, r)$ for some $r \geq 2$. 
The main result of this chapter is that in bipartite tournaments, Hamiltonicity implies even dicycle extendability, with some exceptions, and hence these two properties are almost equivalent.

**Theorem 4.6.** Let $T$ be a bipartite tournament. Then $T$ is Hamiltonian if and only if $T$ is even dicycle extendable, unless $T$ is isomorphic to $T(r, r, r, r)$, for some integer $r \geq 2$.

It is easy to see that in a bipartite tournament a 4-dicycle always exists, and that in fact every vertex is contained in a 4-dicycle. Therefore, in bipartite tournaments, from even dicycle extendability we can derive even pancyclicity, even vertex-pancyclicity and fully even dicycle extendability. Hence we have the following corollary of Theorem 4.6 on the equivalence between all these properties, which implies the above-mentioned result of Beineke and Little, and Theorem 4.5.

**Corollary 4.7.** In a bipartite tournament $T$, the properties of Hamiltonicity, even pancyclicity, even vertex-pancyclicity, even dicycle extendability and fully even dicycle extendability are equivalent, unless $T$ is isomorphic to $T(r, r, r, r)$, for some integer $r \geq 2$.

Furthermore, almost all results on Hamiltonicity and pancyclicity in bipartite tournaments, including the ones listed above, can be generalized to fully even dicycle extendability. For example, we can deduce that any $r$-regular bipartite tournament is fully even dicycle extendable, unless it is isomorphic to $T(r, r, r, r)$.

Cycle extendability is stronger than Hamiltonicity or pancyclicity, and actually among the strongest cycle properties. However, bipartite tournaments are sparser than tournaments. This implies new challenges for the study of even dicycle extendability in bipartite tournaments. In particular, the classical techniques used in the proofs of the results on tournaments (like in [17] and [53]) are not applicable in our proof.

A new tool that we introduce next, which is crucial for our proof, is based on the concept of an in-out graph of a digraph. An in-out graph of a digraph $D$ is a graph that takes the arc set of $D$ as its vertex set, and in which two vertices are joined by a red (green) edge if they share a common head (tail) in
Chapter 4

The concept looks similar to that of line graphs, and actually in-out graphs can be viewed as line graphs of a certain class of bipartite graphs (see Section 4.3). Let \( C_0 \) and \( C_1 \) be two arc-disjoint Hamiltonian dicycles on the same vertex set, and denote the digraph formed by these two bicycles by \( C_0 \cup C_1 \). Let \( L \) be the in-out graph of \( C_0 \cup C_1 \). It is not hard to see that the independent sets of \( L \) correspond to the dipath-dicycle subdigraphs of \( C_0 \cup C_1 \). Therefore, we can use \( L \) to construct and analyze the dipath-dicycle factors of \( C_0 \cup C_1 \), for instance, as in the basic result of Theorem 4.8. The condition of our theorem guarantees the existence of a Hamiltonian dicycle \( H \). In our proof, we need to consider the even extendability of another dicycle \( C \). We perform some contraction operations on \( C \cup H \) to map \( C \) and \( H \) into arc-disjoint Hamiltonian dicycles \( C_0 \) and \( C_1 \) on the same vertex set. Next, we construct \( L \), and color the edges and label the vertices of \( L \). Then, by analyzing \( L \), we derive many structural properties of \( C \cup H \), which imply the even extendability of \( C \). These ideas and techniques have helped us to develop a concise proof of the main result, avoiding the usually tedious case distinctions that are typical in many proofs of cycle properties.

The chapter is arranged as follows. Section 4.2 contains the terminology and notation we need. In Section 4.3, we formally introduce the concept of the in-out graph. Then, as an application of in-out graphs, a theorem on dicycle factors and 1-dipath-dicycle factors of two arc-disjoint Hamiltonian dicycles on the same vertex set is proved. In Section 4.4, we prove Theorem 4.6, whereas other applications of the main theorem appear in Section 4.3. Finally, we complete the chapter with some remarks in Section 4.5.

4.2 Terminology and Notation

In this section, we provide most of the terms and notations used in this chapter. Some of the others terms and notations will be introduced when used in the sequel, for convenience. The concepts that are not explicitly defined are adopted from [12].

We often use \( D \) to denote a digraph, and \( T \) to denote a bipartite tournament. The vertex set and arc set of a digraph \( D \) are denoted by \( V(D) \) and
A($D$), respectively. Let $X \subseteq V(D)$. Then the subdigraph of $D$ induced by $X$ is denoted by $D\langle X \rangle$, or $(X)$ when $D$ is clear. Let $X$ and $Y$ be two disjoint vertex subsets of a digraph $D$, or two disjoint subdigraphs of $D$. If every vertex in $X$ dominates every vertex in $Y$, we say that $X$ dominates $Y$, and denote it by $X \rightarrow Y$. If $X$ or $Y$ contains only one vertex, we use the vertex label instead of the set or the subdigraph, like in $x \rightarrow Y$.

A $k$-dipath-dicycle subdigraph $F$ of a digraph $D$ is a collection of $k$ dipaths and $m$ dicycles such that all dipaths and dicycles are pairwise disjoint. If $F$ spans $D$, we say that it is a $k$-dipath-dicycle factor of $D$. When $k = 0$ in the above definitions, we call $F$ a dicycle subdigraph and a dicycle factor, respectively.

Let $D$ be a digraph with $n$ vertices. If $D$ contains dicycles of length $k$ for every $3 \leq k \leq n$, we say that $D$ is pancyclic. $D$ is vertex-pancyclic (arc-pancyclic) if it has a dicycle of length $k$ containing $v$ (a), for every $3 \leq k \leq n$ and for every vertex $v \in V(D)$ (arc $a \in A(D)$). The concept of even pancyclicity, even vertex-pancyclicity and even arc-pancyclicity are defined analogously, but in this case only dicycles of (all) even length(s) are required.

### 4.3 In-out graph, and dipath-dicycle factors of two arc-disjoint Hamiltonian dicycles

Let $D$ be a digraph. The in-out graph of $D$ is defined as an edge-colored graph, which takes the arc set of $D$ as its vertex set, and in which two vertices are adjacent by a red edge, if they have a common head in $D$, and by a green edge, if they have a common tail in $D$. We denote the in-out graph of $D$ as $L_{\text{io}}(D)$.

In-out graphs are closely related to line graphs. Firstly, for a given digraph $D$, $L_{\text{io}}(D)$ is a subgraph of the line graph of the underlying graph of $D$. Secondly, it is isomorphic to the line graph $L(G)$ of the associated bipartite graph $G$ of $D$, where $G$ is defined as a bipartite graph with vertex set $V' \cup V''$, with $V' = \{v' : v \in V(D)\}$ and $V'' = \{v'' : v \in V(D)\}$, and edge set $\{u'v'' : uv \in E(D)\}$. If we further color every edge $ef$ of $L(G)$ with the color green or red, according to the common endvertex of $e$ and $f$ in $G$ being in $V'$ or $V''$, we have the same coloring as in $L_{\text{io}}(D)$. 
Next, we consider two arc-disjoint Hamiltonian dicycles \( C_0 \) and \( C_1 \) on the vertex set \( \{0, 1, \ldots, k - 1\} \), where \( k \geq 3 \). We denote an arc from vertex \( i \) to vertex \( j \) by \((i, j)\). A dicycle \( C \) (dipath \( P \)) on \( p \) vertices is denoted by \( i_0, i_1, \ldots, i_{p-1}, i_0 \), \( (i_0, i_1, \ldots, i_{p-1}) \) where \( i_t \in \{0, 1, \ldots, k - 1\}, 0 \leq t \leq p - 1 \leq k - 1 \), and \( i_{t+1} \) is the successor of \( i_t \) on \( C \) \((P)\) for \( 0 \leq t \leq p - 1 \) (where the subscripts are taken modulo \( p \)). Without loss of generality, we assume that \( C_0 = 0, 1, \ldots, k - 1, 0 \).

Consider the in-out graph of \( C_0 \cup C_1 \), and let \( L = L_{io}(C_0 \cup C_1) \). Since every arc in \( C_0 \cup C_1 \) has a common head with exactly one arc, and a common tail with exactly one arc, the corresponding vertex has degree two in \( L \) and is associated with one red edge and one green edge. Therefore, \( L \) consists of some mutually disjoint even cycles, the edges of which are red and green alternately. A vertex of \( L \) corresponds to an arc in \( C_0 \cup C_1 \), and is denoted by \((i, j)\) if it is from vertex \( i \) to \( j \) in \( C_0 \cup C_1 \). An edge of \( L \) joining two vertices \((i_0, j_0)\) and \((i_1, j_1)\) is denoted by \((i_0, j_0) - (i_1, j_1)\).

If we properly color \( V(L) \) with two colors \( l_0 \) and \( l_1 \), we partition \( V(L) \) into two sets, any one of which is an independent vertex set of \( L \). It is not hard to see that the subdigraph of \( C_0 \cup C_1 \) consisting of all the arcs in any one of the two sets, is a dicycle factor of \( C_0 \cup C_1 \). If two arcs of \( C_1 \) are in different cycles of \( L \), we can always color them differently, so that we obtain two arc-disjoint dicycle factors of \( C_0 \cup C_1 \), each of which contains exactly one of these two arcs. Furthermore, we prove in the following theorem that, if these two arcs are on the same cycle of \( L \), we can obtain a 1-dipath-dicycle factor of \( C_0 \cup C_1 \), in which the dipath starts with the head of one arc and ends with the tail of the other arc.

**Theorem 4.8.** Let \( k \geq 3 \) be an integer, and let \( C_0 \) and \( C_1 \) be two arc-disjoint Hamiltonian dicycles on the vertex set \( \{0, 1, \ldots, k - 1\} \). \( C_0 = 0, 1, \ldots, k - 1, 0 \) and \( C_1 = i_0, i_1, \ldots, i_{k-1}, i_0 \), where \( i_{t+1} \neq i_t + 1 \), \( 0 \leq t \leq k - 1 \) \((i_k = i_0)\). Let \( L = L_{io}(C_0 \cup C_1) \). \( L \) for any \( 0 \leq i, j \leq k - 1 \) \((\text{possibly } i = j)\), let \((i, j')\) and \((i', j)\) be arcs of \( C_1 \).

1. If \((i, j')\) and \((i', j)\) are in different cycles of \( L \), then there are two arc-disjoint dicycle factors \( F_0 \) and \( F_1 \) of \( C_0 \cup C_1 \), each of which contains exactly one of \((i, j')\) and \((i', j)\).

2. If \((i, j')\) and \((i', j)\) are on the same cycle of \( L \), then there is a 1-dipath-dicycle factor \( F \) of \( C_0 \cup C_1 \), in which the dipath is a \((j, i)\)-dipath.
**Proof.** Statement (1) holds by the above discussion. Now suppose that \((i, j')\) and \((i', j)\) are on the same cycle \(Q\) of \(L\). Deleting the \((i, j') - (i, i+1)\) edge and the \((i', j) - (j-1, j)\) edge of \(Q\), we obtain two paths, each of which contains an odd number of vertices. We properly color all vertices of the resulting graph \(L'\) with colors \(l_0\) and \(l_1\), such that the vertices \((i, j'), (i, i+1), (i', j)\) and \((j-1, j)\) are colored \(l_0\). Now take the subdigraph \(F\) of \(C_0 \cup C_1\) consisting of all arcs that are colored \(l_1\) in \(L'\). In \(F\), \(d^{-}(i) = 1, d^{+}(i) = 0, d^{-}(j) = 0, d^{+}(j) = 1\), and the in-degree and out-degree of all the other vertices are 1. Therefore, \(F\) is a 1-dipath-dicycle factor of \(C_0 \cup C_1\) in which the dipath is a \((j, i)\)-dipath. □

### 4.4 Proof of Theorem 4.6

It suffices to prove that if \(T\) is Hamiltonian, then it is even dicycle extendable, unless it is isomorphic to \(T(r, r, r, r)\) for some \(r \geq 2\).

It is not hard to verify the conclusion for Hamiltonian bipartite tournaments of order at most 8. Suppose that \(T\) is a Hamiltonian bipartite tournament with bipartition \((W, B)\) and \(|V(T)| \geq 10\), which is not even dicycle extendable, but that all Hamiltonian bipartite tournaments of order less than \(|T|\) are even dicycle extendable, or belong to the exceptional class of bipartite tournaments. Let \(C\) be a longest even non-extendable dicycle in \(T\). Since \(T\) is Hamiltonian, \(|C| \leq |T| - 4\).

**Claim 4.1.** \(V(C)\) is not contained in any non-Hamiltonian dicycle \(C'\), such that \(|C'| \geq |C| + 2\).

Assume that such a \(C'\) exists. Since \(C\) is even non-extendable, \(|C'| \geq |C| + 4\). Let \(T' = T \setminus V(C')\). \(T'\) is a bipartite tournament with a Hamiltonian dicycle \(C'\). By our induction hypothesis, \(T'\) is even dicycle extendable, or isomorphic to \(T(r', r', r', r')\), for some integer \(r' \geq 2\). However, \(C\) is even non-extendable in \(T\), and hence even non-extendable in \(T'\). Therefore, \(T'\) is not even dicycle extendable, and so \(T' = T(r', r', r', r')\) for some integer \(r' \geq 2\). By definition, \(V(T')\) can be partitioned into 4 parts \((V_0, V_1, V_2, V_3)\) where \(|V_i| = r'\) and \(V_i \to V_{i+1}\). Without loss of generality, we assume that \(V_0, V_2 \subseteq W\) and \(V_1, V_3 \subseteq B\).

By our selection of \(C\), \(C'\) is even extendable in \(T\). Suppose that \(C'\) can be extended to a dicycle \(C''\), where \(V(C'') = V(C') \cup \{w, b\}\), \(w \in W\) and \(b \in B\).
Suppose that $w$ and $b$ are adjacent in $C''$, say $wb \in A(C'')$. Denote the predecessor of $w$ and the successor of $b$ on $C''$ by $b_0$ and $w_0$, respectively. We must have $b_0 \in V_{2i-1}$ and $w_0 \in V_{2i}$ for $i = 1$ or 2. For if we traverse $C''$ from $w_0$ to $b_0$, we go through $4r'$ vertices, and the vertices must be in $V_{2i}$, $V_{2i+1}$, $V_{2i+2}$ and $V_{2i+3} = V_{2i-1}$ for $i = 1$ or 2, successively and recursively. Without loss of generality, we assume that $b_0 \in V_3$ and $w_0 \in V_0$ as in (A) of Figure 4.1.

Now suppose that $w$ and $b$ are not adjacent in $C''$. Denote the predecessor and the successor of $w$ (b) by $b_0$ and $b_1$ ($w_0$ and $w_1$). Further assume that the predecessor and the successor of $w$ or $b$ are in the same $V_i$ for some $0 \leq i \leq 3$, say $b_0, b_1 \in V_i$. We traverse $C''$ from $b_1$ to $w_0$ to obtain a dipath $P_0$, and from $w_1$ to $b_0$ to obtain a dipath $P_1$. Then, $V(P_0) \cup V(P_1) = V(C'')$. Since $|V_i| = |V_3|$, and their vertices appear on $P_i$ ($i = 0, 1$) alternatingly, by $b_0, b_1 \in V_1$ we have that the predecessor of $w_0$ on $P_0$ and the successor of $w_1$ on $P_1$ must be in $V_3$. Therefore, $w_0 \in V_0$ and $w_1 \in V_2$, as in (B) of Figure 4.1.

Suppose that the predecessor and the successor of $w$ or $b$ are in different $V_i$ for some $0 \leq i \leq 3$, say $w_0 \in V_0$ and $w_1 \in V_2$. Let $P_0$ and $P_1$ be defined as above. Since $|V_i| = |V_3|$, and their vertices appear on $P_i$ ($i = 0, 1$) alternatingly, and since the predecessor of $w_0$ on $P_0$ and the successor of $w_1$ on $P_1$ are both in $V_3$, we can conclude that $b_0, b_1 \in V_1$, as in (B) of Figure 4.1. Therefore, it suffices to consider the two cases in Figure 4.1.

Note that, as $C$ is a dicycle in $T'$, $|C|$ must be divisible by 4, and the vertices of $C$ must be in $V_0$, $V_1$, $V_2$ and $V_3$, successively. We discuss the possible direction of the arcs between $\{w, b\}$ and $V(C)$ below, and extend $C$ in all cases, thus contradicting that $C$ is not even extendable. In Figure 4.2, we use the shadowed region to denote the vertices of $C'$ that are also in $C$.

Consider (A) of Figure 4.1. If there exist $b'_0 \in V(C) \cap V_3$ and $w'_0 \in V(C) \cap V_0$ such that $b'_0 \rightarrow w$ and $b \rightarrow w'_0$, as in (A.1) of Figure 4.2, it is easy to construct a dicycle with vertex set $V(C) \cup \{w, b\}$. Suppose we can not find any vertex $w'_0 \in V(C) \cap V_0$ such that $b \rightarrow w'_0$. Then $V_0 \cap V(C) \rightarrow b$, and the successor $w_0$ of $b$ on $C'$ is in $V_0 \setminus V(C)$, as in (A.2) of Figure 4.2. Since $w_0 \rightarrow V_1 \cap V(C)$, we can extend $C$ by inserting the arc $bw_0$ between two consecutive vertices in $V_0$ and $V_1$ on $C$. The case that we can not find any vertex $b'_0 \in V(C) \cap V_3$ such that $b'_0 \rightarrow w$ can be handled similarly.

Now suppose that $C'$ is extended as in (B) of Figure 4.1. If we can find
Figure 4.1: The two possible ways to extend $C'$

Figure 4.2: Based on (A) of Figure 4.1 to extend $C'$, we extend $C$. 
\[ w_0' \in V_0 \cap V(C), \ w_1' \in V_2 \cap V(C), \text{ and } b_0', b_1' \in V_1 \cap V(C) \text{ such that } w_0' \to b, \]
\[ b \to w_1', \ b_0' \to w \text{ and } w \to b_1', \] then it is not hard to see that we can extend \( C \) by constructing a cycle with vertex set \( V(C) \cup \{w, b\} \). Suppose that we can not find any such \( w_0', w_1', b_0' \text{ or } b_1' \), then similar to the way we handled (A.2) of Figure 4.2, we can find an arc that can be inserted between two consecutive vertices on \( C \), so that we extend \( C \) and complete the proof of Claim 1. \( \Box \)

Let \( C = u_0u_1 \ldots u_{2m-1}u_0 \), where \( u_{2i} \in W \) and \( u_{2i+1} \in B \), \( 0 \leq i \leq m-1 \).

**Claim 4.2.** If \( T - V(C) \) has a spanning dicycle \( Q \), then \(|Q| = 4\) and \( T = T(r, r, r, r) \) for some integer \( r \geq 2 \).

Suppose the condition holds. Let \( Q = v_0v_1 \ldots v_{2k-1}v_0 \), where \( v_{2j} \in W \) and \( v_{2j+1} \in B \), \( 0 \leq j \leq k-1 \). Firstly, we assume that \(|Q| \geq 6\), i.e., \( k \geq 3 \).

By Claim 4.1, there can not be any non-Hamiltonian dicycle longer than \( C \) and containing all vertices of \( C \). Therefore,

(a) if we have \( u_{2i} \to v_{2j-1} \), we must have (with the subscripts of \( u_i \) taken modulo \( 2m \) and the subscripts of \( v_j \) modulo \( 2k \), and the same below)

\[ u_{2i+1} \to v_{2j}, \ u_{2i+1} \to v_{2j+2}, \ldots, u_{2i+1} \to v_{2j-4}, \]

(b) if we have \( u_{2i-1} \to v_{2j} \), we must have

\[ u_{2i} \to v_{2j+1}, \ u_{2i} \to v_{2j+3}, \ldots, u_{2i} \to v_{2j-3}. \]

Since \( T \) is Hamiltonian, there is at least one arc from \( C \) to \( Q \). Without loss of generality, we assume that \( u_0 \to v_1 \). Since \( k \geq 3 \), by (a) we have \( u_1 \to v_2 \) and \( u_1 \to v_4 \). Then, by (b) we deduce that \( u_2 \) sends arcs to every vertex in \( V(Q) \cap B \). Again by (a), we have that \( u_3 \) sends arcs to every vertex in \( V(Q) \cap W \). Applying (a) and (b) alternatingly, we can finally deduce that every vertex on \( C \) sends an arc to every vertex on \( Q \), in different color classes of it. However, there is no arc from \( Q \) to \( C \), contradicting that \( T \) is Hamiltonian. Therefore \(|Q| = 4\), and so \( Q = v_0v_1v_2v_3v_0 \). Then (a) and (b) become

(a) if we have \( u_{2i} \to v_{2j-1} \), we must have \( u_{2i+1} \to v_{2j} \), and

(b) if we have \( u_{2i-1} \to v_{2j} \), we must have \( u_{2i} \to v_{2j+1} \).

Now we prove that \( m \) is even, i.e., \(|C| \) is divisible by 4.
Suppose that $m$ is odd. Applying $(a')$ and $(b')$, by $u_0 \to v_1$ we have $u_1 \to v_2$, and by $u_1 \to v_2$ we have $v_2 \to v_3$. Repeating the process, we then have $u_{2m-1} \to v_2$, since $m$ is odd. And by $u_{2m-1} \to v_2$ we have $u_0 \to v_3$. Applying $(a')$ and $(b')$ repeatedly, by $u_0 \to v_1$ and $u_0 \to v_3$ we finally deduce that every vertex on $C$ sends an arc to every vertex on $Q$, in different color classes of it, again contradicting that $T$ is Hamiltonian. Hence, $m$ is even.

Let $U_i = \{u_{4t+i}, 0 \leq t \leq m/2 - 1\}$, $i = 0, 1, 2, 3$. By $u_0 \to v_1$, repeatedly applying $(a')$ and $(b')$, we have $U_0 \to v_1, U_1 \to v_2, U_2 \to v_3$ and $U_3 \to v_0$. By the above discussion, we must have $v_3 \to u_0$, and by similar arguments we have $v_2 \to u_0$. For any $1 \leq t \leq m/2 - 1$, we have $v_2 \to u_{4t+2}$. If $u_{4t+2} \to u_1$, then we have a dicycle $u_0v_1u_2\ldots u_{4t+2}u_{4t+3}\ldots u_0$, which extends $C$, a contradiction. Therefore, $u_1 \to u_{4t+2}$. Together with $u_1 \to u_2$, we have $u_1 \to U_2$. By similar arguments, we conclude that $U_i \to U_{i+1}$, for all $0 \leq i \leq 3$ (with the subscripts taken modulo 4). Together with $Q$ and the arcs between $Q$ and $C$, we see that $T = T(r, r, r, r)$, where $r = m/2 + 1$. And by $2m = |C| \geq 4$, we have $r \geq 2$.

\textbf{Claim 4.3.} Let $Q$ be a non-spanning dicycle in $T - V(C)$. Then either $C \to Q$, or $Q \to C$.

Suppose the conclusion does not hold. Then there is at least one arc from $C$ to $Q$ and at least one arc from $Q$ to $C$. Then $T' = T(V(C) \cup V(Q))$ is strong. By Theorem 4.4, $T'$ must be Hamiltonian. But a Hamiltonian dicycle of $T'$ is a non-Hamiltonian dicycle of $T$, which contains all vertices of $C$ and is longer than $C$, contradicting Claim 4.1. 

From now on we assume that $T \neq T(r, r, r, r)$ for any $r \geq 2$.

Let $H$ be a Hamiltonian dicycle of $T$. $H \cap C$ consists of some independent dipaths, some of which may degenerate into vertices. We call these dipaths
common dipaths of $H$ and $C$, or just common dipaths when no ambiguity can be caused. Supposing there are $k$ common dipaths, we denote them by $S_0$, $S_1$, $\ldots$, $S_{k-1}$, according to the order in which they appear on $C$. Removing all arcs and internal vertices of the common dipaths from $C$, what left are all the arcs from the terminating vertex of $S_i$ to the starting vertex of $S_{i+1}$, $0 \leq i \leq k - 1$ (the subscripts modulo $k$, and the same below). We call them $C$-arcs, and denote an arc from the terminal vertex of $S_i$ to the starting vertex of $S_{i+1}$ as $a_C(i, i+1)$. Removing all arcs and internal vertices of the common dipaths from $H$, we obtain $k$ independent dipaths, which we call $H$-dipaths. An $H$-dipath starts with the terminating vertex of $S_i$, and terminates with the starting vertex of $S_j$, for some $0 \leq i, j \leq k - 1$. We denote such an $H$-dipath as $S_H(i, j)$. If an $H$-dipath contains no internal vertex, we say that it is trivial; otherwise, we say that it is nontrivial. Note that the number of $C$-arcs and the number of $H$-dipaths are also $k$.

**Claim 4.4.** Let $S = u_i v_0 \ldots v_{t-1} u_j$, $0 \leq i, j \leq 2m - 1$, be a nontrivial $H$-dipath. Then $t \leq 3$.

Suppose to the contrary that $t \geq 4$.

If $t$ is even, consider the arc between $v_0$ and $v_{t-1}$. If $v_0 \rightarrow v_{t-1}$, we can replace $v_0 v_1 \ldots v_{t-1}$ with the arc $v_0 v_{t-1}$ on $H$, and obtain a non-Hamiltonian dicycle which contains all vertices of $C$ and is longer than $C$, contradicting Claim 4.1. If $v_{t-1} \rightarrow v_0$, we have a dicycle $Q = v_0 v_1 \ldots v_{t-1} v_0$ in $T - V(C)$, with one arc from $C$ to $Q$ and one arc from $Q$ to $C$. By Claim 4.3, $Q$ must be a spanning dicycle of $T - V(C)$. But then by Claim 4.2, we must have $T = T(r, r, r, r)$ for some integer $r \geq 2$, contradicting our assumption.

If $t$ is odd, consider the arc between $v_0$ and $v_{t-2}$, and the arc between $v_1$ and $v_{t-1}$. If $v_0 \rightarrow v_{t-2}$ or $v_1 \rightarrow v_{t-1}$, by similar arguments as above, we can obtain a non-Hamiltonian dicycle which contains all vertices of $C$ and is longer than $C$, again contradicting Claim 4.1. Hence, we have $v_{t-2} \rightarrow v_0$ and $v_{t-1} \rightarrow v_1$. Then we have two dicycles $Q_0 = v_0 v_1 \ldots v_{t-2} v_0$ and $Q_1 = v_1 v_2 \ldots v_{t-1} v_1$ in $T - V(C)$, which are not spanning dicycles of $T - V(C)$. Since there is one arc from $C$ to $Q_0$, by Claim 4.3, $C \rightarrow Q_0$. Similarly, $Q_1 \rightarrow C$. However, these are impossible, since $v_1$ is in both $Q_0$ and $Q_1$.

Therefore, we can not have an $H$-dipath with more than three internal vertices. Since there are at least four vertices in $T - V(C)$, there are at least
two nontrivial $H$-dipaths.

**Claim 4.5.** Let $S_H(i,j)$, $0 \leq i, j \leq k - 1$, be a $H$-dipath. Then $j \neq i + 1$.

If $S_H(i,j)$ is trivial, then the claim clearly holds. Suppose that $S_H(i,j)$ is nontrivial, and $j = i + 1$. We can replace the $C$-arc $a_C(i,i+1)$ with $S_H(i,i+1)$ on $C$, obtaining a dicycle $C'$, such that $V(C) \subseteq V(C')$ and $|C'| \geq |C| + 2$. Furthermore, by the above discussion, there is at least one more nontrivial $H$-dipath, whose internal vertices are not contained in $V(C')$, so $C'$ is non-Hamiltonian. This contradicts Claim 4.1. So $j \neq i + 1$.

Let $P = u_i v_0 v_1 v_2 u_j$, $0 \leq i, j \leq 2m - 1$, be an $H$-dipath with three internal vertices. If $v_1 \rightarrow u_{i+1}$, then $C$ can be extended to $u_i v_0 v_1 u_{i+1} C u_i$, a contradiction. Therefore $u_{i+1} \rightarrow v_1$, and similarly $u_{i+2} \rightarrow v_2$, $v_1 \rightarrow u_{j-1}$ and $v_0 \rightarrow u_{j-2}$. Hence, each of $v_0$, $v_1$ and $v_2$ sends and receives some arcs from $C$. Similarly, we can prove that every internal vertex of any nontrivial $H$-dipaths sends and receives some arcs from $C$. So, every vertex in $T - V(C)$ sends and receives some arcs from $C$.

We prove two more claims.

**Claim 4.6.** There is no dicycle in $T - V(C)$.

Suppose that there is a dicycle $Q$ in $T - V(C)$. If $Q$ is spanning, then by Claim 4.2, $B = T(r,r,r,r)$ for some integer $r \geq 2$, contradicting our assumption. If $Q$ is not spanning, by Claim 4.3, either $C \rightarrow Q$, or $Q \rightarrow C$. However, we have just proved that every vertex in $T - V(C)$ sends and receives some arcs from $C$.

**Claim 4.7.** There can not exist a dicycle subdigraph $F$ of $T$, such that $V(C) \subset V(F) \subset V(T)$, where $\subset$ stands for proper inclusion.

Suppose such a dicycle subdigraph $F$ exists. Since every vertex in $T - V(C)$ sends and receives arcs from $C$, $T(V(F))$ must be strong. By Theorem 4.4, $T(V(F))$ is Hamiltonian. Then $V(C)$ is covered by the Hamiltonian dicycle of $T(V(F))$, contradicting Claim 4.1.

We will use Claim 4.6 and Claim 4.7 frequently in the remainder of the proof.
We claim that $k \geq 3$. By the above discussion, there are at least two nontrivial $H$-dipaths, so $k \geq 2$. If $k = 2$, then there are only two common dipaths $S_0$ and $S_1$. The two $C$-arcs must be $a_C(0,1)$ and $a_C(1,0)$, and the two nontrivial $H$-dipaths must be $S_H(0,1)$ and $S_H(1,0)$, contradicting Claim 4.5.

We next define a contraction operation on $C \cup H$. We contract every common dipath $S_i$ into a vertex $i$, $0 \leq i \leq k - 1$. Then, we contract every $H$-dipath $S_H(i,j)$ into an arc $(i,j)$. The resulting digraph consists of two arc-disjoint dicycles on the vertices $\{0, 1, \ldots, k-1\}$. One is $C_0 = 0, 1, \ldots, k-1, 0$, obtained from $C$ by contracting the common paths. The other one, denoted by $C_1$, is formed by all arcs obtained by contracting $H$-dipaths. Formally, we define a mapping $\eta$ from the set of common dipaths, $C$-arcs and $H$-dipaths of $C \cup H$, to the vertex set and arc set of $C_0 \cup C_1$, where

$$\eta(S_i) = i, \quad \eta(a_C(i,i+1)) = (i,i+1), \quad \text{and} \quad \eta(S_H(i,j)) = (i,j).$$

Let $F$ be a subdigraph of $C_0 \cup C_1$. We use $\eta^{-1}(F)$ to denote the subdigraph of $C \cup H$, which consists of the preimages of the vertices and arcs of $F$. We also say that $\eta^{-1}(F)$ is the preimage of $F$.

Let $F_0$ be a dicycle factor of $C_1 \cup C_0$. Then $\eta^{-1}(F_0)$ is a dicycle subdigraph of $C \cup H$ which covers $V(C)$. Let $F_1$ be a 1-dipath-dicycle factor of $C_1 \cup C_0$, in which the dipath is from vertex $i$ to vertex $j$. Then $\eta^{-1}(F_1)$ is a 1-dipath-dicycle subdigraph of $C \cup H$ covering $V(C)$, in which the dipath starts with $S_i$ and terminates with $S_j$.

Let $L = L_{io}(C_0 \cup C_1)$ be the in-out graph of $C_0 \cup C_1$. We will work on $L$ to gain structural properties of $C_0 \cup C_1$ and $C \cup H$ in the rest of our proof. We show in Figure 4.3 an example of $C \cup H$, $C_0 \cup C_1$ and $L_{io}(C_0 \cup C_1)$.

**Claim 4.8.** All arcs of $C_1$ whose preimages are nontrivial $H$-dipaths must be on the same cycle, denoted by $Q$, of $L$.

*Proof.* Suppose to the contrary that there exist two arcs $a_0$ and $a_1$ of $C_1$ whose preimages are nontrivial $H$-dipaths, where $a_0$ and $a_1$ are on different cycles of $L$. By Theorem 4.8, there exists a dicycle factor $F$ of $C_0 \cup C_1$, which contains $a_0$ but does not contain $a_1$. However, $\eta^{-1}(F)$ is a dicycle subdigraph of $C \cup H$, which covers $V(C)$ but does not cover the internal vertices of $\eta^{-1}(a_1)$, contradicting Claim 4.7. \qed
Let \( S_H(i, j_0) \) and \( S_H(i_0, j) \) be two different nontrivial \( H \)-dipaths. Let \( v \) be an internal vertex of \( S_H(i, j_0) \), and \( w \) be an internal vertex of \( S_H(i_0, j) \) such that there is an arc from \( v \) to \( w \). We traverse \( S_H(i, j_0) \) from the starting vertex to \( v \), then go through \( vw \), and traverse \( S_H(i_0, j) \) from \( w \) to the terminating vertex to obtain a dipath, which is uniquely determined by the arc \( vw \) and denoted by \( P(vw) \) (If the arc is from \( w \) to \( v \), the dipath obtained is denoted by \( P(wv) \)).

**Claim 4.9.** \( P(vw) \) must cover all internal vertices of \( S_H(i, j_0) \) and \( S_H(i_0, j) \).

**Proof.** \( S_H(i, j_0) \) and \( S_H(i_0, j) \) are mapped to the arcs \((i, j_0)\) and \((i_0, j)\) by \( \eta \), respectively. By Claim 4.8, both \((i, j_0)\) and \((i_0, j)\) must be on \( Q \). By Theorem 4.8, we can find a 1-dipath-dicycle factor \( F \) of \( C_0 \cup C_1 \) in which the dipath is a \((j, i)\)-dipath. Then, \( \eta^{-1}(F) \) is a 1-dipath-dicycle subdigraph of \( C \cup H \) covering \( V(C) \), in which the dipath \( P \) starts with \( S_j \) and terminates with \( S_i \). Then \( P(vw) \cup P \) is a dicycle of \( C \cup H \). However, \((F \setminus P) \cup \{P(vw) \cup P\} \) is a dicycle subdigraph of \( C \cup H \) covering \( V(C) \), and by Claim 4.7, it must be a dicycle factor of \( T \). Therefore, \( P(vw) \) must cover all internal vertices of \( S_H(i, j_0) \) and \( S_H(i_0, j) \).

Let \( S = u_i v_0 v_1 v_2 u_j, \ 0 \leq i,j \leq 2m-1, \) be an \( H \)-dipath with three internal vertices. Without loss of generality, we may assume that \( v_1 \in B \). Suppose there exists another nontrivial \( H \)-dipath which contains an internal vertex \( w \in W \). Since \( T \) is a bipartite tournament, either \( v_1 \to w \) or \( w \to v_1 \). However, the dipath \( P(v_1 w) \) does not cover \( v_2 \), and the dipath \( P(wv_1) \) does not cover \( v_0 \), both contradicting Claim 4.9. Hence, all other nontrivial \( H \)-dipaths must...
contain only one internal vertex which is in $B$. To keep $T$ balanced, there
must be only one such $H$-dipath $S'$, the internal vertex of which is denoted
by $v_3$. Applying Claim 4.9 on $S$ and $S'$, we have $v_3 \rightarrow v_0$ and $v_2 \rightarrow v_3$. But
then $v_0v_1v_2v_3v_0$ is a dicycle in $T - V(C)$, contradicting Claim 4.6. Therefore,
there can not be a nontrivial $H$-dipath with three internal vertices.

Let $S = u_iv_0v_1u_j$, $0 \leq i, j \leq 2m - 1$, be an $H$-dipath with two internal
vertices. Without loss of generality, we may assume that $v_0 \in W$ and $v_1 \in B$. Assume that there is another nontrivial $H$-dipath $S' = u_i'v_2v_3u'_j$ ($0 \leq i', j' \leq 2m - 1$) with two internal vertices. Suppose $v_2 \in W$ and $v_3 \in B$. Since $T$ is a bipartite tournament, either $v_1 \rightarrow v_3$ or $v_3 \rightarrow v_1$. However, the dipath $P(v_1v_3)$ does not cover $v_2$, and the dipath $P(v_2v_1)$ does not cover $v_0$, both contradicting Claim 4.9. If $v_2 \in W$ and $v_3 \in B$, applying Claim 4.9 on $S$ and $S'$, we must have $v_1 \rightarrow v_2$ and $v_3 \rightarrow v_0$. But then $v_0v_1v_2v_3v_0$ is a dicycle in $T - V(C)$, contradicting Claim 4.6. Therefore, there is at most one nontrivial $H$-dipath with two internal vertices. Furthermore, by $|T| - |C| \geq 4$, there are at least two nontrivial $H$-dipaths with one internal vertex.

![Figure 4.4: An example: $C \cup H$ with black and white colors on the vertices, $C_0 \cup C_1$, and $L_{io}(C_0 \cup C_1)$ with labels on the vertices and colors on the edges (denoted by the labels “g” and “r” on the edges)](image)

To further analyze the structure of $C_0 \cup C_1$ and $C \cup H$, we label the vertices of $L$. Let $a$ be a vertex of $L$, which is an arc of $C_0 \cup C_1$. The preimage of $a$, $\eta^{-1}(a)$, is a dipath (which may degenerate to an arc) in $T$. We assign the labels $l_0$ and $l_1$, denoted as $l_0l_1$, to $a$, where $l_0$, $l_1 \in \{B, W\}$, the starting vertex of $\eta^{-1}(a)$ is in color class $l_0$ of $T$ and the terminating vertex of $\eta^{-1}(a)$
is in color class $l_1$ of $T$. We call $l_0$ the first label and $l_1$ the second label of $a$, and call a vertex with labels $l_0l_1$ an $l_0l_1$-vertex. See Figure 4.4 for an example.

Recall that an edge of $L$ is colored red (green) when the two endvertices of it share a common head (tail) in $C_0 \cup C_1$. Therefore, if two vertices are joined by a red (green) edge, then they have the same second (first) label.

We list some properties of the edge colors and vertex labels in $L$ below. Most of the properties are obvious, but for some we give a short proof.

(1) Since two adjacent vertices in $L$ must have at least one label in common, a $BB$-vertex can never be adjacent to a $WW$-vertex, and a $BW$-vertex can never be adjacent to a $WB$-vertex.

(2) An arc $a = (i, i+1)$ of $C_0$ must be labeled $BW$ or $WB$, $0 \leq i \leq k-1$ (addition modulo $k$). A $WW$-vertex or a $BB$-vertex of $L$ must be an arc of $C_1$, whose preimage is an $H$-dipath with one internal vertex in $C \cup H$. By Claim 4.8, all $WW$- and $BB$-vertices must be on $Q$. A $WW$- or $BB$-vertex must be adjacent to one $WB$-vertex and one $BW$-vertex.

(3) If we traverse $Q$ in one direction, $WW$-vertices and $BB$-vertices must appear alternatingly. Hence, the number of $BB$-vertices and $WW$-vertices must be the same on $Q$.

Proof. If there is no $WW$- or $BB$-vertex on $Q$, then the statement holds. Without loss of generality, assume that we have a $WW$-vertex $a_0$ on $Q$, and traversing $Q$ from $a_0$ in one direction, the next vertex on $Q$ is a $WB$-vertex, which is adjacent to $a_0$ by a green edge. By (1), a $WB$-vertex can never be adjacent to a $BW$-vertex. Therefore, we will keep meeting a $WB$-vertex before we meet the next $WW$- or $BB$-vertex. By (2), the other neighbor of $a_0$ is a $BW$-vertex, so we must have at least one $WW$- or $BB$-vertex other than $a_0$. We denote the first $WW$- or $BB$-vertex we meet after $a_0$ by $a_1$.

Since all $WW$- and $BB$-vertices must be arcs on $C_1$, $a_1$ must be at even distance from $a_0$. And since red edges and green edges appear alternatingly on $Q$, $a_1$ must be adjacent to a $WB$-vertex by a red edge. But then the second label of $a_1$ must be $B$, and therefore it must be a $BB$-vertex.

(4) If we traverse $Q$ in one direction, by the discussion in the proof of (3), the vertices between a $WW$-vertex and a $BB$-vertex that appear consecutively must all be $WB$- or $BW$-vertices. We call the segment of $Q$ consisting of all
such vertices a WB-path (a BW-path), if all these vertices are WB-vertices (BW-vertices). By (2), a WW- or BB-vertex must be adjacent to one WB-vertex and one BW-vertex. Therefore, WB-paths and BW-paths must appear alternatingly on Q.

(5) Let a be a WW-vertex on Q. If we traverse Q from a so that the next vertex is a WB-vertex, then we will meet a WW-vertex, a WB-path, a BB-vertex, and a BW-path successively and recursively, until we return to a. Consider any WW-vertex $a_0$ and any BB-vertex $a_1$. If we delete $a_0$ and $a_1$ from Q, we have two paths $P_0$ and $P_1$, where $P_0$ starts and terminates with WB-vertices, and $P_1$ starts and terminates with BW-vertices. We call $P_0$ the (WB,WB)-path for $a_0$ and $a_1$, and $P_1$ the (BW,BW)-path for $a_0$ and $a_1$.

By the above discussion, in $C \cup H$ there is at most one nontrivial $H$-dipath with two internal vertices, and there are at least two nontrivial $H$-dipaths with one internal vertex. Therefore, there is at least one WW-vertex and one BB-vertex on Q.

**Claim 4.10.** Let $a_0 = (i_0,j_1)$ be a BB-vertex and $a_1 = (i_1,j_0)$ be a WW-vertex on Q. Denote the internal vertex of $S_H(i_0,j_1)$ ($S_H(i_1,j_0)$) by $v_0$ ($v_1$). Then, $v_0 \rightarrow v_1$ ($v_1 \rightarrow v_0$), if and only if all vertices in $V(Q) \setminus \{a_0, a_1\}$ whose preimages are nontrivial $H$-dipaths are on the (BW,BW)-path ((WB,WB)-path) for $a_0$ and $a_1$.

**Proof.** Firstly, since $|T| - |C| \geq 4$, there is at least one more vertex in $V(Q) \setminus \{a_0, a_1\}$ whose preimage under $\eta$ is a nontrivial $H$-dipath.

Assume that $v_0 \rightarrow v_1$. By the proof of Theorem 4.8, we can obtain a 1-dipath-dicycle factor $F$ of $C_0 \cup C_1$ such that the dipath is from $j_0$ to $i_0$. To get $F$, we delete the edges $(i_0,j_1)-(i_0,i_0+1)$ and $(i_1,j_0)-(j_0-1,j_0)$ from Q. Then, we have two paths $P_0$, from $(i_0,j_1)$ to $(i_1,j_0)$, and $P_1$, from $(i_0,i_0+1)$ to $(j_0-1,j_0)$, which is actually the (BW,BW)-path for $a_0$ and $a_1$. We take the arcs in $V(P_0) \cap A(C_0)$ and $V(P_1) \cap A(C_1)$, together with the arcs from other cycles of L to constitute $F$.

The only dipath $P$ in $\eta^{-1}(F)$ starts with $S_{j_0}$ and terminates with $S_{i_0}$. And $P \cup P(v_0v_1)$ is a dicycle in $C \cup H$. Then, $F' = (\eta^{-1}(F) \setminus P) \cup \{P \cup P(v_0v_1)\}$ is a dicycle subdigraph of $T$, which covers $V(C)$, and contains at least the vertices $v_0$ and $v_1$, which are in $V(T) \setminus V(C)$. By Claim 4.7, $F'$ must be a dicycle factor of $T$. However, $F$ does not contain any arc in $V(P_0) \cap A(C_1)$. Therefore, $F'$
does not contain the preimage of any arc in $V(P_0) \cap A(C_1)$. So, the preimage of an arc in $V(P_0) \cap A(C_1)$ must not be a nontrivial $H$-dipath. In other word, all vertices in $V(Q) \setminus \{a_0, a_1\}$ whose preimages are nontrivial $H$-dipaths must be on $P_1$, which is the $(BW, BW)$-path for $a_0$ and $a_1$.

Similarly, if $v_1 \to v_0$, we can conclude that all vertices in $V(Q) \setminus \{a_0, a_1\}$ whose preimages are nontrivial $H$-dipaths must be on the $(WB, WB)$-path for $a_0$ and $a_1$.

Now assume that all vertices in $V(Q) \setminus \{a_0, a_1\}$ whose preimages are nontrivial $H$-dipaths are on the $(BW, BW)$-path for $a_0$ and $a_1$. Since $T$ is a bipartite tournament, exactly one of $v_0 \to v_1$ and $v_1 \to v_0$ holds. If $v_1 \to v_0$ holds, by the above discussion, all vertices in $V(Q) \setminus \{a_0, a_1\}$ whose preimages are nontrivial $H$-dipaths must be on the $(WB, WB)$-path for $a_0$ and $a_1$, a contradiction. Therefore, $v_0 \to v_1$.

The only case that is left can be proved similarly. \hfill \Box

Suppose there are at least six $WW$- or $BB$-vertices on $Q$. Let $a_0$ be a $WW$-vertex on $Q$. Traverse $Q$ in one direction from $a_0$, and denote the first six $WW$- or $BB$-vertices we meet by $a_0$, $a_1$, $a_2$, $a_3$, $a_4$ and $a_5$, according to the order in which they appear. Then, by (3), $a_3$ must be a $BB$-vertex. However, there are $WW$- and $BB$-vertices on both the $(BW, BW)$-path and the $(WB, WB)$-path for $a_0$ and $a_3$, contradicting Claim 4.10. Therefore, there are at most four $WW$- or $BB$-vertices on $Q$. Equivalently, there are at most four nontrivial $H$-dipaths with one internal vertex in $C \cup H$.

Suppose there are four $WW$- or $BB$-vertices on $Q$, which are $a_0$, $a_1$, $a_2$ and $a_3$, according to the order in which they appear in one direction, say clockwise. Denote the internal vertices of $\eta^{-1}(a_i)$ as $v_i$, $0 \leq i \leq 3$. Without loss of generality, we assume that $a_0$ and $a_2$ are $WW$-vertices, $a_1$ and $a_3$ are $BB$-vertices, and if we traverse $Q$ clockwise, the path between $a_0$ and $a_1$ is a $(WB, WB)$-path for $a_0$ and $a_1$. Then, $a_2$ and $a_3$ are on the $(BW, BW)$-path for $a_0$ and $a_1$. By Claim 4.10, we have $v_0 \to v_1$. Similarly, we have $v_1 \to v_2$, $v_2 \to v_3$ and $v_3 \to v_0$. But then we have a dicycle $v_0v_1v_2v_3v_0$ in $T - V(C)$, contradicting Claim 4.6.

Therefore, we can have only one $BB$-vertex $a_0$ and one $WW$-vertex $a_1$ on $Q$. By the above discussion, we have one $WB$- or $BW$-vertex $a_2$, whose preimage is a nontrivial $H$-dipath with two internal vertices. Denote the
internal vertices of $\eta^{-1}(a_0)$ and $\eta^{-1}(a_1)$ by $v_0$ and $v_1$, respectively. Then, $v_0 \in W$ and $v_1 \in B$. Without loss of generality, we assume that $a_2$ is a $BW$-vertex, and denote the internal vertices of $\eta^{-1}(a_2)$ by $v_2 \in W$ and $v_3 \in B$, where $v_2 \rightarrow v_3$. Applying Claim 4.9 on $a_0$ and $a_2$, we have $v_3 \rightarrow v_0$. Applying Claim 4.9 on $a_1$ and $a_2$, we have $v_1 \rightarrow v_2$. Further, $a_2$ is on the $(BW, BW)$-path for $a_0$ and $a_1$, and hence by Claim 4.10, $v_0 \rightarrow v_1$. However, we have a dicycle $v_0v_1v_2v_3v_0$ in $T - V(C)$ then, contradicting Claim 4.6.

This final contradiction completes the proof of Theorem 4.6.

4.5 Conjectures and concluding remarks

The presented results reveal some interesting equivalence relationships between Hamiltonicity, pancyclicity and (even) dicycle extendability in tournaments and bipartite tournaments. Both tournaments and bipartite tournaments are special cases of multipartite tournaments. Therefore, we raise the following conjectures for the more general class of multipartite tournaments.

**Conjecture 4.1.** Let $T$ be a multipartite tournament with a Hamiltonian dicycle. Then $T$ is (vertex) pancyclic, unless $T$ belongs to some exceptional classes that can be characterized.

**Conjecture 4.2.** Let $T$ be a multipartite tournament with a Hamiltonian dicycle, and let $C$ be a non-Hamiltonian dicycle in $T$. Then there exists a dicycle $C'$ such that $V(C) \subseteq V(C')$, and $|C'| = |C| + 1$ or $|C'| = |C| + 2$, unless $T$ belongs to some exceptional classes that can be characterized.

Note that Conjecture 4.1 has been verified for a subclass of multipartite tournaments, that is, extended tournaments.

Let $D$ be a digraph with vertex set $\{v_i : 1 \leq i \leq n\}$, and let $S_1$, $S_2$, ..., $S_n$ be digraphs consisting of independent vertices, which are pairwise vertex-disjoint. The extension of $D$, $D[S_1, S_2, \ldots, S_n]$, is the digraph $F$ with vertex set $V(S_1) \cup V(S_2) \cup \ldots \cup V(S_n)$ and arc set $\{x_ix_j : x_i \in V(S_i), x_j \in V(S_j), v_iv_j \in A(D)\}$. Let $T$ be a tournament. Then any extension of $T$ is called an extended tournament. Let $D$ be a semicomplete digraph. Then any extension of $D$ is called an extended semicomplete digraph. The class of extended tournaments is the intersection of the class of extended semicomplete digraphs and the class of multipartite tournaments. Hence the following
characterization of pancyclic and vertex-pancyclic extended semicomplete digraphs given by Gutin completely determines pancyclic and vertex-pancyclic extended tournaments.

We say that a digraph $D$ is triangular with partition $V_0, V_1$ and $V_2$, if the vertex set of $D$ can be partitioned into three disjoint sets $V_0, V_1$ and $V_2$, with $V_i \rightarrow V_{i+1}$ and there is no arc from $V_{i+1}$ to $V_i$ ($0 \leq i \leq 2$ and the subscripts taken modulo $3$).

**Theorem 4.9.** (Gutin [12]) Let $D$ be a Hamiltonian extended semicomplete digraph of order $n \geq 5$ with $k$ partite sets ($k \geq 3$). Then

(a) $D$ is pancyclic if and only if $D$ is not triangular with a partition $V_0, V_1$ and $V_2$, two of which induce digraphs with no arcs, such that either $|V_0| = |V_1| = |V_2|$ or no $D(V_i)$ ($i = 0, 1, 2$) contains a dipath of length $2$.

(b) $D$ is vertex-pancyclic if and only if it is pancyclic and either $k > 3$ or $k = 3$ and $D$ contains two dicycles $Z$ and $Z'$ of length $2$ such that $Z \cup Z'$ has vertices in the three partite sets.

When we study the relationship between Hamiltonicity, pancyclicity and dicycle extendability, we often have a Hamiltonian dicycle $H$ to start with. Next, we often need to consider another dicycle $C$, which is usually assumed to violate our conclusion and has some maximal or minimal properties. In such cases, the strategies of the current chapter to contract $C \cup H$, construct the in-out graph $L$ of the resulting digraph, and then use $L$ to analyze the dipath-dicycle subdigraph structure of $C \cup H$, can be applied. This may lead to new results in this area.
Chapter 5

On two-paths in random digraphs and their application for cycle and path properties

5.1 Introduction

In this chapter, we consider finite digraphs without loops and parallel arcs. We follow the basic definitions and terminology of [12]. In particular, a directed path $P$ (cycle $C$) is a path (cycle) in which all arcs are pointing in the same direction, and the length of $P$ ($C$) is defined as the number of arcs in $P$ ($C$). From now on all paths and cycles are supposed to be directed, and a path (cycle) of length $k$ is called a $k$-path ($k$-cycle), a $k$-path from $u$ to $v$ is called a $(u, v)$-$k$-path, and a path from $u$ to $v$ is called a $(u, v)$-path. When we talk about a vertex pair, we always mean one with two distinct vertices. The digraphs we consider in this chapter are oriented graphs and tournaments, which are digraphs that have at most one arc and exactly one arc between every vertex pair, respectively. A digraph $D$ is called strongly connected, or strong, if for any vertex pair $\{u, v\} \in V(D)$, there is a $(u, v)$-path and a $(v, u)$-path in $D$. $D$ is called strongly $k$-connected, or $k$-strong, if the removal of any $k – 1$ vertices results in a strong digraph.

We calculate the number of 2-paths between any vertex pair in random oriented graphs and random tournaments. In [99], Reid and Beineke proved
the following theorem. Note that when we say a property $P$ holds for almost all members in a graph class $\mathcal{G}$, we mean that the proportion of labeled graphs on $n$ vertices in $\mathcal{G}$ that have property $P$ tends to 1, as $n$ tends to infinity.

**Theorem 5.1.** (Reid and Beineke [99]) Almost all tournaments are strong.

And they stated the following stronger result.

**Theorem 5.2.** (Reid and Beineke [99]) In almost all tournaments, every pair of vertices lies on a 3-cycle.

In a digraph $D$, if for every arc $uv$ there is a $(v,u)$-2-path (respectively $(u,v)$-2-path), we say that $D$ is arc-3-cyclic (respectively arc-3-anticyclic). Therefore, the statement of Theorem 5.2 is equivalent to saying that almost all tournaments are arc-3-cyclic. It is easy to strengthen this statement to include arc-3-anticyclicity, as the theorem below shows. We list the theorem here without a proof, because we will prove our main result, which is a much stronger result, in Section 5.2.

**Theorem 5.3.** Almost all tournaments are arc-3-cyclic and arc-3-anticyclic.

The existence of and the number of 2-paths between vertex pairs in tournaments have both been considered in many sufficient conditions for cycle and path properties. In particular, in [50], Guo and Volkmann listed them (in terms of arc-3-cyclicity and arc-3-anticyclicity) as one type of main conditions to be used in the study of path-connectivity in tournaments. This sounds appreciable since, first of all the number of 2-paths between every two vertices in a digraph can be computed in polynomial time, and secondly, by Theorem 5.2 and Theorem 5.3, the existence of 2-paths between every vertex pair is guaranteed almost always in random tournaments.

Below, we list some representative results in which the condition on the existence of 2-paths implies the existence of paths and cycles of many lengths. We call a digraph with $n$ vertices pancyclic if it contains cycles of every length from 3 to $n$, and arc-pancyclic if every arc of it is contained in cycles of every length from 3 to $n$. We say that a digraph with $n$ vertices is completely strong path-connected (strongly panconnected, respectively), if for every vertex pair $\{u,v\}$, there are $(u,v)$-paths and $(v,u)$-paths of every length from 2 to $n - 1$ (from 3 to $n - 1$, respectively). The last two definitions are from [109] and [101], respectively.
Theorem 5.4. (Tian, Wu and Zhang [103]) Let $T$ be a tournament which is arc-3-cyclic. Then $T$ is arc-pancyclic, unless $T$ belongs to any of two classes of counterexamples.

We refrain from defining the exceptional classes here, since they are not relevant for our exposition.

Theorem 5.5. (Zhang [109]) Let $T$ be a tournament which is arc-3-cyclic and arc-3-anticyclic. Then $T$ is completely strong path-connected, unless $T$ belongs to one class of counterexamples.

By Theorem 5.4, if a tournament $T$ is arc-3-cyclic, then we have $(v, u)$-paths of every possible length for every arc $uv \in A(T)$. Theorem 5.5 means that, to ensure the existence of paths of every possible length in the other direction, we need both arc-3-cyclic and arc-3-anticyclic. However, the result below, together with Theorem 5.4, shows that arc-3-cyclicity with the additional condition of being 3-strong implies strong panconnectedness, i.e., the existence of paths of every possible length except 2. This indicates that conditions on 2-paths and connectivity may sometimes be combined to guarantee certain properties with respect to path and cycle lengths.

Theorem 5.6. (Guo and Volkmann [50]) Let $T$ be a 3-strong and arc-3-cyclic tournament on $n$ vertices. Then, for every arc $uv$ of $T$, there are $(u, v)$-paths of length $k$ for $3 \leq k \leq n - 1$, unless $T$ is isomorphic to one of two exceptional digraphs.

The conditions and conclusions in the next theorems that follow are relatively “local”, for they involve only paths from $u$ to $v$ for a given vertex pair $\{u, v\}$. They suggest that increasing the number of 2-paths between a vertex pair may help to obtain more sophisticated structures. Also here, the conditions involve certain connectivity assumptions.

Theorem 5.7. (Thomassen [101]) Let $T$ be a 2-strong tournament and $u, v \in V(T)$. If there exist three $(u, v)$-2-paths, then there exist $(u, v)$-k-paths, for every $2 \leq k \leq n - 1$.

Theorem 5.8. (Thomassen [101]) Let $T$ be a 3-strong tournament and $u, v \in V(T)$. If there exist two $(u, v)$-2-paths, then there exist $(u, v)$-k-paths, for every $2 \leq k \leq n - 1$ (except possibly four), unless $T$ is isomorphic to one of two exceptional digraphs.
In this chapter, we generalize Theorem 5.2 and Theorem 5.3, and use the results and other ideas on 2-paths as tools to explore some cycle and path properties in tournaments. Firstly, we observe that in random oriented graphs, which includes random tournaments, we can guarantee more than just the existence of 2-paths between every vertex pair. Specifically, we prove that there can be linearly many (in terms of the order of the graph) 2-paths between every vertex pair in random oriented graphs, where the probability of every arc to exist is a constant. As a special case, we determine a lower bound for $c$ such that almost every tournament on $n$ vertices has at least $cn(u,v)$-2-paths for any vertex pair $\{u,v\}$. We use this results to prove other properties with respect to paths and cycles in tournaments. The idea is that, the more 2-paths between every vertex pair we can guarantee, the stronger properties we may expect. We mainly focus on the property of path extendability (defined below) in tournaments. We conjecture that almost all tournaments are path extendable, and we prove some results that support the conjecture.

We conclude this section with some terminology and notations. Throughout this chapter, we let $\alpha = 0.08365$. Let $D$ be a digraph. We use $V(D)$, $A(D)$ and $n$ to denote the vertex set, the arc set and the order of $D$. Let $u$ be a vertex of $D$. A vertex $v$ is called an in-neighbor (respectively out-neighbor) of $u$ if $vu \in A(D)$ (respectively $uv \in A(D)$). The set of in-neighbors (respectively out-neighbors) of $u$ is called the in-neighborhood (respectively out-neighborhood) of $u$, whose order is called the in-degree (respectively out-degree) of $u$. We also write $u \rightarrow v$ if $uv \in A(D)$. Let $F$ be a subdigraph of $D$. If $u \in V(F)$, the in-neighborhood and out-neighborhood of $u$ in $F$ are denoted by $N_F^-(u)$ and $N_F^+(u)$, respectively. Let $d_F^-(u) = |N_F^-(u)|$ and $d_F^+(u) = |N_F^+(u)|$, where the subscript $F$ will be omitted if $F = D$. If $u \notin V(F)$, the number of arcs from $u$ to $F$ and from $F$ to $u$ are denoted by $d^+(u,F)$ and $d^+(F,u)$, respectively. Note that, when $D$ is a tournament, $d^+(u,F)$ and $d^+(F,u)$ are equal to the number of out-neighbors and in-neighbors of $u$ in $F$, respectively. For two disjoint subdigraph $F_1$ and $F_2$ of $D$, the number of arcs from $F_1$ to $F_2$ is denoted by $d^+(F_1,F_2)$. For $U \subseteq V(D)$, we denote the subdigraph induced by $U$, i.e., the subdigraph with vertex set $U$ and arc set $(U \times U) \cap A(D)$, by $\langle U \rangle$. 
5.2 The number of two-paths between every vertex pair in oriented graphs

In this section, we study 2-paths between every vertex pair in oriented graphs. We use the $\overrightarrow{G}\{n, p\}$ model for random oriented graphs. In this model, we fix $0 \leq p \leq 1/2$, and the model contains all oriented graphs on $n$ vertices, in which for any vertex pair $\{u, v\}$, with probability $p$ we have the arc $uv$, with probability $p$ we have the arc $vu$, and with probability $1 - 2p$ we have no arc between $u$ and $v$. Clearly, $\overrightarrow{G}\{n, 1/2\}$ is then the model for random tournaments on $n$ vertices. Let $\overrightarrow{G}$ be an oriented graph in $\overrightarrow{G}\{n, p\}$.

Let $f(n)$ be an increasing function of $n$, which tends to infinity when $n$ tends to infinity. Let $u_0, v_0 \in V(\overrightarrow{G})$ be two fixed distinct vertices of $\overrightarrow{G}$. We use $\text{Path}_2^{f(n)}(u_0, v_0)$ and $\text{Path}_2^{f(n)}(\overrightarrow{G})$ to denote the events that there are at least $f(n)$ $(u_0, v_0)$-2-paths in $\overrightarrow{G}$, and that there are at least $f(n)$ $(u, v)$-2-paths in $\overrightarrow{G}$ for every vertex pair $\{u, v\}$ of vertices of $\overrightarrow{G}$, respectively.

For any $u \in V(\overrightarrow{G})$, the number of 2-paths with intermediate vertex $u$ is $d^+(u)d^-(u) \leq (n - 1)^2/4$. Therefore, the average number of 2-paths between any vertex pair is at most $n \cdot (n - 1)^2/(4n(n - 1)) = (n - 1)/4$, where equality is only possible to hold when $\overrightarrow{G}$ is a regular tournament. Hence, we always assume that $f(n) \leq (n - 1)/4$.

According to the basic rule that the probability of the union of events is not larger than the sum of the individual probabilities, we have the following relationship between the probability of $\text{Path}_2^{f(n)}(u_0, v_0)$ and $\text{Path}_2^{f(n)}(\overrightarrow{G})$.

$$
P\left(\text{Path}_2^{f(n)}(\overrightarrow{G})\right) = P\left(\bigcup_{u,v \in V(\overrightarrow{G}), u \neq v} \text{Path}_2^{f(n)}(u,v)\right)$$

$$
\leq \sum_{u, v \in V(\overrightarrow{G}), u \neq v} P\left(\text{Path}_2^{f(n)}(u,v)\right)
$$

$$
= n(n - 1)P\left(\text{Path}_2^{f(n)}(u_0,v_0)\right).
$$

We further estimate $P\left(\text{Path}_2^{f(n)}(u_0,v_0)\right)$ as follows. Notice that $\binom{n-2}{r-2}$ is
increasing when \( i \leq (n - 3)/2 \).

\[
P \left( \text{Path}_2^{(n)}(u_0, v_0) \right) = \sum_{i=0}^{f(n)-1} \binom{n-2}{i} p^{2i} (1-p^2)^{n-2-i}
\leq \binom{n-2}{f(n)-1} \sum_{i=0}^{f(n)-1} p^{2i} (1-p^2)^{n-2-i}
= \binom{n-2}{f(n)-1} \frac{(1-p^2)^{n-f(n)-1} \left( (1-p^2)^{f(n)} - p^2 f(n) \right)}{1-2p^2}
= \binom{n-2}{f(n)-1} \frac{(1-p^2)^{n-1}}{1-2p^2} \left( 1 - \left( \frac{p^2}{1-p^2} \right)^{f(n)} \right)
\leq \binom{n-2}{f(n)-1} \frac{(1-p^2)^{n-1}}{1-2p^2}
\tag{5.2}
\]

Applying Stirling’s formula, when \( n \) is sufficiently large, we have the following estimation.

\[
\binom{n-2}{f(n)-1} \sim \frac{1}{\sqrt{2\pi}} \cdot \frac{n^{n-1.5}}{(f(n))^{f(n)-0.5} (n - f(n))^{n-f(n)-0.5}}
\tag{5.3}
\]

By (5.1), (5.2) and (5.3), we have

\[
P \left( \text{Path}_2^{(n)}(T) \right) \leq \frac{n(n-1)}{\sqrt{2\pi}} \cdot \frac{n^{n-1.5}}{f(n)^{f(n)-0.5} (n - f(n))^{n-f(n)-0.5}} \cdot \frac{(1-p^2)^{n-1}}{1-2p^2}
\sim \frac{1}{\sqrt{2\pi} (1-2p^2)} \cdot \frac{n^{n-1.5}}{f(n)^{f(n)-0.5} (n - f(n))^{n-f(n)-0.5}}
\tag{5.4}
\]

Now we can prove the following theorem.

**Theorem 5.9.** For any constant \( 0 < p \leq 1/2 \), almost all oriented graphs in \( \vec{G} \{ n, p \} \) have linearly many 2-paths between every vertex pair.

**Proof.** Letting \( f(n) = cn \), (5.4) becomes
\[ P\left( \text{Path}_2^n(T) \right) \leq \frac{n^{1.5} \sqrt{c(1-c)}}{\sqrt{2\pi}(1-2p^2)(1-p^2)} \cdot \left( \frac{1-p^2}{c^e(1-c)^{1-c}} \right)^n. \] (5.5)

When \( n \) tends to infinity, the right-hand side of (5.5) tends to 0 if and only if
\[ 1 - p^2 < c^e(1-c)^{1-c}. \] (5.6)

The function \( c^e(1-c)^{1-c} \) is decreasing in the range \( (0, 1/2] \), and tends to 1 when \( c \) tends to 0 from above. Therefore, given any \( p_0 \) such that \( 0 < p_0 \leq 1/2 \), there always exists a \( c_0 \) such that for all \( 0 < c \leq c_0 \), \( 1 - p_0^2 < c^e(1-c)^{1-c} \).

That means, almost all \( \overrightarrow{G} \in \mathcal{G}(n, p_0) \) have at least \( c_0n \) \((u, v)\)-2-paths for any of its vertex pairs \{\( u, v \}\}.

In particular, when \( p_0 = 1/2 \), \( \overrightarrow{G} \) becomes a tournament. From (5.6), we have \( c_0 \geq \alpha = 0.08365 \), therefore we obtain the following theorem.

**Theorem 5.10.** Almost all tournaments have at least \( \alpha n \) \((u, v)\)-2-paths for any of its vertex pairs \{\( u, v \)\}.

Since \( \alpha > 1/12 \), we can strengthen the statement of Theorem 5.1.

**Theorem 5.11.** Almost all tournaments are \((n/12)\)-strong.

We now figure out the class of tournaments with the maximum (expected) number of 2-paths between every vertex pair. A *regular tournament* is a tournament with at least three vertices in which all vertices have the same in-degree (and out-degree). Hence, a regular tournament must have an odd order. A regular tournament \( T \) of odd order \( n \) is further called *doubly regular*, if for every vertex pair \{\( u, v \)\}, the number of vertices commonly dominated by \( u \) and \( v \) is a constant, which we denote by \( \lambda \). Simple calculations show that \( \lambda = (n-3)/4 \). Therefore, \( n \equiv 4 \) (mod 3). Furthermore, if \( uv \in A(T) \), then the number of vertices dominating both \( u \) and \( v \), the number of \((u, v)\)-2-paths, and the number of \((v, u)\)-2-paths are \( \lambda \), \( \lambda \) and \( \lambda + 1 \), respectively. Obviously, the least number of 2-paths between every vertex pair in a tournament reaches the maximum value, that is \((n-3)/4 \), in doubly regular tournaments.
5.3 Cycle and path problems in tournaments

Since tournaments are relatively dense, stronger cycle and path properties are expected than in arbitrary orientations. The following results show that strongness is sufficient to imply strong cycle properties in tournaments. Camion ([28]) proved that every strong tournament is Hamiltonian. Moon ([90]) further proved that every strong tournament is vertex-pancyclic. In [57], Hendry restated a result of Moon ([91]) that strong tournaments are cycle extendable, with some exceptions that can be characterized.

However, when we consider panconnectedness and path extendability, it turns out that they are not satisfied by general (strong) tournaments. As we have seen in Chapter 3, even if we restrict our discussion to regular tournaments, we can only ensure paths of every length starting at length 3 between vertex pairs.

**Theorem 5.12.** (Alspach et al. [6], and also Thomassen [101]) Let \(uv\) be an arc of a regular tournament \(T\) with \(n \geq 7\) vertices. Then there exist paths from \(u\) to \(v\) of length \(k\) in \(T\), for every \(3 \leq k \leq n - 1\).

**Theorem 5.13.** (Restated from Theorem 3.11) In a regular tournament \(T\) with at least 7 vertices, every path of length at least 2 is extendable, unless \(T\) belongs to any of three classes of exceptional digraphs.

What is missing in Theorem 5.12 and Theorem 5.13, is paths of length 2 between some vertex pairs. However, by Theorem 5.10, there can be a lot of 2-paths between every vertex pair in random tournaments. Therefore, this gap on 2-paths is somehow bridged if we consider random tournaments. This motivates our interest in strong cycle and path properties in random tournaments.

By Theorem 5.10, almost all tournaments have linearly many 2-paths between every vertex pair. However, in every exceptional digraph related to Theorem 5.4 and Theorem 5.5, there exists only one 2-path between some vertex pairs (we refer to [103]) and [109] for the exceptional digraphs). Therefore, when \(n\) tends to infinity, the proportion of exceptional digraphs for Theorem 5.4 and Theorem 5.5 tends to zero, and hence we have the following two theorems.

**Theorem 5.14.** Almost all tournaments are arc-pancyclic.
Theorem 5.15. Almost all tournaments are completely strong path-connected.

We hope to extend the above results to path extendability as stated in the following conjecture.

Conjecture 5.1. Almost all tournaments are path extendable.

Theorem 5.4 and Theorem 5.5 indicate that, having one 2-path between every vertex pair is sufficient to guarantee arc-pancyclicity and completely strong path-connectedness in a tournament. It turns out that this is far from sufficient to imply path extendability. Let us look at a class of counterexamples.

Let \( \lambda \) be a non-negative integer and \( t = 4\lambda + 3 \). Consider a tournament \( T \) with \( n = 3t = 12\lambda + 9 \) vertices, where \( V(T) = V_0 \cup V_1 \cup V_2 \), \( |V_i| = t \), \( \langle V_i \rangle \) is a doubly regular tournament, and \( V_i \rightarrow V_{i+1} \) for \( i \in \{0, 1, 2\} \), where addition is taken modulo 3. Obviously, any Hamiltonian path of \( \langle V_i \rangle \) is not extendable. Therefore, \( T \) is not path extendable. The least number of 2-paths between a vertex pair in \( T \) is \( \lambda = (n-9)/12 \), which is obtained by \( \{u, v\} \) where \( u, v \in V_i \) and \( uv \in A(T) \), for \( i \in \{0, 1, 2\} \). Hence, a tournament can have as many as \((n-9)/12\) \((u,v)\)-2-paths for any vertex pair \( \{u, v\} \), while still not being path extendable.

Note that \( \alpha \) is slightly larger than 1/12. Furthermore, the above construction of a tournament seems to indicate that increasing the number of 2-paths between vertex pairs any further may result in path extendability of \( T \). Therefore, the examples in some sense support the following conjecture, which together with Theorem 5.10 implies Conjecture 5.1. Following the notation of the previous section, we use \( \text{Path}_{2}^{\alpha n}(T) \) to denote the property that there are at least \( f(n) \) \((u,v)\)-2-paths in \( D \) for every vertex pair \( \{u, v\} \subseteq V(D) \).

Conjecture 5.2. A tournament \( T \) satisfying \( \text{Path}_{2}^{\alpha n}(T) \) is path extendable.

Next we prove two results that support Conjecture 5.1 and Conjecture 5.2. Firstly, we introduce a few more definitions and prove an auxiliary lemma. Let \( P = u_0u_1 \ldots u_{p-1} \) be a non-extendable path of a tournament \( T \), and let \( v \in V(D) \setminus V(P) \). It is easy to see that there can not exist \( 0 \leq i < j \leq p - 1 \) such that \( u_j \rightarrow v \rightarrow u_i \). Therefore, according to the direction of the arcs between \( V(P) \) and \( v \), we can classify \( v \) into three categories, as follows.
Chapter 5

(1) \(v \rightarrow V(P)\),

(2) \(V(P) \rightarrow v\), and

(3) there exists an integer \(s\) with \(1 \leq s \leq p - 1\), such that \(v \rightarrow u_i\) for all \(0 \leq i \leq s - 1\) and \(u_i \rightarrow v\) for all \(s \leq i \leq p - 1\).

We call a vertex \(v \in V(D)\setminus V(P)\) a dominating, dominated or hybrid vertex of \(P\) if \(v\) belongs to category (1), (2) or (3), respectively. Furthermore, for a hybrid vertex \(v\) in category (3) we say that it switches at \(s\). Note that the definitions are applicable to all paths (not necessarily non-extendable paths).

For a path \(P\) and \(x, y \in V(P)\), we use \(P[x, y]\) to denote the segment of \(P\) from \(x\) to \(y\). The below lemma is implied by Theorem 3.16, but we present a much shorter proof than that of Theorem 3.16.

**Lemma 5.16.** Let \(P = u_0u_1 \ldots u_{p-1}\) be a non-extendable path of length \(p\) in a regular tournament \(T\). If \(p \geq 4\), then \(P' = P[u_1,u_{p-2}]\) has at most two hybrid vertices.

**Proof.** Let \(F = T - V(P)\) and \(q = |F|\). Suppose that \(P'\) has a hybrid vertex \(v\). Then \(v\) is also a hybrid vertex of \(P\). Let the number of hybrid vertices of \(P\) be \(h \geq 1\). For every hybrid vertex \(w\) of \(P\), \(u_{p-1} \rightarrow w \rightarrow u_0\), while the other vertices dominate \(V(P)\) or are dominated by \(V(P)\). Therefore, \(d^+(u_{p-1},F) - d^+(u_0,F) = h\).

Since \(p \geq 4\), \(u_1 \neq u_{p-2}\). Since \(v\) is a hybrid vertex of \(P'\), \(u_{p-2} \rightarrow v \rightarrow u_1\). For \(1 \leq i \leq p - 3\), if \(u_0 \rightarrow u_{i+1}\) and \(u_i \rightarrow u_{p-1}\), then the path \(u_0u_{i+1}u_{p-2}u_1u_1 \ldots u_iu_{p-1}\) extends \(P\), a contradiction. Therefore, for \(1 \leq i \leq p - 3\), at most one of \(u_0 \rightarrow u_{i+1}\) and \(u_i \rightarrow u_{p-1}\) holds. Hence,

\[
d^+_p(u_0) + d^-_p(u_{p-1}) \leq (p - 3) + 1 + 1 + 2 = p + 1.
\]

Since \(T\) is regular,

\[
d^+(u_0,F) + d^+(F,u_{p-1}) = (n - 1) - d^+_p(u_0) - d^-_p(u_{p-1}) \geq (n - 1) - (p + 1) = q - 2.
\]

Since \(T\) is a tournament,

\[
d^+(u_{p-1},F) + d^+(F,u_{p-1}) = q.
\]

So

\[
h = d^+(u_{p-1},F) - d^+(u_0,F) \leq 2.
\]

Since every hybrid vertex of \(P'\) is also a hybrid vertex of \(P\), \(P'\) has no more than two hybrid vertices either. \(\square\)
Now we prove the path extendability of doubly regular tournaments, by considering the number of 2-paths between vertex pairs.

**Theorem 5.17.** All doubly regular tournaments on at least seven vertices are path extendable.

**Proof.** Let $T$ be a doubly regular tournament with $n = 4\lambda + 3 \geq 7$ vertices. Suppose to the contrary that $T$ has a non-extendable path $P = u_0u_1 \ldots u_{p-1}$. It is easy to verify that, when $n \in \{7, 11\}$, there is only one doubly regular tournament on $n$ vertices, which is path extendable. Therefore, we may assume $n \geq 15$ and $\lambda \geq 3$. Hence, there are 2-paths between every vertex pair, and so every 1-path, i.e., every arc, in $T$ is extendable. Let $P_1 = u_0u_1u_2$ be a 2-path in $T$. Since $u_0u_1$ is extendable to a path $u_0xu_1$ and $x \neq u_2$, $P_1$ can be extended to the path $u_0xu_1u_2$. Therefore, every 2-path in $T$ is extendable. Similarly, every 3-path in $T$ is extendable. So we may assume that $p \geq 4$.

By Lemma 5.16, $P' = P[u_1, u_{p-2}]$ has at most two hybrid vertices. Let $N^-, N^+$ and $N_h$ be the sets of dominating vertices, dominated vertices and hybrid vertices of $P'$. Then, $N^- \rightarrow u_0$.

We consider the set of 2-paths in $T$ between the vertices in $V(P')$, denoted by $S$. Let $w$ be an intermediate vertex of any 2-path in $S$. Then $w \notin N^- \cup N^+$. Therefore, $w \in V(P) \cup N_h = V(P') \cup \{u_0, u_{p-1}\} \cup N_h$. By Lemma 5.16, $|N_h| \leq 2$. For $w \in \{u_0, u_{p-1}\} \cup N_h$, the number of 2-paths in $S$ with intermediate vertex $w$ is at most $(p'/2)^2$. For $w \in V(P')$, the number of 2-paths in $S$ with intermediate vertex $w$ is at most $(p' - 1)/2$. Therefore,

$$|S| \leq 4 \cdot \left(\frac{p'}{2}\right)^2 + p' \cdot \frac{(p' - 1)^2}{4} = \frac{p'(p' + 1)^2}{4}. \quad (5.7)$$

Since $T$ is doubly regular, for any vertex pair $\{u, v\}$, we have $(n-3)/4 + (n+1)/4 = (n-1)/2$ 2-paths between them in both direction in total. Therefore,

$$|S| = \frac{p'(p' - 1)}{2} \cdot \frac{n-1}{2}. \quad (5.8)$$

By (5.7) and (5.8), we have

$$\frac{p'(p' - 1)}{2} \cdot \frac{n-1}{2} \leq \frac{p'(p' + 1)^2}{4},$$

that is,

$$p'^2 - (n - 3)p' + n \geq 0. \quad (5.9)$$
Since \( n \geq 15 \), solving \( p' \) from (5.9), we have

\[
p' = (n - 3 + \sqrt{(n - 3)^2 - 4n})/2 \geq (n - 3 + (n - 6))/2 = n - 9/2.
\]

Since \( p' \) is an integer, \( p' \geq n - 4 \). Therefore \( p \geq n - 2 \).

By Theorem 5.5, there is a \((u, v)\)-Hamiltonian path in \( T \) for every vertex pair \( \{u, v\} \). Therefore, every path on \( n - 1 \) vertices must be extendable. So, we may assume that \( \alpha = n - 2 \). Let \( V(D) \setminus V(P) = \{v_0, v_1\} \). Since \( T \) is regular and \( n \geq 15 \), \( v_0 \) and \( v_1 \) must be hybrid vertices of \( P \). Suppose \( v_0 \) and \( v_1 \) switch at \( r_0 \) and \( r_1 \), respectively. Without lose of generality, we may assume \( r_0 \leq r_1 \). Then, there is no 2-path from \( v_0 \) to \( v_1 \) in \( T \), a contradiction.

To prove Conjecture 5.2, we need to improve Theorem 5.17 significantly by reducing the lower bound on the number of 2-paths from \( (n - 3)/4 \) to \( \alpha n \). As a step in a possible solution to the conjecture, we prove a structural result which estimates the length of a non-extendable path in \( T \).

**Theorem 5.18.** Let \( T \) be a tournament with \( n \) vertices such that \( \alpha n \geq 1 \), and that \( \text{Path}_2^{\alpha n}(T) \) holds. Assume that \( T \) is not path extendable, and let \( P \) be a non-extendable path of \( T \). Then \(|P| \geq 3\alpha n + 3\).

**Proof.** Let the length of \( P \) be \( p \), and \( P = u_0 u_1 \ldots u_{p-1} \). Furthermore, let \( F = T - V(P) \). For \( u, v \in V(T), u \neq v \), we denote the set of the intermediate vertices of all \((u, v)\)-2-paths in \( T \) by \( V(I(u, v)) \). By the condition of the theorem, \(|V(I(u, v))| \geq \alpha n \geq 1\).

Since \( P \) is not extendable, as we have discussed above, any \( v \in V(F) \) can only be a dominating, dominated or hybrid vertex of \( P \). Therefore, for any integer \( i \) and \( j \) such that \( 0 \leq i < j \leq p - 1 \), there can not exist a vertex \( v \in V(F) \cap V(I(u_i, u_j)) \). Hence, \( V(I(u_i, u_j)) \subseteq V(P) \).

Since \( V(I(u, v)) \geq 1 \) for all vertex pairs \( \{u, v\} \), all 1-paths, i.e., arcs are extendable. Therefore, we may assume \(|P| \geq 3\). By \(|V(I(u_0, u_1))| \geq \alpha n \) and \(|V(I(u_0, u_1))| \subseteq V(P), u_1 \) has at least \( \alpha n \) in-neighbors on \( P \) besides \( u_0 \). So, we have \(|N^+(P)(u_1)| \geq \alpha n + 1\).

Let us consider the set of out-neighbors of \( u_1 \) on \( P, N^+_P(u_1) \). For any \( u_i \in N^+_P(u_1) \), we have \( i > 1 \). Hence, we have \( V(I(u_1, u_i)) \in V(P) \), and furthermore, \( V(I(u_1, u_i)) \subseteq N^+_P(u_1) \). Since \(|V(I(u_1, u_i))| \geq \alpha n, u_i \) has at least \( \alpha n \) in-neighbors in \( N^+_P(u_1) \). This means that in \( \langle N^+_P(u_1) \rangle \) the minimum in-degree is at least \( \alpha n \). Therefore \(|N^+_P(u_1)|(|N^+_P(u_1)| - 1)/2 \geq \alpha n|N^+_P(u_1)| \), from which we have \(|N^+_P(u_1)| \geq 2\alpha n + 1\).
Since $T$ is a tournament, $N_P^-(u_1) \cap N_P^+(u_1) = \emptyset$. We have $p \geq 1 + (\alpha n + 1) + (2\alpha n + 1) = 3\alpha n + 3$.

5.4 Concluding remarks

We have proved that there exist linearly many 2-paths between vertex pairs in almost all oriented graphs, in particular in almost all tournaments. There are quite a number of existing results showing that such conditions on 2-paths between vertex pairs imply strong path and cycle properties in tournaments. Therefore, we expect to establish strong path and cycle properties of random tournaments that are not satisfied by all tournaments. Besides the results we have obtained so far, we are particularly interested in proving path extendability of almost all tournaments. The proofs of the last two theorems in the previous section provide some ideas to try to tackle this problem.

We also hope to explore other properties of tournaments, not limited to properties related to paths and cycles, with the results on 2-path structures we established.
Chapter 6

On the complexity of edge-colored subgraph partitioning problems in network optimization

6.1 Introduction

In the former chapters, we concentrated on structural properties related to the existence of paths and cycles in digraphs (while in Chapter 4 we also worked on bipartite undirected graphs). In this chapter, we will look at edge-colored graphs, from a computational and algorithmic perspective. Firstly, we note that digraphs can be generalized to edge-colored graphs. Let $D$ be a digraph. Replace each arc $uv$ of $D$ by two edges $ux_{uv}$ and $x_{uv}v$ by adding a new vertex $x_{uv}$. Now color the edge $ux_{uv}$ red and the edge $x_{uv}v$ blue. We then have obtained a 2-edge-colored graph, denoted by $G$. It is easy to see that each alternating cycle (path) in $G$ corresponds to a directed cycle (path) in $D$, and vice versa. Of course, by this construction, $G$ belongs to a special family of edge-colored graphs. Hence, it is not surprising that in general edge-colored graphs we can study a greater diversity of structures than in digraphs. Furthermore, edge-colored graphs are more powerful in modeling networks in real-world applications.
Graph based data mining is defined as the science and the art of extracting useful knowledge like patterns and outliers provided by an underlying complex system, in order to draw meaningful conclusions regarding the system’s properties [4,39]. The vertices of a complex network denote the entities in a system, and the edges between the vertices represent some kind of relationship between the entities. Network clustering is an important task, frequently arising with the aim of partitioning a network into clusters of elements with some similar relationship. In many cases, one can investigate specific properties of a data set by detecting special information in the corresponding clusters, for instance, cliques, cycles, spanning trees and connected components. In particular, edge-colored connected components are often used for solving the important clustering problem arising in data mining, which essentially represents partitioning the set of elements of a certain data set into a number of clusters of objects according to some kind of relationship. For example, a major application of edge-colored graph described structures arose in sociometry. Social network analysis has grown to be a field in its own right, with widely accepted methods used in an increasing variety of applications [10,43,80]. Wasserman and Faust [104] describe the main methods and underlying philosophies as well as giving a range of illustrative problems.

6.1.1 Motivation

In social networks, vertices represent people and edges the relations between them. Different kinds of relationship are distinguished with different colors. The social network analysts need to survey each person about their friends, ask for their approval to publish the data and keep a trace of that population for years. Also the applications, implemented on internet, that use the concept of links between friends and friends of friends, like Google+ which is built on this foundation of “Circles”, provide such large structured data sets. One person on Google+ may be connected to many people which can be divided into different circles such as a circle of “Family”, a circle of “Friends”, a circle of “Employees”, and a circle of “Customers”. Colored edges can be used to describe the different relations of circles between members. Since a cluster is typically understood as a “tightly knit” group of elements, the graph theoretic concept of a monochromatic clique, which is a subset of vertices
inducing a monochromatic complete subgraph, is a natural formalization of a cluster that has been used within this context. These monochromatic cliques define cohesive subgroups of some kind of relationship, and provide a useful start to the analysis of the structure of social networks, which gives some basis for the study of information exchange and patterns of influence in social networks. The proposed Socratic query is the following: How many “circles” (or monochromatic cliques) are needed to cover the whole graph? Moreover, colors may have weights which can be assigned by the strength or the level of relevance of relations between members. This might lead to the problem of partitioning the graph into (maximum or) minimum weight-sum clusters with the highest possible level of cohesiveness, which is a natural generalization of the classical clique partitioning problem.

A cycle partition or cycle cover of a graph is a spanning subgraph consisting of some cycles, such that each vertex is contained in exactly one cycle. A special case of the cycle cover problem is the traveling salesman problem (TSP), where the goal is to determine a Hamiltonian cycle of maximum or minimum weight. The problem of cycle partitioning is an important tool for the design of approximation algorithms for different variants of the TSP [30–32,65]. Computing cycle partitions is an important task in the fields of information science, graph theory and combinatorial optimization [75,81].

Although many studies have been carried out to analyze the complexity of cycle partitioning problems and the design of approximation algorithms for it [85,86], few of them considered the more general case of the problem for edge-colored graphs or edge-labeled graphs. For the edge-colored case, the multicolored cycle partitioning problem is to determine the minimum number of vertex-disjoint multicolored cycles (all the edges of any cycle have distinct colors) in $G$ such that every vertex is in at least one multicolored cycle [64].

6.1.2 Related results

A variant of the TSP, called MaxLTSP, where the goal is to determine a Hamiltonian cycle with a maximum number of colors in an edge-colored complete graph, has been considered in [27,36]. It is easy to see that MaxLTSP and the multicolored cycle partitioning problem have a similar relationship to that of TSP and the cycle partitioning problem. The multicolored cycle
partitioning problem models the need for a maximum covering with a certain network structure (in our case such a structure is a multicolored cycle). For example, consider the situation of designing some metropolitan peripheral ring roads, where every color represents a different sub-urban area that a certain link would traverse. In order to minimize the number of peripheral rings such that each of them can cover different sub-urban areas, we wish to use as few multicolored cycles as possible to partition a given edge-colored graph (network).

Labeled network optimization over colored graphs has been extensively studied [54, 55, 71, 89]. Several variations of such problems, and in particular their computational complexity, have been well studied. MacGillivray and Yu [82] studied a general graph partitioning problem including graph coloring, homomorphism to \( H \), conditional coloring, contractibility to \( H \), and partition into cliques as special cases, and investigated their complexity. Yegnanarayanan [107] considered three coloring parameters of a graph \( G \) in connection with their computational complexity, partitions, algebra, projective plane geometry and analysis. Jin et al. [63] investigated the computational complexity of the problem of partitioning complete multipartite 2-edge-colored graphs into the minimum number of vertex-disjoint monochromatic cycles, paths and trees, respectively. For more general coloring and partitioning problems, the readers could refer to Garey and Johnson [45], and Kano and Li [64].

Monochromatic clique and multicolored cycle partitioning problems have important applications in the problems of network optimization arising in information science and operations research mentioned above. We abbreviate the problems of partitioning the vertex set of a (not necessarily properly) edge-colored graph into a minimum number of monochromatic cliques and multicolored cycles to MCLP and MCYP, respectively.

6.1.3 Diamond-free graphs

In graph theory, many important families of graphs can be described by a finite set of individual graphs that do not belong to the family and further exclude all graphs from the family which contain any of these forbidden graphs as (induced) subgraph or minor. Diamond-free graphs belong to such kinds of important families of graphs. The diamond graph is obtained from a complete
graph $K_4$ by removing one edge. A graph is diamond-free if it has no diamond as an induced subgraph. The triangle-free graphs are diamond-free graphs, since every diamond contains a triangle. Much research about diamond-free graphs has focused on graph coloring. Characterizations of (subclasses of) diamond-free graphs and their structural properties have been considered both from a theoretical and applications angle. For example, cactus graphs as well as the family of pseudoforests are diamond-free graphs. The former graph family is downwardly closed under graph minor operations and may be characterized by a single forbidden minor which is the diamond graph [41]. They represent electrical circuits that have useful properties [94] and have also recently been used in comparative genomics as a way of representing the relationship between different genomes or parts of genomes [97]. Pseudoforests also form graph-theoretic models of functions and occur in several algorithmic problems. Pseudoforests are sparse graphs which have very few edges relative to their number of vertices, and their matroid structure allows several other families of sparse graphs to be decomposed as unions of forests and pseudoforests. Pseudoforests also play a key role in parallel algorithms for graph coloring and related problems [47, 72]. Li and Zhang [79] showed that both the problems of determining the minimum number of monochromatic cliques and the minimum number of multicolored cycles that partition $V(G)$ for edge-colored diamond-free graphs are \textbf{NP}-hard.

\section*{6.1.4 Our contribution}

In this chapter, we prove, by reduction from vertex cover in 3-regular connected graphs, that MCLP is APX-hard for graphs that are monochromatic-diamond-free. Previously, it had been shown in [79] that MCLP is \textbf{NP}-hard for diamond-free (and implicitly also for monochromatic-diamond-free) graphs.

We observe that the algorithms from [79] (polynomial algorithm for finding a largest monochromatic clique, and $O(\log m)$-approximation for MCLP, where $m$ is the size of a largest monochromatic clique) do not work for diamond-free graphs, as claimed in [79], but only for monochromatic-diamond-free graphs. Furthermore, the algorithmic ideas of [79] are extended to show
that for monochromatic-diamond-free graphs, one can enumerate all maximal monochromatic cliques in polynomial time and find a \((\log(|V|) + 1)\)-approximation algorithm, even for a weighted version of MCLP.

We prove, by reduction from set cover, that MCYP is \textbf{NP}-hard even for triangle-free graphs. (The graphs constructed in the previous \textbf{NP}-hardness proof from [79] were diamond-free but not triangle-free.)

\section{Preliminaries}

Let \(G = (V, E)\) be a connected undirected simple graph. If \(G\) is assigned a mapping \(\ell : E \rightarrow \mathbb{N}\), we say that \(G\) is an \textit{edge-colored graph}. We call \(\ell(e)\) the color of the edge \(e \in E\), and we use \(\ell(H)\) to denote the number of different colors in the set \(\{\ell(e)|e \in E(H)\}\) for a subgraph \(H\) of \(G\). A \textit{complete graph} is a graph in which every two distinct vertices are adjacent. We denote by \(K_m\) a complete graph on \(m\) vertices, and by \(C_m\) a cycle on \(m\) vertices. A \textit{clique} of \(G\) is a nonempty subset of \(V(G)\) that induces a complete subgraph of \(G\). A clique \(CL\) of \(G\) is called a \textit{monochromatic clique} if all the edges of the corresponding subgraph of \(G\) have the same color. A cycle \(CY\) of \(G\) is called a \textit{multicolored cycle} if \(\ell(CY) = |E(CY)|\), i.e., if no two edges of \(CY\) have the same color. Note that a single vertex can be viewed as a degenerate monochromatic clique or multicolored cycle. We simply call it a \textit{vertex-clique} or \textit{vertex-cycle}.

A subgraph \(H\) of an edge-colored graph \(G\) is called \textit{monochromatic-induced} if \(H\) is monochromatic with edge color \(c\) and for any pair of vertices \(u, v \in V(H)\), \(uv\) is an edge of \(H\) if and only if \(uv\) is an edge of \(G\) with color \(c\). And a graph \(G\) is called \textit{monochromatic-diamond-free} if it does not contain a monochromatic diamond as a monochromatic-induced subgraph. Note that the properties of being diamond-free and monochromatic-diamond-free do not imply each other. For example, a \(K_4\) with one edge colored \(\ell_1\) and the others colored \(\ell_2\) is diamond-free, but not monochromatic-diamond-free. However, a monochromatic cycle on 4 vertices with a chord of a different color is monochromatic-diamond-free, but not diamond-free. A vertex \(u\) is \textit{color-adjacent} to a vertex \(v\) of a monochromatic clique \(CL\) if the edge \(uv\) has the
same color as the edges of \( CL \). A clique \( CL \) of \( G \) is called a \textit{maximal monochromatic clique} if there is no vertex \( u \) of \( G \) color-adjacent to each vertex of \( CL \).

The chapter is organized as follows. In Section 6.3, we show that MCLP is APX-complete on monochromatic-diamond-free graphs with maximum degree 6. In Section 6.4, we generalize MCLP to its weighted version WMCLP, and present a greedy scheme that yields an \( \ln |V(G)| + 1 \)-approximation algorithm for WMCLP on monochromatic-diamond-free graphs. We also provide an example to show that the approximation guarantee is tight. In Section 6.5, we investigate the complexity of MCYP, and show that the corresponding decision problem is NP-complete, even if the input is a triangle-free graph. In the final section, we present some concluding remarks and propose some open problems for further research.

6.3 Inapproximability of MCLP on monochromatic-diamond-free graphs

Given a graph \( G = (V, E) \) and a positive integer \( k \), the \textit{Partition into cliques} (PIC) decision problem consists of deciding whether there exists a partition of \( V \) into \( k \) disjoint subsets such that the subgraph induced by each part of the subsets is a clique of \( G \). Garey and Johnson [45] proved that PIC is NP-complete for \( K_4 \)-free graphs. Cerioli et al. [29] establish both the NP-completeness of PIC for planar cubic graphs and the Max SNP-hardness of PIC for cubic graphs. They also presented a deterministic polynomial time \( 5/4 \)-approximation algorithm for finding clique partitions in maximum degree three graphs. It is easily seen that PIC is a special case of the decision version of MCLP if the graph \( G \) is colored by a unique color. Li and Zhang [79] have proved that the decision version of MCLP is NP-complete, even when the input is restricted to diamond-free graphs. They gave a polynomial algorithm for finding an approximation solution to MCLP in diamond-free graphs with performance ratio \( \ln m + 1 \), where \( m \) is the size of a maximum monochromatic clique in the input graph. However, the algorithm actually works for monochromatic-diamond-free graphs instead of diamond-free graphs.

Hence, if the input graph for MCLP is monochromatic-diamond-free with the size of a maximum monochromatic clique bounded by a constant, this
gives an approximation algorithm with constant performance ratio.

We further investigate the inapproximability of MCLP. Alimonti and Kann have shown in [5] that the Vertex Cover problem restricted to 3-regular connected graphs is APX-complete. This implies that there exists some small \( \epsilon > 0 \) such that the existence of a polynomial time approximation algorithm for finding a minimum cardinality vertex cover in a connected 3-regular graph with performance guarantee \( 1 + \epsilon \) would imply \( \mathbf{P} = \mathbf{NP} \).

An L-reduction is a transformation between optimization problems which linearly preserves approximability features, and it is one type of approximation-preserving reductions. L-reductions in studies of approximability of optimization problems play a similar role to that of polynomial reductions in the studies of computational complexity of decision problems. L-reductions preserve membership in APX for the minimizing case only, as a result of implying AP-reductions. The definition of an L-reduction is given as follows [96].

Let \( f \) be a polynomial-time transformation from a minimization optimization problem \( \Pi \) to a minimization optimization problem \( \Pi' \). We say that \( f \) is an L-reduction if there are constants \( \alpha, \beta > 0 \) such that for each instance \( I \) of \( \Pi \):

a) The optima of \( I \) and \( f(I) \), \( \text{OPT}(I) \) and \( \text{OPT}(f(I)) \) respectively, satisfy \( \text{OPT}(f(I)) \leq \alpha \text{OPT}(I) \).

b) For any solution of \( f(I) \) with cost \( c \), we can find in polynomial time a solution of \( I \) with cost at most \( \text{OPT}(I) + \beta |c - \text{OPT}(f(I))| \).

The constant \( \beta \) will usually be 1. We now give an approximation preserving L-reduction from the Vertex Cover problem in 3-regular connected graphs to MCLP and draw the following conclusions.

**Theorem 6.1.** MCLP is

1. APX-hard on monochromatic-diamond-free graphs, and
2. APX-complete on monochromatic-diamond-free graphs in which the size of a maximum monochromatic clique is bounded by a constant.
6.3.1 Proof of Theorem 6.1

Consider an arbitrary instance of the Vertex Cover problem in 3-regular connected graphs. So let $G = (V, E)$ be a 3-regular connected graph, with $|V| = 2n$ and $|E| = 3n$ for some $n \in \mathbb{Z}^+$. A corresponding MCLP instance on an edge-colored graph $H = (V_H, E_H)$ is constructed from $G$ in the following way. $H$ is obtained from $G$ by replacing every edge $(u, v) \in E$ by a gadget $g(u, v)$ consisting of the vertices $u$ and $v$ as well as two new vertices $e_1^{u,v}$ and $e_2^{u,v}$, and the edges $(u, e_1^{u,v}), (u, e_2^{u,v}), (v, e_1^{u,v})$ and $(v, e_2^{u,v})$. Furthermore, for a vertex $u$ with neighbors $v, w$ and $x$, the vertices $e_1^{u,v}, e_1^{u,w}$ and $e_1^{u,x}$ are made mutually adjacent in $H$. For every vertex $u \in V$, we define a color $\ell(u)$, and for every edge $(u, v) \in E$, we define two colors $\ell(u, e_1^{u,v})$ and $\ell(v, e_1^{u,v})$, where all the colors we define are different. For an edge $(u, v) \in E$, the corresponding edges in $H$ are colored as follows. The edge $(u, e_1^{u,v})$ is assigned color $\ell(u)$ and the edge $(v, e_1^{u,v})$ is assigned color $\ell(v)$. The edge $(u, e_2^{u,v})$ is assigned color $\ell(u, e_1^{u,v})$ and the edge $(v, e_2^{u,v})$ is assigned color $\ell(v, e_1^{u,v})$. For a vertex $u$ with neighbors $v$, $w$ and $x$ in $G$, the edges $(e_1^{u,v}, e_1^{u,w}), (e_1^{u,w}, e_1^{u,x})$ and $(e_1^{u,x}, e_1^{u,v})$ in $H$ are all assigned color $\ell(u)$. This completes the construction and edge-coloring of the graph $H$ (See Figure 6.1). It is easy to observe that a largest monochromatic clique in $H$ corresponds to a $K_4$, and that $H$ is monochromatic-diamond-free, with maximum degree 6. Note that the degree of the vertices $u, v$, and $e_1^{u,v}$ is exactly 6 for every gadget $g(u, v)$.

Let $V_c^*$ be a minimum vertex cover of $G$, and let $P^*$ be a minimum monochromatic clique partition of $H$. Then, we have the following inequalities.

**Lemma 6.2.** $|P^*| \leq 8n \leq 8|V_c^*|$

**Proof.** Since every vertex in $G$ is incident with exactly three edges, $V_c^*$ has at least $|E|/3 = n$ vertices. There are $|V| + 2|E| = 8n$ vertices in $H$, so $H$ can be partitioned into $8n$ vertex-cliques. Hence, $|P^*| \leq 8n \leq 8|V_c^*|$. \quad \square

Suppose $P$ is an arbitrary monochromatic clique partition of $H$. We further have the following conclusion.

**Lemma 6.3.** $P$ can always be turned into a new monochromatic clique partition $P'$ such that $|P'| \leq |P|$ and for every edge $(u, v)$ of $G$, $e_1^{u,v} \in K(u)$ or $e_2^{u,v} \in K(v)$ holds and there is no vertex-clique $u$ or $v$ in $P'$. Here $K(v)$ denotes a vertex-clique $v$ or a (nontrivial) monochromatic clique containing $v$. 

Figure 6.1: Gadgets near the vertex $u$

Proof. First suppose that $K(e^1_{u,v})$ is a vertex-clique in $P$, or is a monochromatic clique with color $\ell(u)$ that does not contain $u$. We can execute one of the following operations on $P$ to merge $K(e^1_{u,v})$ into $K(u)$ or $u$ into $K(e^1_{u,v})$, without increasing the cardinality of $P$. If $u$ forms a vertex-clique or is contained in a monochromatic clique with color $\ell(u)$, then $K(e^1_{u,v})$ can be combined with $K(u)$ to obtain a larger monochromatic clique with color $\ell(u)$. If $u$ is contained in a monochromatic clique with a color different from $\ell(u)$, then $u$ can be taken away from $K(u)$ and combined with $K(e^1_{u,v})$ to form a new clique with color $\ell(u)$.

Therefore, we may assume that $e^1_{u,v} \in K(u)$ or $K(v)$ for all edges $(u, v)$ in $G$. If there exists a vertex-clique $v$ in a gadget $g(u, v)$ after executing the above operations, then $e^1_{u,v} \in K(u)$, and $e^2_{u,v}$ forms a vertex-clique. Hence, $v$ can be combined with $e^2_{u,v}$ to form a new monochromatic clique with color $\ell(v, e_{u,v})$, and the cardinality of the partition is decreased.

After applying the above operations we have obtained a new partition $P'$ with $|P'| \leq |P|$, satisfying the conditions in the lemma.

Let $g(u, v)$ be a gadget in $H$. Without loss of generality we may assume
that $e_{u,v}^1 \in K(u)$ with color $\ell(u)$ in $P'$. Then $e_{u,v}^2$ forms either a vertex-clique, or a clique with $v$ of color $\ell(v, e_{u,v}^2)$.

Let $V_c$ be composed of all the vertices $u \in V$ such that for some edge $(u, v) \in E$, $e_{u,v}^1 \in K(u)$ in $P'$. Since for every edge $(u, v) \in E$, $e_{u,v}^1 \in K(u)$ or $e_{u,v}^1 \in K(v)$ in $P'$, at least one of $u$ and $v$ is in $V_c$. Hence $V_c$ is a vertex cover of $G$.

For every edge $(u, v) \in E$, $e_{u,v}^2$ forms either a vertex-clique or a monochromatic clique together with $v$ or $u$ in $P'$. There are totally $|E|$ such cliques. Each of the other cliques in $P'$ contains exactly one vertex $u \in V$ and at least one vertex $e_{u,v}^1$ for some neighbor $v$ of $u$ in $G$, and hence corresponds to a vertex $u \in V_c$. Consequently,

$$|V_c| = |P'| - |E| = |P'| - 3n \leq |P| - 3n.$$

On the other hand, we can obtain a monochromatic clique partition $\tilde{P}$ of $H$ from a minimum vertex cover $V_c^*$ of $G$, as follows. For a gadget $g(u, v)$ in $H$, if $(u, v)$ is covered by exactly one end vertex in $V_c^*$, say $u$, then let $e_{u,v}^1$ be in the same clique with $u$ in $\tilde{P}$, and hence $K(u)$ is of color $\ell(u)$. If $(u, v)$ is covered by both $u$ and $v$, then let $e_{u,v}^1$ be in the same clique with either $u$ or $v$ in $\tilde{P}$ arbitrarily.

Since every edge is covered by at least one vertex, every vertex of type $e_{u,v}^1$ is contained in either $K(u)$ or $K(v)$. We claim that for every vertex $u \in V_c^*$, $K(u)$ contains at least one vertex of type $e_{u,v}^1$. For if there exists a $u_0 \in V_c^*$ such that $K(u_0)$ contains no vertex $e_{u_0,v}^1$ for every neighbor $v$ of $u_0$ in $G$, then $V_c^* \setminus \{u_0\}$ is a vertex cover of $G$ with cardinality less than $V_c^*$, contradicting the minimality of $V_c^*$. Since no two vertices in $G$ can be in the same clique in $\tilde{P}$, there are exactly $|V_c^*|$ cliques in $\tilde{P}$ containing vertices of type $e_{u,v}^1$.

For any vertex $v$ of $G$ that is not contained in $V_c^*$, let $v$ form a clique in $\tilde{P}$ with a vertex $e_{u,v}^2$ for a neighbor $u$ of $v$ in $G$. Note that such a vertex $e_{u,v}^2$ is always available for $v$, since any neighbor of $v$ in $G$ must be in $V_c^*$.

Finally, we let the remaining vertices of type $e_{u,v}^2$ be vertex-cliques in $\tilde{P}$. $\tilde{P}$ consists of $|E|$ cliques containing vertices of type $e_{u,v}^2$, and $|V_c^*|$ cliques containing vertices of type $e_{u,v}^1$, therefore

$$|V_c^*| = |\tilde{P}| - |E| = |\tilde{P}| - 3n.$$
Thus,

**Lemma 6.4.** \( \tilde{P} \) is a minimum monochromatic clique partition.

*Proof.* If there exists a monochromatic clique partition \( P \) with \( |P| < |\tilde{P}| \), then by the above discussion we can always obtain a vertex cover \( V_c \) of \( G \) with \( |V_c| \leq |P| - 3n < |\tilde{P}| - 3n = |V_c^*| \), contradicting the minimality of \( V_c^* \). \( \square \)

**Lemma 6.5.** The existence of a polynomial time approximation scheme for MCLP restricted to monochromatic-diamond-free graphs with maximum monochromatic clique \( K_4 \) would imply the existence of a polynomial time approximation scheme for the Vertex Cover problem restricted to 3-regular connected graphs.

*Proof.* Given an instance of the Vertex Cover problem restricted to 3-regular connected graphs, we have seen how it can be turned into an instance of MCLP restricted to monochromatic-diamond-free graphs with maximum monochromatic clique \( K_4 \). We can assume that the monochromatic clique partition \( P \) we find satisfies the condition that every vertex of type \( e_{u,v}^1 \) is contained in \( K(u) \) or in \( K(v) \), and that there is no vertex-clique \( u \) or \( v \) in \( P \). Then from \( P \) we can obtain a solution \( V_c \) for the instance of the Vertex Cover problem, in the way we discussed above. We keep using the notations \( P^* \) and \( V_c^* \) to denote the optimal solutions for both problems. We have \( |P| = |V_c| + 3n \) and \( |P^*| = |V_c^*| + 3n \). Further, using \( |P^*| \leq 8|V_c^*| \), we have \( 3n \leq 7|V_c^*| \).

Suppose there exists a small positive \( \epsilon \) such that \( |P| \leq (1 + \epsilon)|P^*| \). Substituting \( V_c \) and \( V_c^* \) into the inequality, we get \( |V_c| + 3n \leq (1 + \epsilon)(|V_c^*| + 3n) \), that is, \( |V_c| \leq (1 + \epsilon)|V_c^*| + 3n(1 + \epsilon) \leq (1 + \epsilon)|V_c^*| + 7\epsilon|V_c^*| = (1 + 8\epsilon)|V_c^*| \).

This completes the proof of the lemma. \( \square \)

Since the Vertex Cover problem restricted to 3-regular connected graphs is APX-complete, we have that MCLP is APX-hard on monochromatic-diamond-free graphs. Finally, we have the algorithm from [79] that works out a solution with a constant approximation ratio for MCLP in monochromatic-diamond-free graphs in which the size of a maximum monochromatic clique is bounded by a constant. Therefore, statement (2) holds.
6.4 An approximation algorithm for WMCLP

We generalize MCLP to its weighted version WMCLP. Let $G$ be an edge-colored graph with colors $\ell(G)$. Each color $c \in \ell(G)$ is associated with a non-negative cost $w(c)$. Every monochromatic clique $CL$ of $G$ with at least two vertices has the same non-negative cost as its color, denoted by $w(CL)$. As any vertex $v$ of $G$ is viewed as a degenerate monochromatic clique, we also assign it a non-negative cost $w(v)$, with $w(v) \leq \min\{w(c)|c \in \ell(G)\}$. WMCLP asks for a monochromatic clique partition such that the sum of the costs of all cliques in the partition is minimal among all the possible partitions. Obviously, MCLP is the special case of WMCLP in which all the costs are equal to 1.

Li and Zhang [79] presented a polynomial algorithm, denoted by Alg(clique), which calculates all maximal monochromatic cliques in a monochromatic-diamond-free graph, and returns a maximum one.

In this chapter, we use Alg(clique) as a subroutine to find all maximal monochromatic cliques in our $(\ln|V(G)| + 1)$-approximation algorithm (Algorithm 1) for solving WMCLP restricted to monochromatic-diamond-free graphs. In Algorithm 1, Alg(clique) is implemented from Step 2 to Step 9.

Let $G$ be a monochromatic-diamond-free graph, and let $CL_1$ and $CL_2$ be two distinct maximal monochromatic cliques in $G$. Suppose that there is at least one common edge $(u, v)$ with color $c$ of $CL_1$ and $CL_2$. Since $CL_1$ and $CL_2$ are maximal, there must be at least one vertex $w \in V(CL_1) \setminus V(CL_2)$ and one vertex $x \in V(CL_2) \setminus V(CL_1)$ such that $w$ and $x$ are not adjacent by an edge of color $c$. But then $u$, $v$, $w$ and $x$ span a monochromatic-diamond in $G$, a contradiction. Therefore, any two distinct maximal monochromatic cliques in a monochromatic-diamond-free graph do not share a common edge.

We note that MCLP can be considered as a variant of the Set Cover problem, in which the (possibly exponentially many) subsets are the vertex sets of all the monochromatic cliques and vertex-cliques of the input graph $G$, and the objective is to find a minimum collection of pairwise disjoint subsets covering the vertex set of $G$. Hence, it is natural that our design of a greedy approximation algorithm for WMCLP is inspired by the greedy algorithm for the weighted Set Cover problem in [34].
Algorithm 1 An approximation algorithm for WMCLP on monochromatic-diamond-free graphs

**Input:** A monochromatic-diamond-free graph $G$;

**Output:** A monochromatic vertex-disjoint clique partition $D$ of $G$;

1. Let $C := \emptyset$, $D := \emptyset$;
2. repeat
3. Select an edge $(v_i, v_j) \in E(G)$;
4. Let $S := \{v_i, v_j\}$;
5. while there is a vertex $v_k$ which is color-adjacent to each vertex of the monochromatic clique $G[S]$ do
6. $S = S \cup \{v_k\}$;
7. end while
8. $C = C \cup \{S\}$, $E(G) = E(G) \setminus E(S)$, where $E(S)$ denotes the edges of $G$ with both end vertices in $S$;
9. until no edge in $E(G)$.
10. Let $Q := V(G) \cup C$;
11. repeat
12. Pick $q \in Q$ such that the ratio $w(q)/|q|$ is minimum, where $w(q)$ denotes the weight of the monochromatic clique $G[q]$;
13. Let $C' := \emptyset$;
14. for all $c \in C$ do
15. $c = c \setminus q$, $C' = C' \cup \{c\}$;
16. end for
17. $D = D \cup \{q\}$, $V(G) = V(G) \setminus q$, $Q = V(G) \cup C'$, $C = C'$;
18. until $V(G) = \emptyset$.
19. return $D$.

**Theorem 6.6.** Algorithm 1 runs in polynomial time and achieves the performance ratio $\ln|V(G)| + 1$ for WMCLP on a monochromatic-diamond-free graph $G$.

**Proof.** In this proof, we do not distinguish between a clique and its vertex set.

First we claim that the set $C$ contains all maximal monochromatic cliques after the execution of the loop from Step 2 to Step 9 in Algorithm 1. Since any two maximal monochromatic cliques do not share an edge in $G$, every edge belongs to one maximal monochromatic clique. Hence, we can start from the end vertices of any edge, and find out the maximal monochromatic clique containing the edge through the loop from Step 5 to Step 7. Then, all edges of
this clique are removed from $E(G)$. Repeating this process until $E(G)$ becomes empty, all maximal monochromatic cliques of $G$ are found. The running time of the loop from Step 2 to Step 9 is at most $O(|E||V|^2) = O(|V|^4)$.

Assume that the loop from Step 11 to Step 17 is iterated $r$ times. Let the vertex-clique or maximal monochromatic clique picked in Step 12 at the $i$-th iteration of the loop be $q_i$, for $1 \leq i \leq r$. Let $G = G_1$ and $G_{i+1} = G_1 \setminus \{q_1 \cup q_2 \cup \ldots \cup q_i\} = G_i \setminus q_i$, for $1 \leq i \leq r - 1$. The algorithm outputs $D = \{q_i, 1 \leq i \leq r\}$ as a solution.

It is easy to prove by induction that $Q$ contains all the vertex-cliques and maximal monochromatic cliques of $G_{i+1}$ at the $i$-th iteration after the execution of Step 16, for $1 \leq i \leq r - 1$. We denote by $P_1$ an optimal monochromatic clique partition of $G_i$ and $w(P_i)$ the cost of $P_i$, for $1 \leq i \leq r$. Note that $P_1$ is an optimal solution of the problem. Let $j$ be an integer such that $1 \leq j \leq r-1$. Let the number of cliques in $P_j$ be $t$, and $P_j = \{p_{j1}, p_{j2}, \ldots, p_{jt}\}$. Then $P'_j = \{p_{ji} \setminus q_j : 1 \leq i \leq t\}$ is a monochromatic clique partition of $G_{j+1}$.

Hence, $w(P_{j+1}) \leq w(P'_j) = \sum_{i=1}^{t} w(p_{ji} \setminus q_j)$. Furthermore, for each $p_{ji} \in P_j$, if $|p_{ji} \setminus q_j| \geq 2$, $w(p_{ji}) = w(p_{ji} \setminus q_j)$; otherwise $w(p_{ji}) \geq w(p_{ji} \setminus q_j)$. Therefore,

$$w(P_{j+1}) \leq w(P'_j) \leq \sum_{i=1}^{t} w(p_{ji}) = w(P_j).$$

Note that for any monochromatic clique $q$ and $q' \subseteq q$ with $|q'| \geq 2$, the relation $\frac{w(q)}{|q|} \leq \frac{w(q')}{|q'|}$ holds. Therefore, the clique $q_i$ picked in Step 12 at the $i$-th iteration has the minimum ratio $\frac{w(q_i)}{|q_i|}$ over all vertex-cliques and monochromatic cliques of $G_i$. So we have $\frac{w(q_i)}{|q_i|} \leq \frac{w(p_{ji})}{|p_{ji}|}$, for $1 \leq j \leq t$. Therefore,

$$w(P_i) = \sum_{j=1}^{t} \frac{|w(p_{ji})|}{|p_{ji}|} |p_{ji}| \geq \frac{w(q_i)}{|q_i|} \sum_{j=1}^{t} |p_{ji}| = \frac{w(q_i)}{|q_i|} |V(G_i)|,$$

or, $w(q_i) \leq \frac{w(P_i)|q_i|}{|V(G_i)|}$. Hence,

$$w(D) = \sum_{i=1}^{r} w(q_i) \leq \sum_{i=1}^{r} \frac{w(P_i)|q_i|}{|V(G_i)|} \leq w(P_1) \sum_{i=1}^{r} \frac{|q_i|}{|V(G_i)|} \leq w(P_1) \sum_{i=1}^{r} \frac{1}{|V(G_i)| - k} = w(P_1) [H(|V(G)|)] \leq w(P_1)(\ln|V(G)| + 1),$$

where $H(|V(G)|)$ is the $|V(G)|$-th harmonic number. \qed
To show that the above approximation ratio is tight, we present an example to demonstrate that the approximation algorithm may find a solution with cost $H(n)$ times the optimum, where $n$ is the number of vertices of the graph.

Let $G$ be an edge-colored complete graph with vertices $v_1, v_2, \ldots, v_n$. And let the cost of every vertex-clique $v_i$ be $\frac{1}{i+\epsilon}$ for $i = 1, 2, \ldots, n$, where $\epsilon$ is a very small positive number. All edges $e \in E(G)$ have the same color of cost 1. It is not difficult to verify that Algorithm 1 finds a solution consisting of all vertex-cliques in the order $v_n, v_{n-1}, \ldots, v_1$, with total cost $\sum_{i=1}^{n} \frac{1}{i+\epsilon}$, and hence arbitrarily close to $H(n)$, whereas the optimal solution picks $G$ directly, with cost 1.

### 6.5 MCYP is NP-complete for triangle-free graphs

We first consider several trivial cases of MCYP. Let $G$ be an edge-colored graph on $n$ vertices. If $G$ is colored with a small number of colors, say $\ell(G) = 1$ or $\ell(G) = 2$, then we can only partition $G$ into $n$ vertex-cycles. When $G$ is colored by the largest possible number of colors, that is, $\ell(G) = |E(G)|$, then finding the minimum multicolored cycle partition of $G$ is at least as hard as finding a Hamiltonian cycle of $G$, which is well-known to be NP-hard. For $\ell(G) \geq 3$, Li and Zhang [79] showed that MCYP is NP-complete, even if the input graph $G$ is diamond-free. Their proof is based on a reduction from the Exact Cover By 3-Sets problem. We achieve a further strengthening by showing that MCYP is NP-complete, even if the input graphs are restricted to triangle-free graphs, a proper subclass of diamond-free graphs.

**Theorem 6.7.** MCYP is NP-complete when restricted to triangle-free graphs.

#### 6.5.1 Proof of Theorem 6.7

MCYP on triangle-free graphs is clearly in NP: a nondeterministic algorithm needs only guess a set of cycles of the input graph, and check in polynomial time whether the cycles in the set are vertex-disjoint multicolored cycles that cover all the vertices of the graph, and whether the number of cycles in the set is no larger than a given positive number.

Our proof of the NP-completeness of MCYP is based on a reduction
from the Minimum Set Cover problem. In an instance of the Minimum Set Cover problem, a universe \( U \) of \( n \) elements, a collection of subsets of \( U \), \( S = \{s_1, \ldots, s_m\} \) where \( \bigcup_{i=1}^{m} s_i = U \), and a positive integer \( k \leq \min\{m, n\} \) are given. The question is whether there exists a subcollection \( C \) of \( S \) with \( |C| \leq k \) that covers all the elements of \( U \).

Suppose now that we are given an instance of the Minimum Set Cover problem, with the universe \( U = \{u_i | 1 \leq i \leq n\} \) and the subset collection \( S = \{s_j | 1 \leq j \leq m\} \), where \( \bigcup_{i=1}^{m} s_i = U \). We construct an edge-colored triangle-free graph \( G \) as follows.

The vertex set of \( G \) is the union of the sets \( U_i = \{u_i^1, u_i^2\} \cup \{u_{ij}^1, u_{ij}^2 | 1 \leq j \leq m\}, 1 \leq i \leq n \), and \( S_i = \{s_{ij} | 0 \leq j \leq n+1\}, 1 \leq i \leq m \).

![Figure 6.2: The graph G constructed from an instance of the Minimum Set Cover problem, in which \( u_j, u_r \in s_i \) and \( j < r \).](image)

Labels on some edges denote the colors of the edges. We assume that the edges \( (s_{ti}, s_{t(n+1)}^{(i)}) \) are colored \( l_0, 1 \leq t \leq m \).

We define the following colors for the edges of \( G \).

1. For \( 1 \leq i \leq n, 1 \leq j \leq 2m + 1 \), define the colors \( c_{ij} \).
2. For \( 0 \leq i \leq n \), define the colors \( \ell_i \).
3. For $1 \leq i \leq m$, define the colors $\ell_{i1}$ and $\ell_{i(n+1)}$.

4. For $1 \leq i \leq m$, supposing the set $s_i$ contains $t_i$ elements, define the colors $d_{i1}, d_{i2}, \ldots, d_{i(t_i-1)}$.

The edges of $G$ and their colors are given below.

1. For $1 \leq i \leq m$, form the cycles $CY(s_i) = s_{i0} s_{i1} \ldots s_{i(n+1)} s_{i0}$. Assign color $\ell_{i1}$ to edge $(s_{i0}, s_{i1})$, and color $\ell_{i(n+1)}$ to edge $(s_{i0}, s_{i(n+1)})$. Each of the other $n$ edges on the cycle is assigned a color from the set $\{c_{j1} : u_j \notin s_i, 1 \leq j \leq n\} \cup \{\ell_0\} \cup \{d_{ij} : 1 \leq j \leq t_i - 1\}$, so that no two edges have the same color. Note that we have exactly $n$ colors in the color set, because $t_i$ is the number of $u_j$'s contained in $s_i$. We denote by $P(s_{i1}, s_{i(n+1)})$ the path $s_{i1} s_{i2} \ldots s_{i(n+1)}$.

2. For $1 \leq i \leq n$, form the cycles $CY(u_i) = u_{i1}^1 u_{i2}^1 \ldots u_{im}^1 u_{i1}^2 u_{i2}^2 \ldots u_{im}^2 u_{i1}^1$. Assign color $\ell_i$ to the edge $(u_{i1}^1, u_{i2}^1)$. And denote the paths $u_{i1}^1 u_{i2}^1 \ldots u_{im}^1 u_{i1}^2 u_{i2}^2 \ldots u_{im}^2 u_{i1}^1$ by $P(u_{i1}^1, u_{i2}^1)$. Assign colors $c_{ij}, 1 \leq j \leq 2m + 1$ to the $2m + 1$ edges of $P(u_{i1}^1, u_{i2}^1)$ successively.

3. For $1 \leq j \leq n$, $1 \leq i \leq m$, if $s_i$ contains $u_j$, join $u_j^1$ and $s_{i1}$ by an edge and assign color $\ell_{i1}$ to the edge, and join $u_j^2$ and $s_{i(n+1)}$ and assign color $\ell_{i(n+1)}$ to the edge.

4. For $1 \leq j < r \leq n$, if there exists a set $s_i$, $1 \leq i \leq m$, containing both $u_j$ and $u_r$, then join $u_j^1$ and $u_r^2$, and $u_j^2$ and $u_r^1$, and assign color $\ell_j$ to the edges.

Figure 2 is an illustration of $G$. It is easy to verify that $G$ is triangle-free and that the construction can be accomplished in polynomial time.

Now suppose that $G$ has a multicolored cycle partition $P$ with $k' + m \leq k + m$ multicolored cycles. We list the following properties of $P$, which are crucial for our proof.
1. For $1 \leq i \leq m$, the vertex $s_{i0}$ either forms a vertex-cycle, or is contained in the cycle $CY(s_i)$ of $P$, and hence we have $m$ cycles in $P$, each containing one $s_{i0}$. To see this, suppose that $s_{i0}$ is contained in a multicolored cycle $CY$. $CY$ must contain the edges $(s_{i0}, s_{i1})$ and $(s_{i(n+1)}, s_{i0})$. Since every edge associated with $s_{i1}$ has color $\ell_{i1}$ except for the edge $(s_{i1}, s_{i2})$, $CY$ must contain $(s_{i1}, s_{i2})$. Similarly $CY$ must contain the edge $(s_m, s_{i(n+1)})$ and hence $CY = CY(s_i)$.

2. For $1 \leq i \leq m$, the path $P(s_{i1}, s_{i(n+1)})$ is contained in a multicolored cycle; otherwise, the vertices $s_{i0}, s_{i1}, \ldots, s_{i(n+1)}$ must form $n+2$ vertex-cycles in $P$. However, we have $m$ multicolored cycles to cover all $s_{i0}$ by Property 1. Together we need at least $m + n + 2 - 1 = m + n + 1 > k' + m$ cycles in $P$, a contradiction.

3. For any $1 \leq i \neq j \leq m$, the two paths $P(s_{i1}, s_{i(n+1)})$ and $P(s_{j1}, s_{j(n+1)})$ cannot be contained in the same multicolored cycle, since both of them have an edge of color $\ell_0$.

4. For $1 \leq i \leq n$, the path $P(u_i^1, u_i^2)$ is contained in a multicolored cycle; otherwise, the $2m$ vertices $u_{ij}^1$ and $u_{ij}^2$, $j = 1, \ldots, m$ would form $2m$ vertex-cycles in $P$. Then, there would be at least $2m + 1 > k' + m$ cycles in $P$, a contradiction.

5. If there is a multicolored cycle $CY$ in $P$ containing only vertices in $\bigcup_{j=1}^n U_j$, then $CY = CY(u_i)$, for some $1 \leq i \leq n$. To see this, let $j_0$ be the smallest index such that $CY$ contains some vertices in $U_{j_0}$. By Property 4, $CY$ contains the paths $P(u_{j_0}^1, u_{j_0}^2)$. If $CY$ contains some more vertices in $\bigcup_{j=1}^n U_j$, then there must be two edges joining $P(u_{j_0}^1, u_{j_0}^2)$ to the other part of $CY$, which must be associated with $u_{j_0}^1$ and $u_{j_0}^2$. However, both edges have the same color $\ell_{j_0}$, contradicting the multicolored property of $CY$. Therefore $CY = CY(u_{j_0})$.

6. If $s_j$ does not contain $u_i$, then the path $P(u_i^1, u_i^2)$ cannot be in a multicolored cycle which contains the path $P(s_{j1}, s_{j(n+1)})$. The reason is that the first edge of $P(u_i^1, u_i^2)$ is colored $c_{i1}$, while there is also an edge of color $c_{i1}$ on the path $P(s_{j1}, s_{j(n+1)})$.

**Lemma 6.8.** There are only four possible kinds of multicolored cycles in the partition $P$, as follows.
(1) The cycles CY(s_i), for some 1 ≤ i ≤ m.

(2) The cycles containing a path P(s_{i1}, s_{i(n+1)}) and several paths P(u^1_j, u^2_j), where u_j ∈ s_i for some 1 ≤ i ≤ m and 1 ≤ j ≤ n.

(3) The vertex-cycles formed by s_0 for some 1 ≤ i ≤ m.

(4) The cycles CY(u_i), for some 1 ≤ i ≤ n.

Proof. By Property 1, every vertex s_0 must be contained in cycle kind of (1) or (3). If s_0 forms a vertex-cycle, then by Property 2 and Property 3, the path P(s_{i1}, s_{i(n+1)}) must form a multicolored cycle with some vertices from \( \bigcup_{i=1}^{n} U_i \) in P. Furthermore, by Property 4 and Property 6, P(s_{i1}, s_{i(n+1)}) must form a multicolored cycle with some paths P(u^1_j, u^2_j), where u_j ∈ s_i for some 1 ≤ j ≤ n. All such cycles belong to cycle kind of (2). Finally, every cycle that contains only vertices from \( \bigcup_{i=1}^{n} U_i \) is of kind (4), by Property 5.

Lemma 6.9. Given a positive integer \( k \leq \min\{m, n\} \), there is a covering \( C \subseteq S \) of \( U \) with no more than \( k \) subsets, if and only if \( G \) has a multicolored cycle partition \( P \) with \( k' + m \leq k + m \) multicolored cycles.

Proof. Suppose \( G \) has a multicolored cycle partition \( P \) with \( k' + m \leq k + m \) vertex-disjoint multicolored cycles. By Property 1, there must be \( m \) cycles of the first or third kind in \( P \), each covering one \( s_0 \), for 1 ≤ i ≤ m. Let \( t \) and \( t' \) denote the number of multicolored cycles in \( P \) of the second kind and the fourth kind, respectively. Then \( t + t' = k' \leq k \). Every multicolored cycle of the second kind contains a path \( P(s_{i1}, s_{i(n+1)}) \) for some \( s_i \in S \); there are \( t' \) such \( s_i \)'s. For every multicolored cycle \( CY(u_i) \) of the fourth kind, we can always find a subset \( s_j \in S \) containing \( u_i \), since \( \bigcup_{j=1}^{n} s_j = U \); there are at most \( t' \) such \( s_j \)'s. The subcollection of \( S \) composed of the \( s_i \)'s and \( s_j \)'s covers \( U \), and has at most \( t + t' \leq k' \leq k \) elements.

Conversely, let there be a covering \( C \) of \( U \) with \( k' \leq k \) subsets. Without loss of generality, let \( C = \{s_1, \ldots, s_k\} \). Let \( s'_1 = s_1 \), and \( s'_i = s_i \setminus \bigcup_{j=1}^{i-1} s_j \), i ≥ 2. Then \( s'_1, \ldots, s'_k \) are \( k' \) disjoint subsets (some of which may be empty) whose union covers \( U \). For every 1 ≤ i ≤ k', if \( s'_i \neq \emptyset \), take a multicolored cycle of the second kind constituted by the path \( P(s_{i1}, s_{i(n+1)}) \) and the paths \( P(u^1_j, u^2_j) \) for all \( u_j \in s'_i \). Since \( \bigcup_{i=1}^{k} s'_i \) covers \( U \), all vertices of \( G \) in \( U_1 \), 1 ≤ i ≤ n, are covered by these multicolored cycles. Finally take \( m \) multicolored cycles of the first kind \( CY(s_i) \) or the third kind \( s_0 \), covering all vertices of \( G \) that are left. Then a partition of at most \( k' + m \leq k + m \) multicolored cycles for \( G \) is obtained.
Since the Minimum Set Cover problem is $\mathbf{NP}$-complete ([45]), by Lemma 6.9, we have that (the decision version of) MCYP with restriction to triangle-free graphs is $\mathbf{NP}$-complete.

6.6 Concluding Remarks

The monochromatic clique and multicolored cycle partitioning problems studied in this chapter have many practical applications in information science and operations research. In this chapter, we obtained results on the inapproximability and complexity of MCLP and MCYP restricted to graphs avoiding some induced subgraphs, and we presented a $(1 + \ln |V(G)|)$-approximation algorithm for WMCLP restricted to monochromatic-diamond-free graphs. A natural suggestion for further research is to consider possible approximation algorithms for MCYP or its weighted version on triangle-free graphs, and alternative algorithms that might improve the approximation ratio $(1 + \ln |V(G)|)$ for WMCLP on monochromatic-diamond-free graphs. Another interesting direction is to study the computational complexity of similar problems, e.g., on minimum monochromatic or multicolored path and tree partition problems, restricted to graphs avoiding some induced subgraphs.
Summary

In this thesis we contribute with new theoretical results and algorithms to the research area related to the existence of cycles and paths in (directed) graphs. In Chapter 1 we briefly present the background, some history as well as the main ideas behind our work. In Chapter 2 we consider sufficient degree sum conditions for the existence of a Hamiltonian cycle in a digraph. These conditions are among the oldest kind of conditions that have been considered in this area. We improve a classical result due to Woodall involving the sum of the in-degree and out-degree of two vertices for which there is no arc from one to another, by characterizing the exceptional digraphs for the weakened condition.

The main themes of Chapter 3 to Chapter 5 are strong cycle and path properties, in the sense that they involve the existence of cycles or paths of many lengths. In Chapter 3, we introduce the concept of path extendability in digraphs, and study path extendability in general digraphs, as well as in tournaments. Several degree and extremal conditions for path extendability are presented, and path extendability of regular tournaments is settled. We also raise several open problems for future research work. In Chapter 4, we study cycle extendability in bipartite tournaments. We prove that in bipartite tournaments, Hamiltonicity, pancyclicity and cycle extendability are equivalent properties, with only one exceptional class that can be clearly described. We propose problems for further consideration involving the equivalence of these properties in the more general class of multipartite tournaments. In Chapter 5, we calculate the number of 2-paths between every vertex pair, a concept
that one often encounters in conditions for cycle and path properties in random oriented graphs and random tournaments. Based on the outcomes of our enumeration of 2-paths, we obtain several results related to path extendability in tournaments.

Finally in Chapter 6, we study computational problems in the context of cycle partitions and clique partitions of edge-colored graphs, which can be viewed as a generalization of digraphs. The results of this chapter deal with computational complexity and algorithmic aspects.

Besides the results we obtained, we would like to emphasize some methods and tools we developed. In Chapter 5, we calculate the number of 2-paths in order to use the outcome in conditions or as a tool for establishing properties related to cycles and paths in tournaments. In Chapter 4, we define the concept of the in-out graph of a digraph. We then use in-out graphs to construct and analyze the structure of cycle factors of two intersecting Hamiltonian cycles, and as such they form a key ingredient in one of our proofs. To establish a degree condition for path extendability in Chapter 3, we introduce a contraction operation to relate path-extendability and cycle-extendability, so that we can use a former result of Hendry to simplify one of our proofs.

The research reported here is mainly focussed on digraphs. In the field of digraphs, abundant research results have been accumulated since the middle of the last century, in particular with respect to paths and cycles. These results have been collected in a comprehensive monograph and numerous surveys. However, we can still find lots of uncultivated areas in this digraph garden, especially when we compare it with the research work on undirected graphs (referred to as graphs below) which constitutes a much larger volume. On the one hand, since graphs can be viewed as special cases of digraphs, many concepts, techniques and results on graphs have been and still can be generalized to digraphs. Because of the more complicated and varied structure of digraphs, such generalizations might lead to difficult challenges, as well as to deep results and surprising discoveries. On the other hand, there are several special types of digraphs, such as oriented graphs, tournaments and their many generalizations, for which we could conduct research that is specific and
without any reference to counterparts on graphs. Most of our works falls into these two categories. The concept of path extendability in digraphs, as well as the degree and extremal conditions for it, can be viewed as generalizations of the counterparts for graphs, whereas the work on cycle extendability, path extendability and 2-paths in tournaments or bipartite tournaments is specific to digraphs.

For the near future, what seems most attractive to us is to further our knowledge on the strong cycle and path properties of tournaments or their generalizations. Conjecture 5.1 that almost all tournaments are path extendable, as well as Conjecture 4.1 and Conjecture 4.2 on pancyclicity and cycle extendability of Hamiltonian multipartite tournaments, will be our main research concern. Meanwhile, we hope to further develop and explore our tools. The concept of in-out graphs has successfully helped to handle cycle extendability of bipartite tournaments. We will next attempt to apply it for extending our knowledge on pancyclicity and cycle extendability of multipartite tournaments. We have also planned a research route to prove path extendability of random tournaments using some ideas related to 2-paths.

As a long-term research direction, paths and cycles in edge-colored graphs seem to usher in a broad prospect for new results. Although edge-colored graphs have been studied for a long time, they have caught much more attention in recent years. Assigning colors to the edges increases the diversity of the structures that one could study. For example, one can investigate structures which are monochromatic, multi-colored, or properly colored, as in Chapter 6. With a color assigned to every edge instead of just a direction, edge-colored graphs have a stronger capability to model various networks appearing in real-world applications, which might need to be characterized with more parameters. Thus edge-colored graphs better meet the needs of the rapidly growing area of network science. In addition to the kind of work on computational complexity and algorithm design as reported in Chapter 6, one could also seek for sufficient conditions for the existence of cycles and paths with restrictions on their coloring in edge-colored graphs.
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Appendix A

Path extendability of examples or exceptional digraphs

In this appendix, we discuss the path extendability (or non-extendability) of the three regular tournaments on 7 vertices, and prove that all digraphs in the exceptional classes of our results in Chapter 3 are not path extendable.

Firstly, we prove that among the three regular tournaments on 7 vertices, the ones in Figure 3.1 (a) and Figure 3.1 (c) are not path extendable, while the one in Figure 3.1 (b) is path extendable.

The tournament in Figure 3.1 (a) is not path extendable, since the arcs $v_i v_{i+5}$, where $0 \leq i \leq 6$ and the addition is modulo 7, is not extendable. The tournament in Figure 3.1 (c) is not path extendable, since the arcs $v_4v_1$, $v_5v_2$ and $v_6v_3$ are not extendable.

The path extendability of the tournament in Figure 3.1 (b) is proved in the theorem below. However we would like to present some general ideas to verify the extendability of short paths first. Let $M(D)$ be the adjacency matrix of a digraph $D$. $D$ is 1-path extendable if and only if $M(D)^2 - M(D)$ is a non-negative matrix. Let $T$ be a tournament with at least 4 vertices. If $T$ is 1-path extendable, then $T$ must be 2-path extendable and 3-path extendable. For,
let $uvw$ be a 2-path in $T$. Then $uv$ can be extended to $uxv$ and $x \neq w$, so $uvw$ is extendable to $uxvw$. Similarly, let $wwx$ be a 3-path in $T$. Then $vw$ can be extended to $vyw$, and $y \neq u$ or $x$, so $wwx$ is extendable to $wwywx$.

**Theorem A.1.** The regular tournament in Figure 3.1 (b) is path extendable.

**Proof.** Let $T$ be the tournament in Figure 3.1 (b). As stated above, $k$-path extendability for $1 \leq k \leq 3$ of $T$ can be proved by verifying that $M(T)^2 - M(T)$ is non-negative.

Suppose that there is a 4-path $P = w_0w_1w_2w_3w_4$ which is not extendable in $T$. Since every arc of $T$ is extendable, the vertices that can be inserted into the arcs of $P$ must be on $P$. Therefore, we can conclude that the arcs $w_0w_3$, $w_1w_4$, $w_2w_0$, $w_3w_1$ and $w_4w_2$ exist. Let the other two vertices in $T$ be $w_5$ and $w_6$. We may assume that $w_5w_2, w_2w_6 \in A(T)$. Since $P$ is not extendable, we must have $w_5v_1, w_5w_0, w_3w_6$ and $w_4w_6 \in A(T)$. Since $d^+(v_1) = 3$, we must have $w_1w_6 \in A(T)$. But then $d^-(w_6) \geq 4$, a contradiction.

To prove 5-path extendability of $T$, we need to verify that $T$ is Hamiltonian-connected. Since $T$ is vertex transitive, we need only verify that $T$ has Hamiltonian paths from $v_0$ to any other vertex. By Theorem 3.10, such paths exist from $v_0$ to $v_1$, $v_2$ and $v_4$, while $v_0v_4v_5v_6v_1v_2v_3$, $v_0v_4v_6v_1v_2v_3v_5$ and $v_0v_1v_5v_2v_3v_4v_6$ are such paths from $v_0$ to $v_3$, $v_5$ and $v_6$, respectively. \square

Next, we will prove that the digraphs in the exceptional classes of our results are not path extendable. Note that $\mathcal{T}_1$ and $\mathcal{T}_2 \setminus \mathcal{T}_2(7)$ are subclasses of $\mathcal{T}_4$, therefore proving that $D \in \mathcal{T}_4$ is not path extendable covers the cases that $D \in \mathcal{T}_1$ and $D \in \mathcal{T}_2 \setminus \mathcal{T}_2(7)$.

**Theorem A.2.** Let $D$ be a digraph with $n$ vertices. $D$ is not path extendable, if $n$ and $D$ satisfy one of the following conditions:

(1) $n \geq 3$ and $D \in \mathcal{D}_3(n)$.
(2) $n \geq 3$ and $D \in \mathcal{D}_4(n)$.
(3) $n = 4$ and $D \in \{D_{5,1}, D_{5,2}\}$.
(4) $n = 2t$ where $t$ is an integer, $t \geq 2$, and $D \in \mathcal{D}_6(t)$.
(5) $n = 6$, $D \in \{D_{7,1}, D_{7,2}\}$.
(6) $n = 3t + 1$ where $t$ is an integer, $t \geq 3$, $D_9(t) \subseteq D \subseteq D_8(t)$, and $d^+_F(u) \geq 1$ for $u \in \{v_1\} \cup V_2$ and $d^-_F(w) \geq 1$ for $w \in V_3 \cup \{v_0\}$, where $F = D \{v_0, v_1\} \cup V_2 \cup V_3$.
(7) $n = 7$ or $n \geq 13$, and $D \in \mathcal{T}_3$. 
Path Extendability

(8) \( n \geq 9 \) and \( D \in \mathcal{T}_4 \).
(9) \( n = 7 \) and \( D = T_2(T) \).

Proof. If \( n \) and \( D \) satisfy (1), then \( D = D_3(n, s) \in D_3(n) \) for some \( 0 \leq s \leq n-2 \). By the definition of \( D_3(n, s) \), we can see that there is no vertex \( u \in V(D) \) such that \( v_0 \to u \) and \( u \to v_1 \). Therefore, \( v_0v_1 \) is a non-extendable path of \( D \), and \( D \) is not path extendable.

If \( n \) and \( D \) satisfy (2), then \( D = D_{4,1}(n) \) or \( D = D_{4,2}(n) \). Suppose that \( D = D_{4,1}(n) \). By the definition of \( D_{4,1}(n) \), \( D \) is obtained from \( D' = K_{n-1} \) by adding a vertex \( u \), which is adjacent to all vertices of \( D' \), together with one arc from a vertex \( w \in V(D') \) to \( u \). Consider any Hamiltonian path \( P \) of \( D' \) terminating at \( w \). Since \( n \geq 3 \), \( |P| \geq 2 \). But \( P \) is not extendable. Therefore \( D \) is not path extendable. The case that \( D = D_{4,2}(n) \) can be proved similarly.

Suppose \( n \) and \( D \) satisfy (3). Both the paths \( v_0v_1v_2 \) in \( D_{5,1} \) and \( D_{5,2} \) are not extendable, therefore \( D \) is not path extendable.

Suppose \( n \) and \( D \) satisfy (4). By the definition of \( D_6(t) \), there is no vertex \( u \in V(D) \) such that \( v_0 \to u \) and \( u \to v_1 \). Therefore, \( v_0v_1 \) is a non-extendable path of \( D \), and so \( D \) is not path extendable.

Suppose \( n \) and \( D \) satisfy (5). Both the paths \( v_0v_1v_2 \) in \( D_{7,1} \) and \( D_{7,2} \) are not extendable, therefore \( D \) is not path extendable.

Suppose \( n \) and \( D \) satisfy (6). Note that in \( D_8(t) \), \( F = \langle V_1 \cup \{v_0, v_1\} \rangle \) is a complete digraph. We can find a Hamiltonian path \( P \) of \( F \) starting at \( v_0 \) and terminating at \( v_1 \). Since \( t \geq 3 \), \( |P| = t + 1 \geq 4 \). In \( D_9(t) \) only the arc \( v_0v_1 \) is removed from \( F \), so \( P \) is also a path of \( D_9(t) \). By \( D_9(t) \subseteq D \subseteq D_8(t) \), \( P \) is a path of \( D \).

If \( P \) is extendable to another path \( P' \), then there exists a vertex \( u \in (V_2 \cup V_3) \cap V(P') \). Suppose \( u \in V_2 \). The only arc from \( V(P) \) to \( u \) is \( v_1u \), and hence \( P' \) can not terminate at \( v_1 \) while containing \( u \), a contradiction. Similarly, assuming \( u \in V_3 \) leads to a contradiction. Therefore \( P \) is not extendable, and so \( D \) is not path extendable.

Suppose \( n \) and \( D \) satisfy (7). The construction of digraphs in \( T_3 \) is based on \( D_{11} \). We firstly prove that in \( D_{11} \), \( P = v_0v_1v_2v_3v_4v_5 \) is not extendable. Suppose to the contrary that \( P \) is extendable to another path \( P' \). From the structure of \( D_{11} \), \( P' \) either (i) starts with the arc \( v_0v_1 \) and terminates with the arc \( v_3v_4 \), or (ii) starts with the arc \( v_0v_3 \) and terminates with the arc \( v_1v_4 \). Suppose (i) holds. The vertices succeeding \( v_1 \) on \( P' \) have to be, in order, \( v_2 \) and \( u_0 \). However, \( u_0 \to v_3 \) and we can not form the path \( P' \) extending \( P \).
Suppose (ii) holds. Since we have only one new vertex in \( P' \), \( v_2 \) must either succeed \( v_3 \) or precede \( v_1 \) on \( P' \). However, \( v_1 \rightarrow v_2 \) and \( v_2 \rightarrow v_3 \). Therefore we can not form the path \( P' \) extending \( P \). Hence, \( P \) is not extendable in \( D_{11} \).

Now we prove that \( P \) is not extendable in \( D \). Note that adding the arc \( u_0u_1 \) or \( u_1u_0 \) in \( D_{11} \) does not help to extend \( P \). Therefore, if \( P \) is extendable to another path \( P' \) in \( D \), then there must exist a vertex \( u \in (N_0^+ \cup N_0^-) \cap V(P') \). However, \( V = V(P) \rightarrow N_0^+ \) and \( N_0^- \rightarrow V \). Hence any vertex from \( N_0^+ \) or \( N_0^- \) can not be in a path extending \( P \). So, \( P \) is not extendable in \( D \), and \( D \) is not path extendable.

Suppose \( n \) and \( D \) satisfy (8). We prove that the path \( P = v_0v_1 \ldots v_{p-1} \) is not extendable. Note that we have \( |P| \geq 3 \). Suppose to the contrary that \( P \) is extendable to \( P' \), and \( V(P') \setminus V(P) = \{u\} \). Since \( V = V(P) \rightarrow N_0^+ \) and \( N_0^- \rightarrow V \), \( u \notin N_0^+ \cup N_0^- \). Suppose \( u \in N_1^+ \). The only arc from \( u \) to \( V \) is \( uv_0 \). Hence \( u \) can not be on \( P' \) which starts with \( v_0 \), a contradiction. Similarly, assuming \( u \in N_1^- \) leads to a contradiction. Therefore \( P \) can not be extendable, and \( D \) is not path extendable.

Suppose \( n \) and \( D \) satisfy (9). We have proven that \( D_2(7) \), which is the same as the graph of Figure 3.1 (a), is not path extendable at the beginning of this section.
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