Spatial flexure mechanism analysis

Energy-based geometric
stiffness modeling

Marijn Nijenhuis

LUVANE

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Dit onderzoek is gefinancierd door de Vernieuwingsimpuls Vidi, project ‘Large range of motion spatial flexure joints — a topology synthesis method’ (14152 NWO TTW), van het ministerie van Onderwijs, Cultuur en Wetenschap.

Cover design door LUVANE.

ISBN: 978-90-365-4944-8

DOI: 10.3990/1.9789036549448

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SPATIAL FLEXURE MECHANISM ANALYSIS
ENERGY-BASED GEOMETRIC STIFFNESS MODELING

PROEFSCHRIFT

ter verkrijging van
de graad van doctor aan de Universiteit Twente,
op gezag van de rector magnificus,
prof. dr. T.T.M. Palstra,
volgens besluit van het College voor Promoties
in het openbaar te verdedigen
op vrijdag 24 januari 2020 om 16.45 uur

door

Marijn Nijenhuis

geboren op 26 februari 1990
te Dinxperlo, Nederland

Dit proefschrift is goedgekeurd door

de promotor

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Samenvatting

Elastische mechanismen worden in de precisietechniek gebruikt voor het sturen van bewegingen met een hoge herhaalbaarheid. Waar de beweging in traditionele mechanismen met starre schakels plaatsvindt dankzij rol- en glijcontacten en daardoor gepaard gaat met hysteresis, vindt deze in elastische mechanismen enkel plaats ten gevolge van elastisch vervormende onderdelen. Elastische mechanismen werken derhalve zonder wrijving, speling, stick-slip en slijtage, met slechts weinig hysteresis als gevolg. De elastische vervorming vormt een uitdaging voor de bredere toepasbaarheid. De bijbehorende spanning beperkt het werkbereik in termen van de vloe- en vermoeiingsgrens, en de veranderende vorm verlaagt de ondersteuningsstijfheid. Dit proefschrift stelt daarom onderzoek voor naar modelleertechnieken en constructieprincipes voor elastische mechanismen om de stijfheidseigenschappen bij grote slag te verbeteren.

Het eerste onderwerp, op het gebied van modelleertechnieken, betreft de bladveer, als zijnde een gangbaar onderdeel van elastische mechanismen. Hoewel de bladveer vaak op vereenvoudigde wijze geanalyseerd wordt, bijvoorbeeld door vlakke vervorming of gelineariseerde stijfheid te veronderstellen, is de vervorming in de praktijk ruimtelijk van aard en dusdanig groot dat niet-lineaire effecten ten gevolge van de geometrische stijfheid aanmerkelijk zijn. Dit proefschrift beschrijft de formulering van een parametrisch en ruimtelijk stijfheidsmodel voor bladveren met gematigde vervorming. Het model wordt gegeven door gesloten uitdrukkingen in termen van ontwerpparameters, zoals de parasitaire beweging, de graden van vrijheid en de aangebrachte belasting. Een gemengd variatieprincipe met een specifieke verzameling van interpolatiefuncties blijkt geschikt te zijn voor de afleiding van de derde-orde termen die het inzakken van de ondersteuningsstijfheid beschrijven. De geometrisch niet-lineaire gevolgen van buiging, afschuiving, verlenging, wringing en welving worden meegenomen, zodat bladveren met een grote breedte-lengteverhouding nauwkeurig beschreven kunnen worden. De op energie gebaseerde discretisatieaanpak wordt tevens uitgebreid ten behoeve van het modelleren van parallelschakelingen van bladveren.

Het bladveermodel wordt numeriek uitgevoerd als een balkelement in multibody-software. Het maakt snellere simulaties mogelijk van elastische mechanismen die bestaan uit een groot aantal bladveren, omdat één dergelijk element per bladveer meestal al toereikend is. Dit wordt aangetoond met verscheidene numerieke modellen van elastische mechanismen met grote slag uit de praktijk.

Het tweede onderwerp betreft het principe van statisch bepaald construeren, wat doorgaans toegepast wordt op elastische mechanismen om deterministisch gedrag te garanderen, maar ten koste gaat van robuustheid, ondersteuningsstijfheid, belastbaarheid en eenvoud. Om de potentie van overbepaald construeren in elastische mechanismen te verkennen, wordt in dit proefschrift de daarmee in verband gebrachte uitlijnfoutgevoeligheid onderzocht. Een simpel twee-bladveren kruisveerscharnier met één

overbepaaldheid dient als voorbeeld in een experimentele, numerieke en analytische analyse van de stijfheidsgevolgen van onbedoelde spanning als gevolg van uitlijnfouten. Het blijkt dat kleine uitlijnfouten in de overbepaalde richting het systeemgedrag aanzienlijk kunnen beïnvloeden. Numerieke modellen laten zien dat het effect voorspeld kan worden met hoge nauwkeurigheid en dat verfijnde aspecten, zoals bladveerdiktevariaties, ongewenste compliantie en welvingsbelemmering, de uitlijnfoutgevoeligheid ten gunste verkleinen. Een analytische knikanalyse biedt een expliciete bovengrens voor de kritische uitlijnfout voor verscheidene kruisveerontwerpen.

Een tweede voorbeeld laat het principe van dynamisch verstijfd statisch bepaald construeren zien. Dit principe maakt het mogelijk om overbepaald te construeren omwille van de hogere dynamische prestaties, door de uitlijnfoutgevoeligheid te verkleinen middels op maat gemaakt visco-elastisch materiaal. Experimenten laten zien dat het elastomeer in staat is om voor ongewenste uitlijnfouten te compenseren zonder wezenlijke spanningsopbouw en daarmee de dynamische prestaties, uitgedrukt in termen van de eerste parasitaire natuurlijke frequentie, te verbeteren. De voorspellingen van analytische en numerieke modellen komen overeen met metingen.

Summary

Flexure mechanisms are used in precision applications for their ability to guide motion with high repeatability. While traditional rigid-link mechanisms move due to rolling or sliding components that induce hysteresis, flexure mechanisms move solely due to elastically deforming components. This means that flexure mechanisms operate without friction, backlash, stick-slip and wear, resulting in low hysteresis. The elastic deformation also presents challenges. The accompanying stress limits the range of motion by yielding and fatigue, and the changing configuration decreases the support stiffness. This thesis presents research on modeling techniques and design principles for flexure mechanisms in order to improve the stiffness characteristics of flexure mechanisms with a large range of motion.

The first thesis topic, on modeling techniques, concerns the sheet flexure, which is a common elastic component of flexure mechanisms. While it is often analyzed in a simplified form, e.g. by assuming planar deformation or linearized stiffness, the deformation in practice is spatial and sufficiently large that nonlinear effects due to the geometric stiffness are significant. This thesis describes the formulation of a parametric spatial stiffness model for sheet flexures at moderate deformations. This closed-form model is expressed in terms of design parameters, such as error motions, degrees of freedom and applied loads. A mixed variational principle with a specific set of interpolation functions is shown to be well suited for the derivation of the third-order terms that govern the support stiffness decrease. The geometrically nonlinear effects of bending, shear, elongation, torsion and warping are included, so that sheet flexures with a large width-to-length ratio can be accurately modeled. The energy-based discretization approach is also extended to model parallel assemblies of sheet flexures.

The sheet flexure model is implemented numerically as a beam element in multibody software. It enables faster simulations of flexure mechanisms containing a large number of sheet flexures, because a single such beam element per sheet flexure generally suffices. This is demonstrated with various numerical models of practical large range of motion flexure mechanisms.

The second thesis topic concerns the exact-constraint design principle, which is commonly applied to flexure mechanisms to ensure deterministic behavior, but at the cost of reduced robustness, support stiffness, load capacity and usually an increased complexity of design. To explore the potential benefit of overconstrained design in flexure mechanisms, this thesis investigates the misalignment sensitivity associated with overconstrained design. An elementary two-flexure cross-hinge with a single overconstraint serves as a case study in an experimental, numerical and analytical analysis of the stiffness effects of inadvertent stress due to misalignment. It is shown that small, typically unintended, misalignments in the overconstrained direction can affect the

system behavior significantly. Numerical models show that the effect can be accurately predicted. The models indicate that detailed aspects, such as flexure thickness variations, parasitic compliance and constrained warping deformation, favorably decrease the misalignment sensitivity. An analytical buckling analysis provides an explicit upper bound for the critical misalignment of a variety of cross-hinge designs.

A second case study demonstrates the principle of dynamically stiffened exact-constraint design. It enables the use of overconstrained design for its increased dynamic performance by reducing the misalignment sensitivity with a custom synthesized viscoelastic material. Experiments show that the elastomer can compensate for unintended misalignments without significant stress buildup, while improving the dynamic performance in terms of a higher first parasitic natural frequency. Analytical and numerical model predictions match with the measurements.

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CHAPTER 1

Introduction

Elastically deforming structures can be exploited as motion guiding mechanisms with low hysteresis and no backlash. These structures are designed as assemblies of generally slender components that provide only low stiffness in the directions of intended motion and high stiffness in the other directions. They are referred to as flexure mechanisms. Unlike rigid-link mechanisms with sliding or rolling contacts, flexure mechanisms have no friction, and are stick-slip and wear free. With these properties, flexure mechanisms enable the highly repeatable motion required for precision manipulation, even in harsh conditions, such as cryogenic and vacuum environments.

From a design point of view, flexure mechanisms pose some unique challenges. Because motion stems from elastic deformation, there is accompanying stress that limits the range of motion by yielding and fatigue. Also, for mechanisms with a larger range of motion, the increased elastic deformation generally leads to stiffness characteristics that change with motion of the mechanism due to the nonlinear geometric stiffness. As a consequence, the motion capability of the mechanism is impacted, since it was derived from the stiffness characteristics in the first place.

The overarching objective of this research is to provide a deeper understanding of the elastic deformation and the configuration-dependent stiffness properties for systematically creating improved precision flexure mechanisms with a large range of motion. This is achieved by advancing the modeling techniques and the design principles for flexure mechanisms.

Several modeling techniques help in understanding the performance of a proposed design. The functional resemblance with rigid-link mechanisms can be used to obtain the kinematics of a flexure mechanism. For a description of the nonlinear stiffness encountered at a large range of motion, numerical models and analytical models exist. Numerical models, such as those from finite element and multibody formulations, generally provide accurate simulations, though at the expense of a high computational cost: modeling the nonlinear geometric stiffness for flexure mechanism components with high aspect ratios requires a large number of elements. The computational cost further increases when the sensitivity with respect to design parameters is desired. While analytical models can offer parametric insight at low computational cost, they are only available for a select number of commonly used flexure mechanism components. The reason is the mathematical complexity that is associated with the derivation of such models, with the useful representation of the model equations, and with the generalization from constituent component to mechanism. The available nonlinear stiffness models are restricted to flexure components with either small cross-sectional dimensions compared to the length or particular load cases. An analytical model for the general load

condition at moderate deformations of the common sheet flexure, also referred to as a leaf spring or flexure strip, is not available. In summary, modeling of flexure mechanisms with a large range of motion is cumbersome: the existing analytical models are limited in applicability and the numerical models are computationally intensive.

Design principles for precision engineering serve to inform design decisions, often by restricting the design space. Some are mere abstractions of intuition and experience, others have a scientific rationale. According to the principle of exact-constraint design, a large class of designs — overconstrained designs — is best avoided to ensure predictability and robustness against static and dynamic misalignments. While this principle is applicable to flexure mechanisms, it has its origins in rigid-link mechanism design and can be reasoned to be overly restrictive for flexure mechanisms. Because the application of the principle reduces a mechanism to a kinematic representation, the distributed nature of the compliance in flexure mechanisms is disregarded instead of exploited. Even when overconstrained, elasticity models still represent flexure mechanisms accurately, allowing for at least a quantitative analysis of the limitations of overconstrained design. In summary, the necessity of exact-constraint design for flexure mechanisms is not well founded; overconstrained design has the potential for higher performance.

The research subgoals of this thesis are

1. modeling the spatial sheet flexure and compound mechanisms
2. exploring the use of overconstrained flexure mechanism design

These subgoals work toward the objective of providing insight into the elastic deformation and stiffness properties for creating improved precision flexure mechanisms with a large range of motion.

OUTLINE

The first research subgoal, on sheet flexure modeling, is treated in chapters 2 and 3. The second research subgoal, on overconstrained design, is treated in chapters 4, 5 and 6. The five chapters have each been written as self-contained scientific papers. While this has created some overlap in the content of this thesis, it should make it easier to read chapters independently. Each chapter contains a separate introduction with the research scope and the relevant literature review. The current introductory chapter only serves to provide the overall research context and the relationship between the various chapters. The reference list for all chapters combined is provided after the last chapter.

Chapter 2 presents a highly reduced nonlinear spatial stiffness model for sheet flexures. This model is formulated as a set of closed-form equations in terms of design parameters. The chapter details the derivation by means of an energy-based discretization approach that allows for the systematic treatment of the geometrically nonlinear effects of bending, elongation, shear, torsion and warping. The model is capable of accurately modeling the nonlinear geometric stiffness of a sheet flexure under a general load condition at moderate deformations. The extension to parallel assemblies of sheet flexures is also detailed. *The contribution of this chapter* is the theory of a spatial stiffness

model for sheet flexures with moderate deformations, as encountered in precision flexure mechanisms. The derivation of an accurate closed-form model for these common components has not been presented before. The chapters that follow use largely the same solid mechanics modeling techniques based on beam theory, though simplified where possible and extended where necessary.

Chapter 3 presents the numerical implementation of the spatial sheet flexure model in the flexible multibody framework of choice, the generalized strain beam formulation. The numerical element can be used for efficient calculations of general flexure mechanisms that consist of a large number of components. *The contribution of this chapter* is a new beam element, based on the theory of chapter 2, for faster numerical simulations of flexure mechanism models that contain sheet flexures. This numerical implementation of the analytical formulation of chapter 2 allows for more practical simulations using multibody analysis software.

Chapters 4 and 5 present the analysis of the consequences of misalignment in a single-overconstraint cross-hinge mechanism. An analytical and numerical buckling analysis provides the operational limit of the mechanism in terms of the allowable misalignment. The analysis is supported by measurements. Chapter 5 provides further details of the effect of constrained warping on the analysis results. *The contribution of these chapters* is the demonstration that an operational limit of the overconstrained cross-hinge design can be predicted. This promotes the use of overconstrained flexure mechanism design in applications with a controlled misalignment tolerance. Since the buckling analysis of this chapter uses largely the same solid mechanics theory as chapter 2, it also provides an experimental verification of the use of geometrically nonlinear beam theory in sheet flexure applications.

Chapter 6 presents the principle of dynamically stiffened exact-constraint design in a case study with an overconstrained parallelogram flexure mechanism and a custom synthesized elastomer. *The contribution of this chapter* is a new design principle that enables the use of overconstrained design for its increased dynamic performance by reducing the misalignment sensitivity with a custom synthesized viscoelastic material. This expands on the overconstraint analysis of chapters 4 and 5 by providing a design solution for the misalignment issue commonly associated with overconstrained design. The agreement between the measurements and the numerical and analytical models also provides a further verification of the modeling tools used in the preceding chapters. From an analytical modeling point of view, the use of case-specific buckling modes in the interpolation functions is shown to lead to a higher accuracy model than the general sheet flexure model of chapter 2: it is accurate up to the bifurcation load.

Chapter 7 presents the conclusions of this thesis.

CHAPTER 2

Third-order stiffness formulation for sheet flexures using an energy-based discretization approach (EDA): 3-D closed-form nonlinear model

Thesis note The paper in this chapter addresses the first subgoal, on flexure mechanism modeling techniques. It presents the derivation, interpretation and verification of a spatial analytical stiffness model for moderate deformations of sheet flexures, which are common elastic components of flexure mechanisms.

Abstract The sheet flexure is commonly used to provide support stiffness in flexure mechanisms for precision applications. While the sheet flexure is often analyzed in a simplified form, e.g. by assuming planar deformation or linearized stiffness, the deformation in practice is spatial and sufficiently large that nonlinear effects due to the geometric stiffness are significant.

This paper presents a compact analytical model for the nonlinear stiffness characteristics of spatially deforming sheet flexures under general 3-D load conditions at moderate deformations. This model provides closed-form expressions in a mixed stiffness and compliance matrix format that is tailored to flexure mechanism analysis. The effects of bending, shear, elongation, torsion and warping deformation are taken into account, so that the in-plane lateral support stiffness is modeled accurately as well. The model is verified numerically against beam and shell-based finite elements. The approach for deriving closed-form solutions in a nonlinear context is detailed in this paper. The Hellinger–Reissner variational principle with a specific physically motivated set of low-order interpolation functions is shown to be well-suited to the geometrically nonlinear analysis of flexures.

An extension of the derivation approach to the nonlinear closed-form analysis of general flexure mechanisms consisting of multiple sheet flexures connected in parallel is presented. This is demonstrated with the case of a spatially deforming parallelogram flexure mechanism and a cross-hinge flexure mechanism.

2.1 INTRODUCTION

Flexure mechanisms are used in precision applications for their predictable behavior [21, 30, 72, 73]. Like traditional rigid-link mechanisms, they are essentially motion guiding mechanisms: they allow motion in some directions, while providing load-carrying support in others. Unlike traditional linkages though, flexure mechanisms rely on the elastic deformation of components. This means that their operation is almost hysteresis-free, backlash-free, limited in range, and governed only by stiffness properties.

Submitted to International Journal of Mechanical Sciences. Co-authors: J.P. Meijaard and D.M. Brouwer.

Figure 2.1: A sheet flexure constraining motion between rigid bodies A and B.

A typical flexure mechanism has a stage, whose motion is dictated by various elastic components that are attached to it in parallel and series configurations. In some directions, the components provide relatively high stiffness in order to constrain the stage motion (and provide load-carrying support), while adding only low stiffness to the directions of the desired stage motion. From a designer's point of view, the elastic components are considered to be constraint elements.

A common elastic component is the *sheet flexure* (also referred to as a leaf spring, flexure strip or flexure blade), depicted in Fig. 2.1 that serves as a constraint element between rigid bodies A and B. Its thickness is typically one or more orders of magnitude smaller than the length and width. As a consequence of that shape, the sheet flexure constrains the relative motion (i.e. provides high stiffness) in the translational x -, z - and rotational y -directions, while enabling the relative motion (i.e. adding only low stiffness) in the rotational x -, z - and translational y -directions. The directions of constraint are also referred to as *support directions*; the directions of intended motion as *degrees of freedom* (DOFs).

Inherent to the sheet flexure is that the desired high stiffness in the support directions decreases when it deflects in a DOF [27], which is referred to as an *elastokinematic* effect [8]. This poses a direct limitation on the performance. A related but less critical observation is that the desired low stiffness in the DOFs changes with the applied loads. These nonideal constraint characteristics of the sheet flexure stem from the configuration-dependent geometric stiffness. They are relevant to the design of flexure mechanisms with a large range of motion, and constitute the topic of this work.

The predictable nature of sheet flexures means that the governing equations of solid mechanics deliver accurate results, e.g. when used in a discretized form in the ubiquitous finite element method. To actually inform design decisions, we seek to develop a parametric model instead. A mathematical model that is accurate and understandable can convey how design parameters affect the nonideal constraint characteristics in ways that the 3-D elasticity differential equations or the finite element analysis of a single design cannot.

For an overview of the solid mechanics theories that govern structures similar to the sheet flexure, the reader is referred to Sen [69]. In the literature on flexure

mechanisms, parametric models for some typical elastic components can be found. The beam constraint model for planar (2-D) sheet flexures considers loads and deformation limited to the x, y -plane of Fig. 2.1 [9, 12]. It captures the relevant nonlinearities due to the geometrical stiffness in a closed-form format. A similar model for wire flexures with isotropic cross-section (e.g. a round or square cross-section) that does consider general spatial (3-D) loads and deformation has been developed [70]. An extension to rectangular cross-sections is also available [13], but limited to small width-to-length ratios, since shear and constrained warping have not been accounted for. As a consequence, it does not accurately model the lateral constraint directions of the sheet flexure, which is designed to provide high stiffness in those directions by means of a large width-to-length ratio.

Parametric models for the spatial sheet flexure are only available for a specific loading condition (a lateral load) and a limited twist angle [54, 58]. For the parallelogram flexure mechanism specifically, parametric expressions for the geometrically nonlinear stiffness are available [16]. A method for obtaining the spatial deformation of a sheet flexure in an analytical-iterative fashion has been proposed, but not applied to the development of a general closed-form parametric model [22].

The contribution of this work is a closed-form parametric model for the general spatial deformation encountered in sheet flexures. In this paper, we present:

1. an energy-based discretization approach for the formulation of this model,
2. the treatment of geometrically nonlinear effects due to bending, torsion, elongation, shear and warping deformation,
3. understandable closed-form expressions that model the nonideal constraint characteristics of sheet flexures in terms of support stiffness, loads in the DOFs, and potential energy,
4. a verification by means of a commercial finite element code,
5. an energy-based procedure for the closed-form analysis of general parallel assemblies of sheet flexures, and
6. the application to the case of a spatially deforming parallelogram flexure mechanism and cross-hinge mechanism.

2.2 METHOD

The development of solid mechanics models involves a trade-off between accuracy and complexity of the resulting formulation. Since we aim to develop a model that can inform design decisions, it should be minimally complex, i.e. be represented by mathematically simple closed-form expressions. To that end, we adopt the reduced model of beam theory (with only a single independent coordinate) rather than that of plate theory (with still two independent coordinates) or full 3-D elasticity theory (with three independent coordinates) [43, 77]. Previous work on flexure mechanism models

Figure 2.2: Kinematic description of a deformed sheet flexure, modeled as a beam. Image reused from the authors' previous work [54, 58].

has also indicated that beam theory is sufficiently accurate for shapes and loads of practical interest at only limited complexity [16, 22, 69]. Furthermore, the current work focuses solely on the static behavior of sheet flexures, so inertia effects are disregarded.

The following section 2.2.1 is an overview of the equations that govern general spatial (3-D) geometrically nonlinear deformation of a beam. These are commonly used and summarized here only to establish the notation, sign and symbol convention that is used in the development of the sheet flexure formulation in the sections after that.

2.2.1 Beam kinematics

Deformed configuration

The configuration of a deformed beam can be represented by its elastic line and the cross-sectional planes attached to it (see Fig. 2.2) [41]. The independent coordinate $s \in [0, L]$ measures the distance along the undeformed elastic line. The orientation of the cross-section at an arbitrary point on the elastic line is described by a set of three orthonormal vectors, as shown in Fig. 2.2(c): $\mathbf{e}_{\bar{x}}$ is normal to the cross-sectional plane, $\mathbf{e}_{\bar{y}}$ and $\mathbf{e}_{\bar{z}}$ are along the principal axes of the cross-sectional plane.

A fixed coordinate frame is attached to the beam at $s = 0$ in the same manner and denoted by base vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ (without overbar). All vectors in the current analysis are resolved in this frame unless noted otherwise. The local ('moving') frame $\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{y}}, \mathbf{e}_{\bar{z}}$ is related to the fixed frame $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$ by a rotation matrix (dependent on s) according to

$$\underbrace{\begin{bmatrix} \mathbf{e}_{\bar{x}} & \mathbf{e}_{\bar{y}} & \mathbf{e}_{\bar{z}} \end{bmatrix}}_{\substack{\text{local frame attached} \\ \text{to deformed cross-section}}} = \mathbf{R}(s) \underbrace{\begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{bmatrix}}_{\text{fixed frame}}. \quad (2.1)$$

The position of the elastic line, shown in Fig. 2.2(b), with respect to the fixed frame is given by

$$\mathbf{r}(s) = \mathbf{r}_0(s) + \mathbf{u}(s) = s\mathbf{e}_x + \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{bmatrix} \begin{bmatrix} u_x(s) \\ u_y(s) \\ u_z(s) \end{bmatrix}, \quad (2.2)$$

where $\mathbf{u}(s)$ is the displacement vector.

As a parametrization for the cross-sectional orientation matrix $\mathbf{R}(s)$, we choose the three Tait–Bryan (Euler) angles $\phi_x(s)$, $\phi_y(s)$ and $\phi_z(s)$ in the specific order

$$\mathbf{R}(s) = \begin{bmatrix} \cos \phi_y & 0 & \sin \phi_y \\ 0 & 1 & 0 \\ -\sin \phi_y & 0 & \cos \phi_y \end{bmatrix} \begin{bmatrix} \cos \phi_z & -\sin \phi_z & 0 \\ \sin \phi_z & \cos \phi_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi_x & -\sin \phi_x \\ 0 & \sin \phi_x & \cos \phi_x \end{bmatrix}. \quad (2.3)$$

The angles describe the rotation with respect to and resolved in the fixed frame denoted by $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$.

The configuration of a deformed beam is thus given by the elastic line position, described by the displacements $u_x(s)$, $u_y(s)$, $u_z(s)$, and the cross-sectional orientation, described by the angles $\phi_x(s)$, $\phi_y(s)$ and $\phi_z(s)$.

Deformation measures

In terms of the six functions that describe the beam configuration (see Sec. 2.2.1), six independent measures of deformation can be identified for a beam that is capable of bending, extension, torsion and shear deformation [63].

The skew-symmetric matrix

$$\boldsymbol{\kappa}(s) = \mathbf{R}^T \mathbf{R}' = \begin{bmatrix} 0 & -\kappa_{\bar{z}} & \kappa_{\bar{y}} \\ \kappa_{\bar{z}} & 0 & -\kappa_{\bar{x}} \\ -\kappa_{\bar{y}} & \kappa_{\bar{x}} & 0 \end{bmatrix} \quad (2.4)$$

contains the three independent curvatures, given by the expressions

$$\begin{aligned} \kappa_{\bar{x}}(s) &= \phi'_x + \phi'_y \sin \phi_z \approx \phi'_x + \phi'_y \phi_z, \\ \kappa_{\bar{y}}(s) &= \phi'_y \cos \phi_x \cos \phi_z + \phi'_z \sin \phi_x \approx \phi'_y + \phi'_z \phi_x, \\ \kappa_{\bar{z}}(s) &= \phi'_z \cos \phi_x - \phi'_y \sin \phi_x \cos \phi_z \approx \phi'_z - \phi'_y \phi_x, \end{aligned} \quad (2.5)$$

truncated after the second-order term. A prime denotes a derivative with respect to the distance s along the undeformed elastic line. As measures of deformation, $\kappa_{\bar{x}}$ represents the specific twist angle of the beam, $\kappa_{\bar{y}}$ the curvature in the $\mathbf{e}_{\bar{x}}$, $\mathbf{e}_{\bar{z}}$ -plane, and $\kappa_{\bar{z}}$ the curvature in the $\mathbf{e}_{\bar{x}}$, $\mathbf{e}_{\bar{y}}$ -plane.

The strains are given by the vector

$$\boldsymbol{\gamma}(s) = \mathbf{R}^T (\mathbf{e}_x + \mathbf{u}' - \mathbf{e}_{\bar{x}}), \quad (2.6)$$

which consists of the three components

$$\begin{aligned} \gamma_{\bar{x}}(s) &\approx u'_x - \phi_y(\phi_y/2 + u'_z) - \phi_z(\phi_z/2 - u'_y), \\ \gamma_{\bar{y}}(s) &\approx u'_y - \phi_z(1 + u'_x) + \phi_x(\phi_y + u'_z), \\ \gamma_{\bar{z}}(s) &\approx u'_z + \phi_y(1 + u'_x) + \phi_x(\phi_z - u'_y), \end{aligned} \quad (2.7)$$

truncated after the second-order terms. Here, $\gamma_{\bar{x}}$ represents the axial strain of the elastic line, $\gamma_{\bar{y}}$ the shear angle in the $\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{y}}$ -plane, and $\gamma_{\bar{z}}$ the shear angle in the $\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{z}}$ -plane.

In their non-linearized form, the finite strain and curvature measures of Eqs. (2.5) and (2.7) are referred to as the compatibility equations and account for geometrically nonlinear effects.

An effect that is not captured by $\boldsymbol{\gamma}$ is the coupling between elongation and torsion, sometimes referred to as the trapeze effect or Wagner effect. It describes the shortening of a beam that is twisted and the torsional buckling due to an axial load [20, 76]. The effect can be accounted for by modifying and augmenting the axial strain $\gamma_{\bar{x}}$ with the term $I_{te}\phi_x'^2/2$, where

$$I_{te} = (w^2 + t^2)/12 \quad (2.8)$$

for a rectangular cross-section with width w and thickness t . This effect was also considered by Sen and Awtar [70] in the model for wire flexures with isotropic cross-sections.

Constitutive relations

Linear material behavior for rectangular cross-sections is governed by

$$\begin{aligned} F_{\bar{x}}(s) &= EA\gamma_{\bar{x}}(s), & M_{\bar{y}}(s) &= EI_y\kappa_{\bar{y}}(s), \\ F_{\bar{y}}(s) &= kGA\gamma_{\bar{y}}(s), & M_{\bar{z}}(s) &= EI_z\kappa_{\bar{z}}(s), \\ F_{\bar{z}}(s) &= kGA\gamma_{\bar{z}}(s), & B(s) &= EI_w\kappa_{\bar{x}}'(s), \end{aligned} \quad (2.9)$$

and the linear ordinary differential equation

$$M_{\bar{x}}(s) = GJ\kappa_{\bar{x}}(s) - EI_w\kappa_{\bar{x}}''(s), \quad (2.10)$$

where $F_{\bar{x}}$ is the axial force, $F_{\bar{y}}$ and $F_{\bar{z}}$ are the transverse forces, $M_{\bar{x}}$ is the torsion moment, $M_{\bar{y}}$ and $M_{\bar{z}}$ are the bending moments, and B is the bimoment. The parameters are Young's modulus E , the cross-sectional area A , the second moments of area about the principal axes I_y and I_z , the shear correction factor k [18], the shear modulus G , Saint-Venant's torsion constant J , and Vlasov's warping constant I_w , which is given by [75]

$$I_w = w^3t^3/144. \quad (2.11)$$

2.2.2 Static equilibrium principles

The static equilibrium configuration that a beam assumes when subjected to applied forces and moments (collectively referred to as loads) is described by various equilibrium principles. One approach for finding the equilibrium configuration is to formulate force and moment balance equations of an infinitesimal-length beam segment; combined with compatibility and constitutive relations, they lead to a coupled system of nonlinear (ordinary) differential equations. Only upon (partial) linearization can exact solutions to these ODEs be obtained. Another approach is to consider the entire beam domain, instead of an infinitesimal segment, and use an energy method, such as the principle of

minimum total potential energy (essentially a virtual work principle for conservative systems) [40].

When the errors associated with a fully linearized beam model are acceptable, the two approaches can be used without mathematical difficulty. However, when geometrically nonlinear effects are relevant, such as in the present work in which stiffness changes with deflection, the governing differential equations can pose difficulties.

Both approaches can be found in the literature on flexure mechanism analysis. In the development of a planar (2-D) model for sheet flexures, Awtar et al. [12] consider a system of partly linearized ODEs that admits an exact solution for the displacement fields, which happens to be transcendental in one of the applied loads. A useful closed-form model has nevertheless been obtained since the exact solution is well approximated by a truncated series expansion. In turn, with that solution, a strain energy expression has been formulated, which is convenient for reducing the mathematical complexity when dealing with multi-flexure mechanisms (as per the principle of virtual work in which constraint forces do no work) [10]. Along the same lines, Sen and Awtar [70] have developed a spatial (3-D) model for slender beam-like flexures restricted to isotropic cross-sections ($I_y = I_z$). Bai et al. [13] have produced an extension to rectangular cross-sections by means of a power series solution to the ODEs; it is restricted to small w/L because it does not account for shear and warping deformation.

Typical sheet flexures, though, are one or more orders of magnitude wider than they are thick, in order to provide adequate lateral support stiffness. In pursuit of a spatial (3-D) model for sheet flexures, Sen [69] has formulated a simplified governing system of ODEs, but with still too much complexity (variable coefficients) for an exact solution. Nijenhuis et al. have formulated a spatial closed-form model for specific cases only: a common lateral loading condition and a general loading condition with limited twist angle [54, 58]. Based on observations in these last publications, we now follow a different approach and construct the desired closed-form spatial sheet flexure model.

When the principle of virtual work is applied to the system's total potential energy functional, the ODEs that are obtained in the variation process (the Euler–Lagrange equations) make the functional stationary and represent the strong form of the equilibrium conditions for the beam. While the ODE is exact, in the sense that it represents the equilibrating solution for each point in the domain, exactness is not a requirement for a useful model in the context of flexure mechanism analysis; given the observed difficulties associated with solutions to such ODEs (and the relative value of an ‘exact’ solution in an approximate theory such as beam theory), we instead use the system's total potential energy functional with a variational principle to approximate the solution by means of suitably chosen interpolation functions (also referred to as test or trial functions). The accuracy and complexity of the resulting sheet flexure formulation are then determined by the choice of interpolation functions and the type of variational principle. An important advantage of the energy-based approximation is the availability of the potential energy scalar, from which the model equations are derived: it can be used to compare the magnitude of effects and provide the justification for the truncation of terms.

In the typical principle of minimum total potential energy, the variations of the functional are carried out with respect to the displacement field. The (kinematically

Figure 2.3: Fixed-free sheet flexure subject to an arbitrary spatial end-load.

admissible) displacement field that makes the potential energy stationary is the physical one. In previous work, it has been observed that the displacement fields of flexure elements become explicitly dependent on applied load parameters when a nonlinear contribution of the geometric stiffness is accounted for. It then makes sense to use a variational principle that not only considers variations of the displacement field, but simultaneous variations of the internal moment field as well. The Hellinger–Reissner variational principle is such a principle [81].

2.2.3 Hellinger–Reissner variational principle

The potential energy expression associated with the Hellinger–Reissner variational principle is given by

$$P = P_{\text{int}} - W_{\text{ext}}, \quad (2.12)$$

where the external work is

$$\begin{aligned} W_{\text{ext}} = & \bar{M}_x^{\text{TB}} \phi_x(L) + \bar{M}_y^{\text{TB}} \phi_y(L) + \bar{M}_z^{\text{TB}} \phi_z(L) \\ & + \bar{F}_x u_x(L) + \bar{F}_y u_y(L) + \bar{F}_z u_z(L) + [\bar{B} - B(L)] \kappa_{\bar{x}}(L) \end{aligned} \quad (2.13)$$

and the internal energy is

$$\begin{aligned}
 P_{\text{int}} = \int_0^L & \left[M_{\bar{x}} \kappa_{\bar{x}} - \frac{(M_{\bar{x}} + B')^2}{2GJ} - \frac{B^2}{2EI_w} \right. && \text{nonuniform} \\
 & + M_{\bar{y}} \kappa_{\bar{y}} - \frac{M_{\bar{y}}^2}{2EI_y} && \text{bending} \\
 & + M_{\bar{z}} \kappa_{\bar{z}} - \frac{M_{\bar{z}}^2}{2EI_z} && \text{bending} \\
 & + F_{\bar{x}} \gamma_{\bar{x}} - \frac{F_{\bar{x}}^2}{2EA} && \text{elongation} \\
 & + F_{\bar{y}} \gamma_{\bar{y}} - \frac{F_{\bar{y}}^2}{2kGA} && \text{shear} \\
 & \left. + F_{\bar{z}} \gamma_{\bar{z}} - \frac{F_{\bar{z}}^2}{2kGA} \right] ds. && \text{shear}
 \end{aligned} \tag{2.14}$$

$\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{z}}$ plane
 $\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{y}}$ plane
 $\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{y}}$ plane
 $\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{z}}$ plane

The potential energy P is a functional of both the displacement (and rotation) fields and moment (and force) fields. The curvatures $\kappa_{\bar{x}}, \kappa_{\bar{y}}, \kappa_{\bar{z}}$ and strains $\gamma_{\bar{x}}, \gamma_{\bar{y}}, \gamma_{\bar{z}}$ in P_{int} are related to the displacements and rotations by Eqs. (2.5) and (2.7). This formulation is based on Timoshenko beam theory with nonuniform torsion.

The Hellinger–Reissner principle states that P is rendered stationary by those displacement and moment fields that are the physical ones [81]. In other words, the beam is in equilibrium when the variation of P with respect to the independent displacement and moment fields vanishes. Compared to the principle of minimum total potential energy, in which the displacements are the only independent fields and variations yield the equilibrium equations, the Hellinger–Reissner principle also encodes the compatibility equations as variations with respect to the moment fields.

A physical interpretation of each term of P_{int} is given alongside the expression. It shows that the internal energy due to the various deformation modes of bending, shear, torsion and elongation can simply be added. The derivation of the terms that account for nonuniform torsion, involving the torsion moment $M_{\bar{x}}$ and bimoment B , is detailed in Appendix 2.8.1.

By interpolating the internal moment and displacement fields using suitable interpolation functions with unknown coefficients, the integral in P can be evaluated. The variations with respect to the unknown displacement coefficients then yield a system of algebraic equilibrium equations; the variations with respect to the unknown moment coefficients yield a system of algebraic compatibility equations. Both systems are expressed in terms of the unknown interpolation coefficients and can potentially be solved for the parameters of interest. This approach will be detailed in the following sections.

Note that the internal loads $M_{\bar{x}}, M_{\bar{y}}, M_{\bar{z}}, F_{\bar{x}}, F_{\bar{y}}, F_{\bar{z}}$ and curvatures in the integrand are with respect to the *local frame* $\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{y}}, \mathbf{e}_{\bar{z}}$ that is attached to the deformed cross-section (see Sec. 2.2.1). Since the displacements in the external work terms are chosen to be defined in the *fixed frame* $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$, the associated (energetically-dual) applied forces

are also resolved in this fixed frame. Since the angles ϕ_x , ϕ_y and ϕ_z are Tait–Bryan angles, the associated moments $\bar{M}_x^{\text{TB}}$, $\bar{M}_y^{\text{TB}}$ and $\bar{M}_z^{\text{TB}}$ are energetically dual to those angles, meaning that these moments are not the same as the components of the moment vector $[\bar{M}_x, \bar{M}_y, \bar{M}_z]$ resolved in the fixed frame. The first-order transformation is given by

$$\begin{aligned}\bar{M}_x^{\text{TB}} &= \bar{M}_x + \bar{M}_y \phi_z(L) - \bar{M}_z \phi_y(L), \\ \bar{M}_y^{\text{TB}} &= \bar{M}_y, \\ \bar{M}_z^{\text{TB}} &= \bar{M}_z + \bar{M}_x \phi_y(L).\end{aligned}\tag{2.15}$$

This transformation cannot be applied directly to W_{ext} at the potential energy level, since $\bar{M}_x$, $\bar{M}_y$ and $\bar{M}_z$ are non-conservative loads; only after taking variations of P can the transformation be applied to the resulting strong form of the equations. In the remainder of this work, the moment vector components will be used, unless noted otherwise.

2.2.4 Order of magnitude estimates

To help choosing the interpolation functions, estimates of the order of magnitude of the terms in the potential energy are useful. The complexity and accuracy of the formulation can then be controlled by retaining only those terms that are significant. We will proceed by stating a small number of assumptions, and apply criteria of maximum stress and buckling to estimate the magnitude of the various geometrical parameters, displacement vector components, deformation measures and loads. These estimates are expressed in terms of a small quantity so that truncation based on the relative powers of this quantity can be justified.

Assumptions

On the geometry, deformation and material of the sheet flexure geometry, it is assumed that

- the width w is the same order as the length L ;
- the transverse end-displacement $u_y(L)$ is limited to $0.15L$. The number

$$u = 0.15 \tag{2.16}$$

is introduced as a small quantity. All estimates in the following section are expressed in terms of powers of u in order to compare relative magnitudes;

- the failure strain ε of the material is at most 0.3%, which is considered a realistic failure strain for high tensile strength metals. In terms of powers of the small quantity u , the magnitude of ε is therefore estimated as u^3 (rather than u^2 or u^4).

Maximum stress criterion

Four different maximum stress cases are examined to estimate, first, the thickness t , the curvature $\kappa_{\bar{y}}$ and the specific twist angle $\kappa_{\bar{x}}$, and, second, the corresponding internal moments. The yield stress εE is considered to be the maximum allowed stress in the material.

1. Bending in the $\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{y}}$ -plane:

The magnitude of the thickness t is such that the maximum allowed stress state is obtained for a transverse displacement $u_y(L)$ of magnitude uL .

Assuming a constant moment load $M_{\bar{z}} = 2EI_z u/L$, the bending stress $M_{\bar{z}} t / 2I_z$ is equal to the yield stress when the thickness has magnitude $L\varepsilon/u = u^2 L$. The corresponding magnitude of $M_{\bar{z}}$ is $EL^3 u^7$.

2. Bending in the $\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{z}}$ -plane:

The magnitude of the curvature $\kappa_{\bar{y}}$ is such that the corresponding bending moment $M_{\bar{y}} = EI_y \kappa_{\bar{y}}$ causes the maximum allowed stress state.

The bending stress $M_{\bar{y}} w / 2I_y$ is equal to the yield stress when the curvature $\kappa_{\bar{y}}$ has magnitude $\varepsilon/L = u^3/L$. The corresponding magnitude of $M_{\bar{y}}$ is $EL^3 u^5$.

3. Torsion:

The magnitude of the specific twist angle $\kappa_{\bar{x}}$ is such that the corresponding torsion moment $M_{\bar{x}} = GJ\kappa_{\bar{x}}$ causes the maximum allowed stress state.

The von Mises stress $\sqrt{3}|\tau_{\max}| = \sqrt{3}Gt|\kappa_{\bar{x}}|$ is equal to the yield stress $\varepsilon E \approx 3\varepsilon G$ when the specific twist angle $\kappa_{\bar{x}}$ has magnitude u/L . The corresponding magnitude of $M_{\bar{x}}$ is $EL^3 u^7$.

4. Elongation:

The magnitude of the axial strain $\gamma_{\bar{x}}$ assumed to be ε at the maximum allowed stress state. The corresponding magnitude of internal axial load $F_{\bar{x}} = EA\gamma_{\bar{x}}$ is $EL^2 u^5$.

Buckling criterion

When loads reach a critical value, elastic instability can occur [76]. With the stress criterion, this poses an additional limit on the magnitude of the loads and serves as a practical estimate. We estimate that the magnitude of the loads is at least one order smaller than the buckling load magnitude.

1. Euler column buckling:

Buckling occurs when the axial load $\bar{F}_x$ reaches the critical value of $-c\pi^2 EI_z / L^2$ (where c depends on the boundary conditions), which has magnitude $EL^2 u^6$.

Force $\bar{F}_x$ is therefore estimated, one order smaller, as $EL^2 u^7$.

2. Lateral buckling due to $\bar{F}_z$:

Buckling occurs when the lateral load $\bar{F}_z$ reaches the critical value of $c\sqrt{EI_z G}/L^2$, which has magnitude EL^2u^6 . Force $\bar{F}_z$ is therefore estimated as EL^2u^7 .

3. Lateral buckling due to $\bar{M}_y$:

Similar to lateral buckling due to $\bar{F}_z$, the magnitude of moment $\bar{M}_y$ is estimated as EL^3u^7 .

Displacements and rotations

Given the maximum transverse displacement $\bar{u}_y$, the small-angle approximation $\phi_z \approx u'_y$ is valid (neglecting shear): the angle ϕ_z has magnitude u . The linear approximations $\kappa_{\bar{x}} \approx \phi'_x$, $\kappa_{\bar{y}} \approx \phi'_y$ and $\kappa_{\bar{z}} \approx \phi'_z$ can be combined with the second-order compatibility relations of Eq. (2.5) to find that

$$\kappa_{\bar{z}} \approx \phi'_z - \phi'_y \phi_x \approx \phi_z/L - \kappa_{\bar{y}} \kappa_{\bar{x}} L \sim u/L, \quad (2.17)$$

$$\phi'_x \approx \kappa_{\bar{x}} - \phi'_y \phi_z \approx \kappa_{\bar{x}} - \kappa_{\bar{y}} \kappa_{\bar{z}} L \sim u/L, \quad (2.18)$$

$$\phi'_y \approx \kappa_{\bar{y}} - \phi'_z \phi_x \approx \kappa_{\bar{y}} - \kappa_{\bar{z}} \kappa_{\bar{x}} L \sim u^2/L, \quad (2.19)$$

where differentiation and integration with respect to s are approximated by division and multiplication with L , respectively. In this fashion, the magnitude of the associated displacements and rotations can be estimated, as summarized in Tbl. 2.1. Then, with the strain relation of Eq. (2.7), it follows that $u'_x \approx \gamma_{\bar{x}} - (u'_y)^2/2 - (u'_z)^2/2 \sim u^2$ and $u_x \sim u^2 L$.

Forces and moments

In applying the maximum stress criterion, the magnitude of the internal loads has been obtained. Note that the internal loads are defined with respect to the local frame. For estimates of the applied loads at $s = L$, which are defined in the fixed frame instead, the equilibrium equations of a finite beam segment can be used.

The relation between the internal moments at s and the applied moments at $s = L$ is given by

$$\begin{bmatrix} M_{\bar{x}}(s) \\ M_{\bar{y}}(s) \\ M_{\bar{z}}(s) \end{bmatrix} = \mathbf{R}^T \left\{ \begin{bmatrix} \bar{M}_x \\ \bar{M}_y \\ \bar{M}_z \end{bmatrix} + \left[\begin{bmatrix} L + u_x(L) \\ u_y(L) \\ u_z(L) \end{bmatrix} - \begin{bmatrix} s + u_x(s) \\ u_y(s) \\ u_z(s) \end{bmatrix} \right] \times \begin{bmatrix} \bar{F}_x \\ \bar{F}_y \\ \bar{F}_z \end{bmatrix} \right\} - \mathbf{f}_{te}(s). \quad (2.20)$$

With a second-order expansion of $\mathbf{R}^T$, it can be approximated by

$$\begin{bmatrix} M_{\bar{x}}(s) \\ M_{\bar{y}}(s) \\ M_{\bar{z}}(s) \end{bmatrix} \approx \begin{bmatrix} 1 - \phi_y^2/2 - \phi_z^2/2 & \phi_z & -\phi_y \\ -\phi_z + \phi_x \phi_y & 1 - \phi_x^2/2 - \phi_z^2/2 & \phi_x + \phi_y \phi_z \\ \phi_y + \phi_x \phi_z & -\phi_x & 1 - \phi_x^2/2 - \phi_y^2/2 \end{bmatrix} \times \quad (2.21)$$

$$\left\{ \begin{bmatrix} \bar{M}_x \\ \bar{M}_y \\ \bar{M}_z \end{bmatrix} - \begin{bmatrix} \bar{F}_y[u_z(L) - u_z(s)] - \bar{F}_z[u_y(L) - u_y(s)] \\ \bar{F}_z[L + u_x(L) - s - u_x(s)] - \bar{F}_x[u_z(L) - u_z(s)] \\ \bar{F}_x[u_y(L) - u_y(s)] - \bar{F}_y[L + u_x(L) - s - u_x(s)] \end{bmatrix} \right\} - \mathbf{f}_{te}(s).$$

In Eq. (2.20), the symbol $\times$ represents the cross product; in Eq. (2.21), it represents the matrix–vector multiplication. Function $\mathbf{f}_{te}(s)$, given by

$$\begin{bmatrix} I_{te} F_{\bar{x}}(s) \phi'_x(s) & 0 & 0 \end{bmatrix}^T, \quad (2.22)$$

represents the correction to the internal torsion moment $M_{\bar{x}}(s)$ because of the modification of the axial strain $\gamma_{\bar{x}}$ to account for the torsion–elongation coupling, as described in Sec. 2.2.1.

Similarly, it follows for the relation between the internal forces and applied forces at $s = L$ that

$$\begin{bmatrix} F_{\bar{x}}(s) \\ F_{\bar{y}}(s) \\ F_{\bar{z}}(s) \end{bmatrix} = \mathbf{R}^T \begin{bmatrix} \bar{F}_{\bar{x}} \\ \bar{F}_{\bar{y}} \\ \bar{F}_{\bar{z}} \end{bmatrix}. \quad (2.23)$$

From these equations, it can be seen what the magnitude of the fixed-frame applied loads can maximally be, so as not to exceed the maximum-stress estimates of the local-frame internal loads. As an example, for $\bar{M}_y$ not to exceed the maximum-stress estimate of $M_{\bar{y}}(s)$, it can be of order $EL^3 u^5$. However, for it not to exceed the maximum-stress estimate of $M_{\bar{x}}(s)$, it can only be of order $EL^3 u^6$, due to the term $\phi_z \bar{M}_y$ in the equilibrium equation for $M_{\bar{x}}(s)$. With similar reasoning, the estimate for $\bar{M}_x$, $\bar{F}_y L$ and $\bar{M}_z$ is $EL^3 u^7$. While the maximum-stress estimate for the loads $\bar{F}_x L$, $\bar{M}_y$ and $\bar{F}_z L$ is $EL^3 u^6$, the buckling criterion ($EL^3 u^7$) is more conservative and should be used instead.

Shear deformation

With the magnitude estimates for the internal loads, it follows that the shear angles (see the constitutive relations of Eq. (2.9)) $F_{\bar{y}}/k_y GA$ and $F_{\bar{z}}/k_z GA$ have magnitudes u^5 and u^3 , respectively. They are at least an order smaller than the estimates for the maximum rotation angles. So, when bending deformation is dominant, the effects of shear deformation can be ignored in the formulation of the current work. In case the sheet flexure is loaded by only a force $\bar{F}_z$, shear angle $\gamma_{\bar{z}}$ could be significant with respect to rotation angle ϕ_y . Therefore, shear deformation in only the $\bar{z}$ -direction is included in this work. As a consequence of assuming that $\gamma_{\bar{y}} \approx 0$, the transverse displacement u_y and rotation ϕ_z are not independent, but related by $u'_y = \phi_z$.

2.2.5 Choice of interpolation functions

In using the Hellinger–Reissner variational principle to obtain an approximate solution, interpolation functions for the internal loads ($M_{\bar{x}}$, $M_{\bar{y}}$, $M_{\bar{z}}$, $F_{\bar{x}}$, $F_{\bar{y}}$, $F_{\bar{z}}$, B) and the associated displacement fields (ϕ_x , ϕ_y , ϕ_z , u_x , u_y , u_z) can be chosen. With the order of magnitude estimates of Sec. 2.2.4, the order of the leading terms in the integrand of P can be estimated; it can be seen that they are all of the order $EL^2 u^8$. Given the exact expressions for the internal moments in Eq. (2.20) and curvatures in Eq. (2.5), we choose interpolation functions that consist of only the largest terms (i.e. of order $EL^2 u^8$) in those expressions, meaning that terms in the integrand of P of the order

External forces EL^2	$\bar{F}_x$ u^7	$\bar{F}_y$ u^7	$\bar{F}_z$ u^7	Displacements L	u_x u^2	u_y u	u_z u^2
Internal forces EL^2	$F_{\bar{x}}(s)$ u^5	$F_{\bar{y}}(s)$ u^7	$F_{\bar{z}}(s)$ u^5	Rotations	ϕ_x u	ϕ_y u^2	ϕ_z u
External moments EL^3	$\bar{M}_x$ u^7	$\bar{M}_y$ u^7	$\bar{M}_z$ u^7	Curvatures $1/L$	$\kappa_{\bar{x}}$ u	$\kappa_{\bar{y}}$ u^3	$\kappa_{\bar{z}}$ u
Internal moments EL^3	$M_{\bar{x}}(s)$ u^7	$M_{\bar{y}}(s)$ u^5	$M_{\bar{z}}(s)$ u^7	Strains	$\gamma_{\bar{x}}$ u^3	$\gamma_{\bar{y}}$ u^5	$\gamma_{\bar{z}}$ u^3

Table 2.1: Summary of the order of magnitude estimates in terms of powers of u .

EL^2u^9 and lower are neglected. The displacement fields are interpolated by the simplest polynomial that is in agreement with the boundary conditions and coincidentally the solution of the fully linearized problem.

The load interpolation functions are then given by

$$\begin{aligned}
M_{\bar{x}}(s) &= \tilde{M}_x + \tilde{F}_z[\tilde{u}_y - u_y(s)] + u'_y(s)[\tilde{M}_y - \tilde{F}_z(L-s)] - I_{te}\tilde{F}_x\phi'_x(s), \\
M_{\bar{y}}(s) &= \tilde{M}_y - \tilde{F}_z(L-s), \\
M_{\bar{z}}(s) &= \tilde{M}_z + \tilde{F}_y(L-s) - \tilde{F}_x[\tilde{u}_y - u_y(s)] - \phi_x(s)[\tilde{M}_y - \tilde{F}_z(L-s)], \\
F_{\bar{x}}(s) &= \tilde{F}_x, \\
F_{\bar{y}}(s) &= 0, \\
F_{\bar{z}}(s) &= \tilde{F}_z.
\end{aligned} \tag{2.24}$$

Tilde superscripts denote the unknown coefficients. The leading terms in the curvature and strain expressions are

$$\begin{aligned}
\kappa_{\bar{x}} &= \phi'_x, \\
\kappa_{\bar{y}} &= \phi'_y + \phi_x\phi'_z \approx \phi'_y + \phi_x u''_y, \\
\kappa_{\bar{z}} &= \phi'_z \approx u''_y, \\
\gamma_{\bar{x}} &= u'_x - \phi_z(\phi_z/2 - u'_y) + I_{te}\phi'^2_x/2 \approx u'_x + u'^2_y/2 + I_{te}\phi'^2_x/2, \\
\gamma_{\bar{y}} &= 0, \\
\gamma_{\bar{z}} &= u'_z + \phi_y.
\end{aligned} \tag{2.25}$$

The displacement interpolation functions ($\xi = s/L$) are

$$\begin{aligned}
 \phi_x(\xi) &= \tilde{\phi}_x(3\xi^2 - 2\xi^3), \\
 \phi_y(\xi) &= \frac{1}{1+\Phi} \tilde{u}_z/L(-6\xi + 6\xi^2) + \frac{1}{1+\Phi} \tilde{\phi}_y(-2\xi + 3\xi^2 + \Phi\xi), \\
 u_x(\xi) &= \tilde{u}_x\xi, \\
 u_y(\xi) &= \tilde{u}_y(3\xi^2 - 2\xi^3) + L\tilde{\phi}_z(-\xi^2 + \xi^3), \\
 u_z(\xi) &= \frac{1}{1+\Phi} \tilde{u}_z(3\xi^2 - 2\xi^3 + \Phi\xi) + \frac{1}{2+2\Phi} L\tilde{\phi}_y(2\xi^2 - 2\xi^3 + \Phi\xi - \Phi\xi^2),
 \end{aligned} \tag{2.26}$$

where $\Phi = 12EI_y/(kGAL^2)$.

The load interpolation functions of Eq. (2.24) can be seen to take the deformed configuration of the beam into account. The loads that are associated with relatively low stiffness, i.e. the internal torsion moment $M_{\bar{x}}$ and bending moment $M_{\bar{z}}$, are dependent on the deformed shape of the sheet flexure as given by the displacement and rotation associated with relatively low stiffness, i.e. $u_y(s)$ and $\phi_x(s)$.

For an interpolation of the varying bimoment that is consistent with this approximation of the deformed sheet flexure configuration, we use the torsion ODE from Eq. (2.10) with the interpolation of the internal torsion moment $M_{\bar{x}}(s)$ from Eq. (2.24) in order to determine $\kappa_{\bar{x}}(s)$ given the constrained warping boundary conditions

$$\kappa_{\bar{x}}(0) = \kappa_{\bar{x}}(L) = 0. \tag{2.27}$$

In turn, the internal bimoment interpolation follows from the constitutive relation for $B(s)$, as given by Eq. (2.9). This way, the bimoment interpolation does not introduce additional unknown coefficients, which otherwise would need to be determined as well. The bimoment interpolation is a function of the same unknown coefficients as $M_{\bar{x}}(s)$, which it is based on, and of the dimensionless parameter

$$\lambda = \sqrt{\frac{GJL^2}{EI_w}} \approx \sqrt{\frac{24}{1+\nu}} \frac{L}{w}, \tag{2.28}$$

which can be regarded as a spatial decay rate for the warping constraint. The full expression is provided in Appendix 2.8.2.

The interpolation functions clearly reflect that we consider the sheet flexure as an elastic constraint element with three DOFs and three support directions, as a principal consequence of the first assumption of Sec. 2.2.4 that the width is the same order as the length. It can be seen that the strain and curvature approximations of Eq. (2.25) in the support directions, i.e. $\kappa_{\bar{y}}$ and $\gamma_{\bar{x}}$, have a second-order term, whereas the strain and curvature approximations associated with the DOFs, i.e. $\kappa_{\bar{x}}$ and $\kappa_{\bar{z}}$, only have the linear term. Conversely, the load interpolation of Eq. (2.24) in the support directions, i.e. $F_{\bar{x}}$, $M_{\bar{y}}$ and $F_{\bar{z}}$, only consists of the linear solution, whereas the load interpolation associated with the DOFs, i.e. $M_{\bar{x}}$ and $M_{\bar{z}}$, contains a coupling with the large displacements u_y and ϕ_x and the support direction loads $\tilde{F}_x$, $\tilde{M}_y$ and $\tilde{F}_z$ to take the deformed configuration into account.

2.2.6 Equation format

The interpolation functions of Eqs. (2.24) and (2.26) contain 6 unknown displacement (and rotation) coefficients and 6 unknown moment (and force) coefficients, whose equilibrium value can be obtained by application of the variational principle. Since the interpolation functions are based on physical insight, the notation of the superscript tilde serves to indicate that the coefficient is unknown while also showing what physical quantity the coefficient is based on. For instance, $\tilde{F}_y$ is the actual applied force in y -direction that does external work, while $\tilde{F}_y$ denotes a coefficient that is unknown and based on the physical force in y -direction. A consequence of the physical basis for the interpolation functions is that the moment interpolation functions depend on both the moment coefficients and the displacement coefficients, as per the deformed-configuration equilibrium of Eq. (2.20). It is this physically-motivated coupling that enables the relatively low-order polynomials to capture the relevant higher-order energy terms, including some fourth-order energy terms that lead to the third-order elastokinematic terms in the final model equations.

Substitution of Eqs. (2.24) and (2.26) in the potential energy expression Eq. (2.12) yields a polynomial scalar expression for the Hellinger–Reissner potential energy P , which can be used for further calculations. It is given in full in Appendix 2.8.3. In addition to the 12 unknown coefficients denoted by a superscript tilde, the six loads $\tilde{M}_x, \tilde{M}_y, \tilde{M}_z, \tilde{F}_x, \tilde{F}_y, \tilde{F}_z$ applied at the free end are present.

Since the integrand of P_{int} in Eq. (2.14) is at most quadratic in the moment fields and the moment interpolation functions of Eq. (2.24) are linear in the moment coefficients, it follows that P is at most quadratic in the moment coefficients. Taking variations with respect to the moment coefficients then yields a linear system of equations (representing the compatibility equations) in the unknown moment coefficients, meaning that they can be determined. In principle, this solution could be used to eliminate the moment coefficients from P and reduce it to a standard potential energy formulation expressed in terms of only the displacement coefficients. While that does reduce the number of equations and unknown coefficients, it yields an energy scalar that is a rational function instead of a polynomial. We have found that some calculations are considerably easier to carry out with the larger polynomial, especially when they involve differentiation-type operations, even when using analytical solver software. This is elaborated on in Sec. 2.5 on the analysis of assemblies of multiple sheet flexures.

The distinction between the high and low stiffness deformation modes of a sheet flexure is also manifested in the structure of polynomial P . Because the support direction loads $\tilde{F}_x, \tilde{M}_y$ and $\tilde{F}_z$ are coupled with the DOFs $\tilde{\phi}_x, \tilde{u}_y$ and $\tilde{\phi}_z$, and because the same DOFs appear in the second-order terms for the strains and curvatures, polynomial P contains fourth-order terms in these coefficients. Conversely, though, P is at most quadratic in the loads in the direction of the DOFs, i.e. $\tilde{F}_x, \tilde{M}_y$ and $\tilde{F}_z$, and the displacements and rotation in the support direction, i.e. $\tilde{u}_x, \tilde{\phi}_y$ and $\tilde{u}_z$.

As a consequence, the variations with respect to the six moment coefficients can be solved for the combination of $\tilde{M}_x, \tilde{F}_y, \tilde{M}_z, \tilde{u}_x, \tilde{\phi}_y$ and $\tilde{u}_z$ and used for eliminating $\tilde{M}_x, \tilde{F}_y, \tilde{M}_z$ from P , while retaining its polynomial structure. P then still depends on the 6

displacement coefficients, the 3 moment coefficients $\tilde{F}_x$, $\tilde{M}_y$ and $\tilde{F}_z$, in addition to the material parameters, geometry parameters and the 6 applied loads. This expression of P is used in the following sections.

It can be seen that the Hellinger–Reissner variational principle offers the advantage that the desired fourth-order energy terms consist of a mix of displacement and moment variables, which is at most quadratic in the displacement and the moment variables. It means that the process of taking variations then yields relatively simple equations that consist of at most third-order terms that are linear in either the displacement or the moment variables.

In line with the observation in Tbl. 2.1 that error motions $\tilde{u}_x$, $\tilde{\phi}_y$ and $\tilde{u}_z$ are an order smaller than DOFs $\tilde{\phi}_x$, $\tilde{u}_y$ and $\tilde{\phi}_z$ (the DOFs), we choose to express the model as functions of these three DOFs and the three loads in the remaining directions, further exploiting the distinction between the deformation modes of high and low stiffness. It means that the mathematical expressions of the model are a representation of the actuation loads $\tilde{M}_x$, $\tilde{F}_y$ and $\tilde{M}_z$, and the error motions $\tilde{u}_x$, $\tilde{\phi}_y$ and $\tilde{u}_z$.

The closed-form solution can be obtained by first taking variations with respect to $\tilde{F}_x$, $\tilde{M}_y$ and $\tilde{F}_z$ and solving for the three error motions $\tilde{u}_x$, $\tilde{\phi}_y$ and $\tilde{u}_z$. Second, the variations of P with respect to the three error motions can be used to eliminate $\tilde{M}_x$, $\tilde{F}_y$ and $\tilde{M}_z$. Third, the moments dual to the Tait-Bryan angles can be transformed to moment vector components, according to Eq. (2.15). Fourth, the variations with respect to the DOFs $\tilde{\phi}_x$, $\tilde{u}_y$, $\tilde{\phi}_z$ can be solved for the actuation loads $\tilde{M}_x$, $\tilde{F}_y$ and $\tilde{M}_z$.

2.3 RESULTS

The expressions that constitute the sheet flexure model are shown in a mixed compliance–stiffness matrix format. The three load–displacement relations are given by

$$\begin{aligned} \tilde{M}_x = c_1 \frac{GJ}{L} \tilde{\phi}_x &+ \tilde{M}_y [c_3 \quad c_4] \begin{bmatrix} \tilde{u}_y/L \\ \tilde{\phi}_z \end{bmatrix} + \tilde{F}_z [c_5 \quad c_6] \begin{bmatrix} \tilde{u}_y/L \\ \tilde{\phi}_z \end{bmatrix} \\ &+ c_2 \tilde{F}_x \frac{I_{te}}{L} \tilde{\phi}_x, \end{aligned} \quad (2.29)$$

$$\begin{aligned} \tilde{F}_y = \frac{EI_z}{L^2} [12 \quad -6] \begin{bmatrix} \tilde{u}_y/L \\ \tilde{\phi}_z \end{bmatrix} &+ \tilde{F}_x [6/5 \quad -1/10] \begin{bmatrix} \tilde{u}_y/L \\ \tilde{\phi}_z \end{bmatrix} + c_3 \tilde{M}_y \tilde{\phi}_x/L \\ &+ c_5 \tilde{F}_z \tilde{\phi}_x, \end{aligned} \quad (2.30)$$

$$\begin{aligned} \tilde{M}_z = \frac{EI_z}{L} [-6 \quad 4] \begin{bmatrix} \tilde{u}_y/L \\ \tilde{\phi}_z \end{bmatrix} &+ \tilde{F}_x L [-1/10 \quad 2/15] \begin{bmatrix} \tilde{u}_y/L \\ \tilde{\phi}_z \end{bmatrix} \\ &+ (1 + c_4) \tilde{M}_y \tilde{\phi}_x + c_6 \tilde{F}_z L \tilde{\phi}_x. \end{aligned} \quad (2.31)$$

These expressions describe how the actuation loads depend on the applied loads and the DOFs. Besides the common terms from linear analysis, there are terms that depend on both the DOF and the applied load. As in the 2-D case, these capture load-stiffening (and load-softening) behavior [12]. The solution of the variational problem also yields terms in Eqs. (2.29)–(2.31) that are quadratic in the applied loads; these have been

discarded at only a small error for parameter values in accordance with the earlier estimates. The coefficients c_i are functions of only the warping parameter λ and given in Appendix 2.8.4.

The error motion relations are given by

$$\begin{aligned} \tilde{u}_x/L &= \bar{F}_x \frac{1}{EA} \\ &+ [\tilde{\phi}_z \quad \tilde{u}_y/L] \begin{bmatrix} -1/15 & 1/20 \\ 1/20 & -3/5 \end{bmatrix} \begin{bmatrix} \tilde{\phi}_z \\ \tilde{u}_y/L \end{bmatrix} + c_7 I_{te} \tilde{\phi}_x^2/L^2 \\ &+ \bar{F}_x \frac{L^2}{EI_z} [\tilde{\phi}_z \quad \tilde{u}_y/L] \begin{bmatrix} 11/6300 & -1/1400 \\ -1/1400 & 1/700 \end{bmatrix} \begin{bmatrix} \tilde{\phi}_z \\ \tilde{u}_y/L \end{bmatrix} + \bar{F}_x c_{11} \frac{I_{te}^2}{GJL^2} \tilde{\phi}_x^2 \quad (2.32) \\ &+ \bar{M}_y \frac{L}{EI_z} \tilde{\phi}_x \left(-\frac{\tilde{u}_y}{700L} + \frac{\tilde{\phi}_z}{1400} \right) + \bar{M}_y \frac{I_{te}}{GJL} \tilde{\phi}_x (-c_{13} \tilde{\phi}_z - c_{11} \tilde{u}_y/L) \\ &+ \bar{F}_z \frac{L^2}{EI_z} \tilde{\phi}_x \left(\frac{\tilde{u}_y}{1400L} - \frac{\tilde{\phi}_z}{300} \right) + \bar{F}_z \frac{I_{te}}{GJ} \tilde{\phi}_x (c_{18} \tilde{\phi}_z - c_{13} \tilde{u}_y/L), \end{aligned}$$

$$\begin{aligned} \tilde{\phi}_y &= -\frac{1}{2} \bar{F}_z \frac{L^2}{EI_y} + \bar{M}_y \frac{L}{EI_y} \\ &+ (c_9 - 1) \tilde{\phi}_x \tilde{\phi}_z + c_8 \tilde{\phi}_x \tilde{u}_y/L \\ &+ \bar{F}_x \frac{L^2}{EI_z} \tilde{\phi}_x \left(-\frac{\tilde{u}_y}{700L} + \frac{\tilde{\phi}_z}{1400} \right) + \bar{F}_x \frac{I_{te}}{GJ} \tilde{\phi}_x (-c_{13} \tilde{\phi}_z - c_{11} \tilde{u}_y/L) \quad (2.33) \\ &+ \bar{M}_y \frac{L}{GJ} [\tilde{\phi}_z \quad \tilde{u}_y/L] \begin{bmatrix} c_{12} & c_{13} \\ c_{13} & c_{14} \end{bmatrix} \begin{bmatrix} \tilde{\phi}_z \\ \tilde{u}_y/L \end{bmatrix} + \bar{M}_y \frac{L}{EI_z} \frac{1}{700} \tilde{\phi}_x^2 \\ &+ \bar{F}_z \frac{L^2}{GJ} [\tilde{\phi}_z \quad \tilde{u}_y/L] \begin{bmatrix} c_{19} & c_{20} \\ c_{20} & c_{13} \end{bmatrix} \begin{bmatrix} \tilde{\phi}_z \\ \tilde{u}_y/L \end{bmatrix} - \bar{F}_z \frac{L^2}{EI_z} \frac{1}{1400} \tilde{\phi}_x^2, \end{aligned}$$

$$\begin{aligned} \tilde{u}_z/L &= \frac{1}{3} \bar{F}_z \frac{L^2}{EI_y} - \frac{1}{2} \bar{M}_y \frac{L}{EI_y} + \bar{F}_z \frac{1}{k_z GA} \\ &+ c_{10} \tilde{\phi}_x \tilde{\phi}_z + c_9 \tilde{\phi}_x \tilde{u}_y/L \\ &+ \bar{F}_x \frac{I_{te}}{GJ} \tilde{\phi}_x (c_{18} \tilde{\phi}_z - c_{13} \tilde{u}_y/L) + \bar{F}_x \frac{L^2}{EI_z} \tilde{\phi}_x \left(\frac{\tilde{u}_y}{1400L} - \frac{\tilde{\phi}_z}{300} \right) \quad (2.34) \\ &+ \bar{M}_y \frac{L}{GJ} [\tilde{\phi}_z \quad \tilde{u}_y/L] \begin{bmatrix} c_{19} & c_{20} \\ c_{20} & c_{13} \end{bmatrix} \begin{bmatrix} \tilde{\phi}_z \\ \tilde{u}_y/L \end{bmatrix} - \bar{M}_y \frac{L}{EI_z} \frac{1}{1400} \tilde{\phi}_x^2 \\ &+ \bar{F}_z \frac{L^2}{GJ} [\tilde{\phi}_z \quad \tilde{u}_y/L] \begin{bmatrix} c_{15} & c_{16} \\ c_{16} & c_{17} \end{bmatrix} \begin{bmatrix} \tilde{\phi}_z \\ \tilde{u}_y/L \end{bmatrix} + \bar{F}_z \frac{L^2}{EI_z} \frac{43}{6300} \tilde{\phi}_x^2. \end{aligned}$$

These expressions describe how the error motions in the constraint directions depend on the applied loads and the DOFs. For the purpose of interpretation and verification

of the model, Eqs. (2.29)–(2.34) are summarized as

$$\begin{bmatrix} \bar{M}_x \\ \bar{F}_y \\ \bar{M}_z \end{bmatrix} = [\mathbf{K}_1 + \mathbf{K}_2(\bar{F}_x, \bar{M}_y, \bar{F}_z)] \begin{bmatrix} \tilde{\phi}_x \\ \tilde{u}_y \\ \tilde{\phi}_z \end{bmatrix}, \quad (2.35)$$

$$\begin{bmatrix} \tilde{u}_x \\ \tilde{\phi}_y \\ \tilde{u}_z \end{bmatrix} = [\mathbf{C}_1 + \mathbf{C}_2(\tilde{\phi}_x, \tilde{u}_y, \tilde{\phi}_z)] \begin{bmatrix} \bar{F}_x \\ \bar{M}_y \\ \bar{F}_z \end{bmatrix} + \mathbf{p}_1(\tilde{\phi}_x, \tilde{u}_y, \tilde{\phi}_z) \quad (2.36)$$

to highlight the structure. Matrices $\mathbf{K}_1$ and $\mathbf{C}_1$ account for the common linear stiffness and compliance contributions. $\mathbf{K}_2$ is only linearly dependent on the applied loads and accounts for changes in the actuation stiffness due to applied loads, sometimes referred to as geometric load-stiffening and load-softening effects. Quadratic vector-function $\mathbf{p}_1$ captures a purely kinematic relation between the error motions and the DOFs; it describes an error motion incurred from motion in the DOFs. Matrix $\mathbf{C}_2$ is still a (quadratic) function of the DOFs and governs the decrease in support stiffness that accompanies motion in the DOFs. It is referred to as an elastokinematic effect in literature [12, 70] and tends to limit the performance of large range of motion precision mechanisms consisting of sheet flexures. The terms associated with matrix $\mathbf{C}_2$ are *third-order* terms in the model variables, i.e. the DOFs and the applied loads. Matrix $\mathbf{C}_2$ can be considered a principal result of the current analysis: while the other terms in Eqs. (2.35)–(2.36) could be found with a less detailed analysis, $\mathbf{C}_2$ is only obtained when the fourth-order energy terms are considered. For this reason, the current model is referred to as a third-order stiffness formulation.

A consequence of the structure of Eqs. (2.35) and (2.36) is that the actuation stiffness is modeled as a quantity independent of the DOFs and the support stiffness independent of the applied loads.

2.4 VERIFICATION

2.4.1 Prior art

For a verification of the model described by Eqs. (2.29)–(2.34), we first note the relation with prior art. The planar beam constraint model considers a 2-D sheet flexure, which has DOFs $\tilde{\phi}_z$ and $\tilde{u}_y$, and applied load $\bar{F}_x$ [12]. The planar model constitutes two expressions for the actuation loads $\bar{F}_y$ and $\bar{M}_z$, and one expression for the error motion $\tilde{u}_x$. All of the terms in those expressions can be identified within the current spatial model. The same terms can be found in the expressions derived for 2-D sheet flexures that are subjected to axial loads large with respect to the critical load [60]. Also, parts of the expressions for $\tilde{\phi}_y$ and $\tilde{u}_z$ can be found in the earlier work on the error motions of 3-D sheet flexures that are limited in twist angle [54], when the effects of constrained warping present in the current model are disregarded. Bai et al. [13] have developed a 3-D beam model that assumes bending curvatures $\kappa_{\bar{y}}$ and $\kappa_{\bar{z}}$ of equal magnitude, whereas the assumptions in Sec. 2.2.4 of the current work imply that $\kappa_{\bar{z}} \gg \kappa_{\bar{y}}$ for

what we consider a typical sheet flexure constraint element. As a consequence, the model of Bai et al. does not discriminate between a bending plane for high and for low stiffness; $\tilde{u}_z$ is not necessarily a constraint direction and $\tilde{\phi}_y$ and $\tilde{u}_z$ are not necessarily error motions; their model therefore captures e.g. the effect of large $\tilde{u}_z$ and $\tilde{\phi}_y$ on $\tilde{u}_x$ as parasitic motion, unlike the current work. The essential difference between the two models lies in the allowable values for w/L . For the model of Bai et al., w/L should remain small since torsion-elongation (Wagner) coupling, constrained warping and shear deformation have not been included and all become significant when $w \rightarrow L$. These effects are considered in the current work.

2.4.2 Verification procedure

To facilitate the verification of the model, which could be regarded as a 6×6 system of nonlinear stiffness relations with 21 unique components, the equation structure represented by Eqs. (2.35)–(2.36) can be utilized: by first establishing the validity of the structure for the parameter range of interest, only the accuracy of the function $\mathbf{p}_1$ and the unique components of $\mathbf{K}_2$ and $\mathbf{C}_2$ needs to be verified.

For the comparison with finite element analysis, a model in ANSYS Mechanical (with two-node BEAM188 and four-node SHELL181 elements) and SPACAR (with two-node BEAMW elements) is used with all nonlinear analysis options enabled [34, 36, 37]. The parameter values are based on the order of magnitude estimates of Sec. 2.2.4 for a sheet flexure with a length of 100 mm. Tbl. 2.2 lists the corresponding dimensions, material properties and maximum values for the applied loads and DOFs. The equilibrium configuration in both programs is determined for various combinations of the applied loads and DOFs. By means of an additional perturbation analysis, the stiffness matrix at the free end in the deformed configuration can be calculated and cast into the mixed format

$$\begin{bmatrix} \Delta \tilde{M}_x \\ \Delta \tilde{F}_y \\ \Delta \tilde{M}_z \\ \Delta \tilde{u}_x \\ \Delta \tilde{\phi}_y \\ \Delta \tilde{u}_z \end{bmatrix} = \mathbf{B} \begin{bmatrix} \Delta \tilde{F}_x \\ \Delta \tilde{M}_y \\ \Delta \tilde{F}_z \\ \Delta \tilde{\phi}_x \\ \Delta \tilde{u}_y \\ \Delta \tilde{\phi}_z \end{bmatrix} \quad (2.37)$$

so that the numerical entries of matrix $\mathbf{B}$ can be compared with the components of $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{C}_1$ and $\mathbf{C}_2$.

2.4.3 Equation structure validity

To test the linear relationship between the actuation loads and the DOFs that Eq. (2.35) portrays, the ANSYS model with plate elements is loaded by the maximum values from Sec. 2.2.4 for all applied loads simultaneously. The three actuation loads are observed as the three DOFs are incremented individually to their maximum values. It is found that the maximum relative error with respect to a least squares fit of a linear function is at most 0.7% for eight of the nine cases. The only significant deviation is observed for

the torsion moment $\bar{M}_x$ as a function of the torsion angle $\tilde{\phi}_x$. With increasing $\tilde{\phi}_x$, the Wagner torque [78],

$$\frac{1}{360} Ew^5 t (\phi'_x)^3, \quad (2.38)$$

which is a nonlinear function of the specific twist angle acting as a stiffening effect, starts to dominate the response: the deviation is 10% for 0.05 radians and 34% for 0.1 radians. Note that this Wagner torque is different from the Wagner term (the torsion–elongation coupling), which is taken into account by $\gamma_{\bar{x}}$ in Sec. 2.2.1. The Wagner torque depends strongly on the sheet flexure width and is not captured by the used beam-based finite elements (which predict a linear response that is in agreement with the current analytical model). The torque could readily be incorporated into the model expression for $\bar{M}_x$ by accounting for the associated strain energy per unit length term [48]

$$\frac{1}{1440} Ew^5 t \kappa_{\bar{x}}^4 \quad (2.39)$$

in P_{int} in Eq. (2.14). The additional term in $\bar{M}_x$ is then found to be

$$\frac{1}{175} Ew^5 t \tilde{\phi}_x^3 / L^3. \quad (2.40)$$

Its effect on the error motions is considerably more involved because of the interplay with the nonuniform torsion (warping) in the deformed configuration. The additional term of Eq. (2.38) renders the differential equation Eq. (2.10) nonlinear. It is unlikely that the increased complexity of the bimoment or torsion moment interpolation that the inclusion in the variational framework would require can be captured in the compact format of the current formulation. Additionally, since the effect is clearly identifiable, and much less pronounced in sheet flexures with smaller width-to-length ratios, omission in the model is deemed acceptable. With a numerical element implementation of this formulation, the effect of the Wagner torque on the error motions can be captured by simply accounting for Eq. (2.40) in the element description and using multiple elements to model a sheet flexure.

2.4.4 Model accuracy

Since the presented model is an approximation based on beam theory, errors are incurred in two stages. These can be identified by comparing against

- beam finite elements, to assess the accuracy of the approximation (consisting of interpolation and truncation),
- plate (shell) finite elements, to expose relevant effects outside beam theory and assess the validity of the use of beam theory.

Due to the large width-to-length ratio of some sheet flexures, beam theory may not be able to capture all relevant effects. Besides the mentioned Wagner torque, two

L [mm]	w [mm]	t [mm]	E [GPa]	ν [-]	
100	100	2.25	200	0.3	
$\bar{F}_x$ [kN]	$\bar{M}_y$ [Nm]	$\bar{F}_z$ [kN]	$\tilde{\phi}_x$ [-]	$\tilde{u}_y$ [mm]	$\tilde{\phi}_z$ [-]
3.42	342	3.42	0.15	15	0.15

Table 2.2: Verification parameters: dimensions, material properties, and maximum values for the applied loads and DOFs.

Figure 2.4: Deviation of the actuation load derivatives with respect to nonlinear ANSYS shell elements.

facets of anticlastic curvature play a role. Primarily, the anticlastic curvature of the cross-section manifests itself as a bending stiffness that increases with bending deformation, dependent on the ratio $w^2/(Rt)$, where R is the bending radius [7]. In the limit case of plane strain, the bending stiffness increases by a factor of $1/(1 - \nu^2)$. For sheet flexures, the common physical embodiment of the interfaces at the two ends (e.g. clamping blocks) prevents the local anticlastic curvature, leading to an increase in actuation stiffness $\mathbf{K}_1$ that is present even for infinitesimal deformation and dependent on the ratio w/L [47].

For the verification of the symmetric matrix $\mathbf{K}_1 + \mathbf{K}_2$, the deviation of the six unique components with respect to the shell elements in ANSYS is depicted in Fig. 2.4 for increasing values of the three applied loads (and zero values of the DOFs). It can be seen that the deviation of the derivatives with respect to $\bar{F}_y$ and $\bar{M}_z$ is at most 7.8%, whereas the deviation of the derivatives with respect to $\bar{M}_x$ is at most 5.1%. This difference is already present for zero applied load and can be attributed to the effect of constrained anticlastic curvature: the deviation reduces to only 1.6% at most when the model is compared to the geometrically nonlinear beam element in ANSYS and SPACAR, which does not model anticlastic curvature but does capture the load-stiffening effect of $\mathbf{K}_2$.

Figure 2.5: Deviation of the error motion derivatives with respect to nonlinear ANSYS shell elements. Small inset figures in the graphs represent the load cases visually.

In case the effect of anticlastic curvature is of interest, it could be taken into account by means a modification of $\mathbf{K}_1$ according to the procedure of Meijaard [47].

For the verification of the symmetric matrix $\mathbf{C}_1 + \mathbf{C}_2$, we first note that $\mathbf{C}_2$ is a quadratic function of the three DOFs. Out of the various possible combinations of the DOFs, some interesting cases are

1. a nonzero $\tilde{\phi}_x$ only,
2. nonzero $\tilde{u}_y$ and $\tilde{\phi}_z$ only (the planar case),
3. a combination of $\tilde{\phi}_x$ and nonzero $\tilde{u}_y$ or $\tilde{\phi}_z$, and
4. combined nonzero values for all three DOFs.

Figure 2.6: Deviation of the error motions with respect to nonlinear ANSYS shell elements. Small inset figures in the graphs represent the load cases visually.

Fig. 2.5 shows the deviation of the six unique components for these four cases with respect to shell elements. When only $\tilde{\phi}_x \neq 0$ (Fig. 2.5(a)), the maximum deviation is 6.0% and occurs at the maximum value of the twist angle $\tilde{\phi}_x$. In the planar case, Fig. 2.5(b), the maximum deviation is 3.7%. In Fig. 2.5(c), when $\tilde{\phi}_z$ is fixed to a nonzero value, it is observed that the deviation of the derivative $\partial \tilde{u}_x / \partial \tilde{M}_y$ increases strongly with $\tilde{\phi}_x$ starting from a value of 13%. A similar trend is observed in the derivatives $\partial \tilde{\phi}_y / \partial \tilde{M}_y$ and $\partial \tilde{\phi}_y / \partial \tilde{F}_z$, although the deviation remains limited to 18% at the maximum twist angle. This is explained by the torsion moment needed in this case, which shows that the Wagner torque is dominant for the larger $\tilde{\phi}_x$. When additionally $\tilde{u}_y = 15 \text{ mm}$ (Fig. 2.5(d)), the qualitative behavior is the same; up to $\tilde{\phi}_x = 0.1$ radians, the maximum deviation is 12%, except for $\partial \tilde{u}_x / \partial \tilde{M}_y$. To exclude the effect of the Wagner torque, a comparison with beam elements shows that for the four cases the maximum deviation is only 3.7% for all derivatives, except $\partial \tilde{u}_x / \partial \tilde{M}_y$ and $\partial \tilde{u}_x / \partial \tilde{F}_z$: those two have a maximum deviation of 7.8%. Also, when shell elements are used and the width of the sheet flexure is reduced to $3L/4$, the deviations of $\partial \tilde{u}_x / \partial \tilde{M}_y$ for the third and fourth case are already 48% and 22% smaller, respectively.

For the verification of the quadratic vector-valued function $\mathbf{p}_1$, the deviation of the error motions themselves with respect to ANSYS shell elements is depicted in Fig. 2.6 for various combinations of the DOFs. For the two cases of Figs. 2.6(a) and 2.6(c), the maximum deviation is only 1.9%. When $\tilde{u}_y$ is incremented at a constant $\tilde{\phi}_x = 0.15$ (as shown in Fig. 2.6(b)), the maximum deviation is 14%. When compared to beam elements, the maximum deviation for all cases is only 0.1%.

The verification of the individual components of $\mathbf{K}_2$ and $\mathbf{C}_2$ for various load cases is sensitive and exposes deviations easily. The close match with the solution obtained with numerical beam elements indicates that the geometrically nonlinear effects within

Figure 2.7: Parallelogram flexure mechanism; loads are applied in the center of compliance.

beam theory are described accurately by the model. From the comparison with shell elements, we conclude that the beam-based analytical model performs well, when the Wagner torque and, to a lesser extent, the anticlastic curvature can be disregarded. This is the case when mainly the DOFs $\tilde{u}_y$ and $\tilde{\phi}_z$ are used and the torsion angle $\tilde{\phi}_x$ remains limited. It helps that the Wagner torque actually leads to a higher support stiffness or at least a smaller decrease in support stiffness with DOF motion. This means that the current model only underestimates the support stiffness at the larger torsion angles, especially since common precision applications demand a minimum value of support stiffness. In case the torsion angle is large and the quantitative effect on the torsion moment or the support stiffness is desired, it should be modeled numerically.

2.5 PARALLEL ASSEMBLIES OF SHEET FLEXURES

The variational principle used for deriving the spatial sheet flexure model can readily be applied to assemblies of sheet flexures, especially when all sheet flexures are connected in parallel by a rigid body. The procedure is to formulate the total potential energy of the system, take into account the kinematic constraints, take variations with respect to the unknowns, and solve the system for the parameters of interest.

This section demonstrates the procedure by using the presented sheet flexure formulation for obtaining a closed-form model for a parallelogram flexure mechanism and a cross-hinge flexure mechanism, which both consist of two identical sheet flexures, as shown in Figs. 2.7 and 2.8. The total potential energy in both cases is given by

$$P^{\text{tot}} = P^l + P^r - W^e, \quad (2.41)$$

where the superscripts l and r denote the left and right sheet flexure, respectively. The external work W^e is due to loads that are applied in the center of compliance of the mechanism, and given by

$$\bar{M}_x^{\text{TB}} \phi_x^c + \bar{M}_y^{\text{TB}} \phi_y^c + \bar{M}_z^{\text{TB}} \phi_z^c + \bar{F}_x u_x^c + \bar{F}_y u_y^c + \bar{F}_z u_z^c. \quad (2.42)$$

The superscript c is used to denote displacements and rotations of the compliance center, resolved in the global frame. In principle, P denotes the expression of the potential energy of the sheet flexure that is given by P_{int} from Eq. (2.14). The scalar polynomial presented in Sec. 2.2.6 and Eq. (2.76), without the external work due to the applied forces, is used. By construction, P is dependent on the six rotation and displacement parameters (as in a conventional displacement-based potential energy formulation) and, as a consequence of the Hellinger–Reissner formulation, three additional internal force and moment parameters for each sheet flexure, which need to be solved for and ultimately eliminated.

The shuttle imposes kinematic constraints on the two sheet flexure ends, as derived in Appendix 2.8.5. In terms of the coordinates of the compliance center, the translational kinematic constraints are given by

$$\begin{bmatrix} -h_x \\ d_y/2 \\ h_z \end{bmatrix} + \begin{bmatrix} u_x^c \\ u_y^c \\ u_z^c \end{bmatrix} - \mathbf{R}^s \begin{bmatrix} -h_x \\ d_y/2 \\ h_z \end{bmatrix} - \mathbf{R}^i \begin{bmatrix} u_x^l \\ u_y^l \\ u_z^l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad (2.43)$$

$$\begin{bmatrix} -h_x \\ -d_y/2 \\ -h_z \end{bmatrix} + \begin{bmatrix} u_x^c \\ u_y^c \\ u_z^c \end{bmatrix} - \mathbf{R}^s \begin{bmatrix} -h_x \\ -d_y/2 \\ -h_z \end{bmatrix} - (\mathbf{R}^i)^T \begin{bmatrix} u_x^r \\ u_y^r \\ u_z^r \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (2.44)$$

Parameter d_y denotes the shuttle length, and h_x and h_z the location of the center of compliance (as detailed in Appendix 2.8.5). Rotation matrix $\mathbf{R}^i$ rotates the global frame about the z -axis to the frame attached to the undeformed sheet flexure. It is given by

$$\begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2.45)$$

and reduces to the identity matrix in the case of the parallelogram flexure mechanism, for which $\alpha = 0$. The rotational kinematic constraints due to the shuttle, expressed in terms of the coordinates of the compliance center, are given by

$$\mathbf{R}^s = \mathbf{R}^i \mathbf{R}^l (\mathbf{R}^i)^T \quad \text{and} \quad \mathbf{R}^s = (\mathbf{R}^i)^T \mathbf{R}^r \mathbf{R}^i. \quad (2.46)$$

Rotation matrices $\mathbf{R}^s$, $\mathbf{R}^l$ and $\mathbf{R}^r$ are given by $\mathbf{R}(s)$ from Eq. (2.3). $\mathbf{R}^s$ is parametrized by ϕ_x^c , ϕ_y^c and ϕ_z^c . Eqs. (2.43)–(2.46) can be solved for the displacements and rotations at the two sheet flexure ends. These relations can then be used to eliminate the sheet flexure displacements and rotations from P^{tot} , so that P^{tot} only depends on the six coordinates of the shuttle and the three additional Hellinger–Reissner internal load parameters per sheet flexure. For simplicity, the torsion–elongation coupling is ignored in this section.

Application of the Hellinger–Reissner principle to P^{tot} yields 12 algebraic equations for the 12 unknown coefficients. The six loads acting in the center of compliance, i.e. $\bar{F}_x$, $\bar{F}_y$, $\bar{F}_z$, $\bar{M}_x^{\text{TB}}$, $\bar{M}_y^{\text{TB}}$, $\bar{M}_z^{\text{TB}}$, are additional parameters in this system of equations. At this stage, the transformation of the moments, detailed in Sec. 2.2.3, can be performed.

In line with the equation format for the sheet flexure, we choose to solve this system in terms of the DOFs and the loads applied in the five constraint directions. An adequate asymptotic solution can be obtained as a truncated series, since the exact solution of this system of equation requires considerable mathematical labor (even with analytical solver software), the added value of an exact solution is limited in the context of approximate theories such as the present beam theory, and the truncation of the series expansion can be physically motivated. By means of successive implicit differentiation of the 12 algebraic governing equations, recursive equations linear in the derivatives can be obtained for an arbitrary order and used to construct a series approximation. Since we choose to format the equations of the model such that smaller quantities (the error motions and actuation loads) are expressed as functions of larger quantities (the applied loads and, in particular, the DOFs), convergence of the series is less unlikely.

2.5.1 Parallelogram mechanism

For the parallelogram flexure mechanism, angle $\alpha = 0$, shuttle length $d_y = d$, and the location of the compliance center is given by $h_x = L/2$ and $h_z = 0$. The single displacement u_y^c is the DOF. By carrying out the implicit differentiation to the third order, simplifying terms by discarding EI_z and GJ with respect to $kGAL^2$ and EAL^2 , neglecting third-order terms in a single variable, and constructing the series, the expression for the actuation force is obtained as

$$\bar{F}_y = u_y^c \left(\frac{24EI_z}{L^3} + \frac{6}{5L} \bar{F}_x \right). \quad (2.47)$$

The error motions in the five constraint directions are obtained as

$$u_x^c/L = \bar{F}_x \left(\frac{1}{2EA} + \frac{1}{1400EI_z} (u_y^c)^2 \right) - \frac{3}{5L^2} (u_y^c)^2, \quad (2.48)$$

$$u_z^c/L = \bar{F}_z \left(\frac{1}{2kGA} + \frac{L^2}{24EI_y} + \frac{p_1}{56GJ} (u_y^c)^2 \right), \quad (2.49)$$

$$\phi_x^c = \bar{M}_x \frac{L}{d^2} \left(\frac{2}{kGA} + \frac{L^2}{6EI_y} + \frac{p_1}{14GJ} (u_y^c)^2 \right), \quad (2.50)$$

$$\phi_y^c = \bar{M}_y/L \left(\frac{L^2}{2EI_y} + \frac{p_2}{10GJ} (u_y^c)^2 \right), \quad (2.51)$$

$$\phi_z^c = \bar{M}_z \frac{L}{d^2} \left(\frac{2}{EA} + \frac{1}{350EI_z} (u_y^c)^2 \right), \quad (2.52)$$

where coefficients p_1 and p_2 are related to the warping properties of the sheet flexures. The coefficients are trigonometric functions of λ only and provided in full in Appendix 2.8.6. Convenient approximations are given by

$$p_1 = -0.0845 + 0.0493\lambda - 0.00114\lambda^2 + 0.0000101\lambda^3, \quad (2.53)$$

$$p_2 = -0.134 + 0.0627\lambda - 0.00160\lambda^2 + 0.0000151\lambda^3, \quad (2.54)$$

with at most a 2.4% deviation for $4.3 \leq \lambda \leq 43$, which corresponds with $1 \leq L/w \leq 10$ when $\nu = 0.3$.

The error motions u_z^c , ϕ_x^c and ϕ_y^c are the result of the spatial character of this analysis. The expressions have the common constant stiffness term. In addition, they also have a contribution due to torsion, quadratically dependent on the degree of freedom u_y^c , and referred to as an elastokinematic effect. It is this term that describes the observed increase in error motion, when the mechanism deflects in its DOF under a constant applied $\bar{F}_z$, $\bar{M}_x$ or $\bar{M}_y$ load. As a corollary, this term quantifies how the support stiffness $d\bar{F}_z/du_z^c$, $d\bar{M}_x/d\phi_x^c$ and $d\bar{M}_y/d\phi_y^c$ decreases over the range of motion of the mechanism.

For a verification by means of literature, we note that Brouwer et al. [16] have also derived the expressions (including constrained warping) for ϕ_y^c and u_z^c , though by means of a derivation specific to only the parallelogram flexure mechanism (unlike the current general derivation). They have validated the expressions extensively against FEM with a maximum deviation of 6.0% for $u_y^c = 0.1L$.

Error motion u_x^c exhibits a structure similar to u_z^c and ϕ_y^c , but also includes a term independent of load $\bar{F}_x$; it is a purely kinematic contribution to displacement u_x^c , also referred to as the shortening effect [74]. Actuation load $\bar{F}_y$ has a term linear in $\bar{F}_x$, which describes the change in actuation stiffness due to applied load $\bar{F}_x$. The expressions for u_x^c , ϕ_z^c and $\bar{F}_y$ are not dependent on the out-of-plane parameters $\bar{M}_x$, $\bar{F}_z$ or $\bar{M}_y$. The fact that the current spatial analysis has resulted in these expressions implies that spatial effects have no significant contribution to u_x^c , ϕ_z^c and $\bar{F}_y$ and that a planar analysis would suffice. In fact, these terms have been derived before in earlier work on the planar analysis of the parallelogram flexure mechanism [12, 61].

For a verification by means of a numerical simulation, a SPACAR model of the parallelogram mechanism with BEAMW elements is used. Since these elements are beam-based, missing effects due to non-beam-like behavior are not revealed. These have been elaborated on in Sec. 2.4. This verification only serves to demonstrate the accuracy of the approximation of the geometrically nonlinear effects within beam theory. Similar to Eq. (2.37) for the single sheet flexure, the numerical compliance matrix in the deformed configuration is cast into the mixed format

$$\begin{bmatrix} \Delta \bar{F}_y \\ \Delta u_x^c \\ \Delta u_z^c \\ \Delta \phi_x^c \\ \Delta \phi_y^c \\ \Delta \phi_z^c \end{bmatrix} = \mathbf{B} \begin{bmatrix} \Delta u_y^c \\ \Delta \bar{F}_x \\ \Delta \bar{F}_z \\ \Delta \bar{M}_x \\ \Delta \bar{M}_y \\ \Delta \bar{M}_z \end{bmatrix}, \quad (2.55)$$

consistent with the format of the model equations in Eqs. (2.47)–(2.52). The equations provide the diagonal components of $\mathbf{B}$, which are dependent on the single DOF u_y^c . For the sheet flexure dimensions and material according to Tbl. 2.2 and a shuttle length of 100 mm (equal to the sheet flexure length), the largest deviation between the SPACAR model and the analytical solution is 5.9% when $u_y^c = 0.15L$. When the width is reduced and instead $L = 100$ mm, $w = 25$ mm, $t = 0.3$ mm and $u_y^c = 0.1L$, the largest deviation

Figure 2.8: Cross-hinge flexure mechanism; loads are applied in the center of compliance. Force $\bar{F}_z$ and moment $\bar{M}_z$ along the positive z -axis are not shown.

is 2.4%. Even though L/w is now larger than one, the effect of constrained warping is still considerable: when free warping end-conditions are used in the SPACAR model, compared to constrained warping conditions in the analytical model, the deviation in the components $B_{3,3}$, $B_{4,4}$, and $B_{5,5}$ is still at least 41%.

2.5.2 Cross-hinge mechanism

For the cross-hinge mechanism of Fig. 2.8, we set the shuttle length d_y to

$$2L(1 - b/L) \sin \alpha \quad (2.56)$$

in order to introduce the dimensionless crossing length b/L as a parameter. The location of the compliance center is given by

$$h_x = L(1 - b/L) \cos \alpha, \quad h_z = w/2. \quad (2.57)$$

The single rotation ϕ_z^c of the compliance center is the DOF. Compared to the parallelogram mechanism, the mathematical labor for obtaining an asymptotic solution to the third order is considerably more involved because of the additional geometric parameters α and b/L . Unless the analytical sensitivities with respect to α and b/L are desired, it is recommended to choose values for α and b/L before carrying out the implicit differentiation.

For a doubly symmetric cross-hinge mechanism with $\alpha = 45^\circ$ and $b = L/2$, the process of implicit differentiation, series construction, and simplification of small terms yields the expressions for the actuation load

$$\bar{M}_z = \phi_z^c \left(\frac{2EI_z}{L} - \frac{L}{3\sqrt{2}} \bar{F}_x \right), \quad (2.58)$$

the planar error motions

$$u_x^c/L = \bar{F}_x \left(\frac{1}{EA} + \frac{h_z^2}{EI_y} + \frac{L^2}{720EI_z} (\phi_z^c)^2 + \frac{h_z^2 p_3}{12GJ} (\phi_z^c)^2 \right) + \bar{M}_x \left(\frac{h_z}{EI_y} + \frac{h_z p_3}{12GJ} (\phi_z^c)^2 \right) + \frac{1}{6\sqrt{2}} (\phi_z^c)^2, \quad (2.59)$$

$$u_y^c/L = \bar{F}_y \left(\frac{1}{EA} + \frac{h_z^2}{EI_y} + \frac{L^2}{720EI_z} (\phi_z^c)^2 + \frac{h_z^2 p_3}{12GJ} (\phi_z^c)^2 \right) - \bar{M}_y \left(\frac{h_z}{EI_y} + \frac{h_z p_3}{12GJ} (\phi_z^c)^2 \right), \quad (2.60)$$

and the spatial error motions

$$u_z^c/L = \bar{F}_z \left(\frac{1}{2kGA} + \frac{L^2}{24EI_y} + \frac{L^2 p_2}{1440GJ} (\phi_z^c)^2 \right), \quad (2.61)$$

$$\phi_x^c = \bar{M}_x \left(\frac{L}{EI_y} + \frac{L p_3}{12GJ} (\phi_z^c)^2 \right) + \bar{F}_x h_z \left(\frac{L}{EI_y} + \frac{L p_3}{12GJ} (\phi_z^c)^2 \right), \quad (2.62)$$

$$\phi_y^c = \bar{M}_y \left(\frac{L}{EI_y} + \frac{L p_3}{12GJ} (\phi_z^c)^2 \right) - \bar{F}_y h_z \left(\frac{L}{EI_y} + \frac{L p_3}{12GJ} (\phi_z^c)^2 \right). \quad (2.63)$$

Warping coefficient p_2 already appeared in the parallelogram mechanism analysis of Sec. 2.5.1; coefficient p_3 is new and given by

$$p_3 = 1 + \frac{12 - 6\lambda}{\lambda^2} + \frac{12}{\lambda - \lambda \exp(\lambda)}. \quad (2.64)$$

Similar to the error motions and actuation load for the sheet flexure and parallelogram mechanism, we have obtained the well-known linear terms, a load-stiffening term in $\bar{M}_z$ due to applied load $\bar{F}_x$, elastokinematic terms linear in the applied loads and quadratic in ϕ_z^c , and a kinematic term in u_x^c that is quadratic in ϕ_z^c . This term describes what is referred to as pivot shift in literature: the deviation of the location of the effective rotation axis of the mechanism [22, 27]. Although pivot shift in the x -direction is dominant for $\phi_z^c = 0.15$, it does occur in the y -direction as well, as an effect proportional to the cube of ϕ_z^c [28]. The term does not appear in this analysis because we chose to exclude all third-order terms in a single variable when constructing the series approximation due to the mathematical complexity (even with analytical solver software).

A verification with SPACAR, similar to the one for the parallelogram mechanism, shows that the deviation in the derivatives of the model equations of Eqs. (2.58)–(2.63) with respect to the applied loads and the DOF is at most 1.8% for the sheet flexure dimensions and material according to Tbl. 2.2 and $\phi_z^c = 0.15$. Similar observations for the influence of the warping constraints hold.

2.6 CONCLUSION

An accurate approximate closed-form model has been obtained for the spatially deforming sheet flexure subject to a general end-load that includes bending, shear, elongation, torsion and warping deformation. This model has been derived by means of the Hellinger–Reissner variational principle and a suitable set of interpolation function tailored to the geometrically nonlinear constraint behavior of a sheet flexure. The resulting model expressions are presented in a format suited for flexure mechanism analysis, in terms of degrees of freedom, error motions, applied loads, and actuation loads. The model captures the (nonlinear) geometrical stiffness as kinematic, elastokinematic, and load-stiffening effects. The sheet flexure model is verified numerically against FEM. Based on the provided energy scalar for a single sheet flexure, an approach for obtaining closed-form nonlinear models for arbitrary parallel assemblies of sheet flexures is presented. The validity of this approach is illustrated with the cases of the spatial deformation of a parallelogram flexure mechanism and a cross-hinge mechanism.

2.7 ACKNOWLEDGMENTS

This research was funded by the Innovative Research Incentives Scheme VIDI (Stichting voor de Technische Wetenschappen) (14152 NWO TTW) of the Ministry of Education, Culture and Science of the Netherlands.

2.8 APPENDIX OF CHAPTER 2

2.8.1 Hellinger–Reissner functional for nonuniform torsion

In a potential energy formulation for a fixed–free beam, the total potential energy due to nonuniform torsion is given by

$$\int_0^L \left(\frac{1}{2} GJ \phi_x'^2 + \frac{1}{2} EI_w \phi_x''^2 \right) ds - \bar{M}_x \phi_x(L) - \bar{B} \phi_x'(L), \quad (2.65)$$

where $\bar{M}_x$ represents the torsion moment and $\bar{B}$ the bimoment applied at the free end [76]. Due to the end being fixed, both the rotation and the cross-sectional warping vanish at $s = 0$: $\phi_x(0) = 0$ and $\phi_x'(0) = 0$. Taking variations with respect to the angle of twist $\phi_x(s)$ yields the stationarity conditions

$$GJ \phi_x'' - EI_w \phi_x^{(4)} = 0, \quad (2.66)$$

$$GJ \phi_x'(L) - EI_w \phi_x'''(L) = \bar{M}_x, \quad (2.67)$$

$$EI_w \phi_x''(L) = \bar{B}, \quad (2.68)$$

which represent the equilibrium equations for the beam with constant torsional rigidity GJ and constant warping rigidity EI_w . Boundary condition (2.67) shows how the torsion moment $\bar{M}_x$ is composed of the uniform torsion part $GJ \phi_x'$, which produces shear stress only, and the warping part $-EI_w \phi_x'''$, which plays a role when the cross-sections are not free to warp at the ends, the cross-sectional geometry changes over the beam length, or a torque is applied within the domain $0 < s < L$. In those cases, the warping varies along the beam during torsion and is accompanied by axial strain and hence normal stress and additional shear stress. The finite deformation of a sheet flexure, in which internal loads depend on the deformed configuration, belongs to the last case.

For the derivation of a Hellinger–Reissner-based energy formulation that also explicitly considers the distribution of the internal torsion moment and bimoment along the beam, the total potential energy formulation (in which only variations with respect to ϕ_x are considered) serves as the starting point. The compatibility equations

$$\kappa_x = \phi_x', \quad (2.69)$$

$$\kappa_w = \phi_x'' \quad (2.70)$$

are included in Eq. (2.65) by means of Lagrange multipliers λ_t and λ_w to yield

$$\int_0^L \left(\frac{1}{2} GJ \kappa_x^2 + \frac{1}{2} EI_w \kappa_w^2 + \lambda_t (\phi_x' - \kappa_x) + \lambda_w (\phi_x'' - \kappa_w) \right) ds - \bar{M}_x \phi_x(L) - \bar{B} \phi_x'(L). \quad (2.71)$$

Variations with respect to κ_x and κ_w then show that $\lambda_t = GJ \kappa_x$ and $\lambda_w = EI_w \kappa_w$, lending λ_t the interpretation of the uniform torsion moment M_{xt} and λ_w the bimoment B . By elimination of κ_x and κ_w , Eq. (2.71) becomes

$$\int_0^L \left(-\frac{M_{xt}^2}{2GJ} + M_{xt} \phi_x' - \frac{B^2}{2EI_w} + B \phi_x'' \right) ds - \bar{M}_x \phi_x(L) - \bar{B} \phi_x'(L), \quad (2.72)$$

representing a Hellinger–Reissner-type energy functional for the independent fields M_{xt} , B and ϕ_x . It is more convenient to have a formulation in terms of the torsion moment instead of M_{xt} . To this end, Eqs. (2.66)–(2.68) can be used to replace M_{xt} by $M_{\bar{x}} + B'$, yielding

$$\int_0^L \left[-\frac{(M_{\bar{x}} + B')^2}{2GJ} + M_{\bar{x}}\phi'_x - \frac{B^2}{2EI_w} \right] ds - \bar{M}_x\phi_x(L) - (\bar{B} - B)\phi'_x(L). \quad (2.73)$$

Sheet flexures generally have warping constraints at $s = 0$ and $s = L$ as a consequence of a clamping construction or monolithic design. The last term of Eq. (2.73) then vanishes. In the case of a folded sheet flexure, which is a single-constraint element that can be regarded as a series assembly of two straight sheet flexures of the current work, the warping of the cross-section is not necessarily constrained at the fold line and the last term may need to be considered.

2.8.2 Bimoment interpolation

The full expression for the bimoment interpolation is given by

$$\begin{aligned} B = & 6\tilde{F}_x I_{te} \tilde{\phi}_x (2\xi - 1)/\lambda^2 + 2\tilde{M}_y [\tilde{u}_y (-6\xi + 3) + L\tilde{\phi}_z (3\xi - 1)]/\lambda^2 \\ & + 2\tilde{F}_z L [-3\tilde{u}_y (4 + \lambda^2 (2\xi^2 - 3\xi + 1)) + L\tilde{\phi}_z (6 + \lambda^2 (3\xi^2 - 4\xi + 1))]/\lambda^4 \\ & - \cosh(\lambda\xi) \operatorname{csch}(\lambda) \left[\tilde{M}_x L/\lambda + 12\tilde{F}_x I_{te} \tilde{\phi}_x / \lambda^3 \right. \\ & \quad \left. + \tilde{M}_y (6L\tilde{\phi}_z + L\lambda^2 \tilde{\phi}_z - 12\tilde{u}_y)/\lambda^3 \right. \\ & \quad \left. + \tilde{F}_z (4L^2 \tilde{\phi}_z - 6L\tilde{u}_y)/\lambda^3 \right] \\ & + \cosh(\lambda - \lambda\xi) \operatorname{csch}(\lambda) \left[\tilde{M}_x L/\lambda + 12\tilde{F}_x I_{te} \tilde{\phi}_x / \lambda^3 \right. \\ & \quad \left. + \tilde{M}_y (6L\tilde{\phi}_z - 12\tilde{u}_y)/\lambda^3 \right. \\ & \quad \left. + \tilde{F}_z (-8L^2 \tilde{\phi}_z + 18L\tilde{u}_y + L\lambda^2 \tilde{u}_y)/\lambda^3 \right], \end{aligned} \quad (2.74)$$

where $\cosh(\xi)$ denotes the hyperbolic cosine function, $\operatorname{csch}(\xi) = 1/\sinh(\xi)$ the hyperbolic cosecant function, λ the dimensionless parameter

$$L\sqrt{GJ/(EI_w)}, \quad (2.75)$$

and ξ the dimensionless coordinate s/L .

2.8.3 Energy expression

After substitution of the interpolation functions, the full expression for the potential energy of the sheet flexure is given by

$$\begin{aligned}
P = & (\tilde{M}_x - \tilde{M}_x^{\text{TB}})\tilde{\phi}_x + (\tilde{M}_y - \tilde{M}_y^{\text{TB}})\tilde{\phi}_y + (\tilde{M}_z - \tilde{M}_z^{\text{TB}})\tilde{\phi}_z \\
& + (\tilde{F}_x - \tilde{F}_x)\tilde{u}_x + (\tilde{F}_y - \tilde{F}_y)\tilde{u}_y + (\tilde{F}_z - \tilde{F}_z)\tilde{u}_z \\
& - \frac{\tilde{F}_x^2 L}{2EA} - \frac{\tilde{F}_z^2 L}{2kGA} - \frac{w_6 \tilde{M}_x^2 L}{2GJ} - \frac{\tilde{F}_y^2 L^3}{6EI_z} - \frac{\tilde{F}_y \tilde{M}_z L^2}{2EI_z} - \frac{\tilde{M}_z^2 L}{2EI_z} - \frac{\tilde{M}_y^2 L}{2EI_y} + \frac{\tilde{M}_y \tilde{F}_z L^2}{2EI_y} - \frac{\tilde{F}_z^2 L^3}{6EI_y} \\
& + \tilde{F}_x L \left(\frac{3}{5L^2} [-I_{\text{te}} \tilde{\phi}_x^2 - \tilde{u}_y^2] + \frac{\tilde{u}_y \tilde{\phi}_z}{10L} - \frac{\tilde{\phi}_z^2}{15} \right) + \tilde{F}_z L \left(\frac{\tilde{\phi}_x \tilde{\phi}_z}{5} - \frac{\tilde{\phi}_x \tilde{u}_y}{10L} \right) + \tilde{M}_y \left(\frac{6\tilde{\phi}_x \tilde{u}_y}{5L} - \frac{\tilde{\phi}_x \tilde{\phi}_z}{10} \right) \\
& + \frac{\tilde{M}_z L}{EI_z} \left(\frac{1}{2} \tilde{F}_x \tilde{u}_y + \frac{1}{12} \tilde{F}_x \tilde{\phi}_z L - \frac{3}{20} \tilde{F}_z \tilde{\phi}_x L + \frac{1}{2} \tilde{M}_y \tilde{\phi}_x \right) \\
& + \frac{\tilde{F}_y L^2}{EI_z} \left(\frac{7}{20} \tilde{F}_x \tilde{u}_y + \frac{1}{30} \tilde{F}_x \tilde{\phi}_z L - \frac{1}{15} \tilde{F}_z \tilde{\phi}_x L + \frac{3}{20} \tilde{M}_y \tilde{\phi}_x \right) \\
& + \frac{\tilde{M}_x L}{GJ} \left([w_4 - 1] \tilde{F}_z \tilde{u}_y - \frac{w_5}{6} \tilde{F}_z \tilde{\phi}_z L - w_5 \tilde{M}_y \tilde{u}_y / L + [w_4 - 1] \tilde{M}_y \tilde{\phi}_z + w_5 \tilde{F}_x I_{\text{te}} \tilde{\phi}_x / L \right) \\
& + \frac{\tilde{F}_x L^3}{EI_z} \left(-\frac{13}{70L^2} \tilde{u}_y^2 - \frac{13}{420L} \tilde{u}_y \tilde{\phi}_z - \frac{1}{210} \tilde{\phi}_z^2 \right) - \tilde{F}_x^2 \frac{I_{\text{te}}^2}{GJL} \left(\frac{3}{5} \tilde{\phi}_x^2 w_3 \right) \\
& + \frac{\tilde{F}_x \tilde{F}_z L^3}{EI_z} \left(\frac{9}{140L} \tilde{\phi}_x \tilde{u}_y + \frac{1}{60} \tilde{\phi}_x \tilde{\phi}_z \right) + \tilde{F}_x \tilde{F}_z \frac{I_{\text{te}}}{GJ} \left(\frac{1}{5} L \tilde{\phi}_x \tilde{\phi}_z w_3 - \frac{1}{10} \tilde{\phi}_x \tilde{u}_y w_2 \right) \\
& + \frac{\tilde{F}_x \tilde{M}_y L^2}{EI_z} \left(-\frac{9}{70L} \tilde{\phi}_x \tilde{u}_y - \frac{11}{210} \tilde{\phi}_x \tilde{\phi}_z \right) + \tilde{F}_x \tilde{M}_y \frac{I_{\text{te}}}{GJ} \left(-\frac{1}{10} \tilde{\phi}_x \tilde{\phi}_z w_2 + \frac{6}{5} \tilde{\phi}_x \tilde{u}_y w_3 / L \right) \\
& + \frac{\tilde{F}_z^2 L^3}{EI_z} \left(-\frac{19}{1260} \tilde{\phi}_x^2 \right) + \frac{\tilde{F}_z \tilde{M}_y L^2}{EI_z} \left(\frac{3}{35} \tilde{\phi}_x^2 \right) + \frac{\tilde{M}_y^2 L}{EI_z} \left(-\frac{13}{70} \tilde{\phi}_x^2 \right) \\
& + \frac{\tilde{F}_z^2 L^3}{GJ} \left(-\frac{3w_{11}}{70L^2} \tilde{u}_y^2 + \frac{w_{10}}{105L} \tilde{u}_y \tilde{\phi}_z - \frac{2w_9}{105} \tilde{\phi}_z^2 \right) + \frac{\tilde{F}_z \tilde{M}_y L^2}{GJ} \left(\frac{w_2}{10L^2} \tilde{u}_y^2 - \frac{w_8}{5L} \tilde{u}_y \tilde{\phi}_z + \frac{w_7}{30} \tilde{\phi}_z^2 \right) \\
& + \frac{\tilde{M}_y^2 L}{GJ} \left(-\frac{3w_3}{5L^2} \tilde{u}_y^2 + \frac{w_2}{10L} \tilde{u}_y \tilde{\phi}_z - \frac{w_1}{15} \tilde{\phi}_z^2 \right).
\end{aligned} \tag{2.76}$$

The coefficients w_i are solely functions of λ and given by Eq. (2.77).

$$\begin{aligned}
w_1 &= \frac{180(\lambda^2 + 6) \operatorname{csch}(\lambda) + 2\lambda(\lambda^4 + 30\lambda^2 + 270) - 15(\lambda^4 + 12\lambda^2 + 72) \coth(\lambda)}{2\lambda^5}, \\
w_2 &= \frac{\lambda(\lambda^4 + 720) - 120(\lambda^2 + 12) \tanh\left(\frac{\lambda}{2}\right)}{\lambda^5}, \\
w_3 &= \frac{\lambda^5 - 10\lambda^3 + 120\lambda - 240 \tanh\left(\frac{\lambda}{2}\right)}{\lambda^5}, \\
w_4 &= \frac{\lambda(\lambda^2 - 6) + (\lambda^2 + 12) \tanh\left(\frac{\lambda}{2}\right)}{\lambda^3}, \\
w_5 &= \frac{\lambda^3 - 12\lambda + 24 \tanh\left(\frac{\lambda}{2}\right)}{\lambda^3}, \\
w_6 &= \frac{\exp(\lambda)(\lambda - 2) + \lambda + 2}{(\exp(\lambda) + 1)\lambda}, \\
w_7 &= \frac{120(\lambda^2 - 6) \coth(\lambda) + 240(\lambda^2 + 3) \operatorname{csch}(\lambda) + (\lambda^2 - 30)\lambda^3}{\lambda^5}, \\
w_8 &= \frac{\lambda^5 - 10\lambda^3 + 5(\lambda^4 + 24\lambda^2 + 120) \operatorname{csch}(\lambda) + 180\lambda - 600 \coth(\lambda)}{\lambda^5}, \\
w_9 &= \frac{\lambda^6 - 14\lambda^4 + 420\lambda^2 - 420\lambda(5 \cosh(\lambda) + 4) \operatorname{csch}(\lambda) + 3780}{\lambda^6}, \\
w_{10} &= \frac{\lambda^6 + 63\lambda^4 + 5040\lambda^2 - 420\lambda(\lambda^2 + 2(\lambda^2 + 21) \cosh(\lambda) + 30) \operatorname{csch}(\lambda) + 30240}{\lambda^6}, \\
w_{11} &= -\frac{420\lambda(\lambda^2 + 18) \operatorname{csch}(\lambda) + 35\lambda(\lambda^4 + 36\lambda^2 + 360) \coth(\lambda) - 3(\lambda^6 + 84\lambda^4 + 1540\lambda^2 + 6720)}{3\lambda^6}.
\end{aligned} \tag{2.77}$$

The definition of these coefficients is such that their limit is 1 when $\lambda \rightarrow \infty$.

2.8.4 Warping coefficients sheet flexure model

The model expressions for the sheet flexure contain the coefficients c_i that account for the warping deformation. Their definition in terms of the w_i from Eq. (2.77) is given by

$$\begin{array}{ll}
c_1 & 1/w_6 \\
c_2 & -6/5 + 2w_5/w_6 \\
c_3 & -w_5/w_6 \\
c_4 & (w_4 - 1)/w_6 \\
c_5 & (w_4 - 1)/w_6 \\
c_6 & -w_5/(6w_6) \\
c_7 & 3/5 - w_5/w_6 \\
c_8 & w_5/w_6 \\
c_9 & (w_4 - 1)/w_6 \\
c_{10} & w_5/(6w_6) \\
c_{11} & 6w_3/5 - w_5^2/w_6 \\
c_{12} & 2w_1/15 - (w_4 - 1)^2/w_6 \\
c_{13} & -w_2/10 + (w_4 - 1)w_5/w_6 \\
c_{14} & 6w_3/5 - w_5^2/w_6 \\
c_{15} & -w_5^2/(36w_6) + 4w_9/105 \\
c_{16} & (w_4 - 1)w_5/(6w_6) - w_{10}/105 \\
c_{17} & -(w_4 - 1)^2/w_6 + 3w_{11}/35 \\
c_{18} & -w_3/5 + w_5^2/(6w_6) \\
c_{19} & (w_4 - 1)w_5/(6w_6) - w_7/30 \\
c_{20} & [-30(w_4 - 1)^2 - 5w_5^2 + 6w_6w_8]/(60w_6)
\end{array} \tag{2.78}$$

2.8.5 Derivation of kinematic constraints due to the shuttle

Fig. 2.9 shows the vectors used in Eq. (2.43) for the kinematic relation that the shuttle enforces between the two sheet flexure ends of the cross-hinge flexure mechanism. Parameter d_y is the shuttle length. Parameters h_x and h_z represent the distance between the sheet flexure ends and the center of compliance, as measured along the x -axis and z -axis, respectively.

Figure 2.9: Kinematic constraints due to the shuttle of the cross-hinge mechanism.

2.8.6 Warping coefficients parallelogram mechanism

The full expressions for the warping coefficients of the parallelogram mechanism of Sec. 2.5.1 are given by

$$p_1 = 1 + \frac{16128}{\lambda^6} + \frac{2688}{\lambda^4} + \frac{588}{5\lambda^2} - \coth(\lambda/2) \left(\frac{8064}{\lambda^5} + \frac{672}{\lambda^3} + \frac{14}{\lambda} \right), \quad (2.79)$$

$$p_2 = \frac{120 + 60\lambda + 12\lambda^2 + \lambda^3 + \exp(\lambda)(-120 + 60\lambda - 12\lambda^2 + \lambda^3)}{2\lambda^2 + \lambda^3 + \lambda^2 \exp(\lambda)(-2 + \lambda)}. \quad (2.80)$$

CHAPTER 3

Third-order stiffness formulation for sheet flexures using an energy-based discretization approach (EDA): numerical beam element

Thesis note The paper in this chapter continues with the first subgoal, on flexure mechanism modeling techniques. The numerical implementation of the analytical sheet flexure model of chapter 2 as a beam element is described. The element offers a more practical way to simulate flexure mechanisms compared to the analytical model. It provides shorter computation times compared to existing beam elements for sheet flexures. The element can also be used to model the Wagner torque accurately, which was shown to be unfeasible in an analytical fashion.

Abstract This paper presents the stiffness formulation of a beam element with the relevant third-order nonlinear geometric effects for relatively wide and thin rectangular beams, in particular when loaded in the plane and simultaneously deformed out of the plane. The formulation is based on Timoshenko beam theory with nonuniform torsion and Wagner effects. The element is incorporated into the generalized strain beam formulation for multibody systems. Numerical simulations of precision flexure mechanisms show that the use of a single third-order element per flexible member can already yield adequate performance, at a significant reduction of the necessary degrees of freedom and the computation time, compared to a conventional second-order element.

3.1 INTRODUCTION

In the stiffness analysis of deforming structures, the third-order terms in the load-displacement relationship can have a significant effect, especially in high aspect ratio structures that are designed to provide a high stiffness in one plane and undergo moderate elastic deformation in another plane, enabled by a much lower stiffness [12, 58]. Examples are the so-called sheet flexures in flexure mechanisms used for precision manipulation [21, 30, 72, 73]. Sheet flexures have rectangular cross-sections that are wide and thin. The design optimization of flexure mechanisms is computationally more involved when the sheet flexures need to be modeled by multiple elements for an accurate description of the nonlinear stiffness characteristics. Sheet flexures generally have one dimension that is two orders of magnitude smaller than the other dimensions, meaning that plate elements seem appropriate; beams are used instead to keep the number of degrees of freedom limited and the models simple.

Submitted to Computational Mechanics.

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In this paper, we present the stiffness formulation for a numerical element with the third-order effects that are relevant for wide and thin rectangular beams. The stiffness formulation is based on the analytical closed-form model that was presented in Ref. 57. The element is incorporated into the generalized strain beam formulation for multibody systems [14, 33]. A key aspect of the formulation is the use of generalized strain and stress parameters, alongside the common nodal coordinates and loads. The generalized strain beam formulation is suited for the simulation of multibody systems that consist of both rigid and flexible elements, since the retention of the discretized strains (the generalized strains) in the equations to be solved enables a natural specification of element rigidity (by setting the generalized strains to a zero value) and element flexibility (by providing constitutive relations between the generalized strains and stresses). Various types of elements have been formulated, such as truss, hinge, tube, wheel and beam elements [46, 68]. The beam element with a second-order stiffness description, based on Timoshenko beam theory, serves as a basis for the current work [34, 36, 49].

The derivation of the third-order terms in the load–displacement relations is carried out in a corotational frame by means of the Hellinger–Reissner variational principle [81] with a specific set of physically-motivated displacement and stress interpolation functions: an explicit dependency of the stress interpolation on the displacement shape functions enables the use of low-order polynomials. In order to model beams with a large width to length ratio accurately, elongation, bending, transverse shear, non-uniform torsion (in particular constrained warping) and Wagner effects are included. The interpolation of the bimoment (the load associated with non-uniform torsion) takes into account the deformed configuration and uses hyperbolic shape functions. While this derivation yields discretized equations that are expressed in both the Hellinger–Reissner stress variables and the nodal coordinates, they are linear in the stress variables and easily reduced to a nodal coordinates-only set. The use of mixed-field variational principles for the stiffness formulation of elements in multibody and finite element methods is well established in literature [5, 6, 19, 23, 24, 66]. To the knowledge of the authors, the use of this particular set of interpolation functions for a refined description of the geometric stiffness as encountered in sheet flexures is new.

In Sec. 3.2, an overview of the stiffness aspect of the generalized strain beam formulation is presented, so that the new element can be introduced in Sec. 3.3. In Sec. 3.4, several examples of numerical simulations are provided to demonstrate the benefits that the third-order element offers.

3.2 GENERALIZED STRAIN BEAM FORMULATION

In this section, we will first formulate the discrete equations for a beam element with second-order geometrical effects. While this element has been presented before and has been implemented in the computer program SPACAR [37], the derivation based on a mixed-field variational principle is new. The resulting discrete equations of this section are needed in the derivation of the third-order element in Sec. 3.3.

The formulation of an elastic beam element involves the discretization of a beam continuum. The governing equations are described first, followed by the linearized

Figure 3.1: Deformed beam configuration, given by the elastic line position and cross-sectional orientation. The boundary conditions are depicted as simply supported ones.

solution and the specific choices for the interpolation functions. The discretization is carried out by means of the Hu–Washizu variational principle [81], which allows for the interpolation of the displacements, as well as the strains and stresses, yielding the equilibrium, compatibility and constitutive equations.

3.2.1 Beam continuum

The undeformed straight reference configuration of a beam of length L is given by the position of the elastic line $\mathbf{r}_0(s)$, where $0 \leq s \leq L$ is the material coordinate. The coordinate frame is represented by the orthogonal unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$. The beam is oriented such that the undeformed cross-sectional plane is normal to $\mathbf{e}_x$; the principal axes of the cross-sections are aligned with $\mathbf{e}_y$ and $\mathbf{e}_z$. Vectors and matrices in the following analysis are expressed as components in the coordinate frame represented by $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$, unless noted otherwise.

In the deformed configuration, shown in Fig. 3.1, the position of the elastic line is

$$\mathbf{r}(s) = \mathbf{r}_0(s) + \mathbf{u}(s) = s\mathbf{e}_x + \mathbf{u}(s), \quad (3.1)$$

where $\mathbf{u}$ represents the displacement. The orientation of the cross-section is represented by the orthogonal triad of unit vectors

$$\begin{bmatrix} \mathbf{e}_{\bar{x}} & \mathbf{e}_{\bar{y}} & \mathbf{e}_{\bar{z}} \end{bmatrix} = \mathbf{R}(s) \begin{bmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \end{bmatrix}, \quad (3.2)$$

which is rotated with respect to the undeformed configuration by rotation matrix $\mathbf{R}$. It can be parametrized in various ways. Since the deformation in this work remains limited, the rotations with respect to the undeformed configuration are small and the singularities of Euler angles pose no problems. We choose to parametrize $\mathbf{R}$ by the Euler angles according to

$$\mathbf{R}(s) = \begin{bmatrix} \cos \phi_y & 0 & \sin \phi_y \\ 0 & 1 & 0 \\ -\sin \phi_y & 0 & \cos \phi_y \end{bmatrix} \begin{bmatrix} \cos \phi_z & -\sin \phi_z & 0 \\ \sin \phi_z & \cos \phi_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi_x & -\sin \phi_x \\ 0 & \sin \phi_x & \cos \phi_x \end{bmatrix}. \quad (3.3)$$

The adopted rotation order has no particular significance on the derivation in this section. The angles represent rotation with respect to and resolved in the coordinate frame denoted by $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$. For small angles, $\phi_x(s)$ is the twist angle, $\phi_y(s)$ the bending angle in the $\mathbf{e}_x, \mathbf{e}_z$ -plane, and $\phi_z(s)$ the bending angle in the $\mathbf{e}_x, \mathbf{e}_y$ -plane.

The curvatures $\boldsymbol{\kappa}$ and strains $\boldsymbol{\gamma}$ of the beam are given by [63]

$$\boldsymbol{\kappa}(s) = \mathbf{R}^T \mathbf{R}' = \begin{bmatrix} 0 & -\kappa_{\bar{z}} & \kappa_{\bar{y}} \\ \kappa_{\bar{z}} & 0 & -\kappa_{\bar{x}} \\ -\kappa_{\bar{y}} & \kappa_{\bar{x}} & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\gamma}(s) = \mathbf{R}^T (\mathbf{e}_x + \mathbf{u}' - \mathbf{e}_{\bar{x}}), \quad (3.4)$$

where the prime denotes differentiation with respect to s . These strain–displacement and curvature–rotation relations are collectively referred to as compatibility equations in the following sections. The components of skew-symmetric matrix $\boldsymbol{\kappa}$ are

$$\begin{aligned} \kappa_{\bar{x}} &= \phi'_x + \phi'_y \sin \phi_z, \\ \kappa_{\bar{y}} &= \phi'_y \cos \phi_x \cos \phi_z + \phi'_z \sin \phi_x, \\ \kappa_{\bar{z}} &= \phi'_z \cos \phi_x - \phi'_y \sin \phi_x \cos \phi_z. \end{aligned} \quad (3.5)$$

The warping curvature, energetically dual to the bimoment, is given by [76]

$$\kappa_w = \frac{d}{ds} (\phi'_x + \phi'_y \sin \phi_z). \quad (3.6)$$

The components of vector $\boldsymbol{\gamma}$ are

$$\begin{aligned} \gamma_{\bar{x}}(s) &\approx u'_x - \phi_y(\phi_y/2 + u'_z) - \phi_z(\phi_z/2 - u'_y), \\ \gamma_{\bar{y}}(s) &\approx u'_y - \phi_z(1 + u'_x) + \phi_x(\phi_y + u'_z), \\ \gamma_{\bar{z}}(s) &\approx u'_z + \phi_y(1 + u'_x) + \phi_x(\phi_z - u'_y), \end{aligned} \quad (3.7)$$

truncated after the second-order terms. Axial strain $\gamma_{\bar{x}}$ can be augmented with the term

$$\frac{1}{2} I_{te} \phi'^2_x, \quad \text{where } I_{te} = \frac{1}{12} (w^2 + t^2), \quad (3.8)$$

to account for a coupling between torsion and elongation for beams with a rectangular cross-section of width w and thickness t . This term is sometimes referred to as the trapeze effect or Wagner term. It models the torsional buckling due an axial load and the shortening of a beam that is twisted [20, 76].

The elastic energy for an initially straight beam with linear elastic material is given by

$$P_{\text{elastic}} = \frac{1}{2} \int_0^L [EA\gamma_{\bar{x}}^2 + k_y GA\gamma_{\bar{y}}^2 + k_z GA\gamma_{\bar{z}}^2 + GJ\kappa_{\bar{x}}^2 + EI_y\kappa_{\bar{y}}^2 + EI_z\kappa_{\bar{z}}^2 + EI_w\kappa_w^2] ds, \quad (3.9)$$

where E is Young's modulus, A the cross-sectional area, k_y and k_z the shear correction factors, G the shear modulus, J Saint-Venant's torsion constant, I_y and I_z the second moments of area about the principal axes, and I_w Vlasov's warping constant [80].

3.2.2 Linear solution

The linear solution is part of the interpolation scheme of the higher-order elements, as will be detailed in Sec. 3.2.3 and 3.3.3. It can be obtained by using the linearized compatibility relations

$$\begin{aligned} \gamma_{\bar{x}} &= u'_x, & \gamma_{\bar{y}} &= u'_y - \phi_z, & \gamma_{\bar{z}} &= -u'_z - \phi_y, \\ \kappa_{\bar{x}} &= \phi'_x, & \kappa_{\bar{y}} &= \phi'_y, & \kappa_{\bar{z}} &= \phi'_z, & \kappa_w &= \phi''_x. \end{aligned} \quad (3.10)$$

The ends of the beam are referred to as node p (at $s = 0$) and node q (at $s = L$). With the simply supported boundary conditions

$$\begin{aligned} s = 0: & \begin{cases} u_x = 0, & \phi_x = 0, \\ u_y = 0, & \phi_y = \phi_y^p, & \phi'_x = W^p, \\ u_z = 0, & \phi_z = \phi_z^p, \end{cases} \\ s = L: & \begin{cases} u_x = u_x^q, & \phi_x = \phi_x^q, \\ u_y = 0, & \phi_y = \phi_y^q, & \phi'_x = W^q, \\ u_z = 0, & \phi_z = \phi_z^q, \end{cases} \end{aligned} \quad (3.11)$$

eight local nodal coordinates $\mathbf{x}_l$ are introduced: in node p , there are the two bending angles ϕ_y^p and ϕ_z^p , and the specific twist angle W^p . In node q , there are the two bending angles ϕ_y^q and ϕ_z^q , the specific twist angle W^q , as well as the torsion angle ϕ_x^q and the axial displacement u_x^q . Other boundary conditions could have been used as well. Simply supported boundary conditions are used here, because they introduce some symmetry and make the two nodes more or less interchangeable.

The linear solution is obtained as

$$u_x = u_x^q \xi, \quad (3.12)$$

$$\begin{aligned} u_y &= L\phi_z^p(\xi - 2\xi^2 + \xi^3) + L\phi_z^q(-\xi^2 + \xi^3) \\ &\quad + \Phi_y(L\phi_z^p + L\phi_z^q)(-\xi + 3\xi^2 - 2\xi^3)/(2 + 2\Phi_y), \end{aligned} \quad (3.13)$$

$$\begin{aligned} u_z &= L\phi_y^p(-\xi + 2\xi^2 - \xi^3) + L\phi_y^q(\xi^2 - \xi^3) \\ &\quad + \Phi_z(L\phi_y^p + L\phi_y^q)(\xi - 3\xi^2 + 2\xi^3)/(2 + 2\Phi_z), \end{aligned} \quad (3.14)$$

$$\phi_y = -u'_z + \Phi_z(\phi_y^p + \phi_y^q)/(2 + 2\Phi_z), \quad (3.15)$$

$$\phi_z = u'_y + \Phi_y(\phi_z^p + \phi_z^q)/(2 + 2\Phi_y), \quad (3.16)$$

with dimensionless coordinate $\xi = s/L$ and dimensionless shear parameters

$$\Phi_y = \frac{12EI_z}{k_y GAL^2}, \quad \Phi_z = \frac{12EI_y}{k_z GAL^2}. \quad (3.17)$$

The solution for ϕ_x contains hyperbolic functions of ξ , given in full in Appendix 3.7.1. A simpler polynomial approximation, consistent with the boundary conditions, is given by

$$\phi_x = \phi_x^q(3\xi^2 - 2\xi^3) + LW^p(\xi - 2\xi^2 + \xi^3) + LW^q(-\xi^2 + \xi^3) \quad (3.18)$$

and used instead.

3.2.3 Second-order beam formulation

The basis for the geometrically nonlinear effects is provided by the second-order terms in the compatibility relations of Eq. (3.4) and given by

$$\begin{aligned} \kappa_{\bar{x}} &= \phi'_x + \phi'_y\phi_z, & \gamma_{\bar{x}} &= u'_x + u_y'^2/2 + u_z'^2/2 + I_{te}\phi_x'^2/2, \\ \kappa_{\bar{y}} &= \phi'_y + \phi'_z\phi_x, & \gamma_{\bar{y}} &= u'_y - \phi_z, \\ \kappa_{\bar{z}} &= \phi'_z - \phi'_y\phi_x, & \gamma_{\bar{z}} &= u'_z + \phi_y. \end{aligned} \quad (3.19)$$

The relations have been slightly simplified, because the axial stiffness and transverse shear stiffness are much larger than the bending and torsional stiffness of a beam. As a consequence, the axial displacement, u_x , and the shear strains, $\gamma_{\bar{y}}$ and $\gamma_{\bar{z}}$, are regarded as second-order effects.

The Hu-Washizu principle is used for obtaining the discretized equilibrium, compatibility and constitutive equations in terms of the displacements, strains and stresses [81]. The corresponding functional is given by

$$\begin{aligned} P_{\text{HW}} = P_{\text{elastic}} + \int_0^L \bigg[& F_{\bar{x}}(u'_x + u_y'^2/2 + u_z'^2/2 + I_{te}\phi_x'^2/2 - \gamma_{\bar{x}}) \\ & + F_{\bar{y}}(u'_y - \phi_z - \gamma_{\bar{y}}) + F_{\bar{z}}(u'_z + \phi_y - \gamma_{\bar{z}}) \\ & + M_{\bar{y}}(\phi'_y + \phi'_z\phi_x - \kappa_{\bar{y}}) + M_{\bar{z}}(\phi'_z - \phi'_y\phi_x - \kappa_{\bar{z}}) \\ & + (M_{\bar{x}} + B')(\phi'_x + \phi'_y\phi_z - \kappa_{\bar{x}}) + B(\phi_x'' - \kappa_w) \bigg] ds, \end{aligned} \quad (3.20)$$

with P_{elastic} from Eq. (3.9). Besides the displacements (and rotations), this functional introduces the strains (and curvatures), and the stresses. The stresses are the beam stress resultants: the internal axial force $F_{\bar{x}}$, internal shear forces $F_{\bar{y}}$ and $F_{\bar{z}}$, internal torsion moment $M_{\bar{x}}$, internal bending moments $M_{\bar{y}}$ and $M_{\bar{z}}$, and internal bimoment B . The stress components are expressed in the moving frame denoted by $\mathbf{e}_{\bar{x}}$, $\mathbf{e}_{\bar{y}}$, $\mathbf{e}_{\bar{z}}$ in Fig. 3.1. The derivation of the terms involving the bimoment can be found in Ref. 57 in Appendix 2.8.1.

Discretization involves the selection of sets of discrete parameters, in terms of which the displacement (and rotation), stress, and curvature (and strain) fields can then be interpolated. The resulting discretized equations are expressed in terms of those sets of discrete parameters and approximate the exact solution. The discrete stress parameters are referred to as generalized stresses; the discrete curvature and strain parameters as generalized strains.

For the interpolation of the displacements and rotations, we use the linear solution of Eqs. (3.12)–(3.16) together with Eq. (3.18). The local nodal coordinates $\mathbf{x}_l$ are the discrete parameters.

For the curvatures and strains, we construct interpolation functions by combining the linearized compatibility relations of Eq. (3.10) with the interpolation for the displacement and rotations of Eqs. (3.12)–(3.18), and make the specific choice to introduce eight generalized strains by carrying out the replacements

$$\begin{aligned} u_x^q &\rightarrow \varepsilon_1, & -L\phi_y^p &\rightarrow \varepsilon_3, \\ L\phi_x^q &\rightarrow \varepsilon_2, & L\phi_y^q &\rightarrow \varepsilon_4, \\ L^2W^p &\rightarrow \varepsilon_7, & -L\phi_z^p &\rightarrow \varepsilon_5, \\ L^2W^q &\rightarrow \varepsilon_8, & L\phi_z^q &\rightarrow \varepsilon_6. \end{aligned} \quad (3.21)$$

With this particular choice, we obtain strain and curvature interpolations that are constructed according to the linear beam solution, dependent only on the generalized strains ε_1 to ε_8 , and given by

$$\gamma_{\bar{x}} = \varepsilon_1/L, \quad (3.22)$$

$$\gamma_{\bar{y}} = \Phi_y(\varepsilon_5/L - \varepsilon_6/L)/(2 + 2\Phi_y), \quad (3.23)$$

$$\gamma_{\bar{z}} = \Phi_z(\varepsilon_4/L - \varepsilon_3/L)/(2 + 2\Phi_z), \quad (3.24)$$

$$\kappa_{\bar{x}} = \varepsilon_2/L^2(6\xi - 6\xi^2) + \varepsilon_7/L^2(1 - 4\xi + 3\xi^2) + \varepsilon_8/L^2(-2\xi + 3\xi^2), \quad (3.25)$$

$$\kappa_w = \varepsilon_2/L^3(6 - 12\xi) + \varepsilon_7/L^3(-4 + 6\xi) + \varepsilon_8/L^3(-2 + 6\xi), \quad (3.26)$$

$$\kappa_{\bar{y}} = \varepsilon_3/L^2(4 - 6\xi) + \varepsilon_4/L^2(-2 + 6\xi) + \Phi_z(\varepsilon_3/L^2 - \varepsilon_4/L^2)(-3 + 6\xi)/(1 + \Phi_z), \quad (3.27)$$

$$\kappa_{\bar{z}} = \varepsilon_5/L^2(4 - 6\xi) + \varepsilon_6/L^2(-2 + 6\xi) + \Phi_y(\varepsilon_5/L^2 - \varepsilon_6/L^2)(-3 + 6\xi)/(1 + \Phi_y). \quad (3.28)$$

Parameters $\boldsymbol{\varepsilon}$ have the dimension of length and can be interpreted as measures of deformation: ε_1 measures the elongation strain, ε_2 , ε_7 and ε_8 nonuniform torsion (warping), ε_3 and ε_4 bending in the $\mathbf{e}_{\bar{x}}$, $\mathbf{e}_{\bar{z}}$ -plane, and ε_5 and ε_6 bending in the $\mathbf{e}_{\bar{x}}$, $\mathbf{e}_{\bar{y}}$ -plane.

For the interpolation of the stresses, we use eight generalized stresses σ_1 to σ_8 that are constructed specifically to be energetically dual to the generalized strains $\boldsymbol{\varepsilon}$. This means that the generalized stresses produce the virtual work $-\boldsymbol{\sigma}^T \delta \boldsymbol{\varepsilon}$. In turn, with the virtual work $\mathbf{F}^T \delta \mathbf{x}_l$ due to the applied nodal forces $\mathbf{F}$, the equilibrium condition states that

$$\boldsymbol{\sigma}^T \delta \boldsymbol{\varepsilon} = \mathbf{F}^T \delta \mathbf{x}_l \quad (3.29)$$

for all kinematically admissible virtual variations. Given the choice of the generalized strains $\boldsymbol{\varepsilon}$ in Eq. (3.21), it then follows that the generalized stresses are

$$\begin{aligned} \sigma_1 &= F_{\bar{x}}(L), & \sigma_3 &= M_{\bar{y}}(0)/L, \\ \sigma_2 &= M_{\bar{x}}(L)/L, & \sigma_4 &= M_{\bar{y}}(L)/L, \\ \sigma_7 &= -B(0)/L^2, & \sigma_5 &= M_{\bar{z}}(0)/L, \\ \sigma_8 &= B(L)/L^2, & \sigma_6 &= M_{\bar{z}}(L)/L. \end{aligned} \quad (3.30)$$

Interpolation functions for the stresses can then be constructed as

$$\begin{aligned} F_{\bar{x}}(s) &= \sigma_1, & M_{\bar{x}}(s) &= L\sigma_2, \\ F_{\bar{y}}(s) &= \sigma_5 - \sigma_6, & M_{\bar{y}}(s) &= \sigma_3(L - s) + \sigma_4s, & B(s) &= L\sigma_7(s - L) + L\sigma_8s, \\ F_{\bar{z}}(s) &= \sigma_4 - \sigma_3, & M_{\bar{z}}(s) &= \sigma_5(L - s) + \sigma_6s. \end{aligned} \quad (3.31)$$

Application of the Hu–Washizu principle implies carrying out the variations with respect to the three independent fields: the displacements (and rotations), the stresses and the strains (and curvatures). By taking into account the arbitrary character of the virtual variations, three sets of equations for the unknown parameters are obtained.

Taking variations with respect to the generalized strains $\boldsymbol{\varepsilon}$ yields the constitutive equations

$$\sigma_1 = EA\varepsilon_1/L, \quad (3.32)$$

$$\begin{bmatrix} \sigma_2 \\ \sigma_7 \\ \sigma_8 \end{bmatrix} = \frac{GJ}{30L^3} \begin{bmatrix} 36 & -3 & -3 \\ -3 & 4 & -1 \\ -3 & -1 & 4 \end{bmatrix} \begin{bmatrix} \varepsilon_2 \\ \varepsilon_7 \\ \varepsilon_8 \end{bmatrix} + \frac{EI_w}{L^5} \begin{bmatrix} 12 & -6 & -6 \\ -6 & 4 & 2 \\ -6 & 2 & 4 \end{bmatrix} \begin{bmatrix} \varepsilon_2 \\ \varepsilon_7 \\ \varepsilon_8 \end{bmatrix}, \quad (3.33)$$

$$\begin{bmatrix} \sigma_3 \\ \sigma_4 \end{bmatrix} = \frac{EI_y}{(1 + \Phi_z)L^3} \begin{bmatrix} 4 + \Phi_z & -2 + \Phi_z \\ -2 + \Phi_z & 4 + \Phi_z \end{bmatrix} \begin{bmatrix} \varepsilon_3 \\ \varepsilon_4 \end{bmatrix}, \quad (3.34)$$

$$\begin{bmatrix} \sigma_5 \\ \sigma_6 \end{bmatrix} = \frac{EI_z}{(1 + \Phi_y)L^3} \begin{bmatrix} 4 + \Phi_y & -2 + \Phi_y \\ -2 + \Phi_y & 4 + \Phi_y \end{bmatrix} \begin{bmatrix} \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}, \quad (3.35)$$

representing the stiffness matrix of the system in terms of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$, abbreviated as $\boldsymbol{\sigma} = \mathcal{S}(\boldsymbol{\varepsilon}) = \mathcal{S}\boldsymbol{\varepsilon}$.

Taking variations with respect to the generalized stresses $\boldsymbol{\sigma}$ yields the compatibility equations

$$\begin{aligned} \varepsilon_1 &= u_x^q + L \left[2(\phi_y^p)^2 + 2(\phi_y^q)^2 + 2(\phi_z^p)^2 + 2(\phi_z^q)^2 - \phi_y^p\phi_y^q - \phi_z^p\phi_z^q \right] / 30 \\ &\quad + I_{te} \left\{ 18(\phi_x^q)^2 + L^2 [2(W^p)^2 - W^pW^q + 2(W^q)^2] \right. \\ &\quad \left. - 3L\phi_x^q(W^p + W^q) \right\} / (30L), \end{aligned} \quad (3.36)$$

$$\varepsilon_2 = L\phi_x^q - L(\phi_y^p + \phi_y^q)(\phi_z^p - \phi_z^q)/2, \quad (3.37)$$

$$\varepsilon_3 = -L\phi_y^p - L\phi_x^q(\phi_z^p - 2\phi_z^q)/10 - L^2W^p\phi_z^p/10 + L^2W^q(\phi_z^p - \phi_z^q)/30, \quad (3.38)$$

$$\varepsilon_4 = L\phi_y^q + L\phi_x^q(2\phi_z^p + 9\phi_z^q)/10 + L^2W^p(\phi_z^q - \phi_z^p)/30 - L^2W^q\phi_z^q/10, \quad (3.39)$$

$$\varepsilon_5 = -L\phi_z^p + L\phi_y^q(\phi_y^p - 2\phi_y^q)/10 + L^2W^p\phi_y^p/10 + L^2W^q(\phi_y^q - \phi_y^p)/30, \quad (3.40)$$

$$\varepsilon_6 = L\phi_z^q - L\phi_x^q(2\phi_y^p + 9\phi_y^q)/10 + L^2W^p(\phi_y^p - \phi_y^q)/30 + L^2W^q\phi_y^q/10, \quad (3.41)$$

$$\varepsilon_7 = L^2W^p - L(\phi_y^p + \phi_y^q)(\phi_z^p - \phi_z^q)/2, \quad (3.42)$$

$$\varepsilon_8 = L^2W^q - L(\phi_y^p + \phi_y^q)(\phi_z^p - \phi_z^q)/2, \quad (3.43)$$

Figure 3.2: Global and local coordinate frames. Nodes p and pu are depicted as separate nodes; their positions coincide.

where the terms involving shear have been ignored. These equations represent the discrete analog to the nonlinear strain and curvature relations of Eq. (3.19).

The two sets of discrete equations, Eqs. (3.32)–(3.35) and Eqs. (3.36)–(3.43), reflect the character of the considered beam continuum: geometrically nonlinear behavior and physically linear material behavior.

When taking variations with respect to the displacements and rotations (in the presence of some external work due to applied loads), the equilibrium equations are obtained. These are not listed here explicitly since Eq. (3.29) can be used directly. In Sec. 3.2.5, this is detailed further in the context of systems with multiple interconnected elements.

3.2.4 Transformation to global coordinates

In Eqs. (3.36)–(3.43), the generalized strains are expressed as functions of the displacement and rotations in a simply supported coordinate frame. For the assembly of multiple elements and the description of rigid body motion, the transformation to a global coordinate frame is necessary.

Given the unit base vectors $\mathbf{n}_x$, $\mathbf{n}_y$ and $\mathbf{n}_z$ of a global coordinate frame, the position of the two nodes with respect to the global frame is denoted by $\mathbf{r}_0^p$ and $\mathbf{r}_0^q$. The orientation of the cross-sections at the nodes with respect to the global frame is described by the rotation matrices $\mathbf{R}_0^p$ and $\mathbf{R}_0^q$, which rotate the global frame to the triad connected to the deformed cross-sections. Since the rigid body motion can be large, the parametrization of $\mathbf{R}_0^p$ and $\mathbf{R}_0^q$ should be free from singularities. Euler parameters (unit quaternions) could be used for this purpose.

The undeformed reference configuration of the beam is indicated schematically between nodes pu and qu in Fig. 3.2. The local nodal coordinates have been defined with respect to this configuration (see Sec. 3.2.2). In this section, $\mathbf{r}_i^j$ and $\mathbf{R}_i^j$ denote the vector and rotation matrix from node i to node j . The rotational local nodal coordinates ϕ_x^q , ϕ_y^p , ϕ_y^q , ϕ_z^p and ϕ_z^q are expressed in the frame attached to pu . They can be found from $\mathbf{R}_{pu}^p$ and $\mathbf{R}_{qu}^q$, which are given by $\mathbf{R}(0)$ and $\mathbf{R}(L)$, respectively, from Eq. (3.3).

Loop-closure type constraint equations can be used to find the relation between the local and global nodal coordinates. For the translations, the constraint equation is

$$\mathbf{r}_0^p + \mathbf{r}_p^{pu} + \mathbf{r}_{pu}^{qu} + \mathbf{r}_{qu}^q + \mathbf{r}_q^0 = 0. \quad (3.44)$$

Accounting for the simply supported boundary conditions of Eq. (3.11), it follows that $\mathbf{r}_p^{pu} = 0$ and

$$\mathbf{r}_0^p + \mathbf{R}_0^p \mathbf{R}_p^{pu} [L + u_x^q \ 0 \ 0]^T + \mathbf{r}_q^0 = 0. \quad (3.45)$$

This equation can be rearranged to

$$(\mathbf{R}_p^p)^T [L + u_x^q \ 0 \ 0]^T = (\mathbf{R}_0^p)^T (\mathbf{r}_q^q - \mathbf{r}_0^p), \quad (3.46)$$

which expresses the relation between global nodal coordinates on the right-hand side and the local nodal coordinates $\mathbf{R}_{pu}^p$ and u_x^q on the left-hand side. From this equation, u_x^q , ϕ_y^p and ϕ_z^p can be extracted.

For the rotations, the constraint equation is

$$\mathbf{R}_0^p \mathbf{R}_p^{pu} \mathbf{R}_{pu}^{qu} \mathbf{R}_{qu}^q \mathbf{R}_q^0 = \mathbf{I}. \quad (3.47)$$

Since the reference configuration is straight, $\mathbf{R}_{pu}^{qu} = \mathbf{I}$. Rearranging then yields

$$(\mathbf{R}_{qu}^q)^T = (\mathbf{R}_0^q)^T \mathbf{R}_0^p (\mathbf{R}_{pu}^p)^T, \quad (3.48)$$

which expresses the remaining local nodal coordinates $\mathbf{R}_{qu}^q$ as functions of the known quantities on the right-hand side. From this equation, the rotations at node q , i.e. ϕ_x^q , ϕ_y^q and ϕ_z^q , can be extracted.

The warping-associated coordinates W^p and W^q are independent of the rigid body motion of the beam; no transformation is needed.

The combination of Eqs. (3.36)–(3.43) with Eqs. (3.46) and (3.48) yields

$$\boldsymbol{\varepsilon} = \mathcal{D}(\mathbf{x}). \quad (3.49)$$

This is the deformation function that expresses the generalized strains in terms of the global coordinate vector $\mathbf{x}$, which contains $\mathbf{r}_0^p$, $\mathbf{r}_0^q$ and the rotation parameters of $\mathbf{R}_0^p$ and $\mathbf{R}_0^q$.

3.2.5 Multibody systems of interconnected elements

In the generalized strain beam formulation, multibody systems are modeled by finite elements that have nodes for the connection to other elements and the interaction with the environment. An element is defined by the deformation function $\boldsymbol{\varepsilon} = \mathcal{D}(\mathbf{x})$, a constitutive model (or stress function) $\boldsymbol{\sigma} = \mathcal{S}(\boldsymbol{\varepsilon})$ and inertial properties. In the current work, only a refinement of the stiffness is considered. The details of a consistent mass description for the discussed beam element can be found in Ref. 36. Other elements, such as truss, hinge, tube and wheel elements have been formulated [46, 68].

Elements in a system are connected by sharing nodal coordinates. The equations of motion are determined from the element deformation and stress functions, and the application of the principle of virtual work. Besides the nodal coordinates, the generalized strains can also be used as degrees of freedom, prescribed a zero value to model rigidity, a nonzero value to prescribe relative motion, or prescribed as a function of time in dynamic simulations. The equations of motion can be linearized about an arbitrary state of motion to determine buckling modes, natural modes, compliances, and state-space relations [35, 36]. A kinematic mobility analysis can be performed [2]. An arc-continuation method has been described to follow the solution path in static simulations [45].

3.3 THIRD-ORDER STIFFNESS FORMULATION

In this section, the analytical nonlinear stiffness model tailored to wide and thin rectangular beams of Ref. 57 is converted to a numerical element for use in the generalized strain beam formulation.

The second-order effects, i.e. the third-order terms in the energy functional, in the previous section stem solely from the nonlinear terms in the compatibility equations. The displacement and stress interpolation is based on the linearized solution. For a description of third-order effects, i.e. fourth-order terms in the energy functional, we will make use of the knowledge of the exact beam equilibrium equations. It turns out that this way, the displacement (and rotation) interpolation based on the linear solution is still sufficiently accurate. While this is not necessarily the only way to find the fourth-order terms, it is a way that does not require a refinement of the displacement interpolation.

The continuum model based on the Hellinger–Reissner variational principle is described, along with suitable interpolation functions for the discretization. This follows the approach of Ref. 57, Sec. 2.2, with the difference that here the equations are formulated for simply supported boundary conditions instead of fixed–free ones, in order to match with the results of Sec. 3.2 of this paper. After that, a procedure for the numerical implementation is presented.

3.3.1 Hellinger–Reissner variational principle

The continuum description is based on the Hellinger–Reissner two-field variational principle, in which the displacements and stresses are the independent fields [81]. The derivation could also be carried out with the Hu–Washizu three-field principle, but the additional strain interpolation is an unnecessary requirement or freedom that would introduce confusing redundancy.

The Hellinger–Reissner functional can be obtained from the Hu–Washizu principle by using the constitutive equations to eliminate the strains and curvatures from Eq. (3.20). The variational principle then only provides the equilibrium and compatibil-

ity equations. The corresponding functional is given by

$$P_{\text{HR}} = \int_0^L \left[(M_{\bar{x}} + B')\kappa_{\bar{x}} + M_{\bar{y}}\kappa_{\bar{y}} + M_{\bar{z}}\kappa_{\bar{z}} + F_{\bar{x}}\gamma_{\bar{x}} + F_{\bar{y}}\gamma_{\bar{y}} + F_{\bar{z}}\gamma_{\bar{z}} + B\kappa'_{\bar{x}} \right. \\ \left. - \frac{(M_{\bar{x}} + B')^2}{2GJ} - \frac{M_{\bar{y}}^2}{2EI_y} - \frac{M_{\bar{z}}^2}{2EI_z} - \frac{F_{\bar{x}}^2}{2EA} - \frac{F_{\bar{y}}^2}{2k_yGA} - \frac{F_{\bar{z}}^2}{2k_zGA} - \frac{B^2}{2EI_w} \right] ds, \quad (3.50)$$

where the strains γ and curvatures κ are dependent on the displacements and rotations, according to Eq. (3.4). The independent fields are the displacements (and rotations) and stresses.

3.3.2 Wagner torque

With increasing torsion angle ϕ_x , there is a nonlinear contribution to the corresponding torsion moment, given by

$$\frac{1}{360}Ew^5t(\phi'_x)^3, \quad (3.51)$$

that acts as a stiffening effect and starts to dominate the response. The effect is referred to as the Wagner torque [78]. Note that it is different from the torsion–elongation coupling (the trapeze or Wagner term) of Eq. (3.8). It is strongly dependent on the cross-sectional width and not captured by the current beam-based formulation that assumes a linear relation between the stress resultants and the measures of beam deformation of Eq. (3.4). The Wagner torque can be incorporated by including the associated strain energy per unit length [48]

$$\frac{1}{1440}Ew^5t\kappa_x^4 \quad (3.52)$$

in the integrand of P_{HR} .

3.3.3 Interpolation

Consider Fig. 3.3 and the exact equilibrium equations for a finite beam segment,

$$\begin{bmatrix} M_{\bar{x}}(s) \\ M_{\bar{y}}(s) \\ M_{\bar{z}}(s) \end{bmatrix} = \mathbf{R}(s)^T \left\{ \begin{bmatrix} \bar{M}_x \\ \bar{M}_y^q \\ \bar{M}_z^q \end{bmatrix} + [\mathbf{r}(L) - \mathbf{r}(s)] \times \begin{bmatrix} \bar{F}_x \\ -(\bar{M}_z^p + \bar{M}_z^q)/(L + u_x^q) \\ (\bar{M}_y^p + \bar{M}_y^q)/(L + u_x^q) \end{bmatrix} \right\}, \quad (3.53)$$

between the internal moments $M_{\bar{x}}, M_{\bar{y}}, M_{\bar{z}}$ expressed in moving frame $\mathbf{e}_{\bar{x}}, \mathbf{e}_{\bar{y}}, \mathbf{e}_{\bar{z}}$ at s , and some externally applied loads $\bar{F}_x, \bar{M}_x, \bar{M}_y^p, \bar{M}_y^q, \bar{M}_z^p, \bar{M}_z^q$ at $s = 0$ and $s = L$, expressed in the fixed frame $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$. The linearized solution, i.e. equilibrium in the undeformed configuration, is obtained when $\mathbf{R}$ is reduced to the identity matrix, and $\mathbf{u} = 0$ so that $\mathbf{r} = \mathbf{r}_0$.

The purpose of the element is to model sheet flexures with the distinct dimensions as they are often encountered in precision flexure mechanisms. We make the specific assumptions that

Figure 3.3: Deformed configuration of a wide and thin beam, depicted with the simply supported boundary conditions of Eq. (3.11). The nodal loads are expressed in the fixed frame $\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z$.

- the width is the same order of magnitude as the length,
- the transverse displacement along the thickness dimension less than 15% of the length, and
- the failure strain of the material is of the order of 0.003, which is a realistic failure strain for high tensile strength metals.

Also, all loads remain sufficiently low that no failure due to buckling or plasticity occurs. These assumptions can be used to estimate the magnitude of the various nonlinear terms in the compatibility equations of Eq. (3.4) and the equilibrium equations of Eq. (3.53), in turn providing magnitude estimates for the terms in the energy functional of Eq. (3.50) and the rationale for the truncation of terms such that only the significant terms up to the fourth order are retained.

We have estimated that the thickness is two orders of magnitude smaller than the length [57, see Sec. 2.2.4]. With the width dimension along the z -axis, $\kappa_{\bar{y}}$ is estimated to be two orders smaller than $\kappa_{\bar{x}}$ and $\kappa_{\bar{z}}$. Also, $u_{\bar{y}}$ is estimated to be two orders larger than $u_{\bar{x}}$ and $u_{\bar{z}}$, and $\phi_{\bar{y}}$ to be two orders smaller than $\phi_{\bar{x}}$ and $\phi_{\bar{z}}$. Furthermore, it follows that the leading order terms in the energy functional, including the fourth-order ones, are retained when the exact equilibrium equations of Eq. (3.53) are reduced to

$$\begin{aligned} M_{\bar{x}}(s) &= \tilde{M}_x + u_{\bar{y}}(s)(\tilde{M}_y^p + \tilde{M}_y^q)/L + u_{\bar{y}}'(s)[\tilde{M}_y^p(\xi - 1) + \tilde{M}_y^q\xi], \\ M_{\bar{y}}(s) &= \tilde{M}_y^p(\xi - 1) + \tilde{M}_y^q\xi, \\ M_{\bar{z}}(s) &= \tilde{M}_z^p(\xi - 1) + \tilde{M}_z^q\xi + u_{\bar{y}}(s)\tilde{F}_x - \phi_{\bar{x}}(s)[\tilde{M}_y^p(\xi - 1) + \tilde{M}_y^q\xi], \end{aligned} \quad (3.54)$$

and the exact compatibility equations of Eq. (3.4) are reduced to

$$\begin{aligned} \kappa_{\bar{x}} &= \phi_{\bar{x}}', & \gamma_{\bar{x}} &= u_{\bar{x}}' + u_{\bar{y}}'^2/2, \\ \kappa_{\bar{y}} &= \phi_{\bar{y}}' + \phi_{\bar{x}}\phi_{\bar{z}}' \approx \phi_{\bar{y}}' + \phi_{\bar{x}}u_{\bar{y}}'', & \gamma_{\bar{y}} &= 0, \\ \kappa_{\bar{z}} &= \phi_{\bar{z}}' \approx u_{\bar{y}}'', & \gamma_{\bar{z}} &= u_{\bar{z}}' + \phi_{\bar{y}}. \end{aligned} \quad (3.55)$$

Here, $\xi = s/L$ and the transverse shear along the thickness dimension $\gamma_{\bar{y}}$ is ignored so $\phi_z = u_y'$. In Eq. (3.54), the six unknown Hellinger–Reissner stress variables $\tilde{M}_x$, $\tilde{M}_y^p$, $\tilde{M}_y^q$, $\tilde{M}_z^p$, $\tilde{M}_z^q$ and $\tilde{F}_x$, denoted by a superscript tilde, have been introduced. Displacement $u_y(s)$ and rotation $\phi_x(s)$ are interpolated according to the linear solution of Eqs. (3.12)–(3.18), disregarding the effect of shear deformation on $u_y(s)$.

For a bimoment interpolation that is consistent with the torsion interpolation $M_{\bar{x}}$ of Eq. (3.54), and which takes the deformed configuration into account, the differential equation

$$GJ\kappa_{\bar{x}} - EI_w\kappa_{\bar{x}}'' = M_{\bar{x}}(s), \quad \kappa_{\bar{x}}(0) = W^p, \quad \kappa_{\bar{x}}(L) = W^q \quad (3.56)$$

can be used to calculate first $\kappa_{\bar{x}}(s)$ and then the bimoment $B(s) = EI_w\kappa_{\bar{x}}'(s)$. The bimoment interpolation obtained this way contains hyperbolic functions of s and is dependent on the unknown Hellinger–Reissner stress variables $\tilde{M}_x$, $\tilde{M}_y^p$, $\tilde{M}_y^q$, the nodal coordinates ϕ_x^q , ϕ_z^p , ϕ_z^q , W^p , W^q , and the dimensionless parameter

$$\lambda = L\sqrt{GJ/(EI_w)}. \quad (3.57)$$

It is given in full in Appendix 3.7.2.

The assumption of having only one small dimension is reflected in Eqs. (3.54) and (3.55): since the axial stiffness and the in-plane bending stiffness for this type of beam are much larger than the torsional stiffness and the out-of-plane bending stiffness, only the nonlinear terms in the elongation strain $\gamma_{\bar{x}}$ and in-plane curvature $\kappa_{\bar{y}}$ are needed. Also, only the deformed-configuration equilibrium terms need to be accounted for in the torsion moment $M_{\bar{x}}$ and the out-of-plane bending moment $M_{\bar{z}}$. These terms in the expressions for $M_{\bar{x}}$ and $M_{\bar{z}}$ constitute the third-order effects of the current work.

With the stress interpolation functions of Eq. (3.54), the compatibility equations of Eq. (3.55), and the displacement and rotation interpolation simply according to the linear solution of Eqs. (3.12)–(3.18), the Hellinger–Reissner functional of Eq. (3.50) can be expressed in terms of the six stresses $\tilde{F}_x$, $\tilde{M}_x$, $\tilde{M}_y^p$, $\tilde{M}_y^q$, $\tilde{M}_z^p$, $\tilde{M}_z^q$ and the nodal coordinates u_x^q , ϕ_x^q , ϕ_y^p , ϕ_y^q , ϕ_z^p , ϕ_z^q and W^p , W^q .

Taking variations with respect to the nodal coordinates yields the discretized equilibrium equations; taking variations with respect to the stresses yields the discretized constitutive equations. An important observation is the fact that the exact equilibrium equation of Eq. (3.53) is linear in the applied loads and the energy functional of Eq. (3.50) is at most quadratic in the stresses. As a consequence, the variations with respect to the stresses are always linear in the stresses whenever some form of the equilibrium equation of Eq. (3.53) is used. Hence, the stress variables can always be solved for and eliminated, yielding in turn a standard displacements-only based discretization.

3.3.4 Closed-form solution

It has been shown that the internal energy P_{HR} of Eq. (3.50) with the interpolation functions from Sec. 3.3.3 can also be used to develop an analytical parametric model for sheet flexures [57, see Sec. 2.2]. Such a model can be useful in a design context as it offers parametric insight.

This model can be obtained by combining the internal energy P_{HR} with some external work due to applied loads at the nodes, such as

$$\bar{F}_x u_x^q + \bar{M}_x \phi_x^q + \bar{M}_y^p \phi_y^p + \bar{M}_y^q \phi_y^q + \bar{M}_z^p \phi_z^p + \bar{M}_z^q \phi_z^q. \quad (3.58)$$

The variations of the total energy expression yield a nonlinear system of equations that happens to admit a closed-form polynomial solution when, first, the six Hellinger–Reissner stresses are eliminated and, second, the remaining smaller variables are expressed as functions of the larger variables, as

$$\left[u_x^q, \phi_y^p, \phi_y^q, \bar{M}_x, \bar{M}_z^p, \bar{M}_z^q \right]^T = \mathbf{f}(\phi_x^q, \phi_z^p, \phi_z^q, \bar{F}_x, \bar{M}_y^p, \bar{M}_y^q). \quad (3.59)$$

Variables $u_x^q, \phi_y^p, \phi_y^q$ are small because they correspond with motion in the directions of relatively high stiffness; variables $\bar{M}_x, \bar{M}_z^p, \bar{M}_z^q$ are small because they correspond with loads in the directions of relatively low stiffness. The opposite holds for the right-hand side variables. This solution can be found in closed form, precisely because the interpolation is based on the assumption that there is (only) one small dimension. This leads to a nonlinear coupling between only certain variables.

When transformed to cantilever boundary conditions, the analytical solution in this particular format provides a parametric nonlinear model for the effects due to the geometric stiffness as encountered in e.g. flexure mechanisms in precision manipulation [57, see Sec. 2.5]. The closed-form solution of Eq. (3.59) is used in part in Sec. 3.3.5 as well.

3.3.5 Numerical implementation

Numerical elements in the generalized strain beam formulation are defined by the deformation function $\boldsymbol{\varepsilon} = \mathcal{D}(\mathbf{x})$ and the stress function $\boldsymbol{\sigma} = \mathcal{S}(\boldsymbol{\varepsilon})$. The equilibrium equations are formulated in terms of the deformation function. For the second-order beam element in Sec. 3.2, we have shown that both functions can be obtained as the discretized compatibility and constitutive equations of a beam continuum. The third-order effects of this section stem from the consideration of the deformed-configuration equilibrium in combination with nonlinear compatibility equations. To account for these effects in the generalized strain beam formulation, we choose to use a refinement of the stress function and use the same deformation function as for the second-order element. It means that the third-order effects are accounted for solely by a nonlinear constitutive relation, which facilitates the numerical implementation.

To derive the appropriate constitutive relation, we observe that the variation of the potential energy with respect to the generalized strains is equal to the negative virtual work produced by the generalized stresses, i.e.

$$\delta P = \frac{\partial P}{\partial \varepsilon_i} \delta \varepsilon_i = \sigma_i \delta \varepsilon_i, \quad (3.60)$$

so that the generalized stresses can be found as

$$\sigma_i = \frac{\partial P}{\partial \varepsilon_i}. \quad (3.61)$$

This relation cannot be used directly with P_{HR} , since P_{HR} is a function of six Hellinger–Reissner stress variables and the local nodal coordinates $\mathbf{x}_l$, i.e.

$$P_{\text{HR}} = f_1(\tilde{F}_x, \tilde{M}_x, \tilde{M}_y^p, \tilde{M}_y^q, \tilde{M}_z^p, \tilde{M}_z^q, u_x^q, \phi_x^q, \phi_y^p, \phi_y^q, \phi_z^p, \phi_z^q, W^p, W^q), \quad (3.62)$$

and only indirectly dependent on $\boldsymbol{\varepsilon}$ through Eqs. (3.36)–(3.43). Since the $\boldsymbol{\varepsilon}$ are small quantities, a partial inversion of Eqs. (3.36)–(3.43),

$$u_x^q \approx \varepsilon_1 - (2\varepsilon_5^2 + \varepsilon_5\varepsilon_6 + 2\varepsilon_6^2)/(30L), \quad (3.63)$$

$$\phi_x^q \approx \varepsilon_2/L, \quad (3.64)$$

$$\phi_y^p \approx -\varepsilon_3/L + \varepsilon_2(\varepsilon_5 + 2\varepsilon_6)/(10L^2) + \varepsilon_7\varepsilon_5/(10L^2) - \varepsilon_8(\varepsilon_5 + \varepsilon_6)/(30L^2), \quad (3.65)$$

$$\phi_y^q \approx \varepsilon_4/L + \varepsilon_2(2\varepsilon_5 - 9\varepsilon_6)/(10L^2) + \varepsilon_8\varepsilon_6/(10L^2) - \varepsilon_7(\varepsilon_5 + \varepsilon_6)/(30L^2), \quad (3.66)$$

$$\phi_z^p \approx -\varepsilon_5/L, \quad (3.67)$$

$$\phi_z^q \approx \varepsilon_6/L, \quad (3.68)$$

$$W^p \approx \varepsilon_7/L^2, \quad (3.69)$$

$$W^q \approx \varepsilon_8/L^2, \quad (3.70)$$

can be used to replace the local nodal coordinates with ε_1 to ε_8 . Given that the fourth-order terms in P_{HR} are at most quadratic in the local nodal coordinates, Eqs. (3.63)–(3.70) can be truncated after the second-order terms in $\boldsymbol{\varepsilon}$. Since the axial and in-plane deformation are much smaller than the torsion and out-of-plane deformation, $\varepsilon_1, \varepsilon_3, \varepsilon_4$ may be regarded as second-order terms. By taking the variations with respect to the six stress parameters in P_{HR} , the linear equations

$$\mathbf{b}_1 = [\mathbf{A}_1(\mathbf{x}_l)][\tilde{F}_x \ \tilde{M}_x \ \tilde{M}_y^p \ \tilde{M}_y^q \ \tilde{M}_z^p \ \tilde{M}_z^q]^T \quad (3.71)$$

are obtained, where matrix $\mathbf{A}_1$ is a function of $\mathbf{x}_l$. While it may seem tempting to eliminate all six stress parameters from P_{HR} analytically, the energy expression then turns into a very large rational function and becomes cumbersome to deal with (even with analytical solver software), especially when the derivatives (see Eq. (3.61)) are calculated. Once more, the consequences of the assumption that there is (only) one small dimension can be exploited, in a similar fashion to the closed-form solution of Sec. 3.3.4. Eq. (3.71) can be solved for the specific combination of the small nodal coordinates $u_x^q, \phi_y^p, \phi_y^q$, associated with the high axial and high in-plane bending stiffness, and the loads $\tilde{M}_x, \tilde{M}_z^p, \tilde{M}_z^q$, associated with the low torsion and out-of-plane bending stiffness. The resulting expression then remains a simple polynomial, similar to Eq. (3.59). This polynomial can be used to reduce Eqs. (3.62) and (3.71) to

$$\mathbf{b}_2 = [\mathbf{A}_2(\boldsymbol{\varepsilon})][\tilde{M}_y^p \ \tilde{M}_y^q \ \tilde{F}_x]^T, \quad (3.72)$$

$$P_{\text{HR}} = f_2(\tilde{M}_y^p, \tilde{M}_y^q, \tilde{F}_x, \boldsymbol{\varepsilon}), \quad (3.73)$$

where the partially inverted deformation functions of Eqs. (3.63)–(3.70) have been used to replace $\mathbf{x}_l$ in f_1 and $\mathbf{A}_1$ with $\boldsymbol{\varepsilon}$.

The energy potential is now a function of $\boldsymbol{\varepsilon}$ and the three stress variables $\tilde{M}_y^p$, $\tilde{M}_y^q$ and $\tilde{F}_x$. Expressions for the refined generalized stresses $\boldsymbol{\sigma}$ are obtained by calculating the derivatives of P_{HR} (see Eq. (3.61)) and using the chain rule for the dependency of the stress variables on $\boldsymbol{\varepsilon}$ according to Eq. (3.72), yielding

$$\boldsymbol{\sigma} = \mathcal{S}\left(\boldsymbol{\varepsilon}, \tilde{M}_y^p, \tilde{M}_y^q, \tilde{F}_x, \frac{\partial \tilde{M}_y^p}{\partial \boldsymbol{\varepsilon}}, \frac{\partial \tilde{M}_y^q}{\partial \boldsymbol{\varepsilon}}, \frac{\partial \tilde{F}_x}{\partial \boldsymbol{\varepsilon}}\right). \quad (3.74)$$

This refined stress function can be used to account for the third-order effects that were derived in this section. The vector function is a polynomial of manageable size that still depends on three of the Hellinger–Reissner stress variables and their derivatives with respect to the generalized strains. For the numerical evaluation of $\boldsymbol{\sigma}$ for a given $\boldsymbol{\varepsilon}$, the 3-by-3 linear system of Eq. (3.72) can be solved numerically for $\tilde{M}_y^p$, $\tilde{M}_y^q$ and $\tilde{F}_x$. Their derivatives in turn can be solved numerically from the linear equations

$$\left[\mathbf{A}_2(\boldsymbol{\varepsilon})\right] \begin{bmatrix} \partial \tilde{M}_y^p / \partial \varepsilon_i \\ \partial \tilde{M}_y^q / \partial \varepsilon_i \\ \partial \tilde{F}_x / \partial \varepsilon_i \end{bmatrix} = \mathbf{b}_2 - \left[\partial \mathbf{A}_2 / \partial \varepsilon_i\right] \begin{bmatrix} \tilde{M}_y^p \\ \tilde{M}_y^q \\ \tilde{F}_x \end{bmatrix}, \quad i = 1 \dots 8, \quad (3.75)$$

which were obtained by differentiating the left- and right-hand side Eq. (3.72) with respect to ε_i . Since the eight linear systems contain the same $\mathbf{A}_2$ and $\mathbf{b}_2$, their decompositions can be reused for faster numerical evaluation.

3.4 APPLICATION EXAMPLES

The beam element with third-order stiffness effects has been incorporated into the SPACAR multibody software program [37]. In this section, a comparison with the earlier second-order beam element [34], summarized in Sec. 3.2.3, is carried out for static simulations. The examples are based on flexure mechanisms that are used in precision manipulation; these mechanisms commonly consist of sheet flexures (also referred to as leaf springs or flexure strips), which are wide and thin rectangular beams.

3.4.1 Cantilever beam

Consider the sheet flexure in Fig. 3.4, represented by the cantilever beam with dimensions 100 mm, 100 mm and 2.25 mm, Young's modulus 200 GPa and Poisson's ratio 0.3. The displacement in y -direction at the free end is prescribed a value up to 15% of the length. The rotation about the z -axis at the free end is fixed to a zero value. In the deflected equilibrium configuration, the prescribed displacement and rotation are removed and equivalent forces and moments at the free end are applied. The linearized equilibrium equations are then used to calculate the compliance matrix. Fig. 3.5 shows the diagonal components for the translational x - and z -direction. It can be seen that more than three SPACAR elements with the conventional second-order stiffness formulation are needed for reasonable convergence. The geometrically nonlinear BEAM188 beam element in ANSYS performs similar to the SPACAR element for the x -component

Figure 3.4: Cantilever beam, deflected due to a prescribed displacement u_y . The rotation about the z -axis at the free end is fixed to a zero value. The depicted width-to-length ratio is smaller than simulated.

Figure 3.5: Two diagonal components of the compliance matrix at the free end of a cantilever beam with increasing deflection. Graphs show the new third-order element in SPACAR, the second-order element in SPACAR and the ANSYS BEAM188 element.

Length	Width	Thickness	Young's modulus	Poisson's ratio	Fold angle
33 mm	22 mm	0.4 mm	210 GPa	0.31	97 deg.

Table 3.1: Parameters of the two sheet flexures that make up a single folded sheet flexure in the spherical flexure joint model.

of the compliance; for the z -component, the ANSYS one-element result is closer to the SPACAR three-element result.

A single element with the third-order stiffness formulation of the current work performs better and lies already within 0.8% of the converged value, represented by the 20-element SPACAR and ANSYS solutions, at maximum deflection.

The contribution of the Wagner torque to the torsion moment is relatively straightforward and summarized in Sec. 3.3.2. It can also have a significant effect on various components of the compliance matrix computed in a deformed state. This is illustrated by the same sheet flexure of Fig. 3.4 but with a different loading condition at the free end: transverse displacement u_y is set to $0.15L$; now, also z -axis rotation ϕ_z is set to 0.15 radians and the x -axis twist angle ϕ_x is incremented from 0 up to a value of 0.15 radians. In similar fashion as before, the compliance matrix at the free end is calculated. Models with the third-order element are compared to an ANSYS model with sufficient nonlinear SHELL181 elements for convergence. These plate elements are used to model the Wagner torque, which the ANSYS BEAM181 element does not take into account. It can be seen that the SPACAR models with the third-order beam elements converge to the ANSYS solution. At least three elements are needed to reduce the deviation to 2.4% at the maximum twist angle. When the Wagner torque term is excluded from the element formulation, it can be seen that the deviation increases to 38%, showing the significance of the term on component $\partial u_x / \partial M_y$ of the compliance matrix. Similar observations hold for the other compliance matrix components. With three third-order beam elements, the deviation of all components is at most 6.4% when the Wagner torque is accounted for.

It is noted that this second load case requires multiple third-order elements in order to model the Wagner torque accurately, unlike the previous load case, in which a single third-order element suffices. This can be explained by the refined moment interpolation used for the element: the Hellinger–Reissner moment interpolation accounts for an approximate deformed configuration of the sheet flexure (as evidenced by the presence of $u_y(s)$ and $\phi_x(s)$ in Eq. (3.54)) to enable the use of fewer elements. This approximation does not include the Wagner torque, though, because the relationship is nonlinear and nontrivial.

3.4.2 Spherical flexure joint

Consider the flexible mechanism in Fig. 3.7 as described by Naves et al. [52]. It consists of two sets of three so-called folded sheet flexures (also referred to as folded leaf springs). One set is connected between the fixed base and a rigid intermediate body; the other set

Figure 3.6: Effect of the Wagner torque nonlinearity on a compliance matrix component. Comparison of SPACAR solutions with the third-order beam element and a converged ANSYS solution with nonlinear plate elements. The legend summarizes the relative deviation in percent at $\phi_x = 0.15$ radians with respect to the ANSYS solution.

Figure 3.7: Spherical flexure joint [52]. Center figure shows the six folded sheet flexures without the frames. Right-hand side figure shows an exploded view. Design parameter details can be found in Ref. 52.

Figure 3.8: Support stiffness of the spherical flexure joint from models with the second-order and third-order beam element in SPACAR. The legend summarizes the computation time in seconds (second column) and the maximum of the support stiffness in 10^5 N/m (third column).

between the rigid intermediate body and a rigid end-effector. Each folded sheet flexure can be regarded as two regular sheet flexures, i.e. as two wide and thin rectangular beams, connected at an angle, with dimensions according to Tbl. 3.1. The folded sheet flexures are positioned such that the six fold lines all intersect the same point. This ensures that the mechanism has behavior that is approximately kinematically equivalent to a spherical joint, i.e. the end-effector has three rotational degrees of freedom with respect to the fixed base. Since this flexible embodiment of a spherical joint consists of elastically deforming components, the approximation of that behavior means that the rotations of the end-effector are associated with relatively low stiffness compared to the translations. The high stiffness components are referred to as support stiffness components. The change in support stiffness that accompanies deformation in the directions of low stiffness is an important performance characteristic, since it tends to limit the motion-guiding and load-carrying ability. It is governed by the geometric stiffness.

This system is simulated by 1, 2, 3 and 4 beam elements per sheet flexure, i.e. 2, 4, 6 and 8 elements per folded sheet flexure. Fig. 3.8 shows the support stiffness in z -direction at the center of the end-effector when the end-effector is tilted 30 degrees with respect to an axis in the x, y -plane. That tilt axis is rotated 2π radians about the z -axis in 50 steps. The legend shows the computation times. It can be seen that a single third-order element is already very close to the solution of four second-order elements: the difference in peak stiffness is only 2.3%. While computations with the third-order element itself are a little bit slower, the comparison with the two-element second-order solution of similar performance shows a reduction of 68% in computation time.

Figure 3.9: Schematic overview of the flexure-based hexapod. The spherical flexure joints contain the flexible elements: each of the twelve spherical flexure joints consists of six folded sheet flexures.

Figure 3.10: Performance of various elements in modeling the support stiffness of the flexure-based hexapod.

3.4.3 Flexure-based hexapod

A larger multibody system is a model of the fully flexure-based T-Flex hexapod [53], as shown in Fig. 3.9. This model consists of six actuated legs that each have two links, two spherical flexure joints and one revolute joint at the base. Since this model has twelve spherical flexure joints that each consist of six folded sheet flexures, the model complexity is significant: when using only a single beam element per sheet flexure, the number of elements is already 144 and the number of degrees of freedom is 625. Fig. 3.10 shows the support stiffness in z -direction as a function of the end-effector

displacement in z -direction. It can be seen that a single second-order element per sheet flexure is inadequate. Again, the model with a single third-order element per sheet flexure is very close to the converged solution obtained with multiple second-order elements. Also, it is noted that with increasing number of elements per sheet flexure, the computation time quickly escalates.

3.5 CONCLUSION

This paper presents a third-order stiffness formulation of a geometrically non-linear beam element in the generalized strain beam framework. The new two-node element is based on Timoshenko beam theory with the addition of nonuniform torsion and Wagner effects. The element formulation has been derived by means of the Hellinger–Reissner mixed variational principle and a specific set of interpolation functions. The interpolation of the warping bimoment is based on the differential equation solution with hyperbolic functions. The included third-order effects play a significant role in the description of the in-plane stiffness of wide and thin rectangular beams when deformed out of plane. Numerical simulations of precision mechanism examples that rely on in-plane stiffness show that a single third-order element per sheet flexure can already yield adequate results, which otherwise would have required at least twice as many second-order elements. Hence, the number of degrees of freedom can be reduced, and larger systems can be simulated or optimized. The comparison with two models of 24 and 288 flexible elements demonstrates a reduction in computation time by factors of 3 and 5, respectively.

3.6 ACKNOWLEDGMENTS

This research was funded by the Innovative Research Incentives Scheme VIDI (Stichting voor de Technische Wetenschappen) (14152 NWO TTW) of the Ministry of Education, Culture and Science of the Netherlands.

3.7 APPENDIX OF CHAPTER 3

3.7.1 Hyperbolic torsion angle solution

The full expression for angle ϕ_x , as outlined in Sec. 3.2.2, is given by

$$\begin{aligned}\phi_x = & \phi_x^q \frac{\lambda \xi \cosh(\lambda/2) - \sinh(\lambda/2) + \sinh(\lambda/2 - \lambda \xi)}{\lambda \cosh(\lambda/2) - 2 \sinh(\lambda/2)} \\ & + LW^p \frac{\operatorname{csch}^2(\lambda/2)}{-4\lambda + 2\lambda^2 \coth(\lambda/2)} (\lambda \xi + (\lambda - \lambda \xi) \cosh(\lambda) - \lambda \cosh(\lambda - \lambda \xi) \\ & \quad - \sinh(\lambda) + \sinh(\lambda \xi) + \sinh(\lambda - \lambda \xi)) \\ & - LW^q \frac{\operatorname{csch}^2(\lambda/2)}{-4\lambda + 2\lambda^2 \coth(\lambda/2)} (\lambda - \lambda \xi + \lambda \xi \cosh(\lambda) - \lambda \cosh(\lambda \xi) \\ & \quad - \sinh(\lambda) + \sinh(\lambda \xi) + \sinh(\lambda - \lambda \xi)),\end{aligned}\quad (3.76)$$

where the hyperbolic cosecant function is denoted by $\operatorname{csch}(\xi) = 1/\sinh(\xi)$, the hyperbolic cotangent function by $\coth(\xi) = \cosh(\xi)/\sinh(\xi)$, and the dimensionless parameter λ by

$$L\sqrt{GJ/(EI_w)}. \quad (3.77)$$

3.7.2 Bimoment interpolation

The full expression for the bimoment interpolation, as outlined in Sec. 3.3.3, is given by

$$\begin{aligned}B = & -W^p LGJ \frac{\cosh(\lambda - \lambda \xi) \operatorname{csch}(\lambda)}{\lambda} + W^q LGJ \frac{\cosh(\lambda \xi) \operatorname{csch}(\lambda)}{\lambda} \\ & + \tilde{M}_x L \frac{\operatorname{sech}(\lambda/2) \sinh(\lambda/2 - \lambda \xi)}{\lambda} \\ & + \tilde{M}_y^p L \left\{ \phi_z^p (24 + 6\lambda^2 - 18\lambda^2 \xi + 12\lambda^2 \xi^2) + \phi_z^q (24 + 2\lambda^2 - 12\lambda^2 \xi + 12\lambda^2 \xi^2) \right. \\ & \quad + [-6\phi_z^p - 12\phi_z^q] \cosh(\lambda \xi) \operatorname{csch}(\lambda) \lambda \\ & \quad \left. + [\phi_z^p (-18 - \lambda^2) - 12\phi_z^q] \cosh(\lambda - \lambda \xi) \operatorname{csch}(\lambda) \lambda \right\} \\ & + \tilde{M}_y^q L \left\{ \phi_z^p (24 + 2\lambda^2 - 12\lambda^2 \xi + 12\lambda^2 \xi^2) + \phi_z^q (24 - 6\lambda^2 \xi + 12\lambda^2 \xi^2) \right. \\ & \quad + [-12\phi_z^p + \phi_z^q (-18 - \lambda^2)] \cosh(\lambda \xi) \operatorname{csch}(\lambda) \lambda \\ & \quad \left. + [-12\phi_z^p - 6\phi_z^q] \cosh(\lambda - \lambda \xi) \operatorname{csch}(\lambda) \lambda \right\},\end{aligned}\quad (3.78)$$

where $\operatorname{sech}(\xi) = 1/\cosh(\xi)$ denotes the hyperbolic secant function.

CHAPTER 4

Misalignments in an overconstrained flexure mechanism: a cross-hinge stiffness investigation

Thesis note The paper in this chapter addresses the second subgoal, on flexure mechanism design principles. It presents an analysis of the effects of misalignments in an overconstrained cross-hinge mechanism to explore the use of overconstrained design in pursuit of higher performance. The analytical and numerical models are based on largely the same solid mechanics theory as the preceding chapters, with the addition of a bifurcation buckling analysis.

Abstract The exact-constraint design principle is commonly applied to flexure mechanisms to ensure deterministic behavior, but at the cost of reduced robustness, support stiffness, load capacity and usually an increased complexity of design. To explore the potential benefit of overconstrained design in flexure mechanisms, this paper investigates an elementary two-flexure cross-hinge with a single overconstraint as a case study. The stiffness effects of inadvertent stress due to misalignment are investigated experimentally, numerically and analytically.

A measurement set-up with controllable misalignment has been designed. Measurements show that the first natural frequency of the cross-hinge decreases strongly with misalignment, suggesting that the actuation stiffness decreases due to the misalignment stress, and ultimately vanishes due to bifurcation buckling at a critical misalignment of the order of 0.1 mm for the mechanism at hand.

Simulations with a detailed numerical model support the measurements and expose some additional factors, such as warping and shuttle compliance, which influence the system behavior. Importantly, they also show that the compliance in the support directions of the mechanism increases strongly at the critical misalignment, demonstrating that the mechanism no longer functions at the critical misalignment.

An extensive analytical buckling analysis shows how the stress due to misalignment poses a functional operation limit on the overconstrained mechanism in terms of bifurcation buckling. The analysis serves to corroborate the numerical predictions. An expression is derived for the critical misalignment force and displacement as a function of the geometry and material of general cross-hinge mechanism designs.

4.1 INTRODUCTION

Flexure mechanisms are used in precision applications for their ability to guide motion with high repeatability. While traditional rigid-link mechanisms move due to rolling or sliding components that induce hysteresis, flexure mechanisms move solely due to elastically deforming components. This means that flexure mechanisms operate without

Published in Precision Engineering 62 (2020) pp. 181–195. Co-authors: J.P. Meijaard and D.M. Brouwer.

friction, backlash, stick-slip and wear, resulting in low hysteresis [12, 15, 22, 26, 31, 39, 59, 67, 72, 73].

Deterministic behavior also requires that the influence of manufacturing tolerances and environment variations on static and dynamic performance is limited. This is typically achieved by employing mechanism designs that are exactly constrained [15, 26, 39, 67, 72, 73].

The exact-constraint design principle originates from rigid-link mechanism analysis, in which a mechanism is reduced to a kinematic abstraction: it is regarded as a kinematic assembly of constraint-imposing components that together produce the desired mobility or degrees of freedom (DOF) of the mechanism. When each constraint contributes uniquely to the mechanism mobility, i.e. all constraints are essential, the mechanism is exactly constrained. When the same mobility is achieved with more than just the essential constraints, i.e. there are redundant constraints, the mechanism is overconstrained (also referred to as statically indeterminate) [44].

While the kinematic idealization of a mechanism is not affected by redundant constraints, the physical realization is. Geometric imperfections, such as the ones caused by manufacturing tolerances or thermal expansion differences, essentially turn the redundant constraints into a set of conflicting constraints. This can lead to undesired internal loads and a loss of mobility (also referred to as locking or binding). The rationale for the use of exact-constraint designs is then that the reliance on geometric accuracy can be reduced by simply avoiding all overconstrained designs, and restricting the design space to only the exactly constrained ones. In practice, restriction to exact-constraint design tends to limit the load bearing capacity and produce asymmetric designs.

Exact-constraint design is applied to flexure mechanisms, because its elastic components can also be regarded as constraint-imposing components, albeit with nonideal behavior. By design, the elastic components provide a large ratio of stiffness in various directions. The motion directions of low stiffness are considered the DOFs of the component, relative to the directions of high stiffness, which are considered the constraint (load-carrying, support) directions. Elastic components only approximate the ideal kinematic behavior, since they provide nonzero stiffness (ideally zero) in the DOFs and limited maximum stiffness (ideally infinite) in the constraint directions. While seemingly unfavorable, not strictly enforcing constraints actually implies that the undesired associated loads are lower. Conflicting constraints, e.g. due to geometric imperfections in overconstrained designs, therefore produce lower stress in flexure mechanisms, compromising performance to a lesser degree, with a lower risk of locking.

Since the rationale for exact-constraint design is mostly qualitative and based on rigid-link mechanisms, i.e. neglecting the elastic nature of flexure mechanisms, its use in flexure mechanisms is potentially overly restrictive and an unnecessary limitation on static and dynamic performance. This provides the motivation for the investigation presented in this paper.

In literature, the system behavior of various overconstrained flexure mechanisms has been investigated. Meijaard et al. [50] have studied a parallelogram flexure mechanism, which has a single overconstraint. The effect of a misalignment on eigenfrequency and stiffness is determined analytically, numerically and experimentally. It is shown that

exceeding the critical misalignment can induce bifurcation buckling and compromise system performance. Van de Sande et al. [65] have studied a four-bar mechanism with flexible hinges, which has three overconstraints. A method for establishing the various critical misalignments is presented and validated experimentally.

The potential of overconstrained mechanisms not only to tolerate certain geometric imperfections, but actually achieve greater performance has been investigated as an alternative design paradigm, referred to as elastic averaging [11, 32, 82].

In this paper, we focus on the behavior of an overconstrained flexure mechanism, in order to quantify the performance and the limits of operation. Since flexure mechanisms with a rotational degree of freedom are commonly used in larger precision systems, we choose an elementary two-flexure cross-hinge mechanism to serve as a case study [27, 29, 84]. It has a single overconstraint, limiting the complexity.

The contributions of this paper are a detailed elasticity analysis of the consequences of misalignment in a single-overconstraint cross-hinge mechanism. We present

1. the design of a measurement set-up with controllable misalignment and several features to limit parasitic compliance and hysteresis, and offer repeatable measurements,
2. measurements of eigenfrequencies that show how the system performance is affected by misalignments,
3. numerical simulations that corroborate the measurements, and
4. an analytical buckling analysis of the system. This analysis provides a verification of the numeric model, the theoretical explanation of the observed system behavior and an explicit relation for the critical misalignment as a function of the geometry and material of general cross-hinge mechanism designs.

4.2 CASE DESCRIPTION

4.2.1 Overconstrained flexure mechanism

Fig. 4.1 shows the essence of the two-leaf spring cross-hinge flexure mechanism that serves as the case study of this work. It consists of a shuttle and two attached leaf springs that constrain certain motion of the shuttle. The leaf springs have equal dimensions and cross at a right angle at half their length. In the analytical investigation in Sec. 4.3, this angle and crossing length remain free parameters in order to obtain a more widely applicable result.

A typical leaf spring has a thickness that is one or two orders of magnitude smaller than its width. The length is usually larger than the width. As a consequence of these dimensions, a leaf spring offers a large ratio of stiffness associated with relative motion between its ends. As Fig. 4.2 summarizes for fixed-free boundary conditions, the stiffness is relatively low for motion in the rotational x_1 -, translational x_2 -, and rotational x_3 -directions (associated with torsion, and bending in the plane of lowest rigidity). Contrarily, the stiffness is relatively high for motion in the translational x_1 -, rotational

Figure 4.1: Overconstrained cross-hinge design consisting of two leaf springs. Gray scales in the leaf springs represent the von Mises stress distribution caused by misalignment v_0 due to a force F_0 .

Figure 4.2: DOFs (solid lines) and constraints (dashed lines) of a leaf spring.

x_2 -, and translational x_3 -directions (associated with bending in the plane of highest rigidity, shear and elongation). Due to these stiffness properties, the leaf spring can be considered as a constraint element for relative motion in the three independent directions of high stiffness.

When attached to the shuttle at one end and fixed at the other, the two leaf springs together provide six constraints to the shuttle. Since each leaf spring independently already constrains the vertical position of the shuttle, the system is overconstrained. The two leaf springs constitute an overconstraint for the vertical shuttle position. Only five of the constraints applied to the shuttle are independent and the shuttle has one degree of freedom: it can rotate about the axis that is (for small rotation angles) the intersection of the planes of the two undeformed leaf springs. The intersection line can also be regarded as the overconstrained translational direction. As a mechanism, this configuration is commonly referred to as a cross-hinge or cross-spring flexure or pivot. It guides motion about the rotational axis with low stiffness (referred to as drive or actuation stiffness) and provides high stiffness (referred to as support stiffness or off-axis stiffness) in all other motion directions.

A translational guidance in the vertical direction, represented in Fig. 4.1 by the rollers for the bottom leaf spring, removes the overconstraint and makes the system exactly constrained when the bearing is free to move ($F_0 = 0$). Incidentally, this example also illustrates that exact-constraint design tends to lead to lower support stiffness: since only one leaf spring now carries vertical loads applied to the shuttle, the desired high stiffness of the shuttle in the vertical direction is lower in the exact-constraint design configuration.

4.2.2 Misalignment stress

In the overconstrained configuration, the position of the translational guidance of Fig. 4.1 is essentially prescribed. Since motion in the direction of the overconstraint is associated with the high support stiffness of the leaf springs, a small displacement or misalignment v_0 with respect to the neutral (stress-free) position leads to large internal loads. Such a misalignment can stem from environmental variations, such as manufacturing tolerances or thermal expansion differences. Fig. 4.1 depicts the distribution of the von Mises stress in the leaf springs due to a misalignment v_0 . The leaf springs are essentially loaded by a transverse force under fixed-guided boundary conditions. The dominant stress is normal stress due to bending in the plane of highest rigidity.

One of the ways in which internal loads compromise the performance of the mechanism is by affecting the stiffness. Much like the classical Euler column that loses transverse stiffness under compressive force up to the point of buckling (and, similarly, gains transverse stiffness under tensile force), loads in more complex mechanisms can also increase or decrease the stiffness in various directions. This load-dependent stiffness is sometimes referred to as the geometric stiffness. For the correct operation of the overconstrained cross-hinge, it is crucial that the shuttle provides high support stiffness. Misalignment stress has been shown to cause a decrease of stiffness in other flexure

Length	Width	Thickness	E	ν
80 mm	30 mm	0.3 mm	200 GPa	0.29

Table 4.1: Nominal values of the leaf springs' properties (E is Young's modulus, ν Poisson's ratio).

mechanisms, even up to a critical level at which the leaf springs buckle and almost all stiffness is lost [50]. As such, the occurrence of buckling due to misalignments can be designated as an upper limit of operation for an overconstrained flexure mechanism.

Dynamically, the natural frequencies (also referred to as eigenfrequencies) of the system are affected similarly, since they are governed by the combination of stiffness and misalignment-independent mass. When stiffness in a certain direction decreases, the natural frequency of the mode corresponding to motion in that direction also decreases. When all stiffness is lost, buckling has occurred; the natural frequency then is zero and the corresponding natural mode coincides with the buckling mode of the system. This gives a further interpretation to buckling as an upper limit of operation: the buckling mode is indicative of the motion the system can make without stiffness. The buckling mode is essentially a bifurcation of the equilibrium state, with zero linear stiffness.

4.2.3 Physical realization

A measurement set-up has been designed and fabricated for measuring the effects of misalignments. The nominal dimensions of both leaf springs are given in Tbl. 4.1. We choose to characterize the natural frequencies of the system, instead of the stiffness directly, because precise measurement of the natural frequencies of lowly damped mechanisms is relatively easy, especially for multiple modes (i.e. multiple stiffness components). By design, flexure mechanisms provide widely ranging stiffness in various directions; accurately measuring high stiffness in a certain direction would require an elaborate set-up for avoiding the unintentional contribution of 'off-axis' low stiffness (parasitic compliance).

The use of modal analysis assumes that the system responds linearly to actuation inputs. In the relation between the angle of the shuttle and the applied driving torque, there are nonlinear effects due to large deformation and anti-clastic curvature. These effects are negligibly small (less than 2% effect) for angles of rotation smaller than 0.1 rad and leaf springs with a small width to length ratio [22]. For the current investigation, a modal analysis of the natural frequencies as functions of misalignment contains the necessary information for understanding the system behavior and validating the models.

Although buckling is fundamentally dependent on the load in the structure, we choose to apply the misalignment by means of a prescribed displacement instead of a prescribed force. A primary reason is to avoid excessive deflection that would accompany the occurrence of buckling if the mechanism were loaded by a constant force. This does imply that the exact force in the mechanism is unknown, and only obtained as a quantity derived from measurements of the support stiffness and the prescribed displacement. To provide a constant force to the system, the stiffness should be constant.

Figure 4.3: Photograph of measurement set-up.

Since the support stiffness of the cross-hinge mechanism decreases with rotation of the shuttle, it is important to limit the amplitude of the shuttle motion during the modal analysis. Simulations show that the decrease in support stiffness of this mechanism is less than 5% if the shuttle rotation remains smaller than 0.025 rad, which corresponds to a practically feasible maximum transverse displacement at the end of the leaf springs of 1 mm.

Design characteristics

Fig. 4.3 shows a photograph of the measurement set-up. To reliably control misalignments in the micrometer range, the set-up has several design characteristics that reduce

Figure 4.4: Equivalent stiffness of frame, guidance and cross-hinge in the load-carrying direction. The values of k_{leaf} and k_{shuttle} , making up the effective cross-hinge support stiffness, depend on the cross-hinge configuration parameter b , described in Sec. 4.3.6 and Fig. 4.5(a).

parasitic compliance and hysteresis.

The translational guidance of Fig. 4.1 for setting the misalignment is realized by means of a parallel flexure mechanism, consisting of the guidance stage and guidance leaf springs in Fig. 4.3, to increase repeatability. It has a very high support stiffness; the stroke is limited to 1 mm before yielding. A fine-thread screw is used at the bottom to prescribe the misalignment displacement. The point of application is such that the induced force acts through the center of compliance of the parallel flexure mechanism. This ensures that the stage of the parallel flexure mechanism has no parasitic rotational motion when it moves in its single translational DOF [16, 74]. The parasitic translational motion, i.e. due to the so-called shortening of the leaf springs, is at most 8 μm .

The misalignment displacement is measured at the other side of the parallel flexure mechanism, to avoid the effects of deformation of the guidance stage. Also, the dial gauge is placed co-axially with the misalignment screw, to avoid Abbe errors [3].

Each leaf spring with the two support fillets is made of a single piece of material (stainless steel, AISI 420) by wire EDM, to avoid clamping of the leaf springs and the consequences of strain differences between fillet and leaf spring (micro-slip).

The cross-hinge design consists of only two leaf springs, to have a minimal number of parts. This reduces parasitic compliance due to interface connections. The cross-hinge is oriented such that the rotation axis is vertical. This way the effects of gravity work mainly in the controllable overconstrained direction of the mechanism, and not in the actuation direction.

Equivalent stiffness

Since the cross-hinge mechanism is loaded by a prescribed misalignment displacement v_0 , and not by a prescribed force, it is important that the load-carrying frames provide sufficient stiffness and do not deform significantly. Fig. 4.4 shows a model of stiffness in series and parallel that is representative of the total stiffness between force F_0 and displacement v_0 in the load-carrying part of the set-up. The effective stiffness of the translational guidance, consisting of the guidance stage and guidance leaf springs, is represented by k_{guidance} . The stiffness in the load-carrying direction of the cross-hinge

k_{frame}	k_{leaf}	k_{shuttle}	k_{guidance}
$5 \cdot 10^7$	$2 \cdot 10^6$	$2 \cdot 10^6$	$8 \cdot 10^4$

Table 4.2: Equivalent stiffness in N/m.

mechanism is represented by the series arrangement of cross-hinge leaf spring stiffness k_{leaf} and shuttle stiffness k_{shuttle} . The effective frame stiffness is represented by k_{frame} .

The numerical values are given in Tbl. 4.2. The value of k_{guidance} is obtained by direct measurement (using a calibrated weight with a capacitive displacement sensor). The value of k_{leaf} is the theoretical value due to bending and shear deformation. The value of k_{frame} is obtained by FEM. The effective shuttle stiffness k_{shuttle} is inferred from the measurement of k_{guidance} , the theoretical value of k_{leaf} and the measured total stiffness between F_0 and v_0 . It is concluded that the frame is sufficiently stiff for the purpose of this investigation. Since the shuttle has the same order of magnitude stiffness as the undeflected leaf springs, the shuttle elasticity does have to be accounted for in the numerical model.

4.3 ANALYTICAL INVESTIGATION

For understanding how stress due to misalignment can affect the system behavior, a stability analysis is performed. Based on first principles, it exposes the mechanics that govern the operational limit of the system.

Stability analysis in structural mechanics deals with the character of the response of structures to some load case. A structure is said to have buckled when at some level of the increasing load it assumes a distinctly different equilibrium configuration. As this new configuration can be reached in various ways, different buckling mechanisms, e.g. bifurcation buckling, finite-disturbance buckling and snapthrough buckling, can be distinguished [71]. In the following section, only the bifurcation buckling mechanism will be considered. Various solution approaches are available, such as the classical force equilibrium method, the free-vibration dynamic method, and the minimum potential energy method. We adopt a theory based on the potential energy of the system [17, 38, 40, 79]. This provides a systematic way of modeling the system and deriving the conditions that govern elastic stability. To illustrate some crucial steps in this procedure, a highly simplified example is provided in Appendix 4.8.1.

The buckling of beams with highly different flexural rigidities, loaded in the plane of highest rigidity, is called lateral buckling. This phenomenon was first studied by Michell and Prandtl [51, 62].

4.3.1 Model description

For a balance between model accuracy and mathematical complexity, we choose to account for the 3-D behavior of the leaf springs using a beam model, but ignoring

Figure 4.5: Configuration parameters.

gravity, shuttle elasticity and the effects of constrained warping. Mathematically, we exploit the constraint-based view of leaf springs and assume that the only strain energy is due to deformation modes associated with low stiffness (i.e. torsion and bending in the plane of lowest rigidity). The other deformation modes (i.e. elongation, transverse shear and bending in the plane of highest rigidity) have high stiffness, experience very small deformation, and mainly serve to transfer forces and moments. This is approximated by using Lagrange multipliers to include the conditions that the deformations of high stiffness are zero (i.e. considered infinitely stiff). The shuttle is modeled by a kinematic relation between the two ends of the leaf springs. The angle at which the two leaf springs cross and the length of the shuttle are parameters, such that the analysis can be generalized to a wide variety of cross-hinge designs.

4.3.2 Configuration coordinates

The system consists of two leaf springs of length L , width h and thickness t , and the shuttle of length d_x . The lateral offset between the leaf springs is d_y . The angle between the leaf springs and the normal to the shuttle is α , as indicated in Fig. 4.5(a). In the following analysis, superscripts l and r are used to denote the left and right leaf spring, respectively.

The deformed configuration of the leaf springs is described with respect to the *initial* coordinate frames. These frames are attached to the undeformed leaf springs and given by base vectors $\mathbf{e}_x^l, \mathbf{e}_y^l, \mathbf{e}_z^l$ and $\mathbf{e}_x^r, \mathbf{e}_y^r, \mathbf{e}_z^r$, whose origins coincide with the centroid of the bottom cross-section of the leaf springs. Vectors $\mathbf{e}_x^{l,r}$ and $\mathbf{e}_y^{l,r}$ are oriented along the principal axes of the cross-section and vector $\mathbf{e}_z^{l,r}$ is normal to the cross-sectional plane. The relation with the *global* coordinate frame vectors $\mathbf{n}_x, \mathbf{n}_y, \mathbf{n}_z$ is given by

$$\begin{bmatrix} \mathbf{e}_x^l & \mathbf{e}_y^l & \mathbf{e}_z^l \end{bmatrix} = \mathbf{R}_l \begin{bmatrix} \mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \end{bmatrix}, \quad (4.1)$$

$$\begin{bmatrix} \mathbf{e}_x^r & \mathbf{e}_y^r & \mathbf{e}_z^r \end{bmatrix} = \mathbf{R}_r^T \begin{bmatrix} \mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \end{bmatrix}, \quad (4.2)$$

where

$$\mathbf{R}_i = \begin{bmatrix} \cos \alpha & 0 & -\sin \alpha \\ 0 & 1 & 0 \\ \sin \alpha & 0 & \cos \alpha \end{bmatrix}. \quad (4.3)$$

The deformed configuration of the leaf springs is described by three position and three orientation functions. The position of the deformed neutral line of the leaf springs is given by the vectors

$$\mathbf{r}^l(s) = \mathbf{r}_0^l(s) + \mathbf{u}^l(s) = [u^l(s) \ v^l(s) \ s + w^l(s)], \quad (4.4)$$

$$\mathbf{r}^r(s) = \mathbf{r}_0^r(s) + \mathbf{u}^r(s) = [u^r(s) \ v^r(s) \ s + w^r(s)], \quad (4.5)$$

resolved in the initial frames, where s is the material coordinate from 0 to L along the undeformed leaf springs. For the orientation of the deformed cross-section, a *local* coordinate frame is introduced, see Fig. 4.5(b). It is attached to the deformed cross-section, given by base vectors $\mathbf{e}_x^l, \mathbf{e}_y^l, \mathbf{e}_z^l$ and $\mathbf{e}_x^r, \mathbf{e}_y^r, \mathbf{e}_z^r$, and related to the global frame by

$$[\mathbf{e}_x^l \ \mathbf{e}_y^l \ \mathbf{e}_z^l] = \mathbf{R}_i \mathbf{D}^l [\mathbf{n}_x \ \mathbf{n}_y \ \mathbf{n}_z], \quad (4.6)$$

$$[\mathbf{e}_x^r \ \mathbf{e}_y^r \ \mathbf{e}_z^r] = \mathbf{R}_i^T \mathbf{D}^r [\mathbf{n}_x \ \mathbf{n}_y \ \mathbf{n}_z]. \quad (4.7)$$

Matrices $\mathbf{D}^l$ and $\mathbf{D}^r$ are rotation matrices that describe the rotation of the deformed cross-sections. They can be parametrized in several ways. We choose the Euler angles ϕ^l, ψ^l, χ^l and ϕ^r, ψ^r, χ^r , which are all functions of s , for successive rotation of the initial frames about the x -, z - and y -axes. The full expressions are provided in Appendix 4.8.2. For small deformations, ϕ can be interpreted as the bending angle in the plane of highest rigidity, ψ as the torsion angle and χ as the bending angle in the plane of lowest rigidity of the leaf spring.

The deformation of the beam can be captured by the three curvatures, given as components of the skew-symmetric matrix

$$\tilde{\kappa} = \mathbf{D}^T \mathbf{D}', \quad (4.8)$$

and by the three strains, given as components of the vector

$$\boldsymbol{\gamma} = \mathbf{D}^T \mathbf{r}' - \mathbf{n}_z. \quad (4.9)$$

Since the expressions are the same for the left and right leaf springs, the superscript has been omitted. A prime denotes differentiation with respect to s . In terms of the displacement and rotation functions, a second-order approximation of the six deformation components is given by

$$\kappa_x = \phi' \cos \chi \cos \psi - \psi' \sin \chi \approx \phi' - \psi' \chi, \quad (4.10)$$

$$\kappa_y = \chi' - \phi' \sin \psi \approx \chi' - \phi' \psi, \quad (4.11)$$

$$\kappa_z = \psi' \cos \chi + \phi' \cos \psi \sin \chi \approx \psi' + \phi' \chi, \quad (4.12)$$

$$\gamma_x \approx u' - \chi(1 + w') + \psi(\phi + v'), \quad (4.13)$$

$$\gamma_y \approx v' + \phi(1 + w') - \psi u', \quad (4.14)$$

$$\gamma_z \approx w' - \phi(\phi/2 + v') - \chi(\chi/2 - u'). \quad (4.15)$$

Component κ_x can be interpreted as the curvature in the plane of highest rigidity, κ_y as the curvature in the plane of lowest rigidity, κ_z as the specific twist angle, γ_x and γ_y as the transverse shear strains, and γ_z as the elongation strain.

4.3.3 Kinematic constraints due to the shuttle

The shuttle imposes a fixed distance between the two leaf spring ends, governed by the vector equation

$$\mathbf{R}_i^T \mathbf{u}^r(L) - \mathbf{R}_i \mathbf{u}^l(L) - (\mathbf{R}_i \mathbf{D}^l(L) \mathbf{R}_i^T - \mathbf{I}) \begin{bmatrix} d_x \\ d_y \\ 0 \end{bmatrix} = 0. \quad (4.16)$$

This equation is abbreviated as $[K_x \ K_y \ K_z]^T = 0$ for use in the following sections. In Appendix 4.8.3, the derivation is detailed.

In addition, the shuttle imposes a constraint on the orientation of the two leaf springs, expressed as

$$\mathbf{R}_i \mathbf{D}^l(L) \mathbf{R}_i^T = \mathbf{R}_i^T \mathbf{D}^r(L) \mathbf{R}_i. \quad (4.17)$$

This matrix equality can be converted to the three equations

$$\begin{aligned} \chi^l - \chi^r + \text{quadratic terms} &= 0, \\ \psi^l - \psi^r \cos 2\alpha + \phi^r \sin 2\alpha + \text{cubic terms} &= 0, \\ \phi^l - \phi^r \cos 2\alpha - \psi^r \sin 2\alpha + \text{cubic terms} &= 0, \end{aligned} \quad (4.18)$$

to be satisfied at $s = L$. These equations are abbreviated as $[K_\chi \ K_\psi \ K_\phi]^T = 0$.

4.3.4 Potential energy

The contribution of a single leaf spring to the potential energy of the system is

$$P_{\text{leaf}} = \frac{1}{2} \int_0^L [S_t \kappa_z^2 + S_b \kappa_y^2] ds + \int_0^L [N_u \gamma_x + N_v \gamma_y + N_w \gamma_z + M \kappa_x] ds. \quad (4.19)$$

The first integral represents strain energy due to torsion (with stiffness S_t) and bending (with stiffness S_b) in the plane of lowest rigidity. The torsion and bending stiffness are given by

$$S_t = GJ, \quad S_b = EI_y, \quad (4.20)$$

where G is the shear modulus, J is Saint-Venant's torsion constant, E is Young's modulus, and I_y is the second moment of area about the y -axis of the initial frame (Fig. 4.5(a)). The second integral provides the constraints on the four other deformation components by means of the Lagrange multipliers N_u , N_v , N_w and M , which are still functions of s .

The total potential energy of the system is given by

$$\begin{aligned} P_{\text{tot}} &= P_{\text{leaf}}^l + P_{\text{leaf}}^r - F_0 v^l(0) \\ &\quad + D_x K_x + D_y K_y + D_z K_z + R_\chi K_\chi + R_\psi K_\psi + R_\phi K_\phi. \end{aligned} \quad (4.21)$$

$u^l, v^l, w^l, \chi^l, \psi^l, \phi^l$	configuration functions
$u^r, v^r, w^r, \chi^r, \psi^r, \phi^r$	for left and right leaf spring
N_u^l, N_v^l, N_w^l, M^l	deformation constraint multipliers
N_u^r, N_v^r, N_w^r, M^r	for left and right leaf spring
D_x, D_y, D_z	translational and rotational
R_χ, R_ψ, R_ϕ	shuttle constraint multipliers

Table 4.3: Unknowns that are subject to variation.

Here, P_{leaf}^l and P_{leaf}^r represent the contributions of the left and right leaf spring according to Eq. (4.19). The external work $F_0 v^l(0)$ is the work due to the misalignment at the bottom of the left leaf spring in the vertical direction. Lagrange multipliers D_x, D_y, D_z are used to include the translational shuttle constraint of Eq. (4.16). Lagrange multipliers R_χ, R_ψ and R_ϕ are used to include the rotational shuttle constraint of Eq. (4.18).

4.3.5 Kinematic boundary conditions

The boundary conditions due to the clamping of the leaf springs at the bottom ends are not included in the total potential energy. They are given by

$$\begin{aligned} \text{at } s = 0: \quad & u^l = u^r = v^r = w^l = w^r = 0, \\ & \chi^l = \chi^r = \psi^l = \psi^r = \phi^l = \phi^r = 0, \end{aligned} \quad (4.22)$$

and have to be imposed separately.

4.3.6 Fundamental solution

For the fundamental solution of the system, i.e. the straight undeformed configuration, the expressions for the curvatures, Eqs. (4.10)–(4.12), the strains, Eqs. (4.13)–(4.15), and the shuttle constraints, Eqs. (4.16) and (4.18), can be linearized. Equilibrium requires that the total potential energy of the system is minimized with respect to variations of all unknown functions and variables, summarized in Tbl. 4.3. These are the six functions for the configuration of each leaf spring, the multipliers (functions of s) for the deformation constraints for each leaf spring and the (constant) multipliers for the shuttle constraints.

Taking variations with respect to the unknown functions then yields a system of ordinary differential equations (ODEs) with dynamic boundary conditions. Taking variations with respect to the unknown multipliers again returns the linearized constraints that we sought to impose. The fundamental solution is given by a zero value for all

configuration functions. However, it turns out that some of the multipliers are actually nonzero in the fundamental state and take on the values

$$\begin{aligned} N_v^l &= -F_0, & D_y &= -F_0, \\ N_v^r &= F_0, & R_\phi &= d_x F_0 \cos \alpha \cot 2\alpha, \\ M^l &= -F_0(s-b), & R_\psi &= -d_x F_0 \cos \alpha, \\ M^r &= F_0(s-b), \end{aligned} \quad (4.23)$$

where

$$b = L - (d_x/2) \csc \alpha. \quad (4.24)$$

The cotangent function is denoted by $\cot$ and the cosecant function by $\csc$.

This fundamental solution then is a mathematical reflection of the constraint-based view of leaf springs: since the model only accounts for strain energy due to deformation in modes of low stiffness, and approximates the deformation in modes of high stiffness by a rigid constraint, a force acting in the constrained direction of the leaf springs indeed causes no deformation. So, while application of force F_0 does not lead to deformation, the multipliers that enforce this zero-deformation constraint do increase in value. Multipliers N_v^l and N_v^r can actually be interpreted as the forces that impose $\gamma_y^l = \gamma_y^r = 0$, i.e. as the forces that ensure that the transverse shear is zero, in the presence of the applied load F_0 at $v^l(0)$.

Similarly, multipliers M^l and M^r can be interpreted as the moments required to impose $\kappa_x^l = \kappa_x^r = 0$, i.e. as the moments that ensure zero bending in the plane of highest rigidity. Note that these moments change linearly over the length of the leaf springs. The offset b that shows up can be interpreted as the distance between the bottom end of the leaf spring and the intersection of the undeformed leaf spring axes. It is indicated in Fig. 4.5(a) and referred to as the crossing length in the following sections. The value of b can be positive or negative, depending on whether the intersection lies on (positive) or beyond (negative) the leaf springs. It is as if the misalignment force were applied at this point of intersection.

4.3.7 Support stiffness

Although the leaf spring deformation modes of high stiffness are approximated by a rigid constraint in the fundamental solution of Sec. 4.3.6, the results of that linearized analysis can be used to find an expression for the (finite) support stiffness in the y -direction, associated with the deformation modes of high stiffness. The nonzero multipliers N_v^{lr} and M^{lr} in Eq. (4.23) represent the internal forces and moments that make equilibrium with the applied force F_0 , linearized about the neutral configuration of zero shuttle rotation. Given this equilibrium stress state, the complementary energy of the cross-hinge mechanism can be readily formulated as

$$P_c = -\frac{1}{2kGA} \int_0^L (N_v^{l2} + N_v^{r2}) ds - \frac{1}{2EI_x} \int_0^L (M^{l2} + M^{r2}) ds. \quad (4.25)$$

Figure 4.6: Anti-symmetric (left) and symmetric (right) buckling modes. For visualization purposes, $d_y = 0$, i.e. the leaf springs are depicted as intersecting; the buckling modes are independent of d_y .

Now, the transverse shear γ_y and curvature κ_x are not constrained to zero values, but instead governed by linear constitutive relations, where A is the cross-sectional area, I_x is the second moment of area about the x -axis of the initial frame (Fig. 4.5(a)) and k is the shear correction factor [18]. With Castigliano's second theorem, it follows that

$$v^I(0) = 2F_0 \left[\frac{L}{kGA} + \frac{L^3}{EI_x} \left(\frac{1}{3} - b/L + (b/L)^2 \right) \right], \quad (4.26)$$

meaning that the support stiffness in y -direction is given by

$$k_{\text{supp},y} = \left[\frac{2L}{kGA} + \frac{2L^3}{EI_x} \left(\frac{1}{3} - b/L + (b/L)^2 \right) \right]^{-1}. \quad (4.27)$$

4.3.8 Bifurcation equilibrium

In elastic structures that are prone to bifurcation buckling, a second equilibrium configuration, different from but infinitesimally close to the trivial fundamental solution, is possible due to loads reaching a critical value.

Formally, a bifurcation equilibrium requires that the second variation of the total potential energy with respect to the fundamental solution vanishes. It can be shown that this means that the second-order terms in the potential energy become stationary. This is illustrated in Appendix 4.8.1 by a highly simplified example. Appendix 4.8.4 details the derivation of the equations that govern the bifurcation equilibrium of the actual cross-hinge mechanism.

It follows that the bifurcation configurations are given by nontrivial solutions of two sets of systems: one for a symmetric buckling mode and one for an anti-symmetric

Mode	c_1	c_2	c_3	c_4	c_5	c_6
Anti-symm.	494.9	116.9	202.9	117.6	2086	2602
Symmetric	780.0	26.14	0	59.67	518.2	0

Table 4.4: Numeric values of the coefficients for the geometrical factor β in Eq. (4.29) for the two buckling modes.

buckling mode. These modes are depicted in Fig. 4.6 for a crossing length $b = L/2$. In the anti-symmetric mode, the shuttle moves and both leaf springs deform, with a reversed twist angle. In the symmetric mode, the shuttle remains stationary and both leaf springs deform, with the bending angle and transverse displacement reversed.

The expression for the critical load is given by

$$F_{0,\text{cr}} = \beta(b/L) \frac{\sqrt{S_b S_t}}{L^2}. \quad (4.28)$$

The torsion stiffness S_t and bending stiffness S_b are given by Eq. (4.20). The geometric factor β accounts for the effect of the shuttle length d_x and the angle α on the critical load: it turns out that β is strictly dependent on only the dimensionless crossing length b/L , which is related to d_x and α by Eq. (4.24). The value of the geometric factor β is different for the symmetric and the anti-symmetric mode. It is given in Fig. 4.7 for the two buckling modes for a range of values of b/L .

The relation between β and b/L is approximated by the rational function for β^2 in terms of z^2 ,

$$\beta^2 = c_1 \frac{1 + c_2 z^2 + c_3 z^4}{1 + c_4 z^2 + c_5 z^4 + c_6 z^6}, \quad z = b/L - 1/2, \quad (4.29)$$

for the values of the coefficients given in Tbl. 4.4 with high accuracy (i.e. a maximum relative error of 1.1% and 0.3% in the range $|b/L| < 10$ for the anti-symmetric and symmetric mode, respectively).

For all values of b/L , the geometric factor of the anti-symmetric mode is smallest, meaning that the anti-symmetric mode has the lowest buckling load and represents the critical buckling mode of the system. In both modes, the geometric factor (and hence the buckling load) has a maximum value at $b = L/2$. The value of β is the same for z and $-z$, because they correspond to the cases in which the shuttle and the ground are interchanged.

In the configuration that the leaf springs are placed parallel to each other ($\alpha = 0$), the crossing length $b \rightarrow \infty$ and $\beta \sim 2\pi/z \rightarrow 0$, meaning that the buckling force approaches zero. This matches with the analysis of Meijaard et al. [50] that shows that the overconstrained parallelogram flexure mechanism buckles due to a moment load, whose critical value is $2\pi\sqrt{S_b S_t}/L$.

The critical misalignment displacement can be obtained as

$$v_{\text{cr}}^1(0) = F_{0,\text{cr}}/k_{\text{supp},y}, \quad (4.30)$$

with the critical misalignment force from Eq. (4.28) and support stiffness from Eq. (4.27).

Figure 4.7: Geometric factor β for the critical load.

4.3.9 Design relations

Summarizing the results for practical purposes, an estimate of the critical misalignment of a general cross-hinge mechanism follows from Eq. (4.30) as

$$\frac{v_{cr}^1(0)}{L} \approx 0.55\beta\sqrt{1+\nu}\frac{t^2}{L^2} + \frac{0.94\beta}{\sqrt{1+\nu}}\left(1 - 3\frac{b}{L} + 3\frac{b^2}{L^2}\right)\frac{t^2}{h^2}. \quad (4.31)$$

This estimate provides a conservative lower bound since the effects of warping and frame elasticity, which both increase the critical misalignment, are not included. This expression holds for cross-hinge mechanisms with arbitrary material and geometry properties. The material is accounted for by Poisson ratio ν (assuming isotropic material and a shear correction factor k of 0.85). The geometry is accounted for by crossing length b (indicated in Fig. 4.5(a)), geometric factor β (given in Fig. 4.7(a) and Eq. (4.29) for the critical anti-symmetric buckling mode) and the leaf spring length L , width h and thickness t . Crossing length b is related to the shuttle length d_x and shuttle-leaf spring angle α by Eq. (4.24).

A common class of cross-hinge mechanism designs has leaf springs crossing at half their length ($b = L/2$, $\beta \approx \sqrt{494.9}$) with a Poisson ratio of 0.3. In that case, Eq. (4.31) reduces to

$$\frac{v_{cr}^1(0)}{L} \approx 14\frac{t^2}{L^2} + 4.6\frac{t^2}{h^2}. \quad (4.32)$$

In both Eqs. (4.31) and (4.32), the first term (proportional to t^2/L^2) is the contribution to misalignment due to shear deformation; the second term (proportional to t^2/h^2) is due to bending deformation in the plane of highest rigidity. In case of extreme h/L ratios, one term may be dominant. For wide leaf springs with $h/L > 2.6$, shear deformation is dominant and bending can be neglected (with less than 5% error). For slender leaf springs with $h/L < 0.13$, bending deformation is dominant and shear can be neglected (with less than 5% error).

4.4 NUMERICAL MODEL

The flexible multibody software SPACAR is used for numerical calculations [36, 37]. This program can perform the static and dynamic analysis of mechanisms that consist of interconnected rigid and flexible elements. It can simulate motion for given initial conditions, linearize about an arbitrary state of motion, determine stationary solutions, and with linearized equations, determine the eigenmodes, buckling modes and compliances.

SPACAR includes an element that is based on the geometrically nonlinear Timoshenko–Reissner beam formulation [34]. This two-node beam element captures the effects of bending, elongation, shear, non-uniform torsion, and warping. An element has nodes at both ends. Multiple elements can be interconnected by sharing a common node, much like the standard finite element approach.

A distinctive feature of SPACAR is the definition of element flexibility by means of a set of physically meaningful deformation functions. These modes are unaffected by rigid body motion, so they measure deformation only, and form the discretized analogue of strain and curvature of a continuous beam. Expressed as functions of the nodal coordinates, they can be prescribed a zero value to become the constraint equations that govern element rigidity. Alternatively, one or more deformation modes can be allowed a nonzero value to model flexibility instead, in accordance with a constitutive model.

The various deformation modes are constructed to independently and consecutively represent elongation, torsion, bending in the two principal planes and warping deformation. This means that this formulation matches well with the constraint-based view of leaf springs: an efficient numerical model for a leaf spring is obtained by only allowing nonzero values of the deformation modes associated with relatively low stiffness. This reduces the computational load and avoids numerical problems associated with deformation modes of high stiffness, such as ill-conditioning and reduced maximum time steps.

4.4.1 Simplified model

A simplified numerical model is created with assumptions that are very similar to the analytical investigation. The leaf spring fillets and cross-hinge shuttle are assumed to be rigid. The translational guidance is omitted and replaced by fixed and guided boundary conditions on the end nodes. The only elastic effects modeled are due to the leaf spring deformation. Like in the analytical investigation, the deformation modes of torsion and bending in the plane of lowest rigidity are allowed; elongation, shear and bending in the plane of highest rigidity are ignored. Also, the effects of gravity and constrained warping are neglected.

Each leaf spring is modeled in SPACAR by 20 beam elements of equal length. For the nominal values of the system parameters, given in Tbl. 4.1, the analytical critical load according to Eq. (4.28) is 58.247 N for the anti-symmetric buckling mode and 73.121 N for the symmetric buckling mode. In SPACAR, the same modes are encountered, with corresponding critical loads of 58.589 N and 73.811 N, respectively. This means that

Figure 4.8: Bifurcation diagram illustrating the effect of warping.

the discretization error is less than 1%. More importantly, the analytical solution now serves as a verification of the numerical model, at least to the extent of the effects that this simplified numerical model takes into account

4.4.2 Refined model

Unlike in the analytical investigation, it is feasible to include additional effects in a refined numerical model. The leaf springs are modeled by allowing all deformation modes, meaning that torsion, bending (in both planes), elongation, transverse, warping and torsion-elongation coupling are taken into account. Small variations in thickness (due to manufacturing tolerances) that occur over the length of the leaf springs are modeled. Body forces due to gravity are included. The parallel flexure mechanism serving as the translational guidance is also modeled using beam elements that are fully elastic for the two additional leaf springs, and rigid for the guidance stage. The elasticity of the cross-hinge shuttle is modeled by approximating the mid-section with fully elastic beam elements with a rectangular cross-section.

The misalignment displacement is prescribed and increased in steps of 1 μm . Fig. 4.8 shows the misalignment force as a function of the rotation of the shuttle. The shuttle rotation stays nearly zero until the critical force is established; after this bifurcation of equilibria, the system follows a different (post-buckling) path, for which the shuttle rotates in relation to the applied force. It is found that the practical ‘imperfections’ of shuttle elasticity, gravity and thickness variations only govern the smoothness of the transition from the fundamental path (of zero shuttle rotation) to the stable path after bifurcation. Without the imperfections, the transition is instant and sharp at the critical load; with imperfections, the transition is smoother, as Fig. 4.8 indicates. The imperfections do not affect the critical load or the post-buckling path markedly.

Compared to the simplified model of Sec. 4.4.1, the critical load has increased due to several effects. Gravity and the additional stiffness of the translational guidance (k_{guidance} in Fig. 4.4) that acts in parallel with the cross-hinge together increase the critical load to approximately 77 N. More strikingly, it is found that the effects of warping, i.e. the consequences of constrained warping at the ends of the leaf springs, affect the buckling behavior significantly: the bifurcation load is an additional factor of approximately 1.3

Figure 4.9: Load–deflection curves illustrating the effect of shuttle elasticity.

Figure 4.10: Compliances at the shuttle as functions of misalignment. Rotational compliance c_{drive} in 1/Nm, translational compliances $c_{\text{supp},y}$ and $c_{\text{supp},z}$ in m/N.

larger. A qualitative explanation for this observation is that warping constraints at both ends of the leaf springs increase the effective torsional stiffness; it is as if the length over which torsion deformation can happen is reduced. Since the buckling modes exhibit torsion deformation, the warping constraints effectively increase the bifurcation load.

Fig. 4.9 shows a piecewise linear relation between misalignment displacement $v^l(0)$ and force F_0 . The support stiffness of the mechanism is constant and considerably lower in the buckled state when the critical misalignment is exceeded. It is found that the compliance of the cross-hinge shuttle has a significant effect: it reduces the support stiffness before buckling and increases the critical misalignment displacement by a factor of approximately 1.6. In the buckled state, the compliance of the leaf springs is dominant and the support stiffness of the mechanism is not affected by the shuttle compliance.

Fig. 4.10 shows three compliances calculated for the center of the shuttle, as a function of the misalignment. The rotational compliance in the drive direction of the cross-hinge, c_{drive} , is relatively large compared to the compliance in the other directions. The translational compliances in y - and z -directions, which are two support directions, are given by $c_{\text{supp},y}$ and $c_{\text{supp},z}$, and scaled by factors 10^6 and 10^5 , respectively, to show them in the same figure.

Support direction compliances $c_{\text{supp},y}$ and $c_{\text{supp},z}$ have nearly constant values of $1.9 \cdot 10^{-6}$ and $2.5 \cdot 10^{-7}$ m/N before the critical misalignment of approximately 117.5 μm . When buckling has occurred, the values rapidly increase by factors 19.4 and 275.8, respectively. This indicates that the support stiffness has decreased considerably, meaning that the performance of the mechanism is severely reduced. In the post-buckling regime, the compliances in support directions keep increasing with increasing misalignment and accompanying rotation of the shuttle.

The compliance in the drive direction (the rotational y -direction) increases gradually with misalignment from 2.94 1/Nm to a maximum of 56.0 1/Nm (a factor 19.4 increase) at the bifurcation. Then, it decreases again, with increasing rotation of the shuttle. It is also observed that in the absence of gravity the drive compliance grows unbounded at the bifurcation, on an unstable equilibrium; gravity acts as a perturbation that switches the system to a stable equilibrium path near the bifurcation.

4.5 MEASUREMENTS

For measurement of the first eigenfrequency of the cross-hinge, corresponding to a vibration mode in which the mechanism moves in the DOF, a laser optical displacement sensor is used (Micro-Epsilon optoNCDT 1402-20 with 2 μm resolution). It is located behind the frame of Fig. 4.3 (not visible) and pointed at a side face of the shuttle, through a chamber in the frame. The analog output signal is sampled at 1 kHz and captured using a data acquisition system for further processing. An end-stop is mounted to prevent excessive rotation of the shuttle.

At the start of a measurement, the misalignment displacement is set with the screw at the bottom and read out with the dial gauge at the top. An initial transverse displacement of no more than 0.8 mm at the end of the leaf springs is applied by holding the shuttle against a stop and releasing it to record the free vibrations. Because the vibration mode is only lightly damped, a useful signal is obtained for a time period of at least 20 s. A Fourier transformation is used to extract the frequency content of the signal.

Fig. 4.11 shows the measurement of the eigenfrequency as a function of misalignment. The misalignment is increased in steps of 5 μm . Due to gravity, there is an offset in the misalignment measurement; since the eigenfrequency is highest for zero misalignment, the offset can be found and compensated for. Qualitatively, much like the gradually increasing drive compliance, the first eigenfrequency decreases with misalignment, from a value of 3.48 Hz to 1.02 Hz. It is observed in the experiment that a misalignment of approximately 111 μm leads to a sudden rotation of the shuttle, against the end-stop. This indicates that the cross-hinge has buckled in the anti-symmetric mode. Since the mechanism in this state cannot be actuated by initial nonzero transverse displacements, no measurements could be conducted in the post-buckling regime.

The first eigenfrequency predicted by the refined model in SPACAR as described in Sec. 4.4.2 is also shown. It matches qualitatively and quantitatively with the measurements: the prediction of the critical misalignment is 5.9% larger. After the bifurcation, the eigenfrequency starts increasing again with misalignment.

Figure 4.11: First eigenfrequency as a function of misalignment, comparing measurement and numerical simulations.

As a comparison, a pre-stressed modal analysis was conducted in ANSYS Workbench, which uses solid elements. It predicts similar eigenfrequencies up to the critical misalignment; it lacks post-buckling features and is unable to compute equilibrium in a buckled state.

4.6 CONCLUSION

This work has provided a practical and explicit bound on the misalignment below which overconstrained flexure mechanism designs are feasible, based on bifurcation buckling. A cross-hinge flexure mechanism with a single overconstraint has been analyzed analytically, numerically and experimentally as a case study on the properties of overconstrained flexure mechanisms. It is shown that small, typically unintended, misalignments in the overconstrained direction can affect the system behavior significantly. Measurements show that a misalignment on the order of 0.1 mm is critical for this mechanism, leading to a complete loss of actuation stiffness due to bifurcation buckling. This is supported by a numerical model, which exposes further effects, such as a large decrease in support stiffness at the critical misalignment. An idealized analytical model describes the essential buckling behavior of the system in terms of the buckling modes and an explicit expression for the critical misalignment force and displacement that cause bifurcation. This expression is valid for various cross-hinge mechanism topologies: the shuttle length and the angle between the leaf springs are parameters. Simulations with the numerical model have shown that additional effects due to warping and parasitic compliances explain the difference between the measurements and the analytical model.

4.7 ACKNOWLEDGMENTS

The authors thank Mr. L. Tiemersma for building the experimental set-up and Mr. Z.A.J. Lok for providing the impact modal testing equipment. This research was funded by the Innovative Research Incentives Scheme VIDI (Stichting voor de Technische

Wetenschappen) (14152 NWO TTW) of the Ministry of Education, Culture and Science of the Netherlands.

4.8 APPENDIX OF CHAPTER 4

4.8.1 1-DOF example

The principle of minimum potential energy provides a systematic procedure for finding equilibrium configurations, such as those corresponding to the fundamental (also referred to as trivial) state and to a bifurcation. In the cross-hinge mechanism analysis of the current work, many configuration functions and constraint equations are needed to model the cross-hinge, increasing the complexity. To highlight some crucial steps in the procedure, this appendix provides a simpler example that has only a single degree of freedom.

Figure 4.12: 1-DOF example for illustrating the equilibrium principle.

Fig. 4.12 shows a pinned massless rigid rod of length L connected to a torsion spring with stiffness k . Angle θ fully describes the configuration of this system and can be considered the degree of freedom. A force F acts in the fixed vertical direction along a distance y . The kinematic relation between y and θ is given by

$$y = L(1 - \cos \theta). \quad (4.33)$$

One way to model this system is by formulating the total potential energy (consisting of strain energy and external work), i.e.

$$P_1 = \frac{1}{2}k\theta^2 - Fy, \quad (4.34)$$

and substituting the kinematic relation of Eq. (4.33) such that the total potential energy only depends on the degree of freedom θ . All equilibrium configurations can then be found by requiring that P_1 is stationary, i.e. that the first variation with respect to θ vanishes:

$$\delta P_1 = \frac{dP_1}{d\theta} \delta\theta = (k\theta - FL \sin \theta) \delta\theta = 0. \quad (4.35)$$

The trivial solution is given by $\theta = 0$, for which the rod is exactly vertical. All other solutions are given by the roots of $\sin(\theta)/\theta = k/(FL)$, which is a transcendental expression in θ because of the nonlinear form of the kinematic relation Eq. (4.33). If not all solutions are relevant, but only those close to the trivial one, one could assume that θ is small. The series expansion $\sin \theta \approx \theta$ around $\theta = 0$ can then be used to approximate the nonlinear relation and find that δP_1 is also zero when the force $F = k/L$ and $\theta \neq 0$. In other words, for this critical value of the force, the rod has a nontrivial bifurcation equilibrium configuration.

An alternative approach for finding the fundamental and bifurcation configuration makes use of Lagrange multipliers for including kinematic constraint relations and a series expansion of the total potential energy [40]. Instead of eliminating y from Eq. (4.34) with the kinematic relation of Eq. (4.33), a Lagrange multiplier λ can be used to account for the constraint between y and θ :

$$P_2 = \frac{1}{2}k\theta^2 - Fy + \lambda(y - L + L \cos \theta). \quad (4.36)$$

The three variables θ , y , and λ have to be regarded as independent variables now. Including constraints this way increases the number of variables, but has the benefit that constraint equations without explicit solution (ruling out the former elimination approach) can be included as well.

From a physical point of view, the fundamental configuration is often obvious, except for the value of the Lagrange multiplier. To find it, we use the second-order Taylor expansion of P_2 around the tentative guess $\theta = y = \lambda = 0$, given by

$$\frac{1}{2}k\theta^2 - Fy + \lambda y. \quad (4.37)$$

The first variation is given by

$$k\theta\delta\theta + (\lambda - F)\delta y + y\delta\lambda = 0, \quad (4.38)$$

which, since the three variations are now independent, shows that indeed $\theta_0 = y_0 = 0$. Importantly, it follows that $\lambda_0 = F$ in the fundamental configuration.

For the nearby bifurcation configuration, the total energy expression Eq. (4.36) can be expanded in a truncated series around the fundamental configuration, in order to simplify the nonlinear terms. With the replacement

$$\theta = \theta_0 + \bar{\theta} = \bar{\theta}, \quad y = y_0 + \bar{y} = \bar{y}, \quad \lambda = \lambda_0 + \bar{\lambda} = F + \bar{\lambda}, \quad (4.39)$$

the second-order expansion in $\bar{\theta}$, $\bar{y}$, and $\bar{\lambda}$ is given by

$$\frac{1}{2}(k - FL)\bar{\theta}^2 + \bar{\lambda}\bar{y}. \quad (4.40)$$

It can be seen that the expansion around the nonzero value of λ in the fundamental configuration brings about the new second-order term $-FL\bar{\theta}^2/2$, which is due to $\lambda L \cos \theta$

in the original total potential energy P_2 of Eq. (4.36). Also, all first-order terms have vanished, since the expansion is around an equilibrium configuration. The first variation is now given by

$$(k - FL)\bar{\theta}\delta\bar{\theta} + \bar{\lambda}\delta\bar{y} + \bar{y}\delta\bar{\lambda} = 0. \quad (4.41)$$

Like before, we find the nontrivial solution $\bar{\theta} \neq 0$ and $F = k/L$ (and $\bar{y} = \bar{\lambda} = 0$), corresponding to the bifurcation configuration.

The second approach illustrated here for obtaining equilibrium configurations is used in the cross-hinge mechanism analysis, since it allows for taking into account more complicated constraint equations (such as those given by an implicit relation) and only requires second-order expansions of nonlinear terms. While the basic principle is the same, we use a continuum description for the cross-hinge leaf springs; unlike the single degree of freedom variable here, the potential energy of the cross-hinge depends on several configuration *functions* to accurately model its elastic behavior. Vanishing of the first variation is then expressed as a set of ODEs (Euler–Lagrange equations) and dynamic boundary conditions [17, 40, 71].

4.8.2 Deformation parameters

$$\begin{aligned} \mathbf{D} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\chi & 0 & s\chi \\ 0 & 1 & 0 \\ -s\chi & 0 & c\chi \end{bmatrix} \\ &\approx \begin{bmatrix} 1 - \chi^2/2 - \psi^2/2 & -\psi & \chi \\ \psi + \chi\phi & 1 - \phi^2/2 - \psi^2/2 & -\phi + \chi\psi \\ -\chi + \psi\phi & \phi & 1 - \chi^2/2 - \phi^2/2 \end{bmatrix}, \end{aligned} \quad (4.42)$$

where c is short for the cosine, s for the sine function, and the approximation is truncated after the second-order terms.

4.8.3 Shuttle constraints

Fig. 4.13 shows the vectors that are used for deriving the translational shuttle constraint of Eq. (4.16). Expansion of this vector equation yields

$$\begin{aligned} &(-u^l + u^r + d_y\psi^l)\cos\alpha + (w^l + w^r + d_y\phi^l)\sin\alpha + \text{q-terms} = 0, \\ &-v^l + v^r - (\psi^l + \chi^l\phi^l)d_x\cos\alpha - (\phi^l - \chi^l\psi^l)d_x\sin\alpha + d_y(\phi^{l2} + \psi^{l2})/2 + \text{c-terms} = 0, \\ &d_x\chi^l + (-w^l + w^r - d_y\phi^l)\cos\alpha - (u^l + u^r - d_y\psi^l)\sin\alpha + \text{q-terms} = 0, \end{aligned} \quad (4.43)$$

where q-terms stands for quadratic terms and c-terms for cubic terms.

Figure 4.13: Vectors for deriving the shuttle displacement constraint.

4.8.4 Equations governing bifurcation equilibrium

By expanding the total potential energy of Eq. (4.21) in a series up to second-order terms around the fundamental state from Sec. 4.3.6 and taking variations with respect to the configuration functions, twelve ODEs are obtained. They are summarized as

$$\begin{array}{l|l}
 \chi & -S_b \chi'' - N_u \pm F_0(s-b)\psi' = 0 \\
 \psi & -S_t \psi'' \mp [F_0(s-b)\chi]' \pm F_0 u' = 0 \\
 \phi & -M' + N_v \mp F_0 w' = 0 \\
 u & -N_u' \mp F_0 \psi' = 0 \\
 v & -N_v' = 0 \\
 w & -N_w' \pm F_0 \phi' = 0
 \end{array} \quad (4.44)$$

where $b = L - (d_x/2) \csc \alpha$. Each entry represents two equations: the upper sign corresponds to the left and the lower sign to the right leaf spring equation.

The dynamic boundary conditions are given by

$$\begin{array}{l|l}
 \chi^l & S_b \chi^l - D_{y,0} \phi^l d_x \cos \alpha + D_{y,0} \psi^l d_x \sin \alpha + R_\chi + D_z d_x = 0 \\
 \chi^r & S_b \chi^r - R_\chi = 0 \\
 \psi^l & S_t \psi^l - M_0^l \chi^l + D_{y,0} \chi^l d_x \sin \alpha - D_y d_x \cos \alpha \\
 & + R_\psi + D_{y,0} \psi^l d_y + D_x d_y \cos \alpha + D_z d_y \sin \alpha = 0 \\
 \psi^r & S_t \psi^r - M_0^r \chi^r - R_\phi \sin 2\alpha - R_\psi \cos 2\alpha = 0 \\
 \phi^l & M^l - D_{y,0} \chi^l d_x \cos \alpha - D_y d_x \sin \alpha \\
 & + R_\phi + D_{y,0} \phi^l d_y + D_x d_y \sin \alpha - D_z d_y \cos \alpha = 0 \\
 \phi^r & M^r + R_\psi \sin 2\alpha - R_\phi \cos 2\alpha = 0 \\
 u^l & N_u^l + F_0 \psi^l - D_x \cos \alpha - D_z \sin \alpha = 0 \\
 u^r & N_u^r - F_0 \psi^r + D_x \cos \alpha - D_z \sin \alpha = 0 \\
 v^l & \text{at } s = L: N_v^l - D_y = 0 \\
 & \text{at } s = 0: N_v^l = 0 \\
 v^r & N_v^r + D_y = 0 \\
 w^l & N_w^l - F_0 \phi^l + D_x \sin \alpha - D_z \cos \alpha = 0 \\
 w^r & N_w^r + F_0 \phi^r + D_x \sin \alpha + D_z \cos \alpha = 0
 \end{array} \quad (4.45)$$

to be satisfied at $s = L$. Also, $D_{y,0} = -F_0$, $M_0^l = -F_0(s - b)$ and $M_0^r = F_0(s - b)$ are the nonzero values of the multipliers in the fundamental configuration.

By taking variations with respect to the multipliers M^l , M^r , N_v^r , N_w^l and N_w^r , five first-order ODEs are obtained. These are the linearized versions of the constraints on the deformation components κ_x , γ_y , γ_z in Eqs. (4.10)–(4.15). With the appropriate kinematic boundary conditions, the solution of these ODEs is

$$\phi^l = \phi^r = v^r = w^l = w^r = 0. \quad (4.46)$$

By taking variations with respect to the multipliers R_ϕ , R_χ , R_ψ , D_x , D_y and D_z , the linearized versions of the shuttle constraints of Eqs. (4.16) and Eq. (4.18) are obtained. Assuming that $0 < \alpha < \pi/2$, they can be simplified to

$$\psi^l = \psi^r = 0, \quad (4.47)$$

$$\chi^l = \chi^r, \quad (4.48)$$

$$u^l = u^r, \quad (4.49)$$

$$d_x \chi^l = (u^l + u^r) \sin \alpha, \quad (4.50)$$

$$v^l = v^r, \quad (4.51)$$

at $s = L$. By taking the variation with respect to N_v^l and using Eq. (4.51), it also follows that $v^l(s) = 0$.

The four dynamic boundary conditions for χ^l , χ^r , u^l and u^r can be reduced, by eliminating R_χ , D_x and D_z and using Eq. (4.47), to

$$S_b(\chi^{l'} + \chi^{r'}) + (L - b)(N_u^l + N_u^r) = 0. \quad (4.52)$$

By taking variations with respect to multipliers N_u^l and N_u^r , the linearized versions of the constraints on γ_x^l and γ_x^r ,

$$u^{l'} = \chi^l \quad \text{and} \quad u^{r'} = \chi^r, \quad (4.53)$$

are obtained. Combined with the ODEs for ψ^l and ψ^r , it follows that

$$\chi' = \mp \frac{S_t}{F_0} \frac{1}{s - b} \psi'', \quad (4.54)$$

$$\chi'' = \pm \frac{S_t}{F_0} \frac{1}{(s - b)^2} \psi'' \mp \frac{S_t}{F_0} \frac{1}{s - b} \psi''', \quad (4.55)$$

where the upper sign corresponds to the left and the lower sign to the right leaf spring.

From the ODEs for u^l and u^r , it follows that

$$N_u^l = -F_0 \psi^l + C_0^l, \quad (4.56)$$

$$N_u^r = F_0 \psi^r + C_0^r, \quad (4.57)$$

where C_0^l and C_0^r are unknown integration constants. Substitution of Eqs. (4.55)–(4.57) in the ODEs for χ^l and χ^r yields

$$\pm \frac{S_b S_t}{F_0} \frac{1}{(s-b)^2} \psi'' \mp \frac{S_b S_t}{F_0} \frac{1}{s-b} \psi''' + C_0 \mp F_0 \psi \mp F_0 (s-b) \psi' = 0, \quad (4.58)$$

$$\mp \psi'' \pm (s-b) \psi''' - \frac{F_0}{S_b S_t} (s-b)^2 C_0 \pm \frac{F_0^2}{S_b S_t} (s-b)^2 \psi \pm \frac{F_0^2}{S_b S_t} (s-b)^3 \psi' = 0. \quad (4.59)$$

With a change of variable $x = s - b$, it becomes

$$\pm x \psi''' \mp \psi'' \pm c^2 x^3 \psi' \pm c^2 x^2 \psi - c^2 x^2 \frac{C_0}{F_0} = 0, \quad (4.60)$$

where $c = F_0 / \sqrt{S_b S_t}$. With the integrating factor x^{-2} , the ODE is further reduced to

$$\pm \frac{\psi''}{x} \pm c^2 x \psi - c^2 x \frac{C_0}{F_0} = k_1, \quad (4.61)$$

where k_1 is another integration constant. With the Lommel transformation [42]

$$\psi(x) = X^{\frac{1}{4}} \Psi(X) \pm \frac{C_0}{F_0}, \quad X = \frac{1}{2} c x^2, \quad (4.62)$$

Eq. (4.61) is transformed to a standard non-homogeneous Bessel differential equation in $\Psi(X)$, given by

$$X^2 \Psi'' + X \Psi' + \Psi \left(X^2 - \frac{1}{16} \right) = k_1 \frac{X^{5/4}}{\sqrt{2} c^3}. \quad (4.63)$$

The general solution is

$$\psi(x) = \left(\frac{1}{2} c x^2 \right)^{\frac{1}{4}} \left[C_1 J_{-\frac{1}{4}} \left(\frac{1}{2} c x^2 \right) + C_2 J_{\frac{1}{4}} \left(\frac{1}{2} c x^2 \right) + C_3 H_{\frac{1}{4}} \left(\frac{1}{2} c x^2 \right) \right] \pm \frac{C_0}{F_0}, \quad (4.64)$$

where $J_n(z)$ is the Bessel function of the first kind of order n and $H_n(z)$ the Struve function of order n [4].

With Eq. (4.54), the solution for $\chi(x)$ can be written as

$$\begin{aligned} \chi(x) &= \mp \frac{g}{c} \int_{-b}^x \frac{\psi''}{\bar{x}} d\bar{x} + \chi(-b), \\ &= \mp \frac{g}{c} \{ C_1 [Y_1(x) - Y_1(-b)] + C_2 [Y_2(x) - Y_2(-b)] + C_3 [Y_3(x) - Y_3(-b)] \}, \end{aligned} \quad (4.65)$$

where $g = \sqrt{S_t/S_b}$, and the boundary conditions on χ^l and χ^r at $s = 0$ are satisfied by setting the integration constant to zero. The functions $Y_i(x)$ are given by various

generalized hypergeometric functions (most easily obtained by symbolic manipulation software).

With Eq. (4.53), the solution for $u(x)$ can be written as

$$\begin{aligned} u(x) &= \int_{-b}^x \chi(\bar{x}) d\bar{x} + u(-b) \\ &= \mp \frac{g}{c} \{ C_1 [U_1(x) - U_1(-b) - (x+b)Y_1(-b)] \\ &\quad + C_2 [U_2(x) - U_2(-b) - (x+b)Y_2(-b)] \\ &\quad + C_3 [U_3(x) - U_3(-b) - (x+b)Y_3(-b)] \}, \end{aligned} \quad (4.66)$$

where the boundary conditions on u^l and u^r at $s = 0$ are satisfied by setting the integration constant to zero. Different expressions for $u(x)$ are found for when $b \leq 0$ and $b > 0$. We have been unable to find a single expression that holds for $b \in \mathcal{R}$.

At this point, there are eight unknown constants: the integration constants $C_0^l, C_1^l, C_2^l, C_3^l$, and $C_0^r, C_1^r, C_2^r, C_3^r$. There are also eight equations: the dynamic boundary condition (4.52), the remaining five linearized shuttle constraints (4.47)–(4.50), and the remaining two kinematic boundary conditions $\psi^l = \psi^r = 0$ at $s = 0$.

To reduce the complexity of solving this system further, we choose to split the solution into a symmetric and an anti-symmetric part. This means that we assume that the deformation of the left leaf spring is linked to the deformation of the right leaf spring by a symmetry condition.

For the symmetric case,

$$\chi^l = -\chi^r, \quad \psi^l = \psi^r, \quad u^l = -u^r, \quad N_u^l = -N_u^r. \quad (4.67)$$

The eight equations then reduce to

$$\begin{aligned} \chi(L) &= 0, & \psi(L) &= 0, \\ u(L) &= 0, & \psi(0) &= 0, \end{aligned} \quad (4.68)$$

for unknown coefficients C_0, C_1, C_2 and C_3 .

For the anti-symmetric case:

$$\chi^l = \chi^r, \quad \psi^l = -\psi^r, \quad u^l = u^r, \quad N_u^l = N_u^r. \quad (4.69)$$

The eight equations then reduce to

$$\begin{aligned} \psi(L) &= 0, & u(L) &= (L-b)\chi(L), \\ \psi(0) &= 0, & S_b \chi'(L) &= -(L-b)N_u, \end{aligned} \quad (4.70)$$

for unknown coefficients C_0, C_1, C_2 and C_3 .

Nontrivial solutions for the unknowns of the symmetric and anti-symmetric system require that the determinant of the corresponding coefficient matrix is zero. It can be checked that this is the case when

$$F_0 = \beta \sqrt{S_b S_t} / L^2, \quad (4.71)$$

where β is a factor that only depends on the dimensionless quantity b/L .

CHAPTER 5

Importance of warping in beams with narrow rectangular cross-sections: an analytical, numerical and experimental flexible cross-hinge case study

Thesis note The paper in this chapter details the effect of constrained warping on the overconstraint analysis of chapter 4.

Abstract This paper reports on the significance of warping deformation on the stability analysis of a flexible cross-hinge mechanism, which consists of two leaf springs with rectangular cross-section. The effect of misalignments in this mechanism is studied analytically, numerically and experimentally. An analytical buckling analysis is carried out to determine the theoretical critical load of a generalized cross-hinge mechanism on the basis of first principles. A geometrically nonlinear beam finite element with a non-uniform torsion description is used to model the leaf springs numerically. The change in natural mode frequencies and stiffness as a function of the misalignment is determined by a multibody program. Measurements from a dedicated experimental set-up confirm that the inclusion of warping effects is crucial, even for narrow rectangular cross-sections: it is found that the effects of warping increase the analytical critical buckling load of the system by 55%.

5.1 INTRODUCTION

In precision manipulation mechanisms, the risk of indeterministic behavior is mitigated by the use of flexure joints. Such joints consist of elastically deforming elements, instead of traditional hysteresis-inducing rolling or sliding components. Since the only motion is due to elastic deformation, flexure mechanisms operate without friction, backlash, stick-slip, and wear, resulting in low hysteresis and highly repeatable motion. It also means that the static performance of such mechanisms is characterized by their stiffness properties: in certain directions (typically associated with actuation) low stiffness is desired, whereas especially in load-bearing directions high stiffness is desired. Assemblies of beams with narrow rectangular cross-sections are commonly used to this end.

In pursuit of higher performance (i.e. higher load-bearing stiffness), we are exploring flexure mechanisms that are termed overconstrained. This class of designs is typically avoided altogether because its stiffness properties depend strongly on the correct alignment of components (e.g. due to assembly and manufacturing tolerances). It has been shown that a flexible multibody model can predict this dependency accurately, and actually quantify the mechanism's limit of operation in terms of the allowable

*Published in Proceedings of the 9th ECCOMAS Thematic Conference on Multibody Dynamics (2019) p199–206.
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Figure 5.1: *Left*: Simplified illustration of the flexible cross-hinge mechanism. The von Mises stress distribution due to misalignment displacement v_0 is indicated. *Right*: Photograph of the measurement set-up of the cross-hinge mechanism. The guidance stage serves to apply a controllable misalignment to the cross-hinge mechanism. There are several design characteristics for limiting hysteresis and enabling repeatable measurements [56, see Sec. 4.2.3]. The set-up is used for measuring the mechanism's natural mode frequencies as a function of the misalignment. The natural mode frequencies serve as stiffness measures that are easier to determine experimentally.

misalignment by means of a buckling analysis [50, 56]. In this paper, we show that the effects of constrained warping have a marked influence on the critical load and therefore the allowable misalignment. An analytical lateral buckling analysis provides the theoretical critical load of the system on the basis of first principles and corroborates the measurements and simulations.

5.2 EXPERIMENTAL SET-UP

To study the phenomenon, the flexible cross-hinge in Fig. 5.1(a) serves as a case study. It consists of two leaf springs (80 mm length) with narrow rectangular cross-sections (30 mm width, 0.35 mm thickness) and a shuttle. The shuttle guides motion with respect to the base about the indicated rotation axis; in the other directions, it constrains motion. The overconstrained nature of this particular design manifests itself as a relatively high stress that occurs due to misalignment displacement v_0 (compared to misalignments in the other directions). This stress affects the stiffness properties of the mechanism, therefore its performance, and can even cause bifurcation buckling. The measurement set-up is depicted in Fig. 5.1(b).

5.3 ANALYTICAL ANALYSIS

When the misalignment displacement v_0 or force F_0 exceeds a critical value, buckling occurs, support stiffness is lost and the mechanism no longer functions. To investigate the effects that warping has on the critical load of the system, an analytical buckling analysis is performed. The analysis follows largely the same steps and notation as Nijenhuis et al. [56, see Sec. 4.3], with the addition of a warping model and a change in solution procedure.

The equilibrium conditions of the system are derived using the potential energy of the system, given by

$$P_{\text{tot}} = P_{\text{leaf}}^l + P_{\text{leaf}}^r - F_0 v^l(0) + \begin{bmatrix} D_x & D_y & D_z \end{bmatrix} \begin{bmatrix} K_x \\ K_y \\ K_z \end{bmatrix} + \begin{bmatrix} R_\chi & R_\psi & R_\phi \end{bmatrix} \begin{bmatrix} K_\chi \\ K_\psi \\ K_\phi \end{bmatrix}. \quad (5.1)$$

It consists of strain energy terms, external work $F_0 v^l(0)$ and Lagrange multipliers $D_{x,y,z}$ and $R_{\chi,\psi,\phi}$ to account for the kinematic constraints that the shuttle imposes on the leaf spring ends. The multipliers can be interpreted as the forces and moments needed to enforce the kinematic constraints. Superscripts l and r are used to denote the left and right leaf spring. Shuttle elasticity and gravity are ignored. Using beam theory to describe the leaf springs of length L , width w and thickness t , their contribution to P_{tot} is given by

$$P_{\text{leaf}} = \frac{1}{2} \int_0^L [S_b \kappa_y^2 + S_t \kappa_z^2 + S_w (\kappa'_z)^2] ds + \int_0^L (N_u \gamma_x + N_v \gamma_y + N_w \gamma_z + M \kappa_x) ds. \quad (5.2)$$

We choose to account for the strain energy only due to deformation modes with relatively low stiffness, i.e. bending in the plane of lowest rigidity (stiffness S_b), torsion (stiffness S_t) and warping (stiffness S_w). The deformation modes of relatively high stiffness are modeled as zero-deformation constraints by means of additional Lagrange multipliers $N_{u,v,w}$ and M for respectively zero transverse shear strain γ_x and γ_y , zero elongation strain γ_z and zero bending curvature κ_x in the plane of highest rigidity. Fig. 5.2 shows a schematic overview of the system. The deformed leaf spring configuration is described with respect to the initial coordinate frames x, y, z . Resolved in these frames, the position of the elastic lines is given by

$$\mathbf{r}(s) = \begin{bmatrix} u(s) & v(s) & s + w(s) \end{bmatrix}. \quad (5.3)$$

Also, the orientation of the orthonormal triad attached to the deformed beam cross-section $[\mathbf{m}_x \ \mathbf{m}_y \ \mathbf{m}_z]$ is described by the rotation

$$[\mathbf{m}_x \ \mathbf{m}_y \ \mathbf{m}_z] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi \\ 0 & \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \chi & 0 & \sin \chi \\ 0 & 1 & 0 \\ -\sin \chi & 0 & \cos \chi \end{bmatrix} [\mathbf{i}_x \ \mathbf{i}_y \ \mathbf{i}_z] \quad (5.4)$$

Figure 5.2: For treating a wider variety of cross-hinge designs, angle α , shuttle length d_x and shuttle width d_y are free parameters in this analysis. Crossing length $b = L - (d_x/2) \csc \alpha$ is positive when the point of intersection lies on the leaf springs; negative when it lies beyond (as depicted). For the demonstrator set-up, $b = L/2$. The position vector and orientation triads are only drawn for the left leaf spring.

with respect to the undeformed cross-section triad $[\mathbf{i}_x \ \mathbf{i}_y \ \mathbf{i}_z]$. Then, the shuttle constraints for the rotations are given by

$$\begin{aligned} K_\chi &= \chi^1 - \chi^r = 0, \\ K_\psi &= \psi^1 - \psi^r \cos 2\alpha + \phi^r \sin 2\alpha = 0, \\ K_\phi &= \phi^1 - \phi^r \cos 2\alpha - \psi^r \sin 2\alpha = 0, \end{aligned} \quad (5.5)$$

and for the translations by

$$\begin{aligned} K_x &= (-u^1 + u^r + d_y \psi^1) \cos \alpha + (w^1 + w^r + d_y \phi^1) \sin \alpha = 0, \\ K_y &= -v^1 + v^r - (\psi^1 + \chi^1 \phi^1) d_x \cos \alpha - (\phi^1 - \chi^1 \psi^1) d_x \sin \alpha + d_y (\phi^{12} + \psi^{12})/2 = 0, \\ K_z &= d_x \chi^1 + (-w^1 + w^r - d_y \phi^1) \cos \alpha - (u^1 + u^r - d_y \psi^1) \sin \alpha = 0 \end{aligned} \quad (5.6)$$

at $s = L$. Reissner's relations for the nonlinear beam strains and curvatures energetically dual to the stress resultants are given by [63]

$$\begin{aligned} \kappa_x &= \phi' - \psi' \chi, & \gamma_x &= u' - \chi, \\ \kappa_y &= \chi', & \gamma_y &= v' + \phi(1 + w') - \psi u', \\ \kappa_z &= \psi', & \gamma_z &= w', \end{aligned} \quad (5.7)$$

where the prime denotes differentiation with respect to the independent coordinate s . The kinematic boundary conditions, not included in the total potential energy, are

given by

$$s = 0 : \quad u^l = u^r = v^r = w^l = w^r = \chi^l = \chi^r = \psi^l = \psi^r = \phi^l = \phi^r = 0 \quad (5.8)$$

and

$$s = 0 : \quad \psi^{l'} = \psi^{r'} = \psi^{l''} = \psi^{r''} = 0 \quad (5.9)$$

for the constraints on the warping deformation imposed by the mounting fillets at both ends of the leaf springs.

Application of the principle of minimum potential energy yields the equilibrium conditions for the system in terms of the configuration functions and undetermined multipliers. The fundamental solution branch is given by a zero value for all quantities, except for

$$\begin{aligned} D_y &= -F_0, & N_v^l &= -F_0, & N_v^r &= F_0, \\ R_\phi &= d_x F_0 \cos \alpha \cot 2\alpha, & M^l &= -F_0(s - b), & M^r &= F_0(s - b), \\ R_\psi &= -d_x F_0 \cos \alpha, \end{aligned} \quad (5.10)$$

indicating that load F_0 does not cause deformation in the fundamental solution, but only a “reaction” from the multipliers that enforce the idealized zero-deformation constraints. Bifurcation equilibrium conditions are obtained by requiring that the first variation of P_{tot} with respect to the *fundamental solution* vanishes. This means that only the second-order terms of P_{tot} need to be taken into account (on the basis of which the nonlinear expressions of Eqs. (5.5)–(5.7) were already truncated). The ODEs and natural boundary conditions obtained this way can be solved partially, yielding

$$\phi^l = \phi^r = v^l = v^r = w^l = w^r = 0. \quad (5.11)$$

The addition of the warping term in P_{leaf} leads to coupled variable-coefficient linear ODEs for $\psi^{l,r}$ and $\chi^{l,r}$ that seem to admit no closed-form solution, unlike the original analysis without a warping model [56, see Sec. 4.3]. The remaining boundary conditions, assuming that $0 < \alpha < \pi/2$, are

$$\begin{aligned} s = L : \quad & \psi^l = \psi^r = 0, & d_x \chi^l &= (u^l + u^r) \sin \alpha, \\ & \chi^l = \chi^r, & S_b(\chi^{l'} + \chi^{r'}) &= (b - L)(N_u^l + N_u^r), \\ & u^l = u^r, \end{aligned} \quad (5.12)$$

To obtain a solution, we reduce the complexity by splitting the solution into a symmetric part, for which $\chi^l = -\chi^r$, $\psi^l = \psi^r$, $u^l = -u^r$, and an anti-symmetric part, for which $\chi^l = \chi^r$, $\psi^l = -\psi^r$, $u^l = u^r$. For the symmetric solution, the kinematic boundary conditions and Eqs. (5.12) together become

$$\psi(0) = \psi(L) = \psi'(0) = \psi'(L) = 0, \quad u(0) = u(L) = u'(0) = u'(L) = 0. \quad (5.13)$$

b/L	$w = L/10$ ($\lambda = 42.6$)	$w = L/4$ ($\lambda = 17.2$)	$w = L/2$ ($\lambda = 8.61$)	$w = 3L/4$ ($\lambda = 5.74$)	$w = L$ ($\lambda = 4.31$)
1/2	28.61	31.41	38.32	46.84	56.41
3/4	18.02	19.49	23.34	28.33	34.01
1	11.26	12.02	14.16	17.06	20.41
5/4	7.985	8.471	9.919	11.91	14.23
3/2	6.137	6.493	7.581	9.093	10.86

Table 5.1: Values of the geometric factor β for the symmetric buckling load for $n = 15$.

An approximate solution can be obtained by choosing a set of Fourier shape functions that satisfy the boundary conditions. For the symmetric solution, these can be

$$\psi(s) = \sum_{k=0}^n g_k \sin(\pi s/L) \sin(k\pi s/L), \quad u(s) = \sum_{k=0}^n h_k \sin(\pi s/L) \sin(k\pi s/L), \quad (5.14)$$

and $\chi(s) = u'(s)$. Substituting in P_{tot} and taking variations now yields a system of $2n$ linear algebraic equations for the undetermined coefficients g_k and h_k , whose determinant has to be zero in order to have nontrivial solutions. The smallest positive root corresponds to the critical load of the system, given by

$$F_{0,\text{cr}} = \beta\left(\frac{b}{L}, \lambda\right) \frac{\sqrt{S_b S_t}}{L^2}, \quad \lambda = \sqrt{\frac{S_t L^2}{S_w}} \approx \sqrt{\frac{24}{1 + \nu}} \frac{L}{w}, \quad (5.15)$$

where β is a dimensionless function of the crossing ratio b/L and the spatial decay rate λ , which is a measure of the warping stiffness relative to the torsion stiffness and largely dependent on the w/L ratio of the leaf springs. The structure of $F_{0,\text{cr}}$ is similar to the critical load expression in the original analysis without warping, the difference being the dependency of the geometric factor β on λ as well. Due to the length of the closed-form expression of β , numeric values of β for the symmetric buckling mode are provided in Tbl. 5.1 for practically relevant values of b/L and λ .

5.4 NUMERICAL ANALYSIS

For simulation, a geometrically nonlinear beam finite element has been implemented in the software package SPACAR [37]. This element is based on Timoshenko's bending theory and Reissner's torsion theory in order to capture non-uniform torsion (in addition to bending, shear and elongation) of thin-walled beams with closed, symmetric cross-sections. The element is formulated in the generalized strain beam framework, in which deformation modes are defined to describe both elastic deformations and rigid-body motion of the element. This formulation has the advantages of the co-rotational formulation, while avoiding interpolation of finite rotations. The inertia properties are described with a consistent and lumped mass formulation; the latter is used to model the warping inertia of the beam cross-section [34].

Figure 5.3: *Left:* Convergence of the critical buckling load. *Right:* Measurement and simulation of the first natural frequency f_1 . *No warping* refers to the absence of a warping deformation model (i.e. resembling free warping end-conditions). *Warping* refers to the use of a warping deformation model, used here to model the constrained warping end-condition.

5.5 RESULTS AND DISCUSSION

The idealized analytical buckling analysis for the mechanism at hand ($b = L/2$) predicts $\beta = 34.62$ and a buckling load of 90.63 N, compared to a critical load of 58.25 N in the earlier analysis without warping [56].

In SPACAR, the same idealized conditions can be simulated by excluding the deformation modes of elongation and bending in the plane of highest rigidity, and disabling shear, gravity and shuttle elasticity. Fig. 5.3 (left) shows that the behavior is the same: the critical buckling load of the idealized system increases by 55% from a converged value of 58.60 N to 90.63 N when warping is included in the SPACAR element. A qualitative explanation for the observed behavior is that the warping constraints at the ends of the leaf springs increase the torsional stiffness near the ends, effectively reducing the length over which the leaf spring twists; since the buckling mode exhibits torsion deformation, the buckling load increases.

It is observed that generally the first buckling mode is an anti-symmetric one, in which the shuttle moves and the twist angle of both leaf springs is reversed. The second mode is a symmetric one, in which the shuttle remains stationary, the bending angle of both leaf springs has the same sign and the twist angle is reversed. However, only for the case of the common doubly-symmetric embodiment of the mechanism ($b = L/2$), under the idealized conditions, and with the warping effects included, we see that the anti-symmetric mode stiffens considerably and becomes the second mode. The first mode, which now has multiplicity two, is a symmetric one in which an individual leaf spring buckles and the shuttle remains stationary. For this mode, the analytical buckling load is provided by Eq. (5.15) and Tbl. 5.1.

A refined numerical model is obtained by including all flexible deformation modes and the effects due to the leaf spring thickness variations, shuttle elasticity, the guidance stage and gravity. Fig. 5.3 (right) shows the first natural frequency (in a mode that is

a rotation about the rotation axis) as a function of the misalignment. With increasing misalignment, the rotation stiffness and the natural frequency decrease, and buckling (mechanism failure) is observed at around 0.11 mm misalignment. The numerical SPACAR results with the effects of warping match well with the measurements (the error in critical misalignment is 5.9%) and ANSYS (with solid elements). Unexpectedly, it also shows that the effects of warping play a significant role: with warping the critical misalignment is 35 μm (i.e. 40%) larger.

5.6 CONCLUSION

An analytical buckling analysis is presented for generalized overconstrained flexible cross-hinge mechanisms. The critical load matches with simulations from a geometrically nonlinear beam finite element. It is found that the inclusion of warping deformation of beams with narrow rectangular cross-section is crucial. The results are in good agreement with measurements from a demonstrator set-up.

CHAPTER 6

Improved dynamic performance in flexure mechanisms by overconstraining using viscoelastic material

Thesis note The paper in this chapter continues with the second subgoal, on flexure mechanism design principles. It presents a design solution for mitigating the misalignment sensitivity of overconstrained mechanisms, which was studied in chapters 4 and 5, by means of a custom viscoelastic material. The analytical and numerical models are based on largely the same solid mechanics theory as the preceding chapters, with the addition of the use of buckling modes for a more accurate albeit case-specific interpolation.

Abstract Flexure mechanisms are commonly designed to be exactly constrained to favor determinism, though at the expense of limitations on the maximum parasitic natural frequencies and support stiffness. This paper presents the use of viscoelastic material for providing additional support stiffness in a certain frequency range without the indeterminism commonly associated with overconstraining. This design principle of dynamically stiffened exact-constraint design is exemplified by a parallelogram flexure mechanism. Experiments demonstrate that a custom synthesized elastomer compound can compensate for unintended misalignments without significant internal stress buildup, while improving the dynamic performance in terms of a higher first parasitic natural frequency. An analytical investigation clarifies the relationship between misalignment, internal load, stiffness and natural frequency. Using the buckling modes of the system, the nonlinear geometric stiffness is modeled accurately up to the bifurcation. The measurements and analytical model are corroborated by a nonlinear flexible multibody analysis.

6.1 INTRODUCTION

To ensure determinism, precision flexure mechanisms are typically designed to be exactly constrained [15, 26, 39, 67, 72, 73]. While this mitigates the problems of overconstrained designs, in which tolerances, misalignment errors and temperature gradients lead to internal forces that compromise system behavior, repeatability and predictability, overconstrained flexure mechanisms can offer better dynamic performance (i.e. higher parasitic natural frequencies).

To exploit these benefits and simultaneously avoid the problems of overconstrained designs, we are investigating a new class of flexure mechanisms in which overconstraints are applied by means of viscoelastic material. Since the effective stiffness of the viscoelastic material is frequency-dependent, the overconstraint is only present in a designed frequency range. As a consequence, static loads e.g. due to misalignment errors, for

Submitted to Precision Engineering.

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which the exactly constrained behavior is desired, hardly affect the mechanism. For dynamic loads, for which the overconstrained behavior is desired, the effective stiffness and the first parasitic natural frequency are higher. This improves aspects such as the control bandwidth and tracking error, and lowers vibrations. We are referring to this approach as the principle of dynamically stiffened exact-constrained design.

In the literature, various publications on the merits and problems of overconstrained flexure mechanisms can be found. The effects of misalignments on stiffness, natural frequency and buckling have been investigated for a single-overconstraint parallelogram flexure mechanism, a single-overconstraint cross-hinge mechanism and a triple-overconstraint four-bar mechanism [50, 56, 65]. The potential of overconstrained mechanisms for higher performance has been investigated as an alternative design paradigm, referred to as elastic averaging [11, 32, 82]. The use of viscoelastic material for providing passive damping in mechatronics systems has been investigated [1, 64]. To the knowledge of the authors, there are no earlier publications on the use of viscoelastic material for applying additional constraints in flexure mechanisms in order to mitigate problems normally associated with overconstrained designs.

In this paper, a single-overconstraint parallelogram flexure mechanism is used as a case study. The formulation of a custom synthesised elastomer is detailed. This elastomer has been designed for the purpose of improving the (high frequency) dynamic performance and decreasing the sensitivity to (low frequency) misalignment errors. Measurements on a dedicated demonstrator set-up with controllable misalignment show how these performance attributes vary for the exactly constrained, the conventionally overconstrained and the dynamically stiffened exactly constrained case. A nonlinear analytical model for the misalignment-dependent stiffness and natural frequency is presented. Simulations with a numerical model corroborate the measurements and the analytical model.

6.2 CASE DESCRIPTION

A parallelogram flexure mechanism, consisting of two nominally identical and parallel leaf springs with a connecting shuttle, serves as a case study. This mechanism is considered to have one degree of freedom: a translation of the shuttle in the x -direction of Fig. 6.1, on account of the low stiffness in that direction. Motion of the shuttle in all other directions is associated with a much higher stiffness and considered to be constrained.

In the conventionally overconstrained case, both leaf springs are clamped at the base. This way, both leaf springs constrain the rotation of the shuttle around its longitudinal axis (the rotational x -axis), leading to indeterministic behavior, since a small misalignment at the base (indicated by the fictitious roller bearing that represents the misalignment angle ϕ_0 between both ends around the x -axis) can induce large internal loads. These in turn negatively affect the (geometric) stiffness and natural frequency of the system, as will be detailed in Sec. 6.7. Fig. 6.1 depicts the von Mises stress distribution in the leaf springs due to a misalignment angle ϕ_0 at the base.

Figure 6.1: Von Mises stress due to misalignment ϕ_0 in the overconstrained parallelogram flexure mechanism.

Figure 6.2: Kinematic representation of the constraints. The two fictitious roller bearings are a schematic representation of the boundary conditions of the leaf springs. At the right leaf spring, angle ϕ_0 represents the misalignment. At the left leaf spring, the angle is fixed (overconstrained case), free (exactly constrained case) or governed by the viscoelastic material placed in parallel (dynamically stiffened exactly constrained case).

In the exactly constrained case, the misalignment sensitivity is mitigated by the use of an additional flexure hinge, which removes the overconstraint. The front view of Fig. 6.2 shows a kinematic representation. The additional hinge is placed at the base of the left leaf spring. Now, only one leaf spring constrains the longitudinal rotational motion of the shuttle about the x -axis, meaning that a misalignment angle at the right base principally induces a compensating rotation of the additional flexure hinge at the left base, almost without any internal load and no change in system behavior.

This particular case study has been designed to demonstrate the difference between the overconstrained and exactly constrained case. In the exactly constrained case, the first natural mode describes motion of the shuttle in the degree of freedom (the translational x -direction). The second natural mode describes motion of the shuttle out of the x, z -plane. By design, the overconstrained case has very similar natural modes when it is aligned perfectly, except that the second natural frequency is a factor of about two larger than the exactly constrained case (owing to the specific location of the flexure hinge). Since this frequency tends to limit the achievable vibration attenuation or the bandwidth in a control system, it shows that the overconstrained case has higher dynamic performance. For this reason, we consider the second natural frequency to be a measure of dynamic performance in the current case study. Additionally, in the overconstrained case, there are various natural modes, among which the first, that are strongly affected by misalignment: the corresponding natural frequencies decrease with increasing misalignment. For this reason, we use the first natural frequency as a measure of misalignment sensitivity in the current study. The relationship between natural frequency, misalignment and stiffness will be elaborated on in Sec. 6.7.

In the new dynamically stiffened exactly constrained case, a custom elastomer is placed in parallel with the flexure hinge at the left leaf spring base in order to apply an overconstraint to the system. It allows misalignments (by providing only low stiffness at low frequencies) while increasing the first parasitic natural frequency (by providing high stiffness at high frequencies), on the basis of a clear distinction in the frequency range in which these two otherwise conflicting attributes are desired.

6.3 ELASTOMER COMPOUND FORMULATION

The viscoelastic behavior of polymers is used to apply high stiffness in only a limited frequency range. The temperature- and frequency-dependent glass transition that polymers exhibit may be exploited: the rubbery state is used for misalignment compensation at low frequencies and the glassy state for a stiffness increase at high frequencies. For this application, it means that at operating temperature $T_{\text{op}} = 20$ degrees Celsius, the transition from rubber to glass should occur close before the first parasitic natural frequency of the system, which is considered to be on the order of 100 Hz for the following discussion.

The mechanical characteristics of a viscoelastic material can be obtained from a Dynamic Mechanical Analysis (DMA) test, in which an oscillatory strain $\epsilon_0 \sin(\omega t)$ is applied (in shear, compression or elongation) and the induced stress state $\sigma_0 \sin(\omega t + \delta)$

SBR	137.5	TMQ	1.5
Carbon black N339	50	CBS	2
Zinc oxide	4	Sulfur	15
Stearic acid	1		

Table 6.1: Elastomer compound formulation in Parts per Hundred Rubber (PHR). TMQ is polymerized 2,2,4-trimethyl-1,2-dihydroquinoline. CBS is N-cyclohexyl benzothiazole sulfenamide.

is measured and recorded. Angle δ is the phase lag due to viscosity. Relevant measures are the storage modulus G' and loss modulus G'' , defined by

$$G' = \frac{\sigma_0}{\varepsilon_0} \cos \delta, \quad G'' = \frac{\sigma_0}{\varepsilon_0} \sin \delta, \quad (6.1)$$

which represent the elastic and viscous character of the material, respectively. The complex number

$$G^* = G' + iG'' \quad (6.2)$$

is referred to as the dynamic modulus, whose magnitude $|G^*(\omega)|$ represents σ_0/ε_0 , the frequency-dependent ratio between the amplitude of the harmonic strain and stress profile. This ratio is a measure of the effective material stiffness.

The glass transition can be characterized by means of a glass transition temperature T_g at a certain frequency. T_g is defined here as the maximum of

$$\tan \delta = \frac{G''}{G'}. \quad (6.3)$$

These quantities are temperature- and frequency-dependent. For the current investigation, ideally $T_g < T_{op}$ for a rubbery state in the frequency range of misalignment compensation, and $T_g > T_{op}$ for a glassy state in the higher frequency range of additional support stiffness.

Solution-polymerized styrene-butadiene rubber (S-SBR, SE6233 from Sumitomo Industries, 37.5 Parts per Hundred Rubber oil extended) is selected as the base polymer in the formulation. It is a synthetic rubber with a high styrene content (40% by mass) resulting in a glass transition temperature of -2 degrees Celsius at vanishing frequency. The material is compounded according to the formulation given in Tbl. 6.1. Carbon black is used as reinforcing filler to improve the tensile strength and wear resistance. Zinc oxide and stearic acid are used to speed up the vulcanisation of the elastomer. Stearic acid also improves the processing of the compound, since it acts as a dispersing agent for the carbon black. Antioxidant TMQ is used as a protective agent for the elastomer to improve oxidative heat ageing resistance. The vulcanisation ingredients, i.e. sulfur and the accelerator CBS, are added in a second mixing step. With a relatively high sulfur content and long vulcanisation time of 50 minutes, the cross-link density is increased in order to increase the glass transition temperature [83].

Figure 6.3: Measurements of $\tan \delta$, indicating the glass transition temperature T_g at 1 Hz and 100 Hz.

Figure 6.4: Nomogram for the master curves of the dynamic, storage and loss modulus at a reference temperature of 20 °C. Given a desired frequency on the right-hand side scale, the intersection of a horizontal line with the isotherm of the desired temperature can be determined; next, by extending a vertical line from this intersection point down to the horizontal axis, the corresponding reduced frequency is obtained. The shear moduli are plotted as functions of the reduced frequency.

Fig. 6.3 shows the glass transition temperature for frequencies of 1 Hz and 100 Hz, as the maximum of the $\tan \delta$ curve obtained from a DMA test. Measurements are performed with a static strain of 1% and a dynamic strain of 0.1%, in a temperature range from -15 to 62 degrees Celsius and a frequency range from 1 Hz to 100 Hz. It follows from the figure that at the operating temperature, the compound is in the transition state at 1 Hz and in the desired glassy state at 100 Hz.

The effective stiffness increase of the compound between low frequencies and high frequencies at a given temperature can be obtained by using the time-temperature superposition principle for viscoelastic materials. The principle states that data at one temperature can be superposed on data at another temperature by shifting the curves along the frequency axis. This means that DMA measurements obtained over a practically limited frequency range but at various temperatures can be used to estimate the material properties over a much larger frequency range. For a convenient representation, the effect of both temperature and frequency can be cast into a single variable, referred to as the reduced frequency. It is defined as $f\alpha(T_i)$, where f is the frequency and $\alpha(T_i)$ the shift factor for temperature T_i . At the reference temperature T_0 , the shift factor $\alpha(T_0) = 1$ and the reduced frequency coincides with the frequency. The shift factor for other temperatures needs to be determined from experimental data, e.g. by means of the closed-form t-T-P shifting algorithm [25].

Fig. 6.4 shows the superposed master curves of the shear moduli as a function of the reduced frequency for a reference temperature of 20 degrees Celsius in a nomogram. The graph has an additional frequency scale as the ordinate on the right side. That frequency scale can be used with the oblique isotherms to determine the reduced frequency for a given temperature and frequency. This way, the graph encodes the measurement data for a wide range of temperatures and frequencies.

The ratio of the dynamic modulus magnitude at 100 Hz to the dynamic modulus magnitude at vanishing frequency is found to be 63 at the operating temperature. This is a measure for the effective stiffness increase that can be obtained at the first parasitic natural frequency of the system by using the viscoelastic material. The relatively high sulfur content and long vulcanisation time that were used have a marked effect on this measure: with a lower sulfur content of 1.5 PHR and a shorter vulcanisation time of 18 minutes, the dynamic modulus ratio is only 16.

6.4 MEASUREMENT SET-UP

Figs. 6.5 and 6.6 show a schematic and a photograph of the measurement set-up. It has a screw for controlling the misalignment angle and a dial gauge at the back (not shown) for measuring the misalignment. The base plate, misalignment arm and other frame parts that carry loads are dimensioned such that parasitic compliances are negligible. The leaf springs with support fillets at the ends are made of a single piece of material (Stavax stainless steel, AISI 420) by wire EDM to avoid clamping of the thin leaf springs; the associated strain difference between the clamping blocks and the leaf spring would otherwise lead to micro-slip hysteresis. The leaf springs have a nominal length of 100

Figure 6.5: Schematic representation of the measurement set-up.

Figure 6.6: Photograph of the measurement set-up.

mm, width of 20 mm, and thickness of 0.35 mm. Young's modulus is 200 GPa, the Poisson ratio is 0.29 and the density is 7800 kg/m^3 .

The left notch hinge serves as the flexure hinge that can remove the overconstraint. A detailed view is provided in Appendix 6.11.1. The right notch hinge serves to guide the misalignment arm. The new viscoelastic overconstraint is placed in parallel with the left notch hinge.

The first natural frequency of the system, corresponding to a mode in which the shuttle moves in its degree of freedom, serves as a measure of the misalignment sensitivity of the system, since it is affected strongly by the internal load that develops in the overconstrained case. The first natural frequency is measured by a laser displacement sensor.

The second natural frequency of the system, corresponding to a mode in which the shuttle moves largely in the y -direction of Fig. 6.2, is the performance-limiting first

parasitic natural frequency. It is measured by accelerometers (not shown) on the shuttle. For measurement of the second natural frequency, excessive motion of the shuttle in the degree of freedom is avoided by means of an additional wire constraint (which does not introduce any significant stiffness in the measurement direction).

6.5 NUMERICAL MODEL

Simulations of the system are carried out in the flexible multibody software program SPACAR [37]. Each leaf spring is modeled by eight flexible three-dimensional beam elements that capture the linear and geometrically nonlinear effects associated with bending, shear, torsion, elongation and warping [34, 36]. The slight variations in thickness of the leaf springs over their length as a consequence of the wire EDM process have been measured and accounted for as modified thickness values for the various beam elements. The material characterization from the DMA test is used as the constitutive relation for the elastomer compound model. A standard Kelvin–Voigt model is used with two constant proportionality coefficients that are based on the measured storage and loss modulus from Fig. 6.4.

Since the modal frequency of interest determines the frequency at which the Kelvin–Voigt coefficients should be evaluated, and the value of the coefficients in turn affects the modal frequency, an iterative solver is implemented. During the first iteration, the model is evaluated as an exactly constrained system (so without the elastomer model). The resulting frequency of the mode of interest is input for the Kelvin–Voigt elastomer model in the next iteration, resulting in a corrected modal frequency. This is repeated until some convergence (less than 0.1% deviation in frequency) is obtained.

6.6 RESULTS

Fig. 6.7 shows measurements and simulations of the first natural frequency as a function of the misalignment angle. For all three cases, the measurement and simulation results match well. It can be seen that the first natural frequency decreases strongly with misalignment in the overconstrained case, which agrees with prior art [50, 56]. At only 5 mrad, the internal load exceeds the critical load value and bifurcation buckling occurs, indicating that the actuation stiffness is lost.

In the exactly constrained case, the first natural frequency hardly decreases with misalignment. The small decrease that is observed in both the experiment and the simulation is due to the small but finite stiffness of the notch hinge in the set-up causing just a small internal load.

In the new dynamically stiffened exactly constrained case, a minor decrease in natural frequency is observed, demonstrating the desired low sensitivity to misalignment. In the experiment, the system behavior at the maximum possible misalignment angle of 15 mrad (three times the overconstrained value) is largely unaffected; the simulations predict a critical misalignment angle of 42 mrad.

Figure 6.7: Measurements (solid lines) and simulations (dashed lines) of the first natural frequency for the overconstrained case, the exactly constrained case and the new dynamically stiffened exactly constrained case.

Figure 6.8: Measurements (solid lines) and simulations (dashed lines) of the second natural frequency for the overconstrained case, the exactly constrained case and the new dynamically stiffened exactly constrained case.

The natural frequencies are measured twice: from the stress-free state of zero misalignment to maximum positive misalignment in increments, then to maximum negative misalignment in increments, and back to zero. Successive measurements are performed every two minutes. At the maximum positive and negative misalignment measurement, the wait time is ten minutes; due to stress relaxation, the frequency then increases slightly and a loop in the frequency plot is observed.

Fig. 6.8 shows the second natural frequency as a function of the misalignment. For all three cases, the measurement and simulation results match within 6%. It is observed that the second natural frequency (the first parasitic natural frequency) in the overconstrained case (124 Hz) is a factor of 2.0 larger than in the exactly constrained case (which is 61 Hz). Compared to the exactly constrained case, the new dynamically stiffened case has a natural frequency of 109 Hz, which is a factor of 1.8 larger than the exactly constrained case. This clearly shows the improvement in dynamic performance over the exactly constrained case.

Also, failure of the overconstrained mechanism due to excessive misalignment can be observed: at the critical misalignment angle of 5 mrad, the second natural frequency drops rapidly and the mechanism has buckled. The dynamically stiffened exactly constrained case does not show any negative effect of misalignment in the entire measured range.

6.7 ANALYTICAL INVESTIGATION

In the overconstrained version of the mechanism, misalignments can induce stresses large enough to affect system properties, such as stiffness and natural frequency. In this section, that relationship is investigated in order to clarify the problem that the proposed viscoelastic constraints solve.

An analytical model is developed by formulating the potential and kinetic energy of the overconstrained system. A lumped compliance is introduced at the base of the left leaf spring for modeling the exactly constrained and dynamically stiffened exactly constrained cases as well. The goal is to obtain a model for not only the linear solution of the perfectly aligned overconstrained system, but also the nonlinear misalignment effect. While there are analytical models for the geometric stiffness of leaf springs, they are intended for loads smaller than the critical buckling load and lose validity when the applied loads are the same order. For the particular load case encountered in leaf springs in a parallelogram flexure mechanism, an analytical model valid up to the bifurcation can be formulated.

The formulation of the strain energy of the leaf springs is based on the buckling analysis of the parallelogram flexure mechanism carried out by Meijaard et al. [50], which already provides the critical load and buckling modes of the system. While the buckling analysis is only concerned with the system when all (linear) stiffness is lost, we now use the knowledge of the buckled configuration to augment a linear model and obtain a nonlinear model for the stiffness and natural frequency as general functions of the misalignment, describing how these properties change from their nominally high value to a zero value at the bifurcation. To capture the first three natural frequencies of

Figure 6.9: Definition of coordinate frames and deformation parameters.

the system, we introduce shape functions based on the buckling modes and derive the relevant inertial properties of the system.

The notation convention of the following section largely follows that of the earlier work on overconstrained flexure mechanisms [50, 56].

6.7.1 Description of a single leaf spring

Beam theory is used for modeling the leaf springs of length L , width h , thickness t , and material coordinate s , which is measured from the base. The deformed configuration is given by the position of the elastic line,

$$\mathbf{r}(s) = [u(s) \ v(s) \ s + w(s)]^T, \quad (6.4)$$

and the orientation of an orthogonal triad

$$[\mathbf{e}_x \ \mathbf{e}_y \ \mathbf{e}_z] = \mathbf{R}(s)[\mathbf{n}_x \ \mathbf{n}_y \ \mathbf{n}_z], \quad (6.5)$$

attached to the deformed cross-section at the centroid. This way, matrix $\mathbf{R}(s)$ describes the cross-sectional orientation as a rotation with respect to the global coordinate frame vectors $\mathbf{n}_x$, $\mathbf{n}_y$ and $\mathbf{n}_z$. All vectors in the following section are resolved in this coordinate frame. Matrix $\mathbf{R}(s)$ is parametrized by the three Euler angles ϕ , ψ and χ , according to

$$\mathbf{R}(s) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\chi & 0 & s\chi \\ 0 & 1 & 0 \\ -s\chi & 0 & c\chi \end{bmatrix}, \quad (6.6)$$

where c is short for the cosine and s for the sine function. The three angles are functions of s .

Fig. 6.9 depicts the definition of the coordinate frames and configuration parameters. For small deformations, angle ψ can be interpreted as the torsion angle, ϕ as the bending

angle in the plane of highest rigidity and χ as the bending angle in the plane of lowest rigidity. The strain energy is given by

$$P_{\text{leaf}} = \frac{1}{2} \int_0^L [GJ\kappa_z^2 + EI_y\kappa_y^2 + EI_x\kappa_x^2] ds, \quad (6.7)$$

where G represents the shear modulus, J Saint-Venant's torsion constant, E Young's modulus, and I_x and I_y the second moments of area about the x - and y -axes. The curvatures κ_x , κ_y and κ_z are the components of the skew-symmetric matrix $\mathbf{R}^T \mathbf{R}'$, which are nonlinear functions of the angles ϕ , ψ and χ . The prime denotes differentiation with respect to s .

To obtain a simplified model that captures the essential effects, the distinct geometry of leaf springs is exploited: the thickness is at least two orders of magnitude smaller than the length. Only the energy due to torsion and bending is accounted for; elongation, shear and warping are neglected. For $h/L < 1/4$, shear deformation has a small effect. For h/L ratios approaching unity, constrained warping can have a marked effect on the buckling load [55, see Tbl. 5.15]; for the present case, the effect is minor and only taken into account in the numerical model presented in Sec. 6.5. As a consequence of neglecting shear energy, the cross-sectional plane is normal to the elastic line, so $\chi \approx u'$ and $\phi \approx -v'$.

Since the deformation modes of bending about the y -axis and torsion have a relatively low stiffness EI_y and GJ , respectively, the linear term in the corresponding expressions for the y -axis curvature κ_y and the specific twist κ_z is dominant. Since the deformation mode of bending about the x -axis has a relatively high stiffness EI_x , the first-order term in the corresponding x -axis curvature κ_x is small and the second-order term can be significant as well. It is taken into account, so the curvatures can be approximated as

$$\kappa_z \approx \psi', \quad \kappa_y \approx u'', \quad \kappa_x \approx -v'' + \psi u''. \quad (6.8)$$

The second-order term of κ_x models the geometrically nonlinear effect of lateral buckling due to a load acting in the plane of the leaf spring [50], such as the one due to misalignment ϕ_0 .

The kinetic energy of a leaf spring is modeled as

$$T_{\text{leaf}} = \frac{1}{2} \rho \int_0^L \left[A \left(\frac{\partial u}{\partial t} \right)^2 + (I_x + I_y) \left(\frac{\partial \psi}{\partial t} \right)^2 \right] ds, \quad (6.9)$$

where ρ represents the density and A the cross-sectional area. Only the deformation modes of low stiffness, i.e. bending about the y -axis and torsion, are taken into account here.

6.7.2 Description of the system

The shuttle imposes the kinematic constraints that the leaf spring angles at the ends are equal, i.e.

$$\phi^l(L) = \phi^r(L), \quad \psi^l(L) = \psi^r(L), \quad \chi^l(L) = \chi^r(L), \quad (6.10)$$

and that the translations are related by

$$\mathbf{u}^l(L) + \mathbf{R}(L) \begin{bmatrix} d_x & 0 & 0 \end{bmatrix}^T - \mathbf{u}^r(L) - \begin{bmatrix} d_x & 0 & 0 \end{bmatrix}^T = 0, \quad (6.11)$$

which is approximated to first order by

$$\mathbf{u}^l(L) = \mathbf{u}^r(L), \quad v^l(L) = v^r(L) - d_x \psi(L), \quad w^l(L) = w^r(L) + d_x \chi(L). \quad (6.12)$$

The superscripts l and r are used to denote the left and right leaf spring. The shuttle has length d_x and mass m_s , see Fig. 6.9.

An additional lumped rotational stiffness around the x -axis with value k_n is placed in series with the left leaf spring base. It introduces compliance in the overconstrained system and can be used to represent the exactly constrained and dynamically stiffened exactly constrained cases. At the base, $s = 0$, the boundary conditions are

$$\begin{aligned} u^l &= u^r = v^l = v^r = w^l = w^r = 0, \\ \psi^l &= \psi^r = \chi^l = \chi^r = 0, \\ \phi^r &= \phi_0. \end{aligned} \quad (6.13)$$

In the overconstrained case, $\phi^l(0) = 0$. In the exactly constrained case, $\phi^l(0) = \phi_n$ is unknown and k_n represents the stiffness of the notch hinge (see left notch hinge in Fig. 6.5). In the dynamically stiffened exactly constrained case, $\phi^l(0) = \phi_n$ is unknown as well and k_n represents the combined effective stiffness of the notch hinge and the elastomer at the frequency of interest.

6.7.3 Critical misalignment load and angle

From the buckling analysis by Meijaard et al. [50], we already know that the critical moment

$$M_{cr} = 2\pi \sqrt{GJ EI_y} / L \quad (6.14)$$

applied at the base of the right leaf spring induces buckling of the overconstrained system in two possible modes. In the anti-symmetric mode, depicted in Fig. 6.10(a), the shuttle moves. In the symmetric mode, depicted in Fig. 6.10(b), the shuttle remains stationary [50]. The misalignment angle ϕ_0 is related linearly to the moment applied at the right leaf spring base, so

$$\phi_0 = \frac{2M_0 L}{EI_x}, \quad \phi_{cr} = \frac{2M_{cr} L}{EI_x}. \quad (6.15)$$

6.7.4 Nonlinear model

Instead of deriving the equilibrium conditions of the system as differential equations that render the total potential energy stationary, we first choose a specific set of interpolation functions based on the known solution of the linearized system (referred to as the linear-system solution in this section) and augment it with the buckling modes in order to

Figure 6.10: Buckling modes of the overconstrained system [50]. The front edge of the leaf springs is drawn heavier than the rear edge.

account for the geometrical nonlinearity of buckling. Minimization of the total potential energy with respect to the unknown coefficients of these interpolation functions then yields a set of algebraic equations that is easier to handle.

We choose the interpolation functions

$$\begin{aligned} u &= u_1(3\xi^2 - 2\xi^3) + \frac{u_2}{2\pi}(2\pi\xi - \sin 2\pi\xi) + \frac{u_3}{2\pi}(1 - \cos 2\pi\xi), \\ \psi &= \psi_1\xi + \psi_2 \sin(2\pi\xi) - \psi_3(1 - \cos 2\pi\xi), \\ v &= v_1(3\xi^2 - 2\xi^3) - Lv_2(-\xi^2 + \xi^3) - Lv_3(\xi - 2\xi^2 + \xi^3). \end{aligned} \quad (6.16)$$

The undetermined coefficients are collected in

$$\mathbf{a} = [u_1 \ u_2 \ u_3 \ \psi_1 \ \psi_2 \ \psi_3 \ v_1 \ v_2 \ v_3]^T. \quad (6.17)$$

Coefficient u_1 accounts for the contribution of the linear-system solution to the transverse displacement u of the leaf springs in the low-stiffness x -direction. Coefficient ψ_1 accounts for the contribution of the linear-system solution to the torsion angle ψ . Coefficients u_2, u_3, ψ_2 and ψ_3 account for the new shape function that are based on the first buckling modes of the system: u_2 and ψ_2 are nonzero in the anti-symmetric mode (Fig. 6.10(a)), u_3 and ψ_3 are nonzero in the symmetric mode (Fig. 6.10(b)) [50]. The interpolation of the transverse displacement v in the high-stiffness y -direction is based solely on the linear-system solution. The undetermined coefficients v_1, v_2 and v_3 represent $v(L), \phi(L)$ and $\phi(0)$, respectively. For the right leaf spring, v_3 represents the misalignment ϕ_0 and is considered a known parameter in the analysis. For the left leaf spring, $v_3 = 0$ in the overconstrained case and equal to the unknown coefficient ϕ_n in the exactly and dynamically stiffened exactly constrained cases.

Substitution of Eqs. (6.8) and (6.16) into the strain energy Eq. (6.7) yields an expression that has common quadratic terms in $\mathbf{a}$, but also higher-order terms due the nonlinear relation for the curvature κ_x . The higher-order terms in all coefficients except v_3 can be neglected. The strain energy can then be written as

$$P_{\text{leaf}} = \frac{1}{2} \mathbf{a}^T \mathbf{K}_{\text{leaf}} \mathbf{a}, \quad (6.18)$$

where $\mathbf{K}_{\text{leaf}}$ is the stiffness matrix that still depends on v_3 .

The kinetic energy of a leaf spring is modeled according to Eq. (6.9) as

$$T_{\text{leaf}} = \frac{1}{2} \frac{d\mathbf{a}^T}{dt} \mathbf{M}_{\text{leaf}} \frac{d\mathbf{a}}{dt}, \quad (6.19)$$

where $\mathbf{M}_{\text{leaf}}$ represents the constant mass matrix. Full expressions for the stiffness and mass matrices are provided in Appendix 6.11.2.

On a system level, the kinematic shuttle constraints of Eqs. (6.10) and (6.12) in terms of the components of $\mathbf{a}$ are satisfied when

$$u_1^l + u_2^l = u_1^r + u_2^r, \quad v_1^l = v_1^r - d_x \psi_1^r, \quad v_2^l = v_2^r, \quad \psi_1^l = \psi_1^r. \quad (6.20)$$

The remaining coefficients u_3 , ψ_2 and ψ_3 describe deformation without shuttle motion, so they already satisfy the kinematic constraints. The mass of the shuttle is considerable and modeled as a point mass m_s located midway between the two leaf springs and at a distance d_s above the leaf spring ends at $s = L$. The shuttle mass matrix $\mathbf{M}_{\text{shuttle}}$ is provided in Appendix 6.11.2.

The stiffness and mass matrix of the system, consisting of two leaf springs, the shuttle, and the lumped compliance, can be obtained by a finite element-like assembly of the derived matrices for the single leaf spring and the shuttle. The kinematic shuttle constraints of Eq. (6.20) can be accounted for by direct elimination. In the following sections, we will derive analytical expressions for the first three natural frequencies, since these demonstrate the differences between the overconstrained, exactly constrained and dynamically stiffened cases. To obtain tractable expressions, we only use parts of the full 14×14 system matrices by making suitable assumptions on the dominant mass and motion in the various natural vibration modes. This reduces the mathematical complexity of the eigenvalue problem.

6.7.5 Actuation mode

In the first natural mode, the system moves in the degree of freedom. Depending on the magnitude of ϕ_0 , the system also exhibits some torsion deformation simultaneously. This mode can be modeled on the basis of symmetric leaf spring deformation, the linear-system solution (coefficient u_1) and the anti-symmetric buckling mode (coefficients u_2 and ψ_2). The stiffness is given by

$$\frac{dF_x}{du} = \frac{24EI_y}{L^3} \frac{\phi_{\text{cr}}^2 - (\phi_0 + \phi_n)^2}{\phi_{\text{cr}}^2 - (1 - 6/\pi^2)(\phi_0 + \phi_n)^2}, \quad (6.21)$$

showing the dependency on the misalignment angle ϕ_0 . Notch hinge angle ϕ_n is related to ϕ_0 by

$$\phi_n = \frac{1}{1 + \Phi_1} \phi_0, \quad \Phi_1 = \frac{2Lk_n}{EI_x}. \quad (6.22)$$

Factor Φ_1 is used to account for the notch hinge stiffness k_n relative to the effective rotational x -axis stiffness of the flexure mechanism between the two leaf spring ends at $s = 0$.

When $\phi_0 = 0$, the stiffness dF_x/du of Eq. (6.21) is simply $24EI_y/L^3$, whereas when the misalignment increases, the stiffness decreases. The stiffness reaches a value of zero when $\phi_0 + \phi_n = \phi_{cr}$; the system buckles. The ratio Φ_1 between notch hinge stiffness k_n and $EI_x/2L$ determines the rotation ϕ_n of the notch hinge. When k_n is very large, $\phi_n \rightarrow 0$. Since the shuttle mass m_s is the dominant moving mass in this mode, the first natural frequency can simply be obtained as

$$\omega_1 = \sqrt{\frac{1}{m_s} \frac{dF_x}{du}}. \quad (6.23)$$

6.7.6 Suspension mode

A second natural mode is one in which the shuttle moves out of the x, z -plane due to bending deformation of the leaf springs about the x -axis in a pendulum-like fashion. It is referred to as a suspension mode. In the overconstrained case, the leaf spring deformation is symmetric. In the exactly constrained cases, when the notch hinge stiffness is finite, the shuttle also rotates about a z -axis aligned with the right leaf spring; the leaf springs exhibit some torsion as well and the deformation is no longer symmetric.

This mode can be modeled on the basis of only the linear-system solution (with coefficients v_1, v_2, v_3 and ψ_1). The dominant moving mass is the shuttle mass m_s , so M_{leaf} can be ignored and the natural frequency is

$$\omega_2 = \sqrt{\frac{1}{m_s} \frac{dF_y}{dv}}. \quad (6.24)$$

The natural frequency is affected by the geometric parameters d_s , which gives the distance between the shuttle mass and the leaf spring ends, and d_x , which is the shuttle length. The full expression for the stiffness dF_y/dv is provided in Appendix 6.11.3.

In the overconstrained case, $k_n \rightarrow \infty$ and the effective stiffness is given by

$$\frac{dF_y}{dv} = \frac{6EI_x}{L^3 + 3d_s^2L + 3d_sL^2}, \quad (6.25)$$

which reduces to $6EI_x/L^3$ when also $d_s = 0$.

In the exactly constrained cases with finite notch hinge stiffness, the effective stiffness is given by

$$\frac{dF_y}{dv} = \frac{6EI_x}{L^3} \frac{3 + 4\Phi_2 + \Phi_1(3 + \Phi_2)}{12 + 7\Phi_2 + \Phi_1(3 + \Phi_2)}, \quad \Phi_2 = \frac{L^2GJ}{d_x^2EI_x}, \quad (6.26)$$

when $d_s = 0$. Factor Φ_2 is a measure of the torsion stiffness of the entire flexure mechanism at the shuttle relative to its bending stiffness about the x -axis.

This mode is not affected by misalignment ϕ_0 .

6.7.7 Internal mode

A third natural mode can be obtained on the basis of the symmetric buckling mode, in which the shuttle stays stationary and only the leaf springs deform. It is referred to as an internal mode. We assume that the deformation of the left and right leaf spring is symmetric, so $u_3^l = -u_3^r$ and $\psi_3^l = \psi_3^r$. Since the shuttle does not move, the only moving mass stems from the leaf springs, so the consistent mass matrix $\mathbf{M}_{\text{leaf}}$ is taken into account now. The corresponding eigenvalue problem is given by

$$\begin{pmatrix} -\omega^2 \begin{bmatrix} 3Lm_0/(4\pi^2) & 0 \\ 0 & 3I_0L \end{bmatrix} & \\ & \begin{bmatrix} 4\pi^2 EI_y/L^3 & -(\phi_0 + \phi_n)\pi EI_x/L^2 \\ -(\phi_0 + \phi_n)\pi EI_x/L^2 & 4\pi^2 GJ/L \end{bmatrix} \end{pmatrix} \begin{bmatrix} u_3 \\ \psi_3 \end{bmatrix} = 0, \quad (6.27)$$

where $m_0 = \rho A$ and $I_0 = \rho(I_x + I_y)$. The smallest solution is

$$\omega_3 = \sqrt{\frac{k_3}{\rho AL}}, \quad (6.28)$$

where

$$\begin{aligned} k_3 = & \frac{2}{3}\pi^2 \frac{GJA}{I_x L} + \frac{8}{3}\pi^4 \frac{EI_y}{L^3} \\ & - \frac{2}{3L^3}\pi^2 \sqrt{\left(\frac{GJAL^2}{I_x} - 4\pi^2 EI_y\right)^2 + AE^2 I_x L^2 (\phi_0 + \phi_n)^2}. \end{aligned} \quad (6.29)$$

Since this natural mode exhibits deformation in the low-stiffness deformation of the leaf springs, it is affected, like the first natural mode, by the misalignment as well.

6.7.8 Summary

From this analysis, we conclude that two kinds of natural modes can be distinguished:

1. modes with the low-stiffness bending deformation of the leaf springs and
2. modes with deformation associated with higher stiffness.

The actuation mode (Sec. 6.7.5) and internal mode (Sec. 6.7.7) belong to the first kind. For these modes, the misalignment affects the mode shape and natural frequency. While the relatively large misalignment load M_0 acts solely in the leaf spring plane of high bending stiffness when the leaf spring is undeformed, upon bending (about the y -axis) moment M_0 has a component in the low-stiffness bending plane as well, especially

since the torsional stiffness is relatively small. As a consequence, the mode shape that predominantly features bending when misalignment is zero starts to show an increasing amount of torsion deformation when the misalignment increases.

While the normal elastic restoring force opposes the inertial force of the moving mass and promotes resonance, this component acts in the opposite direction, reducing the effective elastic force and behaving like negative stiffness (also referred to as geometric stiffness), following the same mechanism that governs lateral buckling. For this reason, the corresponding natural frequency decreases with misalignment and the buckling modes can be used to construct the natural modes of the misaligned system.

The second kind of modes do not feature deformation in a low-stiffness direction, meaning that they are not sensitive to parasitic load components.

Given a prescribed misalignment angle, the introduction of the notch stiffness k_n (placed in series with the leaf spring) simply lowers the internal moment and reduces the effect of the geometric stiffness. It also reduces the effective stiffness dF_y/dv in the suspension mode, which lowers the corresponding natural frequency. From a design point of view, the notch stiffness, when sufficiently low, makes the system exactly constrained. Additionally, when the notch stiffness is instead frequency-dependent, it is possible to keep the internal load low and the suspension mode stiffness high at the frequency where it matters, as has been demonstrated with the proposed viscoelastic constraint.

Fig. 6.11 shows the natural frequencies according to the analytical model. Initially, the internal mode has a higher frequency than the suspension mode; around 3.5 mrad, the internal mode decreases to a lower frequency and becomes the second mode of the system. The figure also shows the frequency predictions according to the numerical model of Sec. 6.5 that was used in the comparison with the measurements results in Sec. 6.6. The agreement between the analytical model, the numerical simulations and the measurements is good. A simplified numerical model, subject to largely the same assumptions as the analytical model (i.e. no gravity, warping, shear and leaf spring thickness variations), is visually indistinguishable. The refined numerical model does include these effects and produces only slight quantitative deviations. The analytical model is capable of modeling the geometric stiffness up to the bifurcation, at which the natural frequency of the actuation mode and internal mode goes to zero. Since the bifurcation behavior is very sensitive to perturbations, it can be seen that the refined numerical and the measurements do not portray the vanishing frequency at the bifurcation. Simulations with the numerical model show that this can be attributed to a combination of the effects of gravity, constrained warping and geometric imperfections.

6.8 DISCUSSION

6.8.1 Stress relaxation time

To measure the time scale at which internal stresses (caused by misalignment) decrease in the dynamically stiffened exactly constrained case, a large misalignment of 15 mrad (three times the critical value of the overconstrained case) is applied within 0.2 seconds.

Figure 6.11: Comparison of natural frequencies of the overconstrained system, given by the analytical model of Eqs. (6.23), (6.24) and (6.28), and the numerical model of Sec. 6.5.

Figure 6.12: Change in first natural frequency over time after a sudden initial misalignment, indicating the stress relaxation time scale.

Initially, this causes buckling of the mechanism. In this state, the free vibration of the shuttle, due to an excitation in the first natural mode, is recorded. The change of the first natural frequency over time, due to stress relaxation of the elastomer compound, is shown in Fig 6.12. The natural frequency is affected over a certain time span. Within this time, there will be associated internal stress, deformation and a position error of the shuttle. It can be seen that 60% of the decrease in natural frequency is recovered within 10 seconds. This suggests that the stress relaxation occurs sufficiently fast to negate the internal stresses when a misalignment stems from assembly or manufacturing tolerances. In case the effective misalignment in the mechanism would e.g. vary with the process frequency of a manipulation task, shorter relaxation times or a larger critical misalignment may be needed.

6.8.2 Damping

While this study has focused on the use of viscoelastic material to provide additional stiffness, it is commonly used to provide passive damping. From the modal analysis measurements, the relative damping of the second natural mode can be estimated by means of a logarithmic decrement method. It is found that the relative damping of the second mode is approximately 0.001 in the two cases without viscoelastic material and approximately 0.05 when the material is added in the dynamically stiffened exactly constrained case. Even when the viscoelastic material is present, the relative damping is found to be independent of the misalignment.

6.8.3 Elastomer compound

When the elastomer is exposed to high temperatures, the dynamic modulus may increase further, changing the dynamic performance of the system. For typical flexure mechanism applications, this is likely irrelevant, since such extreme conditions are not expected to occur.

6.9 CONCLUSION

Experimental measurements of a demonstrator set-up of a parallelogram flexure mechanism with a custom synthesised elastomer compound show that

1. the first parasitic natural frequency goes from 61 Hz, without the viscoelastic overconstraint, to 109 Hz, with the viscoelastic overconstraint. This is close to the limit value that can be achieved with a conventionally overconstrained design (which is 124 Hz);
2. the critical misalignment angle, at which buckling occurs and the mechanism no longer functions, goes from only 5 mrad, for the conventional overconstraint, to at least 42 mrad, for the viscoelastic overconstraint.

The measurements are corroborated by a numerical nonlinear multibody analysis combined with DMA (dynamic mechanical analysis) test results of the elastomer compound. An nonlinear analytical model accurately predicts the load-dependent stiffness and first three natural frequencies of the system. The results show that viscoelastic material can be used to significantly increase the dynamic performance of (flexure) mechanisms, while being tolerant to misalignments.

6.10 ACKNOWLEDGMENTS

This research was funded by the Innovative Research Incentives Scheme VIDI (Stichting voor de Technische Wetenschappen) (14152 NWO TTW) of the Ministry of Education, Culture and Science of the Netherlands.

6.11 APPENDIX OF CHAPTER 6

6.11.1 Detailed view of elastomer in the measurement set-up

Figure 6.13: Detailed view of the elastomer in parallel with the left notch hinge.

Fig. 6.13 shows a close-up view of the base of the left leaf spring. The leaf spring fillet is connected on the right-hand side through the notch hinge to the fixed ground. In parallel with the notch hinge, connected on the left-hand side of the fillet is the metal plate with the elastomer (connected in turn to the fixed ground). With the notch hinge free to move, the visco-elastic material provides additional stiffness at high frequencies to the otherwise exactly-constrained system. The system can be made overconstrained by locking the notch hinge.

6.11.2 Leaf spring and shuttle matrices

The stiffness matrix $\mathbf{K}_{\text{leaf}}$ for a single leaf spring is given by

$$\begin{bmatrix} 12S_x & 12S_x & 0 & -2v_3LS_y & -6v_3LS_y/\pi & 6(\pi^2 - 6)v_3LS_y/\pi^2 & 0 & 0 & 0 \\ \cdot & 2\pi^2S_x & 0 & -2v_3LS_y & -\pi v_3LS_y & 9v_3LS_y/2 & 0 & 0 & 0 \\ \cdot & \cdot & 2\pi^2S_x & 6v_3LS_y/\pi & -3v_3LS_y/2 & -\pi v_3LS_y & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & 2\pi^2L^2S_t & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & 2\pi^2L^2S_t & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & L^2S_t & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 12S_y & 6LS_y & 6LS_y \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4L^2S_y & 2L^2S_y \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 4L^2S_y \end{bmatrix}, \quad (6.30)$$

where $S_x = EI_x/L^3$, $S_y = EI_y/L^3$ and $S_t = GJ/L^3$.

The mass matrix $\mathbf{M}_{\text{leaf}}$ for a single leaf spring is given by

$$\begin{bmatrix} \frac{13}{35} L m_0 & \frac{15+5\pi^2+7\pi^4}{20\pi^4} L m_0 & \frac{1}{4\pi} L m_0 & 0 & 0 & 0 & \frac{13}{35} L m_0 & \frac{11}{210} L^2 m_0 & -\frac{13}{420} L^2 m_0 \\ \cdot & \left(\frac{1}{3} + \frac{5}{8\pi^2}\right) L m_0 & \frac{1}{4\pi} L m_0 & 0 & 0 & 0 & \frac{15+5\pi^2+7\pi^4}{20\pi^4} L m_0 & \frac{2+15/\pi^4}{40} L^2 m_0 & -\frac{4+45/\pi^4}{120} L^2 m_0 \\ \cdot & \cdot & \frac{3}{8\pi^2} L m_0 & 0 & 0 & 0 & \frac{1}{4\pi} L m_0 & \frac{3+\pi^2}{24\pi^3} L^2 m_0 & -\frac{3+\pi^2}{24\pi^3} L^2 m_0 \\ \cdot & \cdot & \cdot & I_0 L/3 & -I_0 L/(2\pi) & -I_0 L/2 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & I_0 L/2 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 3I_0 L/2 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 13L m_0/35 & 11L^2 m_0/210 & -13L^2 m_0/420 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & L^3 m_0/105 & -L^3 m_0/140 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & L^3 m_0/105 \end{bmatrix}, \quad (6.31)$$

where $m_0 = \rho A$ and $I_0 = \rho(I_x + I_y)$.

The mass matrix $\mathbf{M}_{\text{shuttle}}$ for the shuttle is given by

$$\begin{bmatrix} m_s & m_s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdot & m_s & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & d_x^2 m_s/4 & 0 & 0 & d_x m_s/2 & -d_s d_x m_s/2 & 0 \\ \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & m_s & -d_s m_s & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & d_s^2 m_s & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix}, \quad (6.32)$$

where m_s is the mass of the shuttle, d_x the length of the shuttle and d_s the distance between the leaf spring end at $s = L$ and the center of mass of the shuttle.

6.11.3 Effective stiffness in suspension mode

The effective stiffness dF_y/dv is given by

$$\frac{6EI_x[3 + 4\Phi_2 + \Phi_1(3 + \Phi_2)]}{3d_s^2 L[6 + 5\Phi_2 + \Phi_1(3 + \Phi_2)] + 3d_s L^2[9 + 6\Phi_2 + \Phi_1(3 + \Phi_2)] + L^3[12 + 7\Phi_2 + \Phi_1(3 + \Phi_2)]}. \quad (6.33)$$

CHAPTER 7

Conclusion

The objective of this thesis is to provide a deeper understanding of the elastic deformation and the configuration-dependent stiffness properties of precision flexure mechanisms for systematically creating improved designs with a large range of motion.

The first research subgoal is the development of modeling techniques for the spatial sheet flexure under general load conditions at moderate deformations. The sheet flexure considered in this work is the component that provides constraints, i.e. high stiffness, in three directions, and allows motion in the other three. For this reason, its width is generally the same order as the length. The maximum transverse displacement is considered to be 15% of the length. At this deformation, sheet flexures exhibit the nonlinear geometric stiffness and enable a large range of motion in flexure mechanisms.

It has been shown that an energy-based discretization approach has several advantages for the formulation of closed-form analytical model expressions (chapter 2) and a numerical element for flexible multibody software (chapter 3). The energy-based discretization approach consists of the use of a variational principle to discretize the beam continuum by means of suitable interpolation functions. The advantage of this approach is that the governing differential equations can be converted to a set of algebraic equations, whose complexity, accuracy, and solvability can be controlled by means of the interpolation function choice. The energy approach allows for a systematic inclusion of the various deformation modes of a beam, i.e. elongation, bending, shear, and nonuniform torsion (which includes warping deformation), in a geometrically nonlinear context. Because the model equations are derived from a potential scalar, the significance of the various effects can be compared. Given some estimates of their order of magnitude, the truncation of terms can be motivated.

In chapter 2, a closed-form solution is presented in an interpretable format: actuation loads and error motions as functions of the applied loads and degrees of freedom. It is verified against beam- and plate-based finite element solutions. The deviations with respect to beam elements are small. It also compares well to plate elements, although there may be significant deviations due to the Wagner torque at larger twist angles. For most precision designs, this effect is not crucial as it affects the torsion moment principally and only underestimates the performance-limiting support stiffness.

In chapter 3, the implementation in the generalized strain beam formulation is detailed. The distinct aspect of this formulation, i.e. the consideration of generalized strains and stresses, is highlighted by first deriving the conventional second-order beam element using a suitable multi-field variational principle. The third-order effects are then accounted for by a nonlinear constitutive relationship that provides very limited

overhead. Because the distribution of the internal moments in the deformed configuration is modeled analytically, it is shown that it is no longer necessary to use a mesh of multiple second-order beam elements per sheet flexure for convergence; instead, a single third-order element per sheet flexure provides adequate results. This leads to a reduction in computation time, especially when more complex flexure mechanisms are simulated. The computation time is reduced by a factor of three for a flexure-based spherical joint simulation and a factor of five for a larger flexure-based hexapod simulation. It is shown that the Wagner torque can also be modeled effectively with the numerical element.

The second research subgoal is exploring the use of overconstrained flexure mechanisms. Two case studies of single-overconstraint flexure mechanisms have been introduced. It has been shown that a quantifiable performance limitation in terms of a critical misalignment can be demonstrated experimentally, simulated numerically and predicated analytically.

In chapter 4, a single-overconstraint cross-hinge mechanism has been introduced to serve as a case study on the limitations of overconstrained design. The effect of a misalignment in the overconstrained direction of the mechanism is analyzed. Bifurcation buckling, due to a critical misalignment displacement that induces the buckling load, can be designated as the quantifiable operational limit of the mechanism. A measurement set-up demonstrates the relationship between misalignment, eigenfrequency and ultimately buckling. The essential mechanics are covered in an analytical buckling analysis, which also provides a practical design formula for the critical misalignment as a function of design parameters. Simulations with a numerical model show that common model idealizations and assumptions tend to underestimate the misalignment tolerance of the mechanism, i.e. reality is more forgiving: only when constrained warping, frame compliance, gravity and leaf spring thickness variations are accounted for do the simulations also match quantitatively with the measurements. The effect of constrained warping is given special attention in chapter 5, because it leads to a considerable increase in the buckling load and hence the critical misalignment. Its importance is often neglected for beams with narrow rectangular cross-sections, such as the sheet flexure. The analytical buckling analysis of chapter 4, which is qualitatively correct, is modified to include constrained warping as well.

In chapter 6, the principle of dynamically stiffened exact-constraint design is introduced, offering a design solution for the problem analyzed in chapters 4 and 5. The principle enables flexure mechanisms that combine the misalignment tolerance of exact-constraint design with the higher dynamic performance of overconstrained design. It is based on the use of viscoelastic material with an appropriate glass transition temperature to provide misalignment compensation at low frequencies and additional support stiffness at the high frequencies associated with parasitic natural modes. Loads due to misalignment are mitigated by the stress relaxation property of the viscoelastic material. The compound formulation of a suitable elastomer is discussed. In the idealized case study of a parallelogram flexure mechanism with controllable misalignment and a selective overconstraint, the dynamic performance of the dynamically stiffened exact-constraint case is on par with the overconstrained case, while the critical misalignment

angle is an order of magnitude larger. Numerical and analytical models match well with the misalignment-dependent measurements of eigenfrequency. The analytical sheet flexure model is refined with respect to the general model of chapters 2 and 3, by including the buckling modes of the system in the set of interpolation functions. While these are specific to the parallelogram mechanism, the approach highlights how a nonlinear analytical model can be constructed that is accurate up to the bifurcation.

7.1 DISCUSSION

7.1.1 Energy-based discretization approach

The energy-based discretization approach for deriving the stiffness model in chapters 2 and 3 makes use of the variational principle due to Hellinger and Reissner. It allows for simultaneous variations of the internal moment fields and the displacement fields. The principle turns out to be well suited to flexure mechanism analysis, because it enables the use of physically-motivated and relatively low-order interpolation functions for finding the fourth-order energy terms that govern the elastokinematic effect. This effect describes the undesired decrease in support stiffness of the sheet flexure as it moves in a degree of freedom. It is not captured by linearized beam theory or by the nonlinear models based on the kinematic equivalence with rigid-link mechanisms. A nonobvious result is that with this principle, displacement interpolation based on linearized beam theory suffices when it is combined with a refined internal moment interpolation based on the known deformed-configuration equilibrium equations. This specific interpolation function choice leads to the nonstandard coupling between displacements and internal moments needed to capture the third-order elastokinematic stiffness effect. The extra unknowns introduced by the separate internal moment interpolation in this fashion turn out to be governed by a set of equations only linear in these unknowns. This means that they can be eliminated in a relatively straightforward manner. An alternative derivation by means of a conventional strain energy based principle may also be possible, but requires refinements of the displacement-field interpolation with respect to linear theory that are not straightforward.

An advantage related to closed-form analysis is that the potential energy scalar now remains a polynomial, albeit it with additional unknowns. Mathematical manipulations, such as successive differentiation steps, are easier to carry out compared to a conventional strain energy based formulation, in which the potential becomes a rational function. This is especially relevant when assemblies of sheet flexures are modeled.

7.1.2 Solutions in closed form

In chapters 2 and 3, the assumptions on the dimensions of the sheet flexure, the material failure strain and the maximum transverse displacement lead to a set of order of magnitude estimates for all physical quantities involved in the stiffness model. It provides the basis for the truncation and ultimately leads to load-displacement relations that can be expressed in closed form. The expressions provide the relationship between error

motions and actuation loads as functions of the applied loads and degrees of freedom. As the relationship is explicit, it can be used for calculations. Still, the inherent complexity and multitude of the effects covered make for lengthy expressions, even though the equation structure is fundamentally only a third-order polynomial (disregarding the trigonometric functions in the warping coefficients). While the model could be refined for higher accuracy by relaxing certain assumptions, and possibly solved or approximated in closed form as well, the increasing length of the expressions progressively decreases the utility of such mathematical equations as further refinements are carried out. The energy method enables a natural extension to models of sheet flexure assemblies, especially for parallel configurations, since the kinematic constraints are then easily accounted for. Although the model equations for the parallelogram flexure mechanism and cross-hinge mechanism are limited in length and readable, it takes considerable effort with analytical solver software to reduce the governing set of algebraic equations to these model equations. It is hard to perform the required analytical manipulations that are effective at reducing the expression size in an algorithmic fashion without compromising the convergence and accuracy of the reduced model.

Similar difficulty has been observed in the analytical buckling analysis of chapters 4 and 5. While the framework for the linear buckling analysis provides the critical load and buckling modes encoded as a system of only linear ordinary differential equations, the mathematical steps required for a closed-form expression are involved. A closed-form expression is derived in chapter 4 with the help of some physical symmetry considerations, dimensional analysis, and careful modeling of only the critical effects. Still, it is shown that for practical purposes, the complicated relationship between the geometric parameters and the critical load is best given by a lookup figure. The issue is compounded with the addition of warping deformation in chapter 5: here, a series solution was used to approximate the solution of the linear buckling problem.

7.1.3 Numerical models

In the development of analytical formulations for both research subgoals, a tool for numerical simulations has proven indispensable, especially one with the ability to switch between degrees of fidelity. It has bridged the gap between the analytical buckling model and the measurements of chapter 4 and helped identify which model idealizations are overly restrictive.

7.1.4 Future potential

The low order of the interpolation functions required in the presented energy-based discretization approach can be exploited in the formulation of nonlinear stiffness models for more complicated flexure mechanism components, at least when the goal is to create a numerical implementation. In combination with design criteria formulated in terms of elastic energy, it seems likely that a computer-aided design optimization procedure for large range of motion flexure mechanisms can be developed.

Dankwoord

Ik zou mijn vrienden, kennissen, collega's en familie willen bedanken voor hun steun de afgelopen jaren. Het heeft me geholpen om van dit proefschrift een mooi boekje te maken.

Dannis, ik wil je bedanken voor je vlotte en vrolijke begeleiding, de goede ideeën, de discussies in de categorie informal learning, de vele conferenties en het goede vertrouwen.

Jaap, bedankt voor het mechanica-onderwijs, de flauwe grappen, de talrijke suggesties en correcties, en het geduld met het lezen van al dat proza.

I would like to thank the defense committee members, Shorya Awtar, Stuart Smith, Bojana Rosić and Marcel Ellenbroek, for their efforts to read my thesis, provide feedback and assess my work. Shorya, your work has been an inspiration. Marcel, bedankt voor je enthousiasme gedurende de hele promotie. De Vidi gebruikerscommissie wil ik bedanken voor hun inbreng.

Mark, Martin, Jan: het was leuk om samen met jullie uit te vogelen hoe flexures werken, hoe SPACAR werkt, hoe studenten werken, hoe confereren werkt, hoe je niet werkt, waarom dingen stuk gaan, hoe je een PhD haalt, wat goede koffie is, en meer. Bedankt!

Ronald, Wouter, Ben, Leo, Johannes, Gert-Willem en andere WA'ers: jullie enthousiaste discussies, tips, en leuke verhalen tijdens pauzes waren zinvolle hulp en afleiding.

Debbie, Martina, Belinda: dank voor het regelen van alles wat geregeld moest worden.

Sven, Jacques, Wilma en Deick: ik heb met veel plezier met jullie aan de masteropdrachten gewerkt.

Jurnan, Boukje, Wout, Martijn, Luuk, Ivor, Koen, Mieke, Venkat, Iqbal, Axel, Leon, Farnaz, Farshad, Luis, Ger en andere Horst-inwoners: bedankt voor de gezelligheid en de samenwerking in de afgelopen jaren.

Tot slot wil ik mijn vader, mijn moeder, Frank en natuurlijk mijn zusje bedanken voor hun steun, aandacht en vrolijkheid.

References

- [1] W. H. T. M. Aangenent, L. F. Koorneef, T. A. M. Ruijl, S. C. J. M. van den Berg, S. H. van der Meulen, J. van Eijk, P. H. G. Wullms, and R. H. A. van Lieshout. “Stage positioning system and lithographic apparatus”. 2018. United States Patent US 9,897,926 B2.
- [2] R. G. K. M. Aarts, J. P. Meijaard, and J. B. Jonker. “Flexible multibody modelling for exact constraint design of compliant mechanisms”. In: *Multibody System Dynamics* 27.1 (2012), pp. 119–133.
- [3] E. Abbe. “Messapparate für Physiker”. In: *Zeitschrift für Instrumentenkunde* 10 (1890), pp. 446–447.
- [4] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*. Vol. 55. Washington, DC: National Bureau of Standards, 1964.
- [5] G. Alfano and F. Marotti de Sciarra. “Mixed finite element formulations and related limitation principles: a general treatment”. In: *Computer Methods in Applied Mechanics and Engineering* 138.1–4 (1996), pp. 105–130.
- [6] R. Alsafadie, M. Hjiar, and J. M. Battini. “Three-dimensional formulation of a mixed corotational thin-walled beam element incorporating shear and warping deformation”. In: *Thin-Walled Structures* 49.4 (2011), pp. 523–533.
- [7] D. G. Ashwell. “The anticlastic curvature of rectangular beams and plates”. In: *The Aeronautical Journal* 54.479 (1950), pp. 708–715.
- [8] S. Awtar. “Analysis and Synthesis of Parallel Kinematic XY Flexure Mechanisms”. PhD thesis. Cambridge, MA: Massachusetts Institute of Technology, 2003.
- [9] S. Awtar and S. Sen. “A Generalized Constraint Model for Two-Dimensional Beam Flexures: Nonlinear Load-Displacement Formulation”. In: *Journal of Mechanical Design* 132.8 (2010), 081008:1–11.
- [10] S. Awtar and S. Sen. “A Generalized Constraint Model for Two-Dimensional Beam Flexures: Nonlinear Strain Energy Formulation”. In: *Journal of Mechanical Design* 132.8 (2010), 081009:1–11.
- [11] S. Awtar, K. Shimotsu, and S. Sen. “Elastic Averaging in Flexure Mechanisms: A Three-Beam Parallelogram Flexure Case Study”. In: *Journal of Mechanisms and Robotics* 2.4 (2010), 041006:1–12.
- [12] S. Awtar, A. H. Slocum, and E. Sevincer. “Characteristics of Beam-Based Flexure Modules”. In: *Journal of Mechanical Design* 129.6 (2007), pp. 625–639.

- [13] R. Bai, S. Awtar, and G. Chen. "A closed-form model for nonlinear spatial deflections of rectangular beams in intermediate range". In: *International Journal of Mechanical Sciences* 160 (2019), pp. 229–240.
- [14] J. F. Besseling. "Non-linear analysis of structures by the finite element method as a supplement to a linear analysis". In: *Computer Methods in Applied Mechanics and Engineering* 3.2 (1974), pp. 173–194.
- [15] D. L. Blanding. *Exact Constraint: Machine Design Using Kinematic Principles*. New York: ASME Press, 1999.
- [16] D. M. Brouwer, J. P. Meijaard, and J. B. Jonker. "Large deflection stiffness analysis of parallel prismatic leaf-spring flexures". In: *Precision Engineering* 37.3 (2013), pp. 505–521.
- [17] B. Budiansky. "Theory of buckling and post-buckling behavior of elastic structures". In: *Advances in Applied Mechanics* 14 (1974), pp. 1–65.
- [18] G. R. Cowper. "The shear coefficient in Timoshenko's beam theory". In: *Journal of Applied Mechanics* 33.2 (1966), pp. 335–340.
- [19] M. Crisfield. *Non-linear Finite Element Analysis of Solids and Structures*. Chichester: John Wiley & Sons, 1991.
- [20] M. S. G. Cullimore. "The Shortening Effect - A Non-linear Feature of Pure Torsion". In: *Engineering Structures* (1949), pp. 153–164.
- [21] F. S. Eastman. "Flexure pivots to replace knife edges and ball bearings". In: *Engineering Experiment Station Bulletin* 86. University of Washington, 1935.
- [22] J. van Eijk. "On the design of plate-spring mechanisms". PhD thesis. Delft: Delft University of Technology, 1985.
- [23] C. A. Felippa. "A survey of parametrized variational principles and applications to computational mechanics". In: *Computer Methods in Applied Mechanics and Engineering* 113.1–2 (1994), pp. 109–139.
- [24] C. A. Felippa. "Parametrized multifield variational principles in elasticity: I. Mixed functionals". In: *Communications in Applied Numerical Methods* 5.2 (1989), pp. 79–88.
- [25] M. Gergesova, B. Zupančič, I. Saprunov, and I. Emri. "The closed form t-T-P shifting (CFS) algorithm". In: *Journal of Rheology* 55.1 (2011), pp. 1–16.
- [26] L. C. Hale. "Principles and Techniques for Designing Precision Machines". PhD thesis. Cambridge, MA: Massachusetts Institute of Technology, 1999.
- [27] J. A. Haringx. "The cross-spring pivot as a constructional element". In: *Applied Scientific Research* 1.1 (1949), pp. 313–332.
- [28] Z. Hongzhe and B. Shusheng. "Accuracy characteristics of the generalized cross-spring pivot". In: *Mechanism and Machine Theory* 45.10 (2010), pp. 1434–1448.
- [29] B. D. Jensen and L. L. Howell. "The modeling of cross-axis flexural pivots". In: *Mechanism and Machine Theory* 37.5 (2002), pp. 461–476.

- [30] R. V. Jones. "A parallel-spring cross-movement for an optical bench". In: *Journal of Scientific Instruments* 33.7 (1956), pp. 279–280.
- [31] R. V. Jones. *Instruments and Experiences, Papers on Measurement and Instrument Design*. Chichester: Wiley, 1988.
- [32] R. V. Jones. "Some uses of elasticity in instrument design". In: *Journal of Scientific Instruments* 39.5 (1962), pp. 193–203.
- [33] J. B. Jonker. "A finite element dynamic analysis of spatial mechanisms with flexible links". In: *Computer Methods in Applied Mechanics and Engineering* 76 (1989), pp. 17–40.
- [34] J. B. Jonker. "Implementation of shear deformable thin-walled beam element for flexible multibody dynamics". In: *Proceedings of the 8th ECCOMAS Thematic Conference on Multibody Dynamics 2017*. Prague, Czech Republic, 2017, pp. 141–162.
- [35] J. B. Jonker, R. G. K. M. Aarts, and J. van Dijk. "A linearized input–output representation of flexible multibody systems for control synthesis". In: *Multibody System Dynamics* 21.2 (2008), pp. 99–122.
- [36] J. B. Jonker and J. P. Meijaard. "A geometrically non-linear formulation of a three-dimensional beam element for solving large deflection multibody system problems". In: *International Journal of Non-Linear Mechanics* 53 (2013), pp. 63–74.
- [37] J. B. Jonker and J. P. Meijaard. "SPACAR — Computer Program for Dynamic Analysis of Flexible Spatial Mechanisms and Manipulators". In: *Multibody Systems Handbook*. Ed. by W. Schiehlen. Berlin, 1990, pp. 123–143.
- [38] W. T. Koiter. "Over de stabiliteit van het elastisch evenwicht". PhD thesis. Amsterdam: Delft University of Technology, H.J. Paris, 1945.
- [39] M. P. Koster. *Constructieprincipes voor het nauwkeurig bewegen en positioneren*. Enschede: Twente University Press, 2000.
- [40] C. Lanczos. *The variational principles of mechanics*. Mathematical Expositions, No. 4. University of Toronto Press, 1949.
- [41] H. Lang, J. Linn, and M. Arnold. "Multi-body dynamics simulation of geometrically exact Cosserat rods". In: *Multibody System Dynamics* 25.3 (2011), pp. 285–312.
- [42] E. Lommel. "Zur Theorie der Bessel'schen Functionen". In: *Mathematische Annalen* 3.4 (1871), pp. 475–487.
- [43] A. E. H. Love. *A treatise on the mathematical theory of elasticity*. Fourth edition. Cambridge: Cambridge University Press, 1927.
- [44] J. C. Maxwell. "On the calculation of the equilibrium and stiffness of frames". In: *Philosophical Magazine* 27.4 (1864), pp. 294–299.

- [45] J. P. Meijaard. "A Method for Calculating and Continuing Static Solutions for Flexible Multibody Systems". In: *Journal of Computational and Nonlinear Dynamics* 13.7 (2018), 071002:1–8.
- [46] J. P. Meijaard. "Fluid-Conveying Flexible Pipes Modeled by Large-Deflection Finite Elements in Multibody Systems". In: *Journal of Computational and Nonlinear Dynamics* 9.1 (2014), 011008:1–7.
- [47] J. P. Meijaard. "Refinements of Classical Beam Theory for Beams with a Large Aspect Ratio of Their Cross-Sections". In: *IUTAM Symposium on Dynamics Modeling and Interaction Control in Virtual and Real Environments*. Budapest, Hungary, 2011, pp. 285–292.
- [48] J. P. Meijaard. "The Importance of Imperfections in Leaf-spring Flexures for the Support Stiffness". In: *Procedia IUTAM* 13 (2015), pp. 82–89.
- [49] J. P. Meijaard. "Validation of flexible beam elements in dynamics programs". In: *Nonlinear Dynamics* 9.1-2 (1996), pp. 21–36.
- [50] J. P. Meijaard, D. M. Brouwer, and J. B. Jonker. "Analytical and experimental investigation of a parallel leaf spring guidance". In: *Multibody System Dynamics* 23.1 (2010), pp. 77–97.
- [51] A. G. M. Michell. "Elastic Stability of Long Beams under Transverse Forces". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 48.292 (1899), pp. 298–309.
- [52] M. Naves, R. G. K. M. Aarts, and D. M. Brouwer. "Large stroke high off-axis stiffness three degree of freedom spherical flexure joint". In: *Precision Engineering* 56 (2019), pp. 422–431.
- [53] M. Naves, R. G. K. M. Aarts, and D. M. Brouwer. "Maintaining high support stiffness". In: *Mikroniek* 59.3 (2019), pp. 17–21.
- [54] M. Nijenhuis and D. M. Brouwer. "A Closed-Form Model for the Support Stiffness of Spatial Flexure Strips With Limited Twist". In: *ASME 2016 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*. Charlotte, NC, USA, 2016, V05AT07A027:1–10.
- [55] M. Nijenhuis, J. B. Jonker, and D. M. Brouwer. "Importance of warping in beams with narrow rectangular cross-sections: an analytical, numerical and experimental flexible cross-hinge case study". In: *Proceedings of the 9th ECCOMAS Thematic Conference on Multibody Dynamics*. Ed. by A. Kecskeméthy and F. Geu Flores. Duisburg, Germany, 2019, pp. 199–206. Chapter 5 of this thesis.
- [56] M. Nijenhuis, J. P. Meijaard, and D. M. Brouwer. "Misalignments in an overconstrained flexure mechanism: a cross-hinge stiffness investigation". In: *Precision Engineering* 62 (2020), pp. 181–195. Chapter 4 of this thesis.
- [57] M. Nijenhuis, J. P. Meijaard, and D. M. Brouwer. "Third-order stiffness formulation for sheet flexures using an energy-based discretization approach (EDA): 3-D closed-form nonlinear model". In: *International Journal of Mechanical Sciences* (). Chapter 2 of this thesis. Submitted.

- [58] M. Nijenhuis, J. P. Meijaard, D. Mariappan, J. L. Herder, D. M. Brouwer, and S. Awatar. “An Analytical Formulation for the Lateral Support Stiffness of a Spatial Flexure Strip”. In: *Journal of Mechanical Design* 139.5 (2017), 051401:1–11.
- [59] J. M. Paros and L. Weisbord. “How to design flexure hinges”. In: *Machine Design* 37.27 (1965), pp. 151–156.
- [60] J. E. Plainevaux. “Etude des déformations d’une lame de suspension élastique”. In: *Il Nuovo Cimento* 4.4 (1956), pp. 922–928.
- [61] J. E. Plainevaux. “Mouvement de tangage d’une suspension élémentaire sur lames élastiques”. In: *Il Nuovo Cimento* 4.5 (1956), pp. 1133–1141.
- [62] L. Prandtl. “Kipperscheinungen, Ein Fall vom instabilem elastischem Gleichgewicht”. PhD thesis. Universität München, 1900.
- [63] E. Reissner. “On one-dimensional large-displacement finite-strain beam theory”. In: *Studies in Applied Mathematics* 52 (1973), pp. 87–95.
- [64] T. A. M. Ruijl and P. Wullms. “Polymer damper technology for improved system dynamics”. In: *Proceedings of the 19th international conference of the european society for precision engineering and nanotechnology*. Bilbao, Spain, 2019, pp. 538–541.
- [65] W. W. P. J. van de Sande, R. G. K. M. Aarts, and D. M. Brouwer. “Effects of misalignments on the static and dynamic behaviour of a multiple overconstrained compliant 4-bar mechanism”. In: *Precision Engineering* 60 (2019), pp. 143–151.
- [66] H. A. F. A. Santos, P. M. Pimenta, and J. P. Moitinho de Almeida. “Hybrid and multi-field variational principles for geometrically exact three-dimensional beams”. In: *International Journal of Non-Linear Mechanics* 45.8 (2010), pp. 809–820.
- [67] P. Schellekens, N. Rosielle, H. Vermeulen, M. Vermeulen, S. Wetzels, and W. Pril. “Design for Precision: Current Status and Trends”. In: *CIRP Annals* 47.2 (1998), pp. 557–586.
- [68] A. L. Schwab. “Dynamics of Flexible Multibody Systems”. PhD thesis. Delft: Delft University of Technology, 2002.
- [69] S. Sen. “Beam Constraint Model: Generalized Nonlinear Closed-form Modeling of Beam Flexures for Flexure Mechanism Design”. PhD thesis. Ann Arbor: University of Michigan, 2013.
- [70] S. Sen and S. Awatar. “A Closed-Form Nonlinear Model for the Constraint Characteristics of Symmetric Spatial Beams”. In: *Journal of Mechanical Design* 135.3 (2013), 031003:1–11.
- [71] G. J. Simitses and D. H. Hodges. “1: Introduction and fundamentals”. In: *Fundamentals of Structural Stability*. Butterworth-Heinemann, 2006.
- [72] A. H. Slocum. *Precision Machine Design*. Englewood Cliffs, Prentice-Hall, 1992.
- [73] S. T. Smith. *Flexures: Elements of Elastic Mechanisms*. CRC Press, 2000.

- [74] H. Soemers. *Design Principles: For Precision Mechanisms*. Enschede: T-pointpress, 2011.
- [75] S. Timoshenko. "On the torsion of a prism, one of the cross-sections of which remains plane". In: *Proceedings of the London Mathematical Society, 2nd series* 20 (1922), pp. 389–397.
- [76] S. Timoshenko and J. M. Gere. *Theory of Elastic Stability*. Second edition. New York: McGraw-Hill, 1961.
- [77] S. Timoshenko and J. N. Goodier. *Theory of Elasticity*. McGraw-Hill, 1969.
- [78] N. S. Trahair. "Nonlinear Elastic Nonuniform Torsion". In: *Journal of Structural Engineering* 131.7 (2005), pp. 1135–1142.
- [79] E. Trefftz. "Zur Theorie der Stabilität des elastischen Gleichgewichts". In: *Zamm Zeitschrift für Angewandte Mathematik und Mechanik* 13.2 (1933), pp. 160–165.
- [80] V. Z. Vlasov. *Thin-walled elastic beams*. Jerusalem: Israel program for scientific translations, 1961.
- [81] K. Washizu. *Variational Methods in Elasticity and Plasticity*. Third edition. Oxford, New York: Pergamon Press, 1982.
- [82] P. Willoughby. "Elastically Averaged Precision Alignment". PhD thesis. Cambridge, MA: Massachusetts Institute of Technology, 2005.
- [83] M. H. Youssef. "Influence of sulfur content and curing time on the glass transition parameters in SBR vulcanizates: an ultrasonic study". In: *Polymer Testing* 22.2 (2003), pp. 235–242.
- [84] S. Zelenika and F. De Bona. "Analytical and experimental characterisation of high-precision flexural pivots subjected to lateral loads". In: *Precision Engineering* 26.4 (2002), pp. 381–388.

List of publications

JOURNAL ARTICLES

First author

- M. Nijenhuis, J.P. Meijaard, D.M. Brouwer. “Third-order stiffness formulation for sheet flexures using an energy-based discretization approach (EDA): 3-D closed-form nonlinear model”. *International Journal of Mechanical Sciences*. Submitted.
- M. Nijenhuis, J.P. Meijaard, M. Naves, R.G.K.M. Aarts, D.M. Brouwer. “Third-order stiffness formulation for sheet flexures using an energy-based discretization approach (EDA): numerical beam element”. *Computational Mechanics*. Submitted.
- M. Nijenhuis, S.T.B. Klein Avink, W.K. Dierkes, J.W.M. Noordermeer, D.M. Brouwer. “Improved dynamic performance in flexure mechanisms by overconstraining using viscoelastic material”. *Precision Engineering*. Submitted.
- M. Naves, M. Nijenhuis, D.M. Brouwer. “The characteristics and optimal design of large-stroke flexure hinges”. Co-first author. In preparation.
- M. Nijenhuis, J.P. Meijaard, D.M. Brouwer. “Misalignments in an overconstrained flexure mechanism: a cross-hinge stiffness investigation”. *Precision Engineering* 62 (2020) pp. 181-195.
- M. Nijenhuis, J.P. Meijaard, D. Mariappan, J.L. Herder, D.M. Brouwer, S. Awtar. “An Analytical Formulation for the Lateral Support Stiffness of a Spatial Flexure Strip”. *Journal of Mechanical Design* 139-5 (2017).

Co-author

- M. Naves, M. Nijenhuis, W.B.J. Hakvoort, D.M. Brouwer. “Flexure-based 60 degrees stroke actuator suspension for a high torque iron core motor”. *Precision Engineering*. Submitted.
- M. Naves, M. Nijenhuis, W.B.J. Hakvoort, D.M. Brouwer. “Design of a large stroke flexure-based suspension for an iron core torque motor”. *Mikroniek*. Submitted.

CONFERENCE CONTRIBUTIONS

First author

- M. Nijenhuis, J.B. Jonker, D.M. Brouwer. “Importance of warping in beams with narrow rectangular cross-sections: an analytical, numerical and experimental flexible cross-hinge case study”. *Proceedings of the 9th ECCOMAS Thematic Conference on Multibody Dynamics* (2019) pp. 199–206. Duisburg, Germany.
- S.T.B. Klein Avink, M. Nijenhuis, W.K. Dierkes, J.W.M. Noordermeer, D.M. Brouwer. “Flexure mechanism with increased dynamic performance by overconstraining using viscoelastic material”. *Proceedings of the 19th international conference of the european society for precision engineering and nanotechnology* (2019) pp. 82–85. Bilbao, Spain.
- D.J. Andree, M. Nijenhuis, D.M. Brouwer. “Design of a meta-material for the torsional reinforcement of a leaf spring”. *Proceedings of The Thirty-Third Annual Meeting of the American Society for Precision Engineering* (2018) pp. 563–568. Las Vegas, NV, USA.
- M. Nijenhuis, J.P. Meijaard, D.M. Brouwer. “A 3-D nonlinear analytical stiffness model for deflected leaf springs”. *Proceedings of The Thirty-Third Annual Meeting of the American Society for Precision Engineering* (2018) pp. 517–520. Las Vegas, NV, USA.
- M. Nijenhuis, J.P. Meijaard, D.M. Brouwer. “A Spatial Parametric Model for the Nonlinear Stiffness Characteristics of Flexure Strips”. *Proceedings of the ASME 2018 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference* (2018) DETC2018-86024. Quebec City, Canada.
- M. Nijenhuis, D.M. Brouwer. “Stiffness consequences of misalignments in an over-constrained flexure design”. *Proceedings of The Thirty-First Annual Meeting of the American Society for Precision Engineering* (2016) pp. 21–25. Portland, OR, USA.
- M. Nijenhuis, D.M. Brouwer. “A Closed-Form Model for the Support Stiffness of Spatial Flexure Strips With Limited Twist”. *Proceedings of the ASME 2016 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference* (2016) DETC2016-59979. Charlotte, NC, USA.
- M. Nijenhuis, J.P. Meijaard, J.L. Herder, S. Awtar, D.M. Brouwer. “An Analytical Formulation for the Lateral Support Stiffness of a Spatial Flexure Strip”. *Proceedings of the ASME 2015 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference* (2015) DETC2015-46591. Boston, MA, USA.

Co-author

- M. Naves, M. Nijenhuis, W.B.J. Hakvoort, D.M. Brouwer. “Design of a large stroke flexure-based suspension for an iron core torque motor”. *Proceedings of the 19th international conference of the european society for precision engineering and nanotechnology* (2019) pp. 52–55. Bilbao, Spain.