Multi-unit Bilateral Trade *

Matthias Gerstgrasser[1] Paul W. Goldberg[1] Bart de Keijzer[2]
Philip Lazos[1] Alexander Skopalik[3]

[1]Department of Computer Science, University of Oxford
{matthias.gerstgrasser, paul.goldberg, filippos.lazos}@cs.ox.ac.uk
[2]School of Computer Science and Electronic Engineering, University of Essex
b.dekeijzer@essex.ac.uk
[3]Department of Applied Mathematics, University of Twente
a.skopalik@utwente.nl

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Abstract

We characterise the set of dominant strategy incentive compatible (DSIC), strongly budget balanced (SBB), and ex-post individually rational (IR) mechanisms for the multi-unit bilateral trade setting. In such a setting there is a single buyer and a single seller who holds a finite number k of identical items. The mechanism has to decide how many units of the item are transferred from the seller to the buyer and how much money is transferred from the buyer to the seller. We consider two classes of valuation functions for the buyer and seller: Valuations that are increasing in the number of units in possession, and the more specific class of valuations that are increasing and submodular.

Furthermore, we present some approximation results about the performance of certain such mechanisms, in terms of social welfare: For increasing submodular valuation functions, we show the existence of a deterministic 2-approximation mechanism and a randomised $e/(1-e)$ approximation mechanism, matching the best known bounds for the single-item setting.

1 Introduction

Auctions form one of the most studied applications of game theory and mechanism design. In an auction setting, a single seller or auctioneer runs a pre-determined procedure or mechanism (i.e., the auction) to sell one or more goods to the buyers, and the buyers then have to strategise on the way they interact with the auction mechanism. An auction setting is rather restrictive in that it involves a single seller that is monopolistic and is assumed to be non-strategic. While this is a sufficient assumption in some cases, there are many applications that are more complex: It is often realistic to assume that a seller expresses a valuation for the items in her possession and that a seller wants to maximise her profit. Such settings in which both buyers and sellers are considered as strategic agents are known as *two-sided markets*, whereas auction settings are often referred to as *one-sided markets*.

The present paper falls within the area of mechanism design for two-sided markets, where the focus is on designing satisfactory market platforms or intermediation mechanisms that enable trade between buyers and sellers. In general, the term “satisfactory” can be tailored to the specific market under consideration, but nonetheless, in economic theory various universal properties have been identified and agreed on as important. The following three are the most fundamental ones:

- *Incentive Compatibility ((DS)IC)*: It must be a dominant strategy for the agents (buyers and sellers) to behave truthfully, hence not “lie” about their valuations for the items in the market. This enables the market mechanism to make an informed decision about the trades to be made.

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- *Individual Rationality (IR)*: It must not harm the utility of an agent to participate in the mechanism.
- *Strong Budget Balance (SBB)*: All monetary transfers that the mechanism executes are among participating agents only. That is, no money is injected into the market, and no money is burnt or transferred to any agent outside of the market.

This paper studies the capabilities of mechanisms that satisfy these three fundamental properties above for a very simple special case of a two-sided market. *Bilateral trade* is the most basic such setting comprising a buyer and a seller, together with a single item that may be sold, i.e., transferred from the seller to the buyer against a certain payment from the buyer to the seller. The bilateral trade setting is a classical one: It was studied in the seminal paper [20] and has been studied in detail in various other publications in the economics literature. Recent work in the Algorithmic Game Theory literature [3, 4, 10] has focused on the welfare properties of bilateral trade mechanisms. These works assume the existence of *prior distributions* over the valuations of the buyer and seller, that may be thought of as modelling an intermediary's beliefs about the buyer's and seller's values for the item.

The present paper studies a generalisation of the classical bilateral trade setting by allowing the seller to hold multiple units initially. These units are assumed to be of a single resource, so that both agents only express valuations in terms of how many units they have in possession. The final utility of an agent (buyer or seller) is then determined by her valuation and the payment she paid or received. We focus our study on characterising which mechanisms satisfy the the above three properties and which of these feasible mechanisms achieve a good social welfare (i.e., total utility of buyer and seller combined).

Due to its simplicity, our setting is fundamental to any strategic setting where items are to be redistributed or reallocated. Our characterisation efforts show that all feasible mechanisms must belong to a very restricted class, already for this very simple setting with one buyer, one seller, and a relatively simple valuation structure. The specific mechanisms we develop are very simple, and suitable for implementation with very little communication complexity.

Our Contribution. Our first main contribution is a full characterisation of the class of truthful, individually rational and strongly budget balanced mechanisms in this setting. We do this separately for two classes of valuation functions: submodular valuations and general non-decreasing valuations. Section 3 presents a high-level argument for the submodular case. A full and rigorous formal proof for both settings is given in Appendix A. Essentially, for the general case, any mechanism that aims to be truthful, strongly budget balanced and individually rational can only allow the agents to trade a single quantity of items at a predetermined price. The trade then only occurs if both the seller and buyer agree to it. This leads to a very clean characterization and has the added benefit of giving a robust, simple to understand mechanism: the agents do not have to disclose their entire valuation to the mechanism, and only have to communicate whether they agree to trade one specific quantity at one specific price. For the submodular case, suitable mechanisms can be characterised as specifying a per-unit price, and repeatedly letting the buyer and seller trade an item at that price until one of them declines to continue.

Secondly, we give approximation mechanisms for the social welfare objective in the Bayesian setting in Section 4, for the case of submodular valuations. Theorem 4.1 presents a 2-approximate deterministic mechanism. For randomised mechanisms, we show a $e/(e-1)$-approximation in Theorem 4.2.

Related Literature. The first approximation result for bilateral trade was presented in [19], where for the single-item case the author proves that the optimal *gain from trade* can be 2-approximated by the *median mechanism*, which is a mechanism that sets the seller's median valuation as a fixed price for the item, and trade occurs if and only if p lies in between the buyer's and seller's valuation and the buyer's valuation exceeds p. The analysis in [19] is done under the assumption that the seller's median valuation does not exceed the median valuation of the buyer. The gain from trade is defined as the increase in social welfare as a result of trading the item. [3] extended the analysis of this mechanism by showing that it also 2-approximates the social welfare without the latter assumption on the medians.

In [3], the authors furthermore consider the classical bilateral trade setting (with a single item) and present various mechanisms for it that approximate the optimal social welfare. Their best mechanism achieves an approximation factor of $e/(e-1)$. As in the present paper, there are prior distributions on the traders' valuations, and the quantity being approximated is the expectation over the priors, of the optimal allocation of the item.

The weaker notion of Bayesian incentive compatibility is considered in [4], where the authors propose a mechanism in which the seller offers a take-it-or-leave-it price to the buyer. They prove that this mechanism approximates the harder *gain from trade* objective within a factor of $1/e$ under a technical albeit often reasonable *MHR condition* on the buyer's distribution.

The class of DSIC, IR, and SBB mechanisms for bilateral trade was characterised in [8] to be the class of *fixed price mechanisms.* In the present work, we characterise this set of mechanisms for the more general multi-unit bilateral trade setting, thereby extending their result. The gain from trade arising from such mechanisms was analysed in [9].

Various recent papers analyse more general two-sided markets, where there are multiple buyers and sellers, who hold possibly complex valuations over the items in the market. [10] analyse a more general scenario with multiple buyers, sellers, and multiple distinct items, and use the same feasibility requirements as ours (DSIC, IR, and SBB). [23] have considered a similar setting but focus on *gains from trade (GFT)* (i.e., the increase in social welfare resulting from reallocation of the items) instead of welfare. They initially considered a multi-unit setting like ours (albeit with multiple buyers and sellers), and they extend their work in [22] to allow multiple types of goods. They present a mechanism that approximates the optimal GFT asymptotically in large markets. [2] designs two-sided market mechanisms for one seller and multiple buyers with a temporal component, where valuations are correlated between buyers but independent across time steps. A good approximation (of factor 1/2) of the social welfare using the more permissive notion of *Bayesian* Incentive Compatibility (BIC) was achieved by [6]. Their optimality benchmark is different from the one we consider as they compare their mechanism to the best possible BIC, IR, and SBB mechanism. A very recent work, [1], proposes mechanisms that achieve social welfare guarantees for both optimality benchmarks. [13] considers optimizing the gains from trade in a two-sided market setting tailored to online advertising platforms, and the authors extend this idea further in [12] by considering two-sided markets in an online setting.

The literature discussed so far aims to maximise welfare under some budget-balance constraints. An alternative natural goal is to maximise the intermediary's profit. This has been studied extensively starting with a paper by Myerson and Satterthwaite [20], which gives an analogue of Myerson's seminal result on optimal auctions, for the independent priors case. Approximately optimal mechanisms for that settings have further been studied. [11, 21] The correlated-priors case has been investigated from a computational complexity perspective by [15], as well as links back to auction theory [14]. Two adversarial, online variants of market intermediation were studied in [16, 18].

2 Preliminaries

In a *multi-unit bilateral trade* instance there is a buyer and seller, where the seller holds a number of units of an item. This number will be denoted by k. The buyer and seller each have a *valuation function* representing how much they value having any number of units in possession. These valuation functions are denoted by v and w, respectively. Precisely stated, a valuation function is a function $v : [k] \cup \{0\} \to \mathbb{R}_{\geq 0}$ where $v(0) = 0$. Note that we use the standard notation $[a]$, for a natural number a, to denote the set $\{1, \ldots, a\}$. We denote by v the valuation function of the buyer, drawn from f, and we denote by w the valuation function of the seller, drawn from g. For $q \in [k]$, the valuation $v(q)$ or $w(q)$ of an agent (i.e., buyer or seller) expresses in the form of a number the extent to which he would like to have q units in his possession.

A mechanism $\mathbb{M}$ interacts with the buyer and the seller and decides, based on this interaction, on an *outcome*. An outcome is defined as a quadruple $(q_B, q_S, \rho_B, \rho_S)$, where q_B and q_S denote the numbers of items allocated to the buyer and the seller respectively, such that $q_B + q_S = k$. Moreover, ρ_B and ρ_S denote the payments that the mechanism charges to the buyer and seller respectively. Note that typically the payment of the seller is negative since he will get money in return for losing some items, while the payment of the buyer is positive since he will pay money in return for obtaining some items. Let $\mathcal{O}$ be the set of all outcomes. For brevity we will often refer to an outcome simply by the number of units traded q_B.

Formally, a mechanism is a function $\mathbb{M} : \Sigma_B \times \Sigma_S \to \mathcal{O}$, where Σ_B and Σ_S denote strategy sets for the buyer and seller. A *direct revelation* mechanism is a mechanism for which Σ_B and Σ_S

consists of the class of valuation functions that we want to consider. That is, in such mechanisms, the buyer and seller directly report their valuation function to the mechanism, and the mechanism decides an outcome based on these reports. We want to define our mechanism in such a way that there is a dominant strategy for the buyer and seller, under the assumption that their valuation functions are in a given class $\mathcal{V}$. It is well known (see e.g. [5]) that then we may restrict our attention to direct revelation mechanisms in which the dominant strategy for the buyer and seller is to report the valuation functions that they hold. Such mechanisms are called *dominant strategy incentive compatible (DSIC)* for $\mathcal{V}$. We will assume from now on that $\mathbb{M}$ is a direct revelation mechanism. In this paper, we consider for $\mathcal{V}$ two natural classes of valuation functions:

- Monotonically increasing submodular functions, i.e., valuation functions v such that for all $x, y \in [k]$ where $x < y$ it holds that $v(x) - v(x-1) \geq v(y) - v(y-1)$ and $v(x) < v(y)$. This reflects a common phenomenon observed in many economic settings involving identical goods: Possessing more of a good is never undesirable, but the increase in valuation still goes down as the held amount increases. For a monotonically increasing submodular function v and number of units $x \in [k]$, we denote by $\tilde{v}(x)$ the *marginal* valuation $v(x) - v(x-1)$. Thus, it holds that $\tilde{v}(x) \geq \tilde{v}(y)$ when $x < y$.
- Monotonically increasing functions, i.e., valuation functions v such that $v(x) < v(y)$ for all $x < y$, where $x, y \in [k]$.

Besides the DSIC requirement, there are various additional properties that we would like our mechanism to satisfy.

- Ideally, our mechanism should be *strongly budget balanced (SBB)*, which means that for any outcome $(q_B, q_S, \rho_B, \rho_S)$ that the mechanism may output it holds that $\rho_B = -\rho_S$. This requirement essentially states that all money transferred is between the buyer and the seller only.
- Additionally, we want that running the mechanism never harms the buyer and the seller. This requirement is known as *(ex-post) individual rationality (IR).* Note that when v and w are the valuation functions of the buyer and the seller, then the initial utility of the buyer is 0 and the initial utility of the seller is $w(k)$. Thus, a mechanism $\mathbb{M}$ is individually rational if for the outcome $\mathbb{M}(v, w) = (q_B, q_S, \rho_B, \rho_S)$ it always holds that $v(q_B) - \rho_B \geq 0$ and $w(q_S) - \rho_S \geq w(k)$.
- We would like the mechanism to return an outcome for which the total utility is high. That is, we want the mechanism to maximise the sum of the buyer's and seller's utility, which is equivalent to maximizing the sum of valuations $v + w$ when strong budget balance holds.

We characterise in Section 3 the class of DSIC, SBB, IR mechanisms for both valuation classes. In Section 4, we subsequently provide various approximation results on the quality of the solution output by some of these mechanisms. For these results, we assume the standard *Bayesian setting*: The mechanism has no knowledge of the buyer's and seller's precise valuation, but knows that these valuations are drawn from known probability distributions over valuation functions. Our approximation results provide mechanisms that guarantee a certain outcome quality (which is measured in terms of *social welfare*, defined in Section 4) for arbitrary distributions on the valuation functions.

Formally, in the Bayesian setting, a multi-unit bilateral trade instance is a pair (f, g, k), where $k \in \mathbb{N}$ is the total number of units that the seller initially has in his possession, and f and g are probability distributions over valuation functions of the buyer and the seller respectively. Note that we do not impose any further assumptions on these probability distributions.

3 Characterisation

In [8] the authors prove that every DSIC, IR, SBB mechanism for classical bilateral trade (i.e. the case where $k = 1$) is a *fixed price mechanism*: That is, the mechanism is parametrised by a price $p \in \mathbb{R}_{\geq 0}$ such that the buyer and seller trade if and only if the buyer's valuation exceeds the price and the price exceeds the seller's valuation. Moreover, in case trade happens, the buyer pays p to the seller. In this paper we characterise the set of DSIC, IR, and WBB mechanisms for multi-unit bilateral trade, and we thereby generalise the characterisation of [8].

Theorem 3.1. *Any mechanism that satisfies DSIC, IR and SBB must be a* sequential posted price mechanism with a fixed per-unit price p, potentially with bundling*, which we will refer to as a* multi-unit fixed price mechanism. *Such a mechanism iteratively proposes a quantity q of units to both the buyer and seller simultaneously, which the seller and buyer can choose to either accept or reject. If both agents accept, q additional units are reallocated from the seller to the buyer, the buyer pays pq to the seller, and the mechanism may then either proceed to the next iteration or terminate. If one of the two agents rejects, the mechanism terminates. Quantity q may vary among iterations, but must be pre-determined prior to execution of the mechanism.*

For increasing submodular valuations, any number of iterations is allowed. For general increasing valuations, the mechanism is further restricted to execute only one iterations (or equivalently, it may only offer one bundle for a fixed price).

In simple terms, our result states that for the submodular valuations case, the only thing to be done truthfully in this setting is to set a fixed per-unit price p, and ask the buyer and seller if they want to trade one or several units of the good at per-unit price p. This repeats until one agent rejects. In the general monotone case this is further restricted to a single such proposed trade. The following is a brief high-level (informally stated) argument of the proof of Theorem 3.1 for the submodular setting. We refer the reader to Appendix A for the complete proof.

Lemma 3.2. *All prices must be fixed in advance, and cannot depend on the bid / valuation of neither the seller nor the buyer.*

Proof. This follows immediately from DSIC and SBB: By DSIC, for any outcome, the price charged to the buyer can't depend on the buyer's bid, otherwise one can construct scenarios in which the price charged by the buyer could be manipulated to the buyer's benefit by misreporting the bid. The same holds for the seller. By SBB the payment of the buyer completely determines the payment of the seller (the payment is simply negated) so neither payment can depend on either's bid. □

Theorem 3.3. *Suppose in a DSIC, SBB, IR mechanism the price for the outcome in which q units are traded is qp for a fixed per-unit price for all potential outcomes. Then the allocation chosen for a given pair of valuation functions is the one arising when asking bidders sequentially if they want to trade one unit (or a bundle of units), until one rejects.*

Proof. To see this, consider the seller's utility function $u_s(q) = q \cdot p + w(k - q)$ and the buyer's utility function $u_b(q) = v(q) - q \cdot p$, if q units would be traded at unit price p. Since both valuation functions are concave, it is easy to see that both utility functions are concave, and each has a single peak (one or more equal adjacent maxima, and no further local maxima). Furthermore they both start at 0, and once either of them becomes negative, it stays negative. Suppose we sequentially ask both bidders if they want to trade one unit for price p, until one rejects. Then the quantity traded is $\min(\text{argmax}(u_s), \text{argmax}\,(u_b))$, i.e. the first of the two peaks. If the mechanism iteratively proposes them bundles $q_1, q_2, \ldots$, then the same expression on the traded quantity would apply, but with the utility functions restricted to the domain $\{0, q_1, q_1 + q_2, \ldots\}$. If we ask them about the big all-k-item bundle, we would choose the bundle outcome iff $u(k) > u(0)$, for both, and 0 if for either of them $u(0) > u(k)$, i.e. if one (the first) of the peaks of the two utility functions restricted to $\{0, k\}$ is at 0.

Now, DSIC means that for any bid of the opposing agent, the agent cannot get anything better than what she gets by telling the truth. If the quantity traded by the mechanism would be larger than $\min(\text{argmax}(u_s), \text{argmax}\,(u_b))$, then the bidder with the lowest peak could improve her utility by claiming that all outcomes higher than her peak are wholly unacceptable (utility less than 0) to them; by IR, the mechanism would then be forced to trade the quantity at the first peak. If, on the other hand, the traded quantity would be less than the quantity of the first peak, then both players would gain by lying, in order to make the mechanism choose to trade a higher quantity (if such a quantity is at all present in the mechanism's set of tradeable quantities.) □

Theorem 3.4. *In a DSIC, SBB, IR mechanism, all potential outcomes, i.e., (quantity,price)-pairs, must have the same per-unit price.*

Proof. Suppose two outcomes have different per-unit prices. W.l.o.g. suppose for $q_1 < q_2$, $p_1/q_1 < p_2/q_2$, i.e. the per-unit price is higher in the larger allocation. Then there exists a valuation function v_{s1} for the seller in which the seller prefers outcome q_2 over q_1, but both give positive utility; and there exists another valuation function v_{s2} that gives negative utility for q_1, but the same utility for q_2. I.e. $0 < u_{s1}(q_1) < u_{s2}(q_2)$ but $u_{s2}(q_1) < 0 < u_{s2}(q_2) = u_{s1}(q_2)$. Now if for a given buyer's valuation, the chosen outcome given v_{s1} is q_1, then the seller would have an incentive to misreport v_{s2}, making outcome q_1 unavailable to the mechanism due to IR, thus making it choose q_2. Vice versa, if per-unit prices are decreasing, the same argument works for the buyer. □

Together, these three results give a full characterisation of the class of DSIC, IR, SBB mechanisms in this setting, although in our full formal proof that we provide in Appendix A, we need to take into account many further technical obstacles and details. There is, in particular, a *tie-breaking rule* present, that takes into account what should happen when the buyer or seller would be indifferent among multiple possible quantities, or when they would get a utility of 0 given the proposed prices and quantities.

For the case of general monotone valuations, any such mechanism must be further restricted to offering only a single outcome (other than no-trades) to the bidders. The complete proof can be found in Appendix A.

4 Approximation Mechanisms

In this section we study the design of DSIC, IR, SBB mechanisms that optimise the social welfare, i.e., the sum of the buyer's and seller's valuation. From Theorem 3.1, our characterization states that such a mechanism needs to be a multi-unit fixed price mechanism, so that the design challenge lies in an appropriate choice of unit-price p and quantities offered at each iteration of the mechanism.

We focus on the case of increasing submodular valuations. Obviously, every item traded can only increase the social welfare. Therefore, given that the objective is to maximise it, we repeatedly offer a single item for trade.[1] The challenge lies thus in determining the right unit price p. It is easy to see that no sensible analysis can be done if absolutely nothing is known about the valuation functions of the buyer and seller. Therefore, we assume a *Bayesian setting*, as introduced in Section 2 in order to model that the mechanism designer has statistical knowledge about the valuations of the two agents: The buyer's (and seller' valuation is assumed to be unknown to the mechanism, but is assumed to be drawn from a probability distribution f (and g) which is public knowledge. We show that we can now determine a unit price that leads to a good social welfare in expectation.

For a valuation function v of the buyer, we write $\hat{v}$ to denote the *marginal increase function* of v: $\hat{v}(q) = v(q) - v(q-1)$ for $q \in [k]$. Thus, $\hat{v}$ is a non-increasing function. Similarly, for a valuation function w of the seller, we write $\breve{w}$ to denote the *marginal decrease function* of w: $\breve{w}(q) = w(k-q+1) - w(k-q)$, for $q \in [k]$, so that $\breve{w}$ is a non-decreasing function. Thus, for all $q \in [k]$, the increase in social welfare as a result of trading q items as opposed to $q-1$ items is $\hat{v}(q) - \breve{w}(q)$. Note that therefore if v and w are increasing submodular valuation functions of the buyer and seller respectively, then the social welfare is maximised by trading the maximum number of units q such that $\hat{v}(q) > \breve{w}(q)$. We measure the quality of a mechanism on a bilateral trade instance (f, g, k) as the factor by which its expected social welfare is removed from the expected optimal social welfare $OPT(f, g, k)$ that would be attained if the buyer and seller would always trade the maximum profitable amount:

$$OPT(f,g,k) = \mathop{\mathbf{E}}_{v\sim f, w\sim g}\left[w(k) + \sum_{q=1}^{\max\{q':\hat{v}(q')>\breve{w}(q')\}} (\hat{v}(q) - \breve{w}(q))\right]$$
$$= \mathop{\mathbf{E}}_{v\sim f, w\sim g}\left[\sum_{q=1}^{k} \breve{w}(q) + \sum_{q=1}^{\max\{q':\hat{v}(q')\geq\breve{w}(q')\}} (\hat{v}(q) - \breve{w}(q))\right]$$

For $q \in [k]$ and a seller's valuation function w, we denote by $GFT(v, w, q)$ the value $\max\{0, \hat{v}(q) - \breve{w}(q)\}$ (where "GFT" is intended to stand for "Gain From Trade"). Note that $GFT(v, w, q)$ is

[1] Also, with respect to our tie-breaking rule mentioned at the end of the last section: We simply employ the tie breaking rule that favours the highest quantity to trade, which is the dominant choice when it comes to maximising social welfare.

non-increasing in q and that $OPT(f, g, k)$ can be written as

$$OPT(f, g, k) = \sum_{q=1}^{k} \mathbf{E}_{w\sim g}[\check{w}(q) + GFT(v, w, q)].$$

Note that a social welfare as high as opt $OPT(f, g, k)$ can typically not be attained by any DSIC, IR, SBB mechanism. However, it is still a natural benchmark for measuring the performance of such a mechanism, and we will see next that there exists such a mechanism that achieves a social welfare that is guaranteed to approximate $OPT(f, g, k)$ to within a constant factor. In particular, for a mechanism $\mathbb{M}$, let $q_{\mathbb{M}}(v, w)$ be the number of items that $\mathbb{M}$ trades on reported valuation profiles (v, w), and define

$$SW(\mathbb{M}, (g, f, k)) = \mathbf{E}_{v\sim f, w\sim g}[v(q_{\mathbb{M}}(v, w)) + v(k - q_{\mathbb{M}(v,w)})]$$

as the expected social welfare of mechanism $\mathbb{M}$. We say that $\mathbb{M}$ achieves an *α-approximation to the optimal social welfare*, for $\alpha > 1$, iff $OPT(g, f, k)/SW(\mathbb{M}, (g, f, k)) < \alpha$.

We show next that the multi-unit fixed price mechanism where p is set such that

$$\sum_{q=1}^{k} \mathbf{Pr}_{w\sim g}[\check{w}(q) \leq p] = k/2$$

achieves a 2-approximation to the optimal social welfare.

Theorem 4.1. *Let (f, g, k) be a multi-unit bilateral trade instance where the supports of f and g contain only increasing submodular functions. Let $\mathbb{M}$ be the multi-unit bilateral trade mechanism where at each step one item is offered for trade at price $p = \sum_{q=1}^{k} \mathbf{Pr}_{w\sim g}[\check{w}(q) \leq p] = k/2$, until either agent reject the offer (informally: p is the price such that the seller is expected to accept to trade half of his units at price p). Mechanism $\mathbb{M}$ achieves a 2-approximation to the optimal social welfare.*

Proof. Let v be an arbitrary buyer's valuation function. We show that the mechanism achieves a 2-approximation if f is the distribution having only v in its support, and hence v is the buyer's valuation with probability 1. It suffices to prove the claim under this assumption, because the unit-price p depends on distribution g only. Hence, if $\mathbb{M}$ achieves the claimed social welfare guarantee for every fixed buyer's valuation function, then it also achieves this guarantee for every distribution on the buyer's valuation. For ease of notation, we will abbreviate $SW(\mathbb{M}, (f, g, k))$ to simply SW and we let $\ell = \max\{q : \hat{v}_k(q) \geq p\}$ be the highest quantity that the buyer would like to trade at unit-price p. In the remainder of the proof, we will omit the subscript $w \sim g$ from the expected value operator.

We first observe that SW can be written as follows, where we write $\mathbf{1}[\cdot]$ to denote the indicator function and E_q for the event that $\hat{v}(q) \geq p \geq \check{w}(q)$.

$$\begin{aligned} SW &= \mathbf{E}\left[\sum_{q=1}^{k}(\check{w}(q) + \mathbf{1}[E_q]GFT(v, w, q))\right] \\ &= \mathbf{E}\left[\sum_{q=1}^{\ell}(\check{w}(q) + \mathbf{1}[E_q]GFT(v, w, q))\right] + \mathbf{E}\left[\sum_{q=\ell+1}^{k}\check{w}(q)\right] \end{aligned} \tag{1}$$

We will bound these last two expected values separately in terms of $OPT(f, g, k)$, and subsequently we will combine the two bounds to obtain the desired approximation factor.

We start with the quantities up to ℓ, for which first rewrite the expression as follows.

$$\begin{aligned} \mathbf{E}\left[\sum_{q=1}^{\ell}(\check{w}(q) + \mathbf{1}[E_q]GFT(v, w, q))\right] &= \sum_{q=1}^{\ell}\mathbf{E}[\check{w}(q)] + \sum_{q=1}^{\ell}\mathbf{Pr}[E_q]\mathbf{E}[GFT(v, w, q)) \mid E_q] \\ &= \sum_{q=1}^{\ell}\mathbf{E}[\check{w}(q)] + \sum_{q=1}^{\ell}\mathbf{Pr}[E_q]\mathbf{E}[GFT(v, w, q)) \mid E_q]. \end{aligned}$$

Now, observe that $\mathbf{Pr}[E_q] = \mathbf{Pr}[p \geq \check{w}(q)]$ for quantities $q \leq \ell$. Since $\sum_{q=1}^{k} \mathbf{Pr}[p \geq \check{w}(q)] = k/2$ and $\mathbf{Pr}[p \geq \check{w}(q)]$ is decreasing in q, this implies that $\sum_{q=1}^{\ell} \mathbf{Pr}[E_q] = \sum_{q=1}^{\ell} \mathbf{Pr}[p \geq \check{w}(q)] \geq \ell/2$. Using additionally the fact that $\mathbf{E}[GFT(v,w,q)) \mid E_q]$ is also non-increasing in q, we obtain the following bound.

$$
\begin{aligned}
\mathbf{E}\left[\sum_{q=1}^{\ell}(\check{w}(q) + \mathbf{1}[E_q]GFT(v,w,q))\right] &\geq \sum_{q=1}^{\ell}\mathbf{E}[\check{w}(q)] + \frac{\sum_{q=1}^{\ell}\mathbf{Pr}[E_q]}{\ell}\sum_{q=1}^{\ell}\mathbf{E}[GFT(v,w,q)) \mid E_q] \\
&\geq \sum_{q=1}^{\ell}\mathbf{E}[\check{w}(q)] + \frac{1}{2}\sum_{q=1}^{\ell}\mathbf{E}[GFT(v,w,q)) \mid E_q] \\
&\geq \sum_{q=1}^{\ell}\mathbf{E}[\check{w}(q)] + \frac{1}{2}\sum_{q=1}^{\ell}\mathbf{E}[GFT(v,w,q))] \\
&\geq \frac{1}{2}\sum_{q=1}^{\ell}\mathbf{E}[\check{w}(q) + GFT(v,w,q)] \qquad (2)
\end{aligned}
$$

For the quantities higher than ℓ, we first observe that non-increasingness of $\mathbf{Pr}[\check{w}(q) < p]$ in the quantity q implies that $\mathbf{Pr}[\check{w}(q) > p]$ is non-decreasing in q. Moreover, $\sum_{q=1}^{k} \mathbf{Pr}[\check{w}(q) \leq p] = k/2$ means that $\sum_{q=1}^{k} \mathbf{Pr}[\check{w}(q) > p] = \sum_{q=1}^{k} \mathbf{Pr}[\check{w}(q) \leq p]$, hence it holds that $\sum_{q=\ell+1}^{k} \mathbf{Pr}[\check{w}(q) > p] \geq \sum_{q=1}^{k} \mathbf{Pr}[\check{w}(q) \leq p]$. Therefore, we derive

$$
\begin{aligned}
\mathbf{E}\left[\sum_{q=\ell+1}^{k}\check{w}(q)\right] &= \frac{1}{2}\sum_{q=\ell+1}^{k}\mathbf{E}[\check{w}(q)] + \frac{1}{2}\sum_{q=\ell+1}^{k}\mathbf{E}[\check{w}(q)] \\
&\geq \frac{1}{2}\sum_{q=\ell+1}^{k}\mathbf{E}[\check{w}(q)] + \frac{1}{2}\sum_{q=\ell+1}^{k}\mathbf{E}[\check{w}(q) \mid \check{w}(q) > p]\mathbf{Pr}[\check{w}(q) > p] \\
&\geq \frac{1}{2}\sum_{q=\ell+1}^{k}\mathbf{E}[\check{w}(q)] + \frac{1}{2}\sum_{q=\ell+1}^{k}\hat{v}(q)\mathbf{Pr}[\check{w}(q) > p] \\
&\geq \frac{1}{2}\sum_{q=\ell+1}^{k}\mathbf{E}[\check{w}(q)] + \frac{1}{2}\sum_{q=\ell+1}^{k}\mathbf{E}[GFT(v,w,q)] \\
&\geq \frac{1}{2}\sum_{q=\ell+1}^{k}\mathbf{E}[\check{w}(q) + GFT(v,w,q)], \qquad (3)
\end{aligned}
$$

where the second inequality holds because $\check{w}(q)$ conditioned on $\check{w}(q) > p$ is always higher than $\hat{v}(q)$ which does not exceed p. Moreover, the third inequality follows because $\mathbf{E}[GFT(v,w,q)] = \mathbf{E}[(\hat{v}(q) - \check{w}(q))\mathbf{1}(\hat{v}(q) > \check{w}(q))] \leq \mathbf{E}[\hat{v}(q)\mathbf{1}(\hat{v}(q) > \check{w}(q))] \leq \mathbf{E}[\hat{v}(q)\mathbf{1}(p > \check{w}(q))] = \hat{v}(q)\mathbf{Pr}[p > \check{w}(q)]$.

We now use (2) and (3) to bound (1) and obtain the desired inequality

$$
SW \geq \frac{1}{2}\sum_{q=1}^{k}\mathbf{E}[\check{w}(q) + GFT(v,w,q)] = \frac{OPT(f,g,k)}{2},
$$

which proves the claim. □

The above 2-approximation mechanism is deterministic. We show next that we can do better if we allow randomisation: Consider the *Generalized Random Quantile Mechanism*, or $\mathbb{M}_G$, which draws a number x in the interval $[1/e, 1]$ where the CDF is $\ln(ex)$ for $x \in [1/e, 1]$. The mechanism then sets a unit price $p(x)$ such that $\mathbf{E}_w[\max\{q : w(q) \geq qp(x)\}] = \sum_{q=1}^{k} \mathbf{Pr}_w[\check{w}(q) \leq p(x)] = xk$, repeatedly offering single item trades as before. In words, the price is set such that the expected number of units that the seller is willing to sell, is an x fraction of the total supply, where x is randomly drawn according to the probability distribution just defined. This randomised mechanism satisfies DSIC, IR, and SBB, because it is simply a distribution over multi-unit fixed price mechanisms. Note that this mechanism is also a generalisation of a previously proposed mechanism: In [3], the authors define the special case of this mechanism for a single item, and call it the *Random Quantile Mechanism*. They show that it achieves a $e/(e-1)$-approximation to the social welfare, and we will prove next that this generalisation preserves the approximation factor, although the proof we provide for it is substantially more complicated and requires various additional technical insights.

Theorem 4.2. *Let (f, g, k) be a multi-unit bilateral trade instance where the supports of f and g contain only increasing submodular functions. The Generalised Random Quantile Mechanism $\mathbb{M}_G$ achieves a $e/(e-1)$-approximation to the optimal social welfare.*

Proof. As in the proof of Theorem 4.1, we fix a valuation function v for the buyer. It suffices to prove the claim under this assumption, because the unit-price p depends on distribution g only. For ease of notation, we will again abbreviate $SW(\mathbb{M}_G, (f, g, k))$ to simply SW.

We first rewrite $OPT(f, g, k)$ as follows:

$$\begin{aligned} OPT(f,g,k) &= \sum_{q=1}^{k} \mathbf{E}_w[\max\{\hat{v}(q), \breve{w}(q)\}] \\ &= \sum_{q=1}^{k} \mathbf{E}_w[\hat{v}(q)] + \sum_{q=1}^{k} \mathbf{E}_w[(\breve{w}(q) - \hat{v}(q))\mathbf{1}[\breve{w}(q) \geq \hat{v}(q)]] \\ &= \sum_{q=1}^{k} \hat{v}(q) + \sum_{q=1}^{k} \mathbf{E}_w[\breve{w}(q) - \hat{v}(q) \mid \breve{w}(q) \geq \hat{v}(q)] \cdot \mathbf{Pr}_w[\breve{w}(q) \geq \hat{v}(q)] \\ &= \sum_{q=1}^{k} \hat{v}(q) + \sum_{q=1}^{k} (\mathbf{E}_w[\breve{w}(q) \mid \breve{w}(q) \geq \hat{v}(q)] - \hat{v}(q)) \cdot \mathbf{Pr}_w[\breve{w}(q) \geq \hat{v}(b)]. \end{aligned} \tag{4}$$

In the remainder of the proof, we will derive a lower bound of $(1 - 1/e)$ times the expression (4) on SW, which implies our claim. We first observe that SW can be bounded and rewritten as follows.

$$\begin{aligned} SW = &\sum_{q=1}^{k} \mathbf{E}_w[\breve{w}(q)\mathbf{1}[\breve{w}(q) \geq \hat{v}(q)]] + \sum_{q=1}^{k} \mathbf{Pr}_w[\breve{w}(q) < \hat{v}(q)]\mathbf{E}_{w,x}[\hat{v}(q)\mathbf{1}[p(x) \in [\breve{w}(q), \hat{v}(q)]] \\ &\quad + \breve{w}(q)\mathbf{1}[p(x) \notin [\breve{w}(q), \hat{v}(q)]] \mid \breve{w}(q) < \hat{v}(q)] \\ \geq &\sum_{q=1}^{k} \mathbf{E}_w[\hat{v}(q)\mathbf{1}[\breve{w}(q) \geq \hat{v}(q)] + (\breve{w}(q) - \hat{v}(q))\mathbf{1}[\breve{w}(q) \geq \hat{v}(q)]] \\ &\quad + \sum_{q=1}^{k} \mathbf{E}_{w,x}[\hat{v}(q)\mathbf{1}[p(x) \in [\breve{w}(q), \hat{v}(q)]] \mid \breve{w}(q) < \hat{v}(q)]] \cdot \mathbf{Pr}_w[\breve{w}(q) < \hat{v}(q)] \\ = &\sum_{q=1}^{k} \hat{v}(q)\mathbf{Pr}_w[\breve{w}(q) \geq \hat{v}(q)] \\ &\quad + \sum_{q=1}^{k} (\mathbf{E}_w[\breve{w}(q) \mid \breve{w}(q) \geq \hat{v}(q)] - \hat{v}(q))\mathbf{Pr}[\breve{w}(q) \geq \hat{v}(q)] \\ &\quad + \sum_{q=1}^{k} \mathbf{E}_{w,x}[\hat{v}(q)\mathbf{1}[p(x) \in [\breve{w}(q), \hat{v}(q)] \mid \breve{w}(q) < \hat{v}(q)]]\mathbf{Pr}_w[\breve{w}(q) < \hat{v}(q)] \\ = &\sum_{q=1}^{k} \hat{v}(q)\mathbf{Pr}_w[\breve{w}(q) \geq \hat{v}(q)] \\ &\quad + \sum_{q=1}^{k} \hat{v}(q)\mathbf{Pr}_{w,x}[p(x) \in [\breve{w}(q), \hat{v}(q)] \mid \breve{w}(q) < \hat{v}(q)]] \cdot \mathbf{Pr}_w[\breve{w}(q) < \hat{v}(q)] \qquad (5) \\ &\quad + \sum_{q=1}^{k} (\mathbf{E}_w[\breve{w}(q) \mid \breve{w}(q) \geq \hat{v}(q)] - \hat{v}(q))\mathbf{Pr}[\breve{w}(q) \geq \hat{v}(q)]. \end{aligned}$$

Next, we bound the first part (5) of the last expression, i.e., excluding the last summation.

$$
\begin{aligned}
(5) &\le \sum_{q=1}^{k} \hat{v}(q)\mathbf{Pr}_w[\check{w}(q) \ge \hat{v}(q)] + \sum_{q=1}^{k} \hat{v}(q)\mathbf{Pr}_w[\check{w}(q) < \hat{v}(q)] \cdot \frac{\int_{1/e}^{z:p(z)=\hat{v}(q)} \mathbf{Pr}_w[\check{w}(q) \le p(x)]\frac{1}{x}dx}{\mathbf{Pr}_w[\check{w}(q) < \hat{v}(q)]} \\
&= \sum_{q=1}^{k} \hat{v}(q)\mathbf{Pr}_w[\check{w}(q) \ge \hat{v}(q)] + \int_{1/e}^{z:p(z)=\hat{v}(q)} \left(\sum_{q=1}^{k} \hat{v}(q)\mathbf{Pr}_w[\check{w}(q) \le p(x)]\right)\frac{1}{x}dx \\
&\ge \sum_{q=1}^{k} \hat{v}(q)\mathbf{Pr}_w[\check{w}(q) \ge \hat{v}(q)] + \int_{1/e}^{z:p(z)=\hat{v}(q)} \sum_{q=1}^{k} \hat{v}(q)\frac{kx}{k}\frac{1}{x}dx \\
&= \sum_{q=1}^{k} \hat{v}(q)\mathbf{Pr}_w[\check{w}(q) \ge \hat{v}(q)] + \sum_{q=1}^{k} \hat{v}(q) \int_{1/e}^{z:p(z)=\hat{v}(q)} 1dx \\
&= \sum_{q=1}^{k} \hat{v}(q)\mathbf{Pr}_w[\check{w}(q) \ge \hat{v}(q)] + \sum_{q=1}^{k} \hat{v}(q)(\mathbf{Pr}[\check{w}(q) < \hat{v}(q)] - \frac{1}{e}) \\
&= (1-1/e)\sum_{q=1}^{k} \hat{v}(q), \qquad (6)
\end{aligned}
$$

where for the inequality we used that both $\hat{v}(q)$ and $\mathbf{Pr}_w[\check{w}(q) < \hat{v}(q)]$ are non-increasing in q, so that replacing all the probabilities by the average probability xk/k yields a lower value. Substituting (5) by (6) and using the expression (4) for OPT then yields the desired bound.

$$
\begin{aligned}
SW \ge (1-1/e)\Big(&\sum_{q=1}^{k} \hat{v}(q) + \sum_{q=1}^{k} (\mathbf{E}_w[\check{w}(q) \mid \check{w}(q) \ge \hat{v}(q)] - \\
&\hat{v}(q))\mathbf{Pr}_w[\check{w}(q) \ge \hat{v}(q)]\Big) \\
= (1-1/e)&OPT(f,g,k).
\end{aligned}
$$

□

Currently we have no non-trivial lower bound on the best approximation factor achievable by a DSIC, IR, SBB mechanism, and we believe that the approximation factor of $e/(e-1)$ achieved by our second mechanism is not the best possible. For our first mechanism, it is rather easy to see that the analysis of the approximation factor of our first mechanism is tight, and that it is a direct extension of the median mechanism of [19], for which it was already shown in [3] that it does not achieve an approximation factor better than 2: The authors show that 2 is the best approximation factor possible for any deterministic mechanism for which the choice of p does not depend on the buyer's distribution.

For the more general class of increasing valuation functions, an approximation factor of $(2e-1)/(e-1) \approx 2.582$ to the optimal social welfare is achieved by a mechanism of [3]: They use a $e/(e-1)$-approximation mechanism for the single-item setting, which yields a $(2e-1)/(e-1)$ approximation mechanism for the multi-unit setting through a conversion theorem which they prove. We note that their conversion theorem is more precisely presented for the setting with a buyer and a seller who holds one *divisible* item. However, their proof straightforwardly carries over to the multi-unit setting. It would be an interesting open challenge to improve this currently best-known bound of $(2e-1)/(e-1)$ for general increasing valuations.

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A Proof of the Characterisation for General Valuations

We denote the class of monotonically increasing submodular functions with domain $[k]$ by $\mathcal{S}_k$. We denote the class of monotonically increasing functions with domain $[k]$ by $\mathcal{I}_k$.

The definition below defines the multi-unit fixed price mechanisms as a direct revelation mechanism. From the point of view of providing a rigorous proof, this is more convenient to work with than the sequential posted price definition given in the main part of the paper.

Definition A.1. *Let $p \in \mathbb{R}_{\geq 0}$, let $S \subseteq [k]$, and let $\tau = (\tau_B, \tau_S, \tau_\cap)$ be a vector of three tie-breaking functions specified below. The* multi-unit fixed price mechanism *$\mathbb{M}_{p,S,\tau}$ is the direct revelation mechanism that returns for a multi-unit bilateral trade instance (f, g, k) an outcome $\mathbb{M}_{p,S,\tau}(v, w) = (q_B, q_S, \rho_B, \rho_S)$ on reported valuation functions v and w, where*

- $\tau_B(v) \subseteq \arg_q \max\{v(q) - qp : q \in S \cup \{0\}\}$ *and* $\tau_B(v) \neq \varnothing$,
- $\tau_S(w) \subseteq \arg_q \max\{w(k - q) + qp : q \in S \cup \{0\}\}$ *and* $\tau_S(w) \neq \varnothing$,
- $\tau_\cap(v, w)$ *is a tie-breaking function that selects an element in* $\tau_B(v) \cap \tau_S(w)$ *in case this intersection is non-empty,*
- $q_B = k - q_S = \begin{cases} \min\{\max \tau_B(v), \max \tau_S(w)\} & \text{if } d_B \cap d_S = \varnothing, \\ \tau_\cap(v, w) & \text{otherwise.} \end{cases},$
- $\rho_B = -\rho_S = q_B p$.

Informally stated, the mechanism offers the buyer and seller a fixed unit price p and a set of quantities S. It then asks the buyer and seller which quantity in $S \cup \{0\}$ they would like to trade when for each unit the buyer would pay p to the seller. The mechanism then makes the buyer and seller trade the minimum of these two demanded numbers at a unit price of p. Typically the preferred quantity is unique for both the buyer and the seller, but in case of indifferences the buyer and seller will specify a set of multiple preferred quantities. In such cases, the tie-breaking functions τ_B, τ_S determine which quantities among the sets of indifferences are considered for trade, and the tie-breaking function $\tau_\cap$ is finally used to determine the traded quantity in case the sets selected by τ_B and τ_S intersect. Otherwise, the minimum of the maximum quantities of τ_B and τ_S is traded.

It turns out that multi-unit fixed price mechanisms characterise the set of all DSIC, IR, and SBB mechanisms with respect to monotonically increasing submodular valuation functions. Moreover, with the additional restriction that S is a singleton set, they characterise the set of all DSIC, IR, and SBB mechanisms with respect to monotonically valuation functions.

We first prove sufficiency.

Theorem A.1. *For all $p \in \mathbb{R}_{\geq 0}$ and $S \subseteq [k]$, the mechanism is IR, SBB, and DSIC with respect to the class of monotonically increasing submodular valuation functions. Moreover, if $|S| = 1$, then $\mathbb{M}_{p,S,\tau}$ is IR, SBB, and DSIC with respect to the class of monotonically increasing valuation functions.*

Proof. First we prove the statement for the class of monotonically increasing submodular valuation functions. The SBB property holds trivially by definition of the mechanism, $\rho_B = -\rho_S$.

Let v and w be increasing submodular valuation functions of the buyer and seller. Let q_B and ρ_B be the quantity given to the buyer and payment made by the buyer under the outcome $\mathbb{M}_{p,S,\tau}(v, w)$. If $\tau_B(v) \cap \tau_S(w)$ is non-empty, then the function $\tau_\cap$ selects a utility maximizing quantity for both the buyer and seller, so IR obviously holds in that case. If $\tau_B(v) \cap \tau_S(w) = \varnothing$, the mechanism $\mathbb{M}_{p,S,\tau}$ is IR for the buyer: his utility is $v(q_B) - \rho_B = v(q_B) - q_B p = v(\min\{\max \tau_B(v), \max \tau_S(w)\}) - \min\{\max \tau_B(v), \max \tau_S(w)\}p$. The value $\max \tau_B(v)$ is defined as a utility-maximizing quantity in S for the buyer, given that the buyer pays p for each unit. If the buyer's valuation function v is submodular, getting any quantity less than $\max \tau_B(v)$ at a price of p per unit will yield the buyer a non-negative utility. Therefore, the buyer's utility is non-negative. For the seller, the argument to establish the IR property is similar: His utility is $w(k - q_B) + \rho_B = w(k - q_B) + q_B p = w(k - \min\{\max \tau_B(v), \max \tau_S(w)\}) + \min\{\max \tau_B(v), \max \tau_S(w)\}p$. The value $\max \tau_S(w)$ is defined as a utility maximizing quantity in S for the seller to give to the buyer, given that the seller receives a payment of p for each unit. As the buyer's valuation function v is submodular, giving any quantity less than $\max \tau_S(w)$ to the buyer at a price of p per unit will yield the seller a non-negative increase utility. Therefore, the seller's utility increase is non-negative.

For the DSIC property, observe that if the mechanism sets $q_B \in \tau_S(w)$, then the mechanism chooses the outcome that is the utility-maximizing one for the seller among all outcomes in the range of the mechanism. On the other hand, if $q_B \in \tau_B(v) \setminus \tau_S(w)$ then the seller can only manipulate the outcome by misreporting a valuation that causes q_B to attain a smaller value, and hence in this case the mechanism will select an outcome where a smaller quantity is traded against a price of p per unit. By increasingness and submodularity of the seller's valuation function, this will result in a lower utility for the seller. Hence, it is a dominant strategy for the seller to not misreport his valuation function. For the buyer, the argument is similar: If the mechanism sets $q_B \in \tau_B(v)$, then the mechanism chooses the outcome that is the utility-maximizing one for the buyer among all outcomes in the range of the mechanism. On the other hand, if $q_B \in \tau_S(w) \setminus \tau_B(v)$ then the buyer can only manipulate the outcome by misreporting a valuation that causes q_B to attain a smaller value, and hence in this case the mechanism will select an outcome where a smaller quantity is traded against a price of p per unit. By increasingness and submodularity of the buyer's valuation function, this will result in a lower utility for the buyer. Hence, it is a dominant strategy for the buyer to not misreport his valuation function.

Next, we prove the statement for the larger class of monotonically increasing valuation functions. Again, the SBB property holds trivially.

As we now work under the assumption that $|S| = 1$, let q be the quantity such that $S = \{q\}$. Let v and w be increasing valuation functions for the buyer and seller respectively. By definition of the mechanism and the increasingness of the valuation functions, it holds that $\tau_B(v) \in \{\{q\}, \{0\}, \{q, 0\}\}$. Likewise, $\tau_S(w) \in \{\{q\}, \{0\}, \{q, 0\}\}$. Therefore, for both the buyer and seller, the traded quantity is 0 or the unique positive quantity q in case he prefers trading q units at least as much as trading 0 units. Hence the buyer and seller both experience a non-negative increase in utility for the outcome decided by the mechanism. This establishes IR. For DSIC, observe that if a positive quantity is traded in the selected outcome under truthful reporting, then the only effect that misreporting can achieve is that a quantity of 0 at a price of 0 is traded instead, which would leave both the buyer and the seller with a 0 increase in utility, hence this will not increase either player's utility. If on the other hand a quantity of 0 is traded at a price of 0, then $\{0\} = \tau_B(v), 0 \notin \tau_S(w)$ or $\{0\} = \tau_S(w), 0 \notin \tau_B(v)$ or $0 \in \tau_B(v), 0 \in \tau_S(w)$. In the first case, clearly the buyer is not incentivised to manipulate the mechanism into producing the alternative outcome where q units are traded, and the seller is unable to manipulate the mechanism into producing that outcome as it selects the minimum of $\tau_S(w)$ and $\tau_B(w)$, where the latter equals $\{0\}$ regardless of the sellers report. For the second case, symmetric reasoning can be applied to conclude that none of the two agents are incentivised to misreport. For the third case, it trivially holds that none of the agents are incentivised to manipulate the mechanism into trading q instead of 0 units. This establishes DSIC. □

Next, we show necessity, i.e., all DSIC, IR, and SBB direct revelation mechanisms are multi-unit fixed price mechanisms.

Theorem A.2. *Let $\mathbb{M}$ be a multi-unit bilateral trade mechanism that is IR, SBB, and DSIC with respect to the class of monotonically increasing submodular valuation functions. Then, there exist*

$p \in \mathbb{R}_{\geq 0}$, $S \subseteq [k]$, and τ such that $\mathbb{M} = \mathbb{M}_{p,S,\tau}$. Moreover, if $\mathbb{M}$ is also IR, SBB, and DSIC with respect to the bigger class of monotonically increasing valuation functions, then $|S| = 1$.

We divide this proof up into several lemmas. We start by proving the theorem for the smaller class of monotonically increasing submodular valuation functions. First, we show that whenever $\mathbb{M}$ trades the same number of items for two distinct pairs of valuation functions, then it must charge the same payments. Second, we extend this by showing that whenever the mechanism trades distinct numbers of items for any two distinct pairs of valuation functions, then the mechanism must charge the same price proportional to the number of items traded. It follows that we may associate to $\mathbb{M}$ a unit price p such that the payment from the buyer to the seller is always $q_B p$, where q_B is the traded quantity. Lastly, we show that there is a set S such that the range of quantities that the seller may let the mechanism trade from (by means of reporting a valuation function to the mechanism), is equal to $(S \cup \{0\}) \cap [\arg_q \max\{v(q) - pq : q \in S \cup \{0\}\}]$. By the fact that the valuation functions are increasing and submodular, and by the fact that $\mathbb{M}$ is DSIC, it follows that truthful reporting of the seller will result in the mechanism trading

$$\arg_q \max\{w(k-q) + pq : q \in (S \cup \{0\}) \cap [\arg_q \max\{v(q) - pq : q \in S \cup \{0\}\}]\}$$

units. This expression is equal to $\min\{\max d_B, \max d_S\}$ if $d_B \cap d_S = \varnothing$ (where d_B and d_S are defined as in Definition A.1) , and otherwise it is a set from which an arbitrary quantity $\tau(v, w)$ may selected. This implies that $\mathbb{M} = \mathbb{M}_{p,S,\tau}$ for the appropriate choices of p, S, and τ.

With respect to the larger class of monotonically increasing valuation functions, the set of DSIC, IR, and SBB mechanisms must be smaller. We prove for this class that whenever the set S consists of more than one quantity, then there must be a pair of valuation functions in which either the buyer or seller is better off by not truthfully reporting his valuation function.

We now proceed by stating and proving formally the claims sketched above. In the proofs of the claims below, we use the following terminology and notation. For ease of exposition, we denote from now on an outcome by a pair (q, p) where q is the traded number of units (i.e., the quantity that the buyer gets assigned) and p is the payment of the buyer, which is equal to the negated payment of the seller by the SBB property. For a reported valuation v of the buyer, let $M_v = \{o \in \mathcal{O} \mid \exists w : \mathbb{M}(v, w) = o\}$ be the menu of outcomes offered to the seller when the buyer reports v. That is, when the buyer reports v, the seller can select one of the outcomes o in M_v by reporting (not necessarily truthfully) some valuation in reply to v. Likewise, we let $N_w = \{o \in \mathcal{O} \mid \exists v : \mathbb{M}(v, w) = o\}$ be the menu of outcomes offered to the buyer when the seller reports v. We let $M = \bigcup_v M_v = \bigcup_w N_w$ be the set of all outcomes that the mechanism can produce, and we let S be the projection of M on the quantity obtainable by the buyer (i.e., the set S consists of all quantities that the mechanism can possibly trade).

The next lemma shows that there is a unique payment that the mechanism charges for every quantity in S, which implies that $\mathbb{M}$ consists of at most k outcomes.

Lemma A.3. *Let $\mathbb{M}$ be IR, SBB, and DSIC with respect to the class of valuation functions $\mathcal{C}$, where $\mathcal{C}$ is either $\mathcal{S}_k$ or $\mathcal{I}_k$. Let (v, w) and (v', w') be two pairs in $\mathcal{C}^2$. Let $\mathbb{M}(v, w) = (q_B, \rho_B)$ and $\mathbb{M}(v', w') = (q'_B, \rho'_B)$. If $q_B = q'_B$, then $\rho_B = \rho'_B$.*

Proof. As $\mathbb{M}$ is DSIC, it is immediate that for every v'' and for every quantity q it holds that there are no two distinct payments p, p' such that (q, p) and (q, p') are both in $M_{v''}$. Also, let (q, p) and (q', p') be in $M_{v''}$, where $q < q'$. Then $p \leq p'$, as otherwise there are valuations of the seller where misreporting results in trading less items at a higher price, which would violate the DSIC property.

Let (v, w) and (v', w') be as in the statement of the lemma, i.e., such that $q_B = q'_B$. When the buyer reports v and the seller reports w, by assumption (q_B, ρ_B) is the outcome, so $(q_B, \rho_B) \in M_v$. Define the valuation function w^* as the function that grows linearly, extremely steeply up to the quantity $k - q_B = k - q'_B$, and grows extremely slowly at a rate of $\epsilon > 0$ from $k - q_B$ onward. We define the function v^* similarly: It grows at an extremely high rate up to the quantity $q_B = q'_B$ and grows at extremely slow rate ϵ from q'_B onward.

We first consider a deviation by the seller from (v, w) to (v, w^*). Let $(q, p) \in M_v$ be the outcome of the mechanism on report (v, w^*). As we have chosen the valuation w^* to be sufficiently steep up to $k - q_B$ items, IR would be violated for a seller with valuation w^* if more than q_B items are traded.

Suppose now that upon report (v, w^*) the mechanism trades strictly less than q_B items. i.e., $q < q_B$. We prove that then, $p = \rho_B$: If we would assume $p < \rho_B$, then a seller with valuation w^*

would misreport w, as in terms of valuation he is practically indifferent between trading q and q_B items (ϵ needs to be chosen small enough for this), and his received payment would increase from p to ρ_B. If on the other hand we would assume that $p > \rho_B$, then a seller with valuation w would misreport w^* as he would retain more items, and receive a higher payment. Thus $p = \rho_B$.

By entirely analogous reasoning, when (v', w^*) is reported to the mechanism, the mechanism also trades $q'_B = q_B$ items or less. Let $q' < q_B$ be the number of traded items under (v', w^*). The payment is equal to ρ'_B, and if less than q_B items are traded, then $w'(k - q') = w'(k - q_B)$.

Next, we define from w^* a valuation function w^{**} for which it holds that under (v, w^{**}) and (v', w^{**}) the same number of items is traded at prices ρ_B and ρ'_B respectively. We do this as follows: If $q' = q$ then we simply let w^{**} be w^*. Otherwise, if $q' \neq q$, assume without loss of generality that $q' > q$ and let $\bar{w}^*$ be the valuation function that grows extremely steeply up to $k - q$ units and increases extremely slowly after $k - q$ units. By considering the deviation by the seller from profile (v, w^*) to profile $(v, \bar{w}^*)$, we see that under $(v, \bar{w}^*)$ at most q' units are traded and the payment is still equal to q_B. Likewise, under $(v', \bar{w}^*)$ at most q' items are traded and the payment is q'_B. Thus, the minimum number of traded items among the pair of strategy profiles $(v, \bar{w}^*)$ and $(v', \bar{w}^*)$ is larger than the minimum number of traded items among the pair $(v, \bar{w}^*)$ and $(v', \bar{w}^*)$. Repeating this operation will thus eventually yield a strategy profile w^{**} such that under (v, w^{**}) and (v', w^{**}) the same number of items is traded at prices ρ_B and ρ'_B respectively.

Now, if we would suppose for contradiction that $\rho_B \neq \rho'_B$, then we may assume without loss of generality that $\rho_B < \rho'_B$. When the seller reports w^{**}, a buyer with valuation function v would now be incentivised to report the valuation function v' instead of v, since then his payment decreases, and he still receives the same number of items. This is a contradiction to the DSIC property. Therefore, $\rho_B = \rho'_B$ which proves our claim. □

By Lemma A.3 there is a unique payment for each quantity $q \in S$, and we denote this payment by $p(q)$. The next lemma extends the previous lemma by stating essentially that payments must grow linearly with the number of allocated items, when one of the players changes his reported valuation.

Lemma A.4. *Let $\mathbb{M}$ be IR, SBB, and DSIC with respect to the class $\mathcal{C}$, where $\mathcal{C}$ is either $\mathcal{S}_k$ or $\mathcal{I}_k$. Let (v, w) and (v', w') be two pairs in $\mathcal{C}^2$. Let $\mathbb{M}(v, w) = (q_B, q_S, \rho_B, \rho_S)$ and $\mathbb{M}(v', w') = (q'_B, q'_S, \rho'_B, \rho'_S)$. If $q_B > 0$, then $\rho'_B = (q'_B/q_B)\rho_B$.*

Proof. First we show that $p(\cdot)$ is a non-decreasing function. Suppose that this is not true, and assume that $(q, p(q))$ and $(q', p(q))$ is the pair of outcomes in M such that (i) $q' > q$ and $p(q) > p(q')$, (ii) there is no $q'' \in S$ such that $q' > q'' > q$ and (iii) q is minimal. Let (v, w) and (v', w') be two valuation profiles that result in these two respective outcomes $(q, p(q))$ and $(q', p(q))$. Let w^* be a valuation function that increases linearly at an extremely high rate up to $k - q'$ and increses extremely slowly afterward. We now see that when (v, w^*) is reported, q' units or less are traded due to IR, and in fact at most q units are traded due to DSIC (because, if q' units were traded, a seller with valuation w^* would misreport w to trade less units for more money), and no less than q'' units are traded due to DSIC, where $q'' \leq q$ is the least quantity such that $p(q'') = p(q)$. (Otherwise the seller with valuation w^* could misreport w and trade more units at almost the same valuation, for significantly less money. We use here that w^* increases sufficiently slowly on the interval $[k - q', k]$). We also observe that when (v', w^*) is reported, (i) q' units or less are traded due to IR, (ii) the traded quantity cannot be any quantity with a higher payment than $p(q')$ since otherwise a seller with valuation w' would misreport w^* if the buyer reports v', and (iii) the traded quantity cannot be any quantity with a lower payment than $p(q')$ since otherwise a seller with valuation w^* would misreport w'. Thus, exactly q' units are traded when (v', w^*) is reported. We conclude that $(q, p(q))$ and $(q', p(q'))$ are both in N_{w^*}, and therefore a buyer with valuation v would have an incentive to misreport v' if the seller reports w^*, which violates DSIC and yields a contradiction. We conclude that the payment function $p(\cdot)$ is non-decreasing.

Also, note that $0 \in S$ and $p(0) = 0$ as otherwise the IR property would be violated for a buyer whose valuation is identically 0 and a seller whose valuation is strictly increasing: When such a valuation profile is reported, the seller's valuation implies that no positive number of items can be traded for payment 0; the buyer's valuation implies that no positive number of items can be traded for a positive payment; so 0 items must be traded for a payment that is not positive (due to IR of the buyer) and not negative (due to IR of the seller).

The claim of this lemma is equivalent to the claim that there exists a unique value p such that $p(q) = pq$ for all $q \in S$. We will demonstrate this by means of contradiction: Suppose that there is no such value p. Let q be the lowest quantity in S such that $p(q'') \neq p(q)q''/q$ for all $q'' \in S, q'' < q$. Let q' be the highest quantity in S such that $p(q'') = p(q')q''/q'$ for all $q'' \in S, q'' < q'$. Note that there is no $q'' \in S$ such that $q' < q'' < q$, and that $p(\cdot)$ behaves linearly up to q', and that q essentially serves as the least witness for the non-linearity of $p(\cdot)$.

We distinguish two cases: The case where $p(q)q'/q > p(q')$, and the case where $p(q)q'/q < p(q')$. First let us assume that $p(q)q'/q > p(q')$. We will derive a contradiction by constructing valuation functions (v^*, w^*) for the buyer and seller such that the following properties are satisfied: (i) outcomes $(q, p(q))$ and $(q', p(q'))$ are in M_{v^*} and N_{w^*}; (ii) the buyer with valuation v^* strictly prefers outcome $(q', p(q'))$ over all outcomes in $N_{w^*} \setminus \{(q', p(q'))\}$; and (iii) the seller with valuation w^* strictly prefers outcome $(q, p(q))$ over all outcomes in $M_{v^*} \setminus \{(q, p(q))\}$. This is a contradiction because (ii) requires that the mechanism outputs $(q', p(q'))$ (otherwise the buyer with valuation v^* would be incentivised to misreport so that $(q', p(q'))$ is output) and (iii) requires that the mechanism outputs $(q, p(q))$ (otherwise the seller with valuation w^* would be incentivised to misreport so that $(q, p(q))$ is output).

Therefore, we will now define the appropriate valuations v^* and w^*. Let v^* be a valuation function that grows linearly at an extremely high rate up to quantity q' and increases extremely slowly afterward. This causes all outcomes where a positive quantity is traded a positive utility for the buyer with valuation v^*, moreover, the maximum utility for such a buyer is achieved at outcome $(q', p(q'))$. This already establishes property (ii). To see that property (i) holds, let w be any seller's valuation so that $(q', p(q')) \in N_w$ (which must exist because $q' \in S$). By the definition of v^*, the mechanism selects outcome $(q', p(q'))$ on report (v^*, w) and therefore $(q', p(q')) \in M_{v^*}$. Now, consider any valuation profile (v, w) that results in outcome $(q, p(q))$, so that $(q, p(q)) \in M_v$. Let w' be the valuation function that grows linearly at an extremely high rate up to quantity $k - q$ and after that point grows linearly at a rate of $p(q)/q - \epsilon/q$ up to quantity k. The initial increase up to point $k - q$ is so steep that the seller can never experience a utility above $w'(k)$ when any quantity higher than q is traded. The value $\epsilon > 0$ is chosen to be so small that the only outcome at which a seller with valuation w' has a positive utility is $(q, p(q))$. Therefore, upon report (v, w') the mechanism outputs $(q, p(q))$ and we may infer that $N_{w'} = \{(q, p(q)), (0, 0)\}$, from which it follows that $(q, p(q)) \in M_{v^*}$, as $(q, p(q))$ must be the selected outcome upon report (v^*, w) (by the DSIC property). This establishes property (i) for v^*.

For valuation function w^*, let w^* increase linearly at an extremely high rate up to quantity $k - q$, and increase extremely slowly afterwards. Clearly, the seller with valuation w^* prefers the outcome $(q, p(q))$ among all outcomes in M, which establishes property (iii). Let (v, w) be any report upon which the mechanism outputs $(q, p(q))$, so that $(q, p(q)) \in M_v$. Then $(q, p(q))$ is also output upon report (v, w^*) which establishes that $(q, p(q)) \in N_{w^*}$. Next, let (v, w) be any report upon which the mechanism outputs $(q', p(q'))$, so that $(q', p(q')) \in N_w$. Let v' be a function that increases at rate $p(q')/q' + \epsilon/q'$ for sufficiently small $\epsilon > 0$ up to quantity q', and increases extremely slowly afterward. Then $(q', p(q'))$ is output when (v', w) is reported, so that $(q', p(q')) \in M_{v'}$. Moreover, trading any quantity higher than q' would yield a negative utility for a buyer with valuation v' (because ϵ is extremely small). Therefore, for $q'' > q'$ it holds that $(q'', p(q'')) \notin M_{v'}$ and in particular $(q, p(q)) \notin M_{v'}$. Thus, when (v', w^*) is reported, the outcome $(q', p(q'))$ is output by the mechanism, and this establishes that $(q', p(q')) \in N_{w^*}$. Note that here we need that $p(q') > 0$, which is the case as the outcome the mechanism returns on (v', w') is IR by assumption and the valuation functions are monotonically increasing. This completes the proof for the case where $p(q)q'/q > p(q')$.

For the case where $p(q)q'/q < p(q')$ we proceed in a similar fashion: Again, we will derive a contradiction by constructing valuation functions (v^*, w^*) for the buyer and seller such that (i) outcomes $(q, p(q))$ and $(q', p(q'))$ are in M_{v^*} and N_{w^*}; (ii) the buyer with valuation v^* strictly prefers outcome $(q, p(q))$ over all options in $N_{w^*} \setminus \{(q, p(q))\}$; and (iii) the seller with valuation w^* strictly prefers outcome $(q', p(q'))$ over all options in $M_{v^*} \setminus \{(q, p(q))\}$. This is a contradiction because (ii) requires that the mechanism outputs $(q, p(q))$ (otherwise the buyer with valuation v^* would be incentivised to misreport so that $(q, p(q))$ is output) and (iii) requires that the mechanism outputs $(q', p(q'))$ (otherwise the seller with valuation w^* would be incentivised to misreport so that $(q', p(q'))$ is output). Note that the difference with the previous case is that here we construct v^* such that the higher of the two quantities q and q' is preferred, instead of the lower one. Likewise, w^* is now constructed such that the lower of the two quantities is preferred instead of the higher one.

We start in this case with the construction of w^*. Let w^* be a valuation function that increases linearly at an extremely high rate up to quantity $k-q$. From $k-q$ to $k-q'$, valuation w^* increases by an amount of $p(q) - p(q') + \epsilon$, where $\epsilon > 0$ is sufficiently small, and w^* increases extremely slowly from $k-q'$ onward. The increase in valuation from quantities $k-q$ to $k-q'$ is slightly higher than the amount by which the payment changes among the quantities q and q', this causes the seller with valuation w^* to encounter a slightly lower (but positive) increase in utility when quantity q is traded instead of quantity q'. Moreover, among all quantities in S up to q', the maximum utility for a seller with valuation w^* is achieved at quantity q', which already establishes property (iii). Lastly, note that due to the extreme steepness of w^* up to $k-q$, the utility of the seller is lower than $w^*(k)$ when any quantity higher than q is traded, so the mechanism will never do so by the IR constraint. It remains to establish property (i). Let (v, w) be any report where the mechanism selects outcome $(q', p(q'))$. It follows by DSIC that outcome $(q', p(q'))$ will also be selected on report (v, w^*), so that $(q', p(q')) \in N_{w^*}$. Next, let (v, w) be any report where the mechanism selects outcome $(q, p(q))$. Let w' be a valuation function that increases extremely steeply up to $k-q$ and increases extremely slowly afterwards, so that the report (v, w') results in $(q, p(q))$ and hence $(q, p(q)) \in N_{w'}$, and because of IR we also infer that $(q'', p(q'')) \notin M_{w'}$ when $q'' > q$. Let v' be a valuation function that increases linearly up to quantity q and increases extremely slowly afterwards, where $v(q) = p(q)+\epsilon$, and $\epsilon > 0$ is sufficiently small. Note that trading a positive quantity lower than q would result in a negative utility for a buyer with valuation v', so that such outcomes are not in $M_{v'}$. Therefore, when (v', w') is reported the outcome selected by the mechanism must be $(q, p(q))$, which shows that $(q, p(q)) \in M_{v'}$. It follows now that the selected outcome upon report (v', w^*) must be $(q, p(q))$ which yields $(q, p(q)) \in N_{w^*}$ and establishes property (i) for w^*.

Lastly, we design v^*. Let v^* simply increase extremely steeply up to the quantity q, and increase extremely slowly afterwards, so that a buyer with valuation v^* experiences positive utility for all outcomes in M, and maximum utility when outcome $(q, p(q))$ is selected. This straightforwardly establishes property (ii). For property (i), let (v, w) be any profile where outcome $(q, p(q))$ results, so that $(q, p(q)) \in N_w$. By DSIC, outcome $(q, p(q))$ is also selected when (v^*, w) is reported, so $(q, p(q)) \in M_{v^*}$. Next, let (v, w) be any profile where outcome $(q', p(q'))$ results, so $(q', p(q')) \in M_v$. Let w' be a function that increases extremely steeply up to quantity $k-q'$, and increases extremely slowly afterwards, so that trading any quantity higher than q' would result in a decrease in utility for a seller with valuation w' (hence the mechanism cannot trade such quantities when w' is reported, by the IR property), and the maximum increase in utility is achieved when $(q', p(q'))$ is chosen. Therefore reporting (v, w') results in outcome $(q', p(q'))$, thus $(q', p(q')) \in N_{w'}$ and $(q'', p(q'')) \notin N_{w'}$ for all $q'' > q'$. Therefore, when (v^*, w') is reported, outcome $(q', p(q'))$ is selected, which establishes property (i) for v^* and completes the proof for the case $p(q)q'/q > p(q')$. □

Let $\mathbb{M}$ be IR, SBB, and DSIC with respect to the class of monotonically increasing submodular valuation functions. From the above it follows that for a mechanism that is IR, SBB, and DSIC with respect to $\mathcal{S}_k$ or $\mathcal{I}_k$, there exists a price $p \in \mathbb{R}_{\geq 0}$ such that for all pairs (v, w) of monotonically increasing submodular valuation functions, the payment charged to the buyer is $q_B p$ (and the payment charged to the seller is $-q_B m$ by SBB). We will refer to p as the *unit price*.

The above corollary establishes the needed properties on the payments of the mechanism. The remaining lemmas use Lemma A.4 by implicitly assuming the existence of the unit price p in their statement, and they characterise the quantities S tradable by the mechanism and the quantities that appear in the menus M_v and N_w. The next lemma states that the utility maximizing outcome in S for a buyer with any valuation function v is always in M_v.

Lemma A.5. *If $\mathbb{M}$ is SBB, IR, and DSIC with respect to $\mathcal{S}_k$, and suppose that unit price p is positive. Then, for all $v \in \mathcal{S}_k$ it holds that $(q, p(q)) \in M_v$ for the lowest q in the set $\arg_q \max\{v(q) - p(q) : q \in S\}$.*

Proof. Let q be the lowest quantity in $\arg_q \max\{v(q) - p(q) : q \in S\}$. Let (v', w') be any report that results in outcome $(q, p(q))$, so that $(q, p(q)) \in N_{v'}$. Let w^* be a valuation function that increases extremely steeply up to the quantity $k-q$ and increases extremely slowly afterwards. Observe that by our assumption that $p > 0$, a seller with valuation w^* strongly prefers outcome $(q, p(q))$ over all other outcomes in S, and trading any quantity larger than q would violate IR. by DSIC, outcome $(q, p(q))$ is thus selected when (v', w^*) is reported, hence $(q, p(q)) \in N_{w^*}$ and $(q', p(q')) \notin N_{w^*}$ for all $q' > q$. So when (v, w^*) is reported, an outcome is selected from N_{w^*} that maximises the utility of the buyer with valuation v, and this outcome is $(q, p(q))$.

□

The following lemma strengthens the previous.

Lemma A.6. *Suppose* $\mathbb{M}$ *is SBB, IR, and DSIC with respect to* $\mathcal{S}_k$ *and suppose that unit price* p *is positive. Let* $v \in \mathcal{S}_k$ *and let* $q \leq \min \arg_{q'} \max\{v(q') - p(q') : q' \in S\}$ *be a quantity not exceeding the least utility-maximizing quantity for a buyer with valuation* v. *It holds that* $\arg_{q'} \max\{v(q') - p(q') : q' \in S, q' \leq q\}$ *is the singleton set containing the quantity* $q' = \max S \cap [q]$, *and that* $q \in M_v$.

Proof. Note that the existence of the unit price p implies that the utility function of the buyer is a submodular function of the traded quantity. Therefore, the utility function for a buyer with valuation v is increasing up to the least utility-maximizing outcome in S, after which it stays constant up to the highest utility-maximizing outcome in S, after which it starts decreasing. Let q be any quantity less than or equal to the least utility-maximizing quantity, i.e., less than $\min \arg \max\{v(q') - p(q') : q' \in S\}$. Then, the utility-maximizing quantity q' in $S \cap [q]$ for a buyer with valuation v is $\max S \cap [q]$. It remains to prove that $(q', p(q'))$ is in M_v. Let (v', w') be any report resulting in outcome $(q', p(q'))$, so that $(q', p(q')) \in N_{v'}$. Let w'' be any function increasing extremely steeply up to quantity $k - q'$, after which it increases extremely slowly. Then $(q', p(q'))$ is the result of report (v', w''), and note that it is not IR to trade a quantity exceeding q' when w'' is reported, so $(q'', p(q'')) \notin N_{w''}$ for $q'' > q'$, and $(q', p(q')) \in N_{w''}$. Therefore, when (v, w'') is reported, a quantity of q' is traded, and no higher quantity. (Note that we use positivity of p here.) Thus, $(q', p(q'))$ is in M_v, which proves the claim. □

The above lemma shows that for a mechanism $\mathbb{M}$ that is SBB, IR, and DSIC with respect to $\mathcal{S}_k$, if $p > 0$, then for any $v \in \mathcal{S}_k$, the menu M_v that the buyer presents to the seller includes the outcomes $(q, p(q))$ such that q is in the subset of S obtained by truncating S at the buyer's least-quantity utility-maximizing outcome.

We can prove the following symmetric lemma for the seller.

Lemma A.7. *Suppose* $\mathbb{M}$ *is SBB, IR, and DSIC with respect to* $\mathcal{S}_k$ *and suppose that the unit price* p *is positive. Let* $w \in \mathcal{S}_k$ *and let* $q \leq \min \arg_{q'} \max\{w(k - q') + p(q') : q' \in S\}$ *be a quantity not exceeding the least utility-maximizing quantity for a seller with valuation* w. *It holds that* $\arg_{q'} \max\{w(k-q') + p(q') : q' \in S, q' \leq q\}$ *is the singleton set containing quantity* $q' = \max S \cap [q]$, *and that* $q \in N_w$.

Proof. Note that the existence of the unit price p implies that the increase in utility of the seller is a submodular function of the traded quantity q. Therefore, the function for a buyer with valuation v is increasing up to the least utility-maximizing outcome in S, after which it stays constant up to the highest utility-maximizing outcome in S, after which it starts decreasing. Let q be any quantity less than or equal to the least utility-maximizing quantity, i.e., less than $\min \arg \max\{w(k - q') + p(q') : q' \in S\}$. Then, the utility-maximizing quantity q' in $S \cap [q]$ for a seller with valuation w is $\max S \cap [q]$. It remains to prove that $(q', p(q'))$ is in N_w. Let (v', w') be any report resulting in outcome $(q', p(q'))$, so that $(q', p(q')) \in N_{w'}$. Let v'' be any function increasing at a rate $p + \epsilon$ up to quantity q', for a sufficiently small $\epsilon > 0$, after which it increases extremely slowly. Then $(q', p(q'))$ is the result of report (v'', w'), and note that it is not IR to trade a quantity exceeding q' when v'' is reported, so $(q'', p(q'')) \notin M_{v''}$ for $q'' > q'$, and $(q', p(q')) \in N_{v''}$. Therefore, when (v'', w) is reported, a quantity of q' is traded, and no higher quantity. Thus, $(q', p(q'))$ is in M_w, which proves the claim. □

The above lemma shows that for any unit price mechanism $\mathbb{M}$ that is SBB, IR, and DSIC with respect to $\mathcal{S}_k$, for any $w \in \mathcal{S}_k$, the menu M_w that the seller presents to the buyer includes the outcomes $(q, p(q))$ such that q is in the subset of S obtained by truncating S at the buyer's least-quantity utility-maximizing outcome.

The last two lemmas combined imply that the menu of buyer consist of the outcomes $(q, p(q))$ in S where the quantity does not exceed the least utility-maximizing outcome, plus an additional arbitrary subset of utility-maximizing outcomes; and the same holds for the seller. We will show that next.

Lemma A.8. *Suppose* $\mathbb{M}$ *is SBB, IR, and DSIC with respect to* $\mathcal{S}_k$ *and suppose that the unit price* p *is positive. Let* $v, w \in \mathcal{S}_k$. *Let* $q = \min \arg_{q''} \max\{v(q'') - p(q'') : q'' \in S\}$ *be the least utility maximizing quantity for the buyer with valuation* v, *then* $M_v = \{(q'', p(q'')) : q'' \in S \cap [q]\} \cup T$ *where*

$T \subseteq \arg_{q''} \max\{v(q'') - p(q'') : q'' \in S\}$. *Similarly let* $q' = \min \arg_{q''} \max\{w(k-q'') + p(q'') : q'' \in S\}$ *be the least utility maximizing quantity for the seller with valuation* w*, then* $N_w = \{(q'', p(q'')) : q'' \in S \cap [q']\} \cup T'$ *where* $T' \subseteq \arg_{q''} \max\{w(k - q'') + p(q'') : q'' \in S\}$.

Proof. Lemmas A.6 and A.7 show that $M_v \supseteq \{(q'', p(q'')) : q'' \in S \cap [q]\}$ and $N_w \supseteq \{(q'', p(q'')) : q'' \in S \cap [q']\}$. Let $\hat{q} \in S$ such that $\hat{q} > \max \arg_{q''} \max\{v(q'') - p(q'') : q'' \in S\}$ and let $\check{q} \in S$ such that $\check{q} > \max \arg_{q''} \max\{w(k - q'') + p(q'') : q'' \in S\}$. It suffices to show that $(\hat{q}, p(\hat{q})) \notin M_v$ and that $(\check{q}, p(\check{q})) \notin N_w$.

Suppose $(\hat{q}, p(\hat{q})) \in M_v$. Let w^* be a valuation function increasing extremely steeply up to quantity $k - \hat{q}$, and increases extremely slowly afterwards. By DSIC, the outcome $(\hat{q}, p(\hat{q}))$ is selected on report (v, w^*), where we use that $p > 0$. However, by Lemma A.7 it holds that $(q, p(q)) \in N_{w^*}$, so that it also must hold by the DSIC property that outcome $(q, p(q))$ is selected, which is a contradiction.

Suppose $(\check{q}, p(\check{q})) \in N_w$. Let v^* be a valuation increasing at extremely high rate up to quantity $\check{q}$, that increases extremely slowly afterwards. By DSIC, the outcome $(\check{q}, p(\check{q}))$ is selected on report (v^*, w). However, by Lemma A.6 it holds that $(q', p(q')) \in N_{v^*}$, so that it also must hold by the DSIC property that outcome $(q, p(q))$ is selected, which is a contradiction. □

We are now finally ready to prove the necessity-part of our characterisation of IR, DSIC, SBB multi-unit bilateral trade mechanisms.

Proof of Theorem A.2. By Lemma A.3, for all $q \in S$ there is a price $p(q)$ such that a payment of $p(q)$ is charged whenever q units are traded. By Lemma A.4, there is a unit price p such that $p(q) = p \cdot q$ for all $q \in S$. This establishes already that the payment function of any IR, DSIC, and SBB mechanism is in accordance with Definition A.1, hence it remains to establish that the traded quantity is also prescribed by Definition A.1.

First we consider the special case $p = 0$. By increasingness of the valuation function of the seller, it follows that the mechanism can only trade a quantity of 0 units in order to satisfy IR. Subsequently it follows by IR and SBB that the mechanism is required to charge a payment of 0. A mechanism that always trades 0 units at price 0 is by definition equal to a mechanism $\mathbb{M}_{0,\varnothing,\tau}$, where τ is arbitrary and irrelevant as there is only a single outcome that the mechanism outputs.

Next, assume that $p > 0$. We prove the claim separately for $\mathcal{S}_k$ and $\mathcal{I}_k$, and we start with $\mathcal{S}_k$. By Lemma A.8, for every pair of valuations (v, w) it holds that $M_v = \{q \in S : q \leq \min \arg_q' \max\{v(q') + p(q')\}\} \cup T$ where T is an arbitrary set of utility-maximizing quantities in S for a buyer with valuation v, and $N_w = \{q \in S : q \leq \min \arg_{q'} \max\{w(k - q') + w(q)\}\} \cup T'$ where T' is an arbitrary set of utility maximizing quantities in S for a seller with valuation w. Let $\tau_S(w)$ be the seller's utility maximizing quantities in N_w and let $\tau_B(v)$ be the buyer's utility-maximizing quantities in M_v. If $\tau_S(w)$ and $\tau_B(v)$ intersect, then by DSIC, mechanism must output any quantity in $\tau_S(w) \cap \tau_B(v)$: call this quantity $\tau_\cap(v, w)$. Otherwise, if $\tau_S(w) \cap \tau_B(v) = \varnothing$, the mechanism must output $\min\{\max \tau_S(w), \max \tau_B(v)\}$, in order to satisfy the DSIC property: Assume that and $\max \tau_B(v) > \max \tau_S(w)$ (the other case is symmetric) and suppose that the mechanism trades any quantity $q \neq \max \tau_S(w)$. Since the traded quantity q must lie in the intersection of M_v and N_v and since $\tau_B(v)$ and $\tau_S(w)$ are sets of highest quantities in M_v and N_w respectively, we have $q < \max \tau_S(w)$. Hence, among the quantities in N_w a quantity less than $\max \tau_S(w)$ is traded, but the buyer prefers quantity $\max \tau_S(w)$ because $\max \tau_S(w)$ is closer to the buyer's set $\max \tau_B(v)$ of utility-maximizing quantities in M_v, which would give the buyer a higher utility due to submodularity. The buyer would thus misreport such that $\max \tau_S(w)$ is output instead. Note the tie-breaking functions $\tau = (\tau_B, \tau_S, \tau_\cap)$ we just established, as well as the derived traded quantity q given to the buyer, agree precisely with those of Definition A.1. We complete the equivalence by noting that $0 \in S$ as we can define a seller's utility function that grows extremely steeply up to quantity k, so that $(0, 0)$ is the only IR outcome. This implies that $\mathbb{M} = \mathbb{M}_{p,S\setminus\{0\},\tau}$.

It remains to prove the claim for $\mathcal{I}$. Suppose for contradiction that there are at least two positive quantities q, q' in S, where $0 < q < q'$. We apply the same technique as in Lemma A.4. Let (v, w) be a valuation profile such that the mechanism selects $(q, p(q))$ when (v, w) is reported, and let (v', w') be a valuation profile such that the mechanism selects $(q', p(q'))$ when (v', w') is reported.

Let v^* be a valuation function that increases extremely slowly up to quantity $q - 1$, then jumps to a value of $pq + 2\epsilon$ at quantity q and proceeds again to grow extremely slowly up to quantity $q' - 1$, and finally jumps to a value of $pq' + \epsilon$ at quantity q' after which it grows extremely slowly onward. The only IR quantities that the mechanism can trade when a buyer reports v^* are 0, q, and q'. As $(q, p(q)) \in N_w$, the mechanism must select the outcome $(q, p(q))$ when (v^*, w) is reported, because of DSIC. So, $(q, p(q)) \in M_{v^*}$. Next, we construct a function w'' for which it holds that $(q, p(q)) \notin N_{w''}$ and $(q', p(q')) \in N_{w''}$: Valuation w'' is defined such that it increases extremely steeply up to the quantity $k - q'$. Subsequently it increases by an amount of $p + (q' + 1)\epsilon$ to quantity $k - q' + 1$, and it increases at a rate of $p - \epsilon$ afterward. Note that the only IR quantities that can be traded under w'' are 0 and q'. We thus have that $(q, p(q)) \notin N_{w''}$ and $(q', p(q')) \in N_{w''}$ because $(q', p(q'))$ in $M_{v'}$ and by DSIC the mechanism must select $(q', p(q'))$ when (v', w'') is reported. Therefore, when (v^*, w') is reported, $(q', p(q'))$ is selected so we see that $(q', p(q')) \in M_{v^*}$.

Let w^* be a valuation function defined as follows. Let $\epsilon > 0$ be sufficiently small. We let $w^*(k) = kp$, and for all $q'' > 0$ not equal to q or q', We let $w^*(k - q'') = p \cdot (k - q'') - \epsilon$, so that the seller's increase in utility for trading q'' units is $pq'' - (w^*(k) - w^*(k - q'')) = pq'' - (pk - p(k - q'') + \epsilon) = -\epsilon$, so when w^* is the valuation of the seller, the mechanism cannot trade q'' items as that would violate IR. Moreover, we define $w^*(k - q) = p \cdot (k - q) + \epsilon$ and $w^*(k - q') = p \cdot (k - q') + 2\epsilon$, so that trading q or q' units leads to an increase in utility for a seller with valuation w^* and so that trading q'' units is the preferred quantity to trade for a seller with valuation w^*. We now see that $(q', p(q')) \in N_{w^*}$, because $(q', p(q'))$ is in $M_{v'}$ so that by DSIC the mechanism outputs $(q', p(q'))$ on report (v', w^*). Next, we construct a function v'' for which it holds that $(q', p(q')) \notin M_{v''}$ and $(q, p(q)) \in M_{v''}$. This function is defined as follows: $v(q'') = pq'' - \epsilon$ for all q'' except q, where $v(q) = pq + \epsilon$. When a buyer reports v'', by IR the mechanism can either trade 0 or q units and no other quantity. On report (v'', w) the mechanism must output $(q, p(q))$ due to DSIC and because $(q, p(q)) \in N_w$ by assumption. Thus $(q, p(q)) \in N_{v''}$ and $(q', p(q')) \notin M_{v''}$, hence when (w^*, v') is reported the mechanism outputs $(q, p(q))$ because of DSIC. This establishes $(p, p(q)) \in N_{w^*}$.

We thus have constructed two functions v^* and w^* for which it holds that both $(q, p(q))$ and $(q', p(q'))$ are in both M_{v^*} and N_{w^*}. Moreover, a buyer with valuation v^* strictly prefers $(q, p(q))$ over $(q', p(q'))$, so by DSIC the mechanism must output the outcome $(q, p(q))$ when (v^*, w^*) is reported. However, a seller with valuation w^* strictly prefers $(q', p(q'))$ over $(q, p(q))$, so by DSIC the mechanism must output the outcome $(q', p(q'))$ when (v^*, w^*) is reported, which is a contradiction. So, we must refute the assumption that there are at least 2 quantities that the mechanism can trade.

Hence, either $(0, 0)$ is always output, in which case the claim is trivial (the mechanism is equal to $\mathbb{M}_{0,\varnothing,\tau}$, where τ is not relevant), or there is a unique positive quantity q such that the mechanism selects either $(q, p(q))$ or $(0, 0)$ and outputs $(q, p(q))$ on at least one valuation profile (v, w). It now suffices to prove, by definition of the mechanism (Definition A.1), that outcome $(q, p(q))$ is selected if both players experience an increase in utility from this outcome. Let (v', w') be an arbitrary valuation profile for which the latter holds. As $(q, p(q)) \in M_v$, we infer that $(q, p(q))$ is output on report (v, w') so that $(p, p(q)) \in M'_w$. Thus, by DSIC, the mechanism must select $(q, p(q))$ on report (v', w') as otherwise a buyer with valuation v' would report v instead. This proves that the mechanism equals $\mathbb{M}_{p,\{(q,p(q))\},\tau}$. □