A Hamiltonian approach to stabilization of nonholonomic mechanical systems

B.M. Maschke*

A.J. van der Schaft†

Abstract

A simple procedure is provided to write the equations of motion of controlled mechanical systems with constraints as controlled Hamiltonian equations with respect to a "Poisson" bracket which does not necessarily satisfy the Jacobi-identity. Based on the Hamiltonian form a stabilization procedure is proposed.

1 Introduction

In a recent paper we have shown that (uncontrolled) mechanical systems with classical constraints can be written as Hamiltonian equations of motion with respect to a generalized type of Poisson bracket, and with respect to a Hamiltonian which is obtained by restricting the internal energy to the constrained state space. This bracket does not necessarily satisfy the Jacobi-identity, which is one of the defining properties of a true Poisson bracket. In fact, the Jacobi-identity is satisfied if and only if the constraints are holonomic. This work was motivated by a paper of Bates & Sniatycki [3] on the Hamiltonian formulation of nonholonomic systems, as well as by our previous work on the Hamiltonian formulation of non-resistive physical systems by network modelling [8], [9].

In the present paper we extend this set-up to *controlled* nonholonomic mechanical systems. Furthermore we show how the Hamiltonian form of the equations (the Jacobi-identity being satisfied or not) may be used for stabilization purposes. Indeed we show how the stabilization procedure for standard Hamiltonian control systems as proposed in [14], [11], see also [5], can be extended to this case. These considerations were very much motivated by the papers [2], [4] on stabilization of controlled nonholonomic systems. We close with our treatment of two well-known simple examples of nonholonomic systems, discussed before in [2].

2 The Hamiltonian formulation of systems with constraints

Let Q be an n -dimensional configuration manifold with local coordinates $q = (q_1, \dots, q_n)$. Consider a smooth Lagrangian function $L : TQ \rightarrow \mathbb{R}$, denoted by $L(q, \dot{q})$, satisfying throughout the usual regularity condition

$$\det \left[\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j} \right] \neq 0 \quad (1)$$

*Lab. d'Automatisme Industriel, Conservatoire National des Arts et Métiers, 21 rue Pinel, 75013 Paris, France

†Dept. of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

(This is e.g. satisfied if L equals kinetic energy with positive definite generalized mass matrix minus potential energy.) Classical constraints are given in local coordinates as

$$A^T(q)\dot{q} = 0 \quad (2)$$

with $A(q)$ a $k \times n$ matrix, $k \leq n$, with entries depending smoothly on q . Throughout we assume that $A(q)$ has rank equal to k everywhere. The constraints (2) determine a k -dimensional *distribution* D on Q , given in every point $q_0 \in Q$ as

$$D(q_0) = \ker A^T(q_0) \quad (3)$$

The constraints (2) are called *holonomic* if the distribution D is involutive, i.e. for any two vectorfields X, Y on Q

$$X \in D, Y \in D \Rightarrow [X, Y] \in D \quad (4)$$

with $[X, Y]$ the Lie-bracket, defined in local coordinates q as $[X, Y](q) = \frac{\partial Y}{\partial q}(q)X(q) - \frac{\partial X}{\partial q}(q)Y(q)$, with $\frac{\partial Y}{\partial q}, \frac{\partial X}{\partial q}$ the

Jacobian matrices. In this case we may find, by Frobenius' theorem, local coordinates $\bar{q} = (\bar{q}_1, \dots, \bar{q}_n)$ such that the constraints (2) are expressed as

$$\dot{\bar{q}}_{n-k+1} = \dots = \dot{\bar{q}}_n = 0, \quad (5)$$

or equivalently $\bar{q}_{n-k+1} = c_{n-k+1}, \dots, \bar{q}_n = c_n$ for certain constants $c_{n-k+1}, \dots, c_n$ determined by the initial conditions, and we may eliminate the coordinates $\bar{q}_{n-k+1}, \dots, \bar{q}_n$. The constraints (2) are called *nonholonomic* if D is *not* involutive, implying that we can *not* use the above elimination procedure.

The equations of motion for the mechanical system on Q with Lagrangian $L(q, \dot{q})$ and constraints (2) are given as (see e.g. [10], [13], [1])

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = A(q)\lambda + B(q)u, \quad (6)$$

$$A^T(q)\dot{q} = 0, \quad \lambda \in \mathbb{R}^k, \quad u \in \mathbb{R}^m$$

where $B(q)u$ are the external forces (controls) applied to the system, with $B(q)$ an $n \times m$ matrix with entries depending smoothly on x . Here $\frac{\partial L}{\partial \dot{q}}$ denotes the column vector $\left(\frac{\partial L}{\partial \dot{q}_1}, \dots, \frac{\partial L}{\partial \dot{q}_n} \right)^T$, and similarly for $\frac{\partial L}{\partial q}$ and subsequent expressions. The *constraint forces* $A(q(t))\lambda(t)$ are determined by the requirement that the constraints $A^T(q(t))\dot{q}(t) = 0$ have to be satisfied for all t .

Defining in the usual way the Hamiltonian $H(q, p)$ by the Legendre transformation

$$H(q, p) = \sum_{i=1}^n p_i \dot{q}_i - L(q, \dot{q}), \quad p_i = \frac{\partial L}{\partial \dot{q}_i}, \quad (7)$$

the constrained Euler-Lagrange equations (6) transform into the constrained Hamiltonian equations on T^*Q

$$\begin{aligned}\dot{q} &= \frac{\partial H}{\partial p}(q, p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p) + A(q)\lambda + B(q)u \\ A^T(q)\frac{\partial H}{\partial p}(q, p) &= A^T(q)\dot{q} = 0\end{aligned}\quad (8)$$

An intrinsic definition of the constrained Hamiltonian equations may be given as follows. The cotangent bundle T^*Q is equipped with its canonical Poisson bracket $\{, \}$, in natural coordinates $(q, p) = (q_1, \dots, q_n, p_1, \dots, p_n)$ for T^*Q expressed as (with F and G smooth functions on T^*Q)

$$\begin{aligned}\{F, G\}(q, p) &= \sum_{i=1}^n \left(\frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} \right)(q, p) = \\ &= \left(\frac{\partial F}{\partial q}^T \frac{\partial F}{\partial p} \right) J \begin{pmatrix} \frac{\partial G}{\partial q} \\ \frac{\partial G}{\partial p} \end{pmatrix}, \quad J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix},\end{aligned}\quad (9)$$

with J the standard Poisson structure matrix. Recall that for any smooth function $H : T^*Q \rightarrow \mathbf{R}$ its Hamiltonian vectorfield X_H on T^*Q is defined in the local coordinates (q, p) as

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} = \begin{pmatrix} -\frac{\partial H}{\partial p} \\ \frac{\partial H}{\partial q} \end{pmatrix}\quad (10)$$

Similarly, for any one-form α on T^*Q we may define the "Hamiltonian" vectorfield Z_α as

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = J \begin{pmatrix} \alpha_1(q, p) \\ \vdots \\ \alpha_n(q, p) \end{pmatrix}\quad (11)$$

where $(\alpha_1(q, p), \dots, \alpha_n(q, p))$ is the local coordinate expression of the one-form α . (Note that $Z_{dH} = X_H$.) Now the columns of $A(q)$ define in local coordinates k one-forms $\alpha^1, \dots, \alpha^k$ on Q . Similarly, the columns of $B(q)$ define m one-forms $\beta^1, \dots, \beta^m$ on Q . Since any one-form on Q may be also regarded as a one-form on T^*Q , we can thus define the vectorfields $Z_{\alpha^1}, \dots, Z_{\alpha^k}, Z_{\beta^1}, \dots, Z_{\beta^m}$ on T^*Q . It can now be readily seen that a coordinate-free description of the first part of (8) is given as (see also [2])

$$\dot{x} = X_H(x) + a(x)\lambda + b(x)u, \quad x \in T^*Q, \quad (12)$$

where $a(x)$ is the matrix with columns $Z_{\alpha^1}, \dots, Z_{\alpha^k}$, and $b(x)$ is the matrix with columns $Z_{\beta^1}, \dots, Z_{\beta^m}$.

The Lagrange multipliers λ may be computed by differentiating $A^T(q)\frac{\partial H}{\partial p}(q, p) = 0$ along (8), i.e.

$$\begin{aligned}\left[\frac{\partial}{\partial q} \left(A^T(q) \frac{\partial H}{\partial p}(q, p) \right) \right]^T \frac{\partial H}{\partial p}(q, p) + A^T(q) \frac{\partial^2 H}{\partial p^2}(q, p) \cdot \\ \left[-\frac{\partial H}{\partial q}(q, p) + B(q)u \right] + A^T(q) \frac{\partial^2 H}{\partial p^2}(q, p) A(q) \lambda = 0\end{aligned}\quad (13)$$

with $\frac{\partial^2 H}{\partial p^2}$ the Hessian matrix with respect to p . This equation may be solved for λ (as function of q, p, u) as long as

$$\det A^T(q) \frac{\partial^2 H}{\partial p^2}(q, p) A(q) \neq 0, \quad (q, p) \in T^*Q, \quad (14)$$

which condition is obviously satisfied because of our standing assumptions (1) and $\text{rank } A(q) = k$. Expressing λ as a function of (q, p, u) and substituting in (8) then leads to the dynamical equations of motion on the constrained state space

$$\mathfrak{X}_r = \{(q, p) \in T^*Q \mid A^T(q) \frac{\partial H}{\partial p}(q, p) = 0\} \quad (15)$$

As shown in [15] a much more efficient and insightful way of obtaining the equations of motion on $\mathfrak{X}_r$ is however the following. Since $\text{rank } A(q) = k$, there exists locally a smooth $n \times (n - k)$ matrix $S(q)$ of rank $n - k$ such that

$$A^T(q)S(q) = 0 \quad (16)$$

(Equivalently, $S(q)$ is such that $D(q) = \text{Im} S(q)$.) Now define $\tilde{p} = (\tilde{p}^1, \tilde{p}^2) = (\tilde{p}_1, \dots, \tilde{p}_{n-k}, \tilde{p}_{n-k+1}, \dots, \tilde{p}_n)$ as

$$\begin{aligned}\tilde{p}^1 &:= S^T(q)p, \quad \tilde{p}^1 \in \mathbf{R}^{n-k} \\ \tilde{p}^2 &:= A^T(q)p, \quad \tilde{p}^2 \in \mathbf{R}^k\end{aligned}\quad (17)$$

It immediately follows from (16) that $(q, p) \mapsto (q, \tilde{p}^1, \tilde{p}^2)$ is a coordinate transformation. The constrained Hamiltonian dynamics (8) in the new coordinates $(q, \tilde{p}^1, \tilde{p}^2)$ take the following form. In the new coordinates $(q, \tilde{p})$ the Poisson structure matrix transforms from (9) into

$$\tilde{J}(q, \tilde{p}) = \begin{pmatrix} (\{q_i, q_j\})_{i,j} & (\{q_i, \tilde{p}_j\})_{i,j} \\ (\{\tilde{p}_i, q_j\})_{i,j} & (\{\tilde{p}_i, \tilde{p}_j\})_{i,j} \end{pmatrix}, \quad i, j = 1, \dots, n \quad (18)$$

and the constrained Hamiltonian dynamics (8) transform into

$$\begin{bmatrix} \dot{q} \\ \dot{\tilde{p}}^1 \\ \dot{\tilde{p}}^2 \end{bmatrix} = \tilde{J}(q, \tilde{p}) \begin{bmatrix} \frac{\partial \tilde{H}}{\partial q} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^1} \\ \frac{\partial \tilde{H}}{\partial \tilde{p}^2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tilde{A}(q) \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ B_r(q) \\ \tilde{B}(q) \end{bmatrix} u \quad (19)$$

$$\tilde{A}(q) \frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$$

with $\tilde{A}(q) := A^T(q)A(q)$ an invertible matrix, and $\tilde{H}(q, \tilde{p})$ the Hamiltonian $H(q, p)$ expressed in the new coordinates $q, \tilde{p}$. Now truncate the transformed Poisson structure matrix $\tilde{J}$ in (18) by leaving out the last k columns and last k rows, and let $\tilde{p}$ satisfy the constraint equation $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$. This defines a $(2n - k) \times (2n - k)$ skew-symmetric matrix J_r on $\mathfrak{X}_r$. An explicit expression for J_r is obtained as follows [15]. Denote the i -th column of $S(q)$ by $S_i(q)$, then

$$J_r = \begin{pmatrix} 0_n & S(q) \\ -S^T(q) & (-p^T[S_i, S_j](q))_{i,j=1, \dots, n-k} \end{pmatrix} \quad (20)$$

where p is expressed as function of $q, \tilde{p}$, with $\tilde{p}$ satisfying $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$. Note that $\text{rank } J_r = 2(n - k)$ everywhere on $\mathfrak{X}_r$. Furthermore, define the reduced Hamiltonian $H_r : \mathfrak{X}_r \rightarrow \mathbf{R}$ as $\tilde{H}(q, \tilde{p})$ with $\tilde{p}$ satisfying $\frac{\partial \tilde{H}}{\partial \tilde{p}^2} = 0$.

Clearly, $(q, \dot{p}^1)$ serve as local coordinates for $\mathfrak{X}_r$. It immediately follows from (19) by disregarding the last equations involving λ and noting that $\frac{\partial \tilde{H}}{\partial \dot{p}^1}(q, \dot{p}) = 0$ that the dynamics on $\mathfrak{X}_r$ in coordinates $(q, \dot{p}^1)$ are described as

$$\begin{pmatrix} \dot{q} \\ \dot{\dot{p}}^1 \end{pmatrix} = J_r(q, \dot{p}^1) \begin{pmatrix} \frac{\partial H_r}{\partial q}(q, \dot{p}^1) \\ \frac{\partial H_r}{\partial \dot{p}^1}(q, \dot{p}^1) \end{pmatrix} + \begin{pmatrix} 0 \\ B_r(q) \end{pmatrix} u \quad (21)$$

These equations are in Hamiltonian format! Indeed, the matrix J_r defines a bracket $\{, \}_r$ on $\mathfrak{X}_r$ by setting

$$\{F_r, G_r\}_r(q, \dot{p}^1) := \left(\frac{\partial F_r^T}{\partial q} \frac{\partial F_r^T}{\partial \dot{p}^1} \right) J_r(q, \dot{p}^1) \begin{pmatrix} \frac{\partial G_r}{\partial q} \\ \frac{\partial G_r}{\partial \dot{p}^1} \end{pmatrix} \quad (22)$$

for any two smooth functions $F_r, G_r : \mathfrak{X}_r \rightarrow \mathbf{R}$. Clearly, this bracket satisfies the first two defining properties of a Poisson bracket (see e.g. [6], [12], [7]),

$$\begin{aligned} (i) \quad \{F_r, G_r\}_r &= -\{G_r, F_r\}_r \\ &\quad (\text{skew-symmetry}) \\ (ii) \quad \{F_r, G_r, H_r\}_r &= \{F_r, G_r\}_r H_r + G_r \{F_r, H_r\}_r \\ &\quad (\text{Leibniz' rule}) \end{aligned} \quad (23)$$

for every $F_r, G_r, H_r : \mathfrak{X}_r \rightarrow \mathbf{R}$. However, for $\{, \}_r$ to be a Poisson bracket also the following property

$$(iii) \quad \{F_r, \{G_r, H_r\}_r\}_r + \{G_r, \{H_r, F_r\}_r\}_r + \{H_r, \{F_r, G_r\}_r\}_r = 0 \quad (\text{Jacobi-identity}) \quad (24)$$

needs to be satisfied. If (24) is satisfied then (21) with $u = 0$ defines a generalized Hamiltonian system with respect to a Poisson bracket, see e.g. [12], [7], [8]. In this case local coordinates $(\bar{q}, \bar{p}, \bar{s})$ for $\mathfrak{X}_r$ may be found such that the system for $u = 0$ takes the form [6], [12], [8]

$$\begin{aligned} \dot{\bar{q}} &= \frac{\partial H_r}{\partial \bar{p}}, \quad \dot{\bar{p}} = -\frac{\partial H_r}{\partial \bar{q}}, \quad \bar{q}, \bar{p} \in \mathbf{R}^{n-k} \\ \dot{\bar{s}} &= 0, \quad \bar{s} \in \mathbf{R}^k. \end{aligned} \quad (25)$$

However, in [15] it has been shown that $\{, \}_r$ satisfies the Jacobi-identity (and thus is a true Poisson bracket) if and only if the constraints $A^T(q)\dot{q} = 0$ are holonomic! This underscores the difficulties of nonholonomic constraints. On the other hand, even if the Jacobi-identity is not satisfied (as in the case for nonholonomic systems), the (pseudo-)Hamiltonian format (21) may still be useful, as we wish to indicate in the next section.

Note that our approach is not unrelated to the approach taken in [4]. Here the Lagrange multipliers λ in the Euler-Lagrange equations (6) are eliminated by premultiplying the equations (6) by the matrix $S^T(q)$, and it is shown that the thus reduced equations can be written as a set of first-order differential equations in q and $\eta \in \mathbf{R}^{n-k}$ with $\dot{q} = S(q)\eta$ parametrizing the admissible velocities $\dot{q}$. This can be regarded as the "Lagrangian counterpart" of our Hamiltonian approach.

3 Stabilization

We note that the dynamics (21) are energy preserving. In fact, by skew-symmetry of J_r we immediately obtain

$$\frac{d}{dt} H_r = \frac{\partial H_r}{\partial \dot{p}^1}(q, \dot{p}^1) B_r(q) u \quad (26)$$

with $\frac{d}{dt}$ denoting differentiation along (21). Suppose now that $(q_0, \dot{p}_0^1)$ is a stationary point of the Hamiltonian H_r ,

i.e. $\frac{\partial H_r}{\partial q}(q_0, \dot{p}_0^1) = 0$, $\frac{\partial H_r}{\partial \dot{p}^1}(q_0, \dot{p}_0^1) = 0$, implying that $(q_0, \dot{p}_0^1)$ is an equilibrium of the uncontrolled constrained dynamics ($u = 0$)

$$\begin{pmatrix} \dot{q} \\ \dot{\dot{p}}^1 \end{pmatrix} = J_r(q, \dot{p}^1) \begin{pmatrix} \frac{\partial H_r}{\partial q}(q, \dot{p}^1) \\ \frac{\partial H_r}{\partial \dot{p}^1}(q, \dot{p}^1) \end{pmatrix} \quad (27)$$

If H_r happens to have a strict minimum in $(q_0, \dot{p}_0^1)$, then it follows from (26) with $u = 0$ that $(q_0, \dot{p}_0^1)$ is a Lyapunov stable equilibrium of (27). On the other hand, as in the case of ordinary Hamiltonian control systems (see e.g. [14], [11]), equation (26) suggests for improved stabilization the smooth state feedback

$$u = -B_r^T(q) \frac{\partial H_r^T}{\partial \dot{p}^1}(q, \dot{p}^1) \quad (28)$$

which results in the monotonous energy decrease

$$\frac{d}{dt} H_r = -\frac{\partial H_r}{\partial \dot{p}^1}(q, \dot{p}^1) B_r(q) B_r^T(q) \frac{\partial H_r^T}{\partial \dot{p}^1}(q, \dot{p}^1) \leq 0 \quad (29)$$

(Note that (28) can be written as $u = -y$, with y the conjugated effort corresponding to the generalized flow u [8].) If H_r has a strict minimum in $(q_0, \dot{p}_0^1)$, then $(q_0, \dot{p}_0^1)$ will thus be at least a Lyapunov stable equilibrium of the closed-loop system (21), (28), and moreover the trajectories will converge to the largest invariant (with respect to (27)) set contained in

$$\{(q, \dot{p}^1) \in \mathfrak{X}_r \mid \frac{\partial H_r}{\partial \dot{p}^1}(q, \dot{p}^1) B_r(q) = 0\} \quad (30)$$

However it can be shown, as in [2], [4], that (21) does not satisfy Brockett's necessary condition, and thus cannot be asymptotically stabilized by a smooth state feedback. Hence this largest invariant set will be always larger than the singleton $\{(q_0, \dot{p}_0^1)\}$.

If H_r does not have a strict minimum in $(q_0, \dot{p}_0^1)$ then, as in the case of ordinary Hamiltonian control systems ([14], [11]), we may try to shape by preliminary feedback the internal energy H_r , if possible, to a function which does have a strict minimum in $(q_0, \dot{p}_0^1)$. Indeed, let H_r be of the form, as usually encountered in applications,

$$H_r(q, \dot{p}^1) = V(q) + \frac{1}{2} (\dot{p}^1)^T G(q) \dot{p}^1, \quad G(q) > 0 \quad (31)$$

(potential energy plus kinetic energy). Necessarily $\dot{p}_0^1 = 0$, and $\frac{\partial V}{\partial q}(q_0) = 0$. Now consider the equation

$$-S^T(q) \frac{\partial \bar{V}}{\partial q}(q) = B_r(q) u \quad (32)$$

For every smooth function $\bar{V}$ such that $S^T(q) \frac{\partial \bar{V}}{\partial q}(q) \in \text{Im } B_r(q)$, for all q , we can determine a smooth feedback $u = \bar{u}(q)$ which solves (32). Application of the feedback $u = \bar{u}(q) + v$, with v the new control variables, will result in a modified system (instead of (21))

$$\begin{pmatrix} \dot{q} \\ \dot{\bar{p}}^1 \end{pmatrix} = J_r(q, \bar{p}^1) \begin{pmatrix} \frac{\partial \bar{H}_r}{\partial q}(q, \bar{p}^1) \\ \frac{\partial \bar{H}_r}{\partial \bar{p}}(q, \bar{p}^1) \end{pmatrix} + \begin{pmatrix} 0 \\ B_r(q) \end{pmatrix} v \quad (33)$$

with $\bar{H}_r(q, \bar{p}^1) = H_r(q, \bar{p}^1) + \bar{V}(q)$. (This results from the special form of J_r given in (20).) If it is possible to find in this manner a function $\bar{V}$ such that $V + \bar{V}$ has a strict minimum in q_0 , then $\bar{H}_r$ will have a strict minimum in $(q_0, 0)$, and thus the additional feedback (28), with u replaced by v , will further stabilize the system. The resulting combined feedback is then given as

$$u = \bar{u}(q) - B_r^T(q) \frac{\partial H_r^T}{\partial \bar{p}^1}(q, \bar{p}^1) \quad (34)$$

with $\bar{u}(q)$ solving (32).

The treatment of [2], [4] corresponds to the special case that $B_r(q)$ has rank $m = n - k$. In this case equation (32) is solvable for every function $\bar{V}(q)$, and thus the potential energy can be shaped in an arbitrary fashion. Therefore, for H_r given by (31), every point $(q_0, 0) \in \mathcal{X}_r$ can be rendered a Lyapunov stable equilibrium by a feedback (34). Note furthermore that in this case the largest invariant (with respect to (27)) set contained in (30) is actually given as

$$\{(q, 0) \in \mathcal{X}_r \mid S^T(q) \frac{\partial(V + \bar{V})}{\partial q}(q) = 0\} \quad (35)$$

(as follows from the form of J_r given in (20)), where $\bar{V}$ is taken such that $V + \bar{V}$ has a strict minimum in q_0 . A similar result has been obtained before in [4] (in the reduced Lagrangian framework) using a different Lyapunov function, and a different feedback control based on this. The main difference is that in our approach the Lyapunov function $\bar{H}_r$ is directly based on the internal energy of the constrained dynamics, and consequently that u given in (34) has a direct physical interpretation. Furthermore, contrary to [4], we consider the stabilization problem for arbitrary B_r and an arbitrary number of controls.

We now treat within our approach two examples of nonholonomic control systems, both of which have been studied before in [2].

Example 3.1 (Knife edge) Consider the control of a knife edge moving in point contact on a plane surface. The constrained Lagrangian equations are given as (all numerical constants are set to unity)

$$\begin{aligned} \ddot{x} &= \lambda \sin \varphi + u_1 \cos \varphi \\ \ddot{y} &= -\lambda \cos \varphi + u_1 \sin \varphi \\ \ddot{\varphi} &= u_2 \end{aligned} \quad (36)$$

with (x, y) Cartesian coordinates of the contact point, φ the heading angle of the knife-edge, u_1 the control in the direction of the heading angle, and u_2 the control torque

about the vertical axis. The nonholonomic constraint is

$$\dot{x} \sin \varphi - \dot{y} \cos \varphi = 0 \quad (37)$$

The total energy H is given as $\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_\varphi^2$, with p_x, p_y, p_φ the corresponding generalized momenta. The constraint (37) can be written as $p_x \sin \varphi - p_y \cos \varphi = 0$. Define as in (17) new coordinates

$$\begin{aligned} p_1 &= p_\varphi \\ p_2 &= p_x \cos \varphi + p_y \sin \varphi \\ p_3 &= p_x \sin \varphi - p_y \cos \varphi \end{aligned} \quad (38)$$

Then $(\varphi, x, y, p_1, p_2)$ are coordinates for $\mathcal{X}_r$, and the dynamics (21) is computed as

$$\begin{aligned} \begin{bmatrix} \dot{\varphi} \\ \dot{x} \\ \dot{y} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cos \varphi \\ 0 & 0 & 0 & 0 & \sin \varphi \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -\cos \varphi & -\sin \varphi & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_r}{\partial \varphi} \\ \frac{\partial H_r}{\partial x} \\ \frac{\partial H_r}{\partial y} \\ \frac{\partial H_r}{\partial p_1} \\ \frac{\partial H_r}{\partial p_2} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \end{aligned} \quad (39)$$

with $H_r(\varphi, x, y, p_1, p_2) = \frac{1}{2}p_1^2 + \frac{1}{2}p_2^2$. Take $\bar{V}(\varphi, x, y) = \frac{1}{2}\varphi^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2$, then the preliminary feedback $\bar{u}(\varphi, x, y)$ is determined by (see (32))

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\cos \varphi & -\sin \varphi \end{bmatrix} \begin{bmatrix} \varphi \\ x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \bar{u} \quad (40)$$

and the resulting combined feedback (34) is

$$\begin{aligned} u_1 &= -x \cos \varphi - y \sin \varphi - p_2 \\ u_2 &= -\varphi - p_1 \end{aligned} \quad (41)$$

The trajectories will converge to the invariant set $\varphi = 0$, $x = 0$, $p_1 = 0$, $p_2 = 0$. A different $\bar{V}$, however, will generally yield a different invariant set. $\square$

Example 3.2 (Rolling vertical wheel) Let x, y be the Cartesian coordinates of the point of contact of the wheel with the plane, φ denotes heading angle, and θ rotation angle. With all constants set to unity, the Lagrangian equations of motion are

$$\begin{aligned} \ddot{x} &= \lambda_1 \\ \ddot{y} &= \lambda_2 \\ \ddot{\theta} &= -\lambda_1 \cos \varphi - \lambda_2 \sin \varphi + u_1 \\ \ddot{\varphi} &= u_2 \end{aligned} \quad (42)$$

with u_1 the control torque about the rolling axis and u_2 the control torque about the vertical axis. The nonholonomic constraints are (rolling without slipping)

$$\dot{x} = \dot{\theta} \cos \varphi, \quad \dot{y} = \dot{\theta} \sin \varphi \quad (43)$$

The total energy H is $\frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}p_\theta^2 + \frac{1}{2}p_\varphi^2$, and the constraints can thus be rewritten as $p_x = p_\theta \cos \varphi$, $p_y = p_\theta \sin \varphi$. Define according to (17) new coordinates

$$\begin{aligned}
p_1 &= p_\varphi \\
p_2 &= p_\theta + p_x \cos \varphi + p_y \sin \varphi \\
p_3 &= p_x - p_\theta \cos \varphi \\
p_4 &= p_y - p_\theta \sin \varphi
\end{aligned} \tag{44}$$

Then $(x, y, \theta, \varphi, p_1, p_2)$ are coordinates for $\mathcal{X}_r$, and the dynamics (21) is computed as (see also [15])

$$\begin{aligned}
\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\varphi} \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} &= \begin{bmatrix} & & & 0 & \cos \varphi \\ & 0_4 & & 0 & \sin \varphi \\ & & & 0 & 1 \\ & & & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ -\cos \varphi & -\sin \varphi & -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H_r}{\partial x} \\ \frac{\partial H_r}{\partial y} \\ \frac{\partial H_r}{\partial \theta} \\ \frac{\partial H_r}{\partial \varphi} \\ \frac{\partial H_r}{\partial p_1} \\ \frac{\partial H_r}{\partial p_2} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{45}
\end{aligned}$$

with $H_r = \frac{1}{2}p_1^2 + \frac{1}{4}p_2^2$. A feedback (34) can be computed as in the preceding example. Note that Example 2 (as well as Example 1) can be easily generalized to a knife edge or rolling wheel on *any* surface. This corresponds to adding a potential energy to H (and to H_r). Furthermore, in both examples one control torque instead of two control torques can be considered. $\square$

4 Conclusions

We have shown, as an extension to [15], that the equations of motion of controlled mechanical systems with constraints may be directly formulated as Hamiltonian equations of motion with respect to a bracket which for nonholonomic constraints does not satisfy the Jacobi-identity, and with respect to a reduced Hamiltonian which is obtained by restricting the total energy to the constrained state space.

Like for ordinary Hamiltonian control systems a stabilization procedure has been proposed, based on the use of the reduced Hamiltonian as a candidate Lyapunov function. However, since Brockett's necessary condition is not satisfied, this will only result in Lyapunov stability, whereas asymptotic convergence is to a non-trivial invariant set. The main challenge is to investigate how the Hamiltonian structure may be used for *asymptotic* stabilization, in which case discontinuous or time-varying feedback is needed ([2], [4]).

References

- [1] V.I. Arnold (Ed.), *Dynamical Systems III* (Encyclopaedia of Mathematical Sciences, Vol. 3), Springer, Berlin, 1988 (translation of the Russian original, 1985).
- [2] A.M. Bloch, M. Reyhanoglu, N.H. McClamroch, Control and stabilization of nonholonomic dynamic systems, *IEEE Trans. Aut. Control*, 37, pp. 1746-1757, 1992.
- [3] L. Bates, J. Sniatycki, Nonholonomic reduction, *Rep. Math. Phys.*, Vol. 32, no. 1, pp. 99-115, 1993.
- [4] G. Campion, B. d'Andrea-Novet, G. Bastin, Controllability and state feedback stabilisability of nonholonomic mechanical systems, in *Advanced Robot Control* (ed. C. Canudas de Wit), LNCIS 162, Springer, Berlin, pp. 106-124, 1991.
- [5] V. Jurdjevic, J.P. Quinn, Controllability and stability, *J. Diff. Equat.* 28, pp. 381-389, 1978.
- [6] P. Libermann, C.M. Marle, *Symplectic Geometry and Analytical Mechanics*, Reidel, Dordrecht, 1987.
- [7] J.E. Marsden, *Lectures on Mechanics*, London Mathematical Society Lecture Note Series 174, Cambridge University Press, Cambridge, 1992.
- [8] B. Maschke, A.J. van der Schaft, P.C. Breedveld, An intrinsic Hamiltonian formulation of network dynamics: non-standard Poisson structures and gyrators, *J. Franklin Institute*, 329, pp. 923-966, 1992.
- [9] B. Maschke, A.J. van der Schaft, P.C. Breedveld, An intrinsic Hamiltonian formulation of the dynamics of LC-circuits, submitted for publication, 1993.
- [10] Y. Neimark, N.A. Fufaev, *Dynamics of Nonholonomic Systems*, Amer. Math. Soc. Translations 33, 1972.
- [11] H. Nijmeijer, A.J. van der Schaft, *Nonlinear Dynamical Control Systems*, Springer-Verlag, New York, 1990.
- [12] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1986.
- [13] R. Rosenberg, *Analytical dynamics*, Plenum Press, New York, 1977.
- [14] A.J. van der Schaft, Stabilization of Hamiltonian systems, *Nonl. An. Th. Meth. Appl.* 10, pp. 1021-1035, 1986.
- [15] A.J. van der Schaft, B.M. Maschke, On the Hamiltonian formulation of nonholonomic mechanical systems, preprint January 1994, to appear in *Rep. Math. Phys.*