Compatible spanning circuits in edge-colored graphs

Zhiwei Guo

Compatible spanning circuits in edge-colored graphs

Zhiwei Guo

COMPATIBLE SPANNING CIRCUITS IN EDGE-COLORED GRAPHS

DISSERTATION

to obtain
the degree of doctor at the University of Twente,
on the authority of the rector magnificus,
Prof. dr. T.T.M. Palstra,
on account of the decision of the Doctorate Board,
to be publicly defended
on Thursday the 17th of September 2020 at 10:45

by

Zhiwei Guo

born on the 23rd of December 1989
in Lingshi Shanxi, P.R. China

This dissertation has been approved by:

Prof. dr. ir. H.J. Broersma	Supervisor
Prof. dr. X. Li	Supervisor
Prof. dr. S. Zhang	Co-supervisor

The research reported in this thesis has been carried out within the framework of the MEMORANDUM OF AGREEMENT FOR A DOUBLE DOCTORATE DEGREE BETWEEN NORTHWESTERN POLYTECHNICAL UNIVERSITY, PEOPLE'S REPUBLIC OF CHINA AND THE UNIVERSITY OF TWENTE, THE NETHERLANDS.

**UNIVERSITY
OF TWENTE.** | **DIGITAL SOCIETY
INSTITUTE**

DSI Ph.D. Thesis Series No. 20-005
Digital Society Institute
PO. Box 217, 7500 AE Enschede, The
Netherlands.

ISBN: 978-90-365-5046-8

ISSN: 2589-7721 (DSI Ph.D. Thesis Series No. 20-005)

DOI: 10.3990/1.9789036550468

Available online at <https://doi.org/10.3990/1.9789036550468>

Typeset with $\LaTeX$

Printed by Ipskamp Printing, Enschede

Cover design by Pengfei Wan and Zhiwei Guo

Copyright ©2020 Zhiwei Guo, Enschede, The Netherlands.

All rights reserved. No part of this thesis may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording, or otherwise, without prior permission from the copyright owner.

Graduation Committee

Chairman/secretary:

Prof. dr. J.N. Kok University of Twente

Supervisors:

Prof. dr. ir. H.J. Broersma University of Twente

Prof. dr. X. Li Nankai University

Co-supervisor:

Prof. dr. S. Zhang Northwestern Polytechnical University

Members:

Prof. dr. M.J. Uetz University of Twente

Dr. W. Kern University of Twente

Prof. dr. N.V. Litvak University of Twente

prof. dr. I. Schiermeyer Technische Universität Bergakademie Freiberg

Prof. dr. ir. D. Paulusma Durham University

Preface

This thesis is based on the research results on the existence of compatible spanning circuits, which were obtained by the author with different collaborators between September 2016 and April 2020. Apart from the introductory chapter (i.e., Chapter 1), the thesis contains six closely related research chapters (i.e., Chapters 2–7). Chapters 2, 6, 7 and part of Chapter 3 are mainly based on the research that the author carried out while he was working as a PhD student at the Northwestern Polytechnical University in Xi'an, P.R. China. Chapter 4 is based on the research that was carried out while the author was visiting the Institute for Discrete Mathematics and Algebra, Technische Universität Bergakademie Freiberg in Freiberg, Germany. The other parts of the thesis are mainly based on the research results of the author at the University of Twente, The Netherlands. This thesis is devoted to developing sufficient conditions for the existence of specific compatible spanning circuits in edge-colored graphs, and to establishing polynomial-time algorithms for finding them. The papers associated with these six research chapters have been listed below and have been published in (or submitted to) scientific journals.

Papers underlying this thesis

- [1] Compatible spanning circuits in edge-colored graphs, *Discrete Mathematics*, **343** (2020), 111908 (with B. Li, X. Li and S. Zhang). (Chapters 2 and 3)
- [2] Almost eulerian compatible spanning circuits in edge-colored graphs, submitted (with H.J. Broersma, B. Li and S. Zhang). (Chapter 3)

- [3] Compatible spanning circuits and forbidden induced subgraphs, submitted (with C. Brause, M. Geißer and I. Schiermeyer). (Chapter 4)
- [4] Some algorithmic results for finding compatible spanning circuits in edge-colored graphs, accepted by *Journal of Combinatorial Optimization* (with H.J. Broersma, R. Li and S. Zhang). (Chapter 5)
- [5] Compatible eulerian circuits in eulerian (di)graphs with generalized transition systems, *Discrete Mathematics*, **341** (2018), 2104–2112 (with X. Li, C. Xu and S. Zhang). (Chapters 6 and 7)

Other recent joint papers by the author

- [1] Linear k -arboricity in product networks, *Journal of Interconnection Networks*, **16** (2016), 1650008 (with Y. Mao, N. Jia and H. Li).
- [2] The k -independence number of graph products, *The Art of Discrete and Applied Mathematics* **1** (2018), 1–19 (with Y. Mao, E. Cheng and Z. Wang).
- [3] Linear k -arboricity of complete bipartite graphs, accepted by *Utilitas Mathematica* (with H. Zhao and Y. Mao).

Contents

Preface	vii
1 Introduction	1
1.1 Some historical examples	2
1.2 Compatible spanning circuits	5
1.3 Terminology and notations	6
1.4 Research background and progress	9
1.5 Main contributions	13
1.6 Thesis outline	14
2 Compatible spanning k-circuits in edge-colored graphs	17
2.1 Introduction	17
2.2 k -CSCs in E-C complete graphs	20
2.3 k -CSCs in E-C complete multipartite graphs	21
2.4 k -CSCs in E-C hamiltonian graphs	23
2.5 Conclusions and future work	26
3 Compatible spanning circuits in edge-colored graphs satisfying certain degree conditions	27
3.1 Introduction	28
3.2 Ore-type degree conditions	29
3.3 Fan-type degree conditions	32
3.4 CSCs in E-C $2(k + 1)$ -edge-connected graphs	36
3.5 CSCs in E-C random graphs	37
3.6 Proofs of key lemmas	42

3.7	Conclusions and future work	76
4	Compatible spanning circuits in edge-colored $\mathcal{H}$-free graphs	77
4.1	Introduction	77
4.2	Preliminaries	79
4.3	CSCs in E-C hamiltonian graphs	82
4.4	CSCs in E-C supereulerian graphs	86
4.5	Conclusions and an open problem	91
5	Algorithmic results for compatible spanning circuits in edge-colored graphs	93
5.1	Introduction	93
5.2	NP-completeness results	94
5.3	Polynomial-time algorithms	96
5.4	Proofs of the two main theorems	97
5.5	Conclusions and future work	106
6	Compatible Euler tours in eulerian graphs with generalized transition systems	109
6.1	Introduction	109
6.2	CETs for generalized transition systems	114
6.3	CETs for weakly generalized transition systems	118
6.4	Conclusions and future work	122
7	Compatible Euler tours in eulerian digraphs with generalized transition systems	123
7.1	Introduction	123
7.2	CETs for generalized transition systems	127
7.3	CETs in arc-colored eulerian digraphs	133
7.4	Conclusions and future work	135
	Summary	137
	Samenvatting (Summary in Dutch)	141
	Bibliography	145

Acknowledgements	157
About the Author	161

Chapter 1

Introduction

Before formally presenting the basic terminology and notations involved in this thesis (see Section 1.3), we start with a brief informal introduction to the topic of this thesis. This part can help the reader, even the non-professional one, in understanding what this thesis concerns.

In many real-world problems, in order to collect the essential information for solving the problem, it is convenient and effective to model the practical situation of a problem by means of a diagram that consists of a set of points and a set of (directed) line segments or curves joining certain pairs of these points. For instance, a printed circuit board can be modeled by a diagram in which its points represent electronic components, and its line segments represent connections between electronic components. As another example, a diagram can be used to model an ecosystem: in this situation, its points represent the species in the ecosystem, and one can draw a directed line segment (or curve) from u to v whenever the species u preys on the species v . In mathematics, such diagrams serving as mathematical abstractions of practical situations of the above mentioned problems are exactly what we refer to as *(di)graphs* consisting of a set of vertices and a set of (arcs) edges joining certain pairs of these vertices. Graphs are so named because they can be represented graphically. The graphical representation can help us understand a lot of essential properties of problems. More generally, the information of a set of objects with certain existing binary relationships between them

may be represented by means of a (di)graph. In that case, the objects are indicated by vertices of the (di)graph, representing the points of the diagram corresponding to the (di)graph, and the (asymmetrical) binary relationships between these objects are indicated by (arcs) edges of the (di)graph, representing the (directed) line segments or curves of the diagram corresponding to the (di)graph.

Graph theory as a vital branch of mathematics focuses on the study of graphs and their properties. Next, we introduce several graph notions that are closely related to the topic of this thesis, by using some historical examples of situations in which graphs have been applied.

1.1 Some historical examples

The foundation stone of graph theory was laid by Leonhard Euler who was one of the most eminent Swiss mathematicians of the eighteenth century. Back in 1736, Euler solved the notable problem of the *Seven Bridges of Königsberg*.

(a)

(b)

Figure 1.1: The map of Königsberg [102] and its corresponding graph.

The old city of Königsberg in Eastern Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and it included two islands (named Kneiphof and Lomse, respectively). The two islands were connected to each

other and to the two mainlands of the city by seven bridges, as shown in Figure 1.1 (a). It is said that the citizens of Königsberg used to try to make a round walk around the city which would cross each of the seven bridges exactly once. After many attempts, a lot of them realized that it is impossible to complete this task, but there was no reasonable proof until 1736. In that year, Euler [41] considered the problem of the Seven Bridges of Königsberg and published a paper on a solution to the problem. This paper is regarded as the first paper in the history of graph theory (see [10]). Euler showed that no such walk is possible by formulating this problem in terms of a graph such that the land masses correspond to the vertices of the graph and each bridge connecting two land masses corresponds to an edge joining the two corresponding vertices (see Figure 1.1 (b)). The essence of the above problem is to find a tour (using only vertices and edges of the graph) that traverses each edge of the graph exactly once and returns to the initial vertex. In honor of Leonhard Euler, such a tour is nowadays called an *Euler tour*. His approach was not only applicable to deal with the particular problem but also to provide a general solution for other problems of the same type.

For the next example, we introduce a mathematical game on a dodecahedron graph (see Figure 1.2), now known as *Hamilton's puzzle*.

Figure 1.2: The dodecahedron graph.

This puzzle, which is somewhat related to the problem of the Seven Bridges of Königsberg, was introduced by the Irish mathematician William

Rowan Hamilton in 1856, under the name “Icosian Game”, referring to the ancient Greek word for twenty (see [10]). The aim of the above game is to find a tour in the dodecahedron graph which passes through (visits) each vertex of the graph exactly once and returns to the initial vertex. In honor of William Rowan Hamilton, such a tour is nowadays called a *Hamilton cycle*. A Hamilton cycle of the dodecahedron graph is indicated by these thick edges in Figure 1.2.

Another well-known practical problem related to the problem of the Seven Bridges of Königsberg and Hamilton’s puzzle is the *Chinese Postman Problem* [117], first introduced and studied by the Chinese mathematician Meigu Guan (i.e., Mei-Ku Kwan, according to the old transcription) in 1960. The simplest version of this problem deals with a postman who wishes to deliver his letters, covering a route with the least possible total distance before returning to the post office. This requires that the postman traverses each road in his route at least once, but should traverse as few roads as possible more than once (if all roads are supposed to have the same length) or the total set of multiple roads with the minimum possible total distance (in the general case). An efficient solution to the Chinese Postman Problem was presented in the book of Bondy and Murty [14].

Motivated by the Chinese Postman Problem, Boesch et al. [11] in 1977 proposed the *Supereulerian Problem* with a goal that is kind of opposite. The purpose of the Supereulerian Problem is to characterize graphs admitting a tour with the same start and end vertex that passes through each vertex of the graph at least once and traverses each edge at most once. It was suggested in [11] that the Supereulerian Problem would be very difficult (and this is indeed the case because it contains the Hamilton Problem restricted to 3-regular graphs, which is known to be NP-complete). Such a tour is called a *spanning circuit*. It is not difficult to see that a spanning circuit can be viewed as a common relaxation of a Hamilton cycle and an Euler tour. Spanning circuits can be applied in many real-life problems, such as in the one-way street problem [88] and in the optimal page retrieval problem [108].

Next, we turn to the main graph theoretical notion of this thesis.

1.2 Compatible spanning circuits

As discussed above, a number of practical and toy problems can be modeled by means of (di)graphs in which its vertices indicate the objects of the problem and its (arcs) edges indicate the (a)symmetrical binary relationships between these objects.

However, in many real-life situations, distinguishing between the existence and nonexistence of an edge or arc between a certain pair of vertices is not enough. Often, it is necessary to introduce additional notions in order to model such situations appropriately and adequately. Consider, for instance, a graph that models a network of relay stations (i.e., radio stations that receive radio signals and retransmit them). In such a graph, each vertex of the graph represents a relay station, and edges (or arcs, depending on the assumptions with respect to the direction signals can pass through the transmission lines) indicate the transmission lines between pairs of relay stations. Suppose that, in order to avoid interference, the frequency at which a radio signal is received by a relay station should be different from the frequency at which the radio signal is retransmitted. Then it is natural to assign labels to the edges (i.e., natural numbers that are usually referred to as colors) to represent the frequencies that are used to receive or transmit a radio signal through the corresponding transmission lines. In such cases, the graph together with the assignment of colors to its edges is usually referred to as an *edge-colored graph*. In the above applications, the colored analogue of a spanning circuit is a *compatible spanning circuit*, defined as a spanning circuit in which any two consecutively traversed edges have distinct colors.

In this thesis, we mainly concentrate our efforts on the existence of compatible spanning circuits visiting each vertex exactly (or at least) a specified number of times in specific edge-colored graphs, from both a sufficient condition perspective and an algorithmic perspective. In addition, we also consider similar problems in (di)graphs for which certain generalizations of (arc-) edge-colorings have been defined.

In the remainder of this introductory chapter, we will present the basic terminology and notations involved in this thesis, together with a survey of the research background and progress on problems related to the topic of the

thesis. We will also provide an overview of our main contributions, as well as an outline of this thesis.

We assume that the reader who attempts to understand the rest of the thesis is familiar with the basic knowledge of mathematics and the essentials of graph theory.

1.3 Terminology and notations

In this section, we formally introduce some basic terminology and notations involved in this thesis. The terminology and notations not defined but used in the thesis are standard and can be found in the most recent version of the textbook of Bondy and Murty [15]. Throughout this thesis, we consider only finite graphs. Thus, we usually use the term “graph” instead of “finite graph”. Unless otherwise specified, the graphs we refer to are always undirected and simple (i.e., without loops or multiple edges).

Let G be a graph. We use $V(G)$ and $E(G)$ to denote the set of vertices of G and the set of edges of G , respectively. We write $v(G) = |V(G)|$ and $e(G) = |E(G)|$; these two basic parameters are called the *order* and *size* of G , respectively. For a vertex v of a graph G , we denote the set of edges of G incident with v by $E_G(v)$, and we denote the set of neighbors of v in G by $N_G(v)$. The *degree* of a vertex v in a graph G , denoted by $d_G(v)$, is defined to be the cardinality of $E_G(v)$. In particular, if G is a simple graph, then we have $d_G(v) = |E_G(v)| = |N_G(v)|$ for a vertex v of G . We let $\Delta(G) = \max\{d_G(v) \mid v \in V(G)\}$, $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ and $\sigma_3(G) = \min\{d_G(u) + d_G(v) + d_G(w) \mid u, v, w \in V(G) \text{ and } uv, uw, vw \notin E(G)\}$, where the vertices u , v and w of G are pairwise distinct. We use $N_i(G)$ and $O(G)$ to denote the set of vertices of G of degree i for a nonnegative integer i and the set of vertices of G of odd degree, respectively. For two vertices u and v of a graph G , a *uv-path* of G refers to a path of G connecting u and v , and the *distance* between u and v in G , denoted by $\text{dist}_G(u, v)$, is defined to be the length of a shortest *uv-path* of G (if it exists). If no ambiguity can arise, we will denote $E_G(v)$, $N_G(v)$, $d_G(v)$ and $\text{dist}_G(u, v)$ by $E(v)$, $N(v)$, $d(v)$ and $\text{dist}(u, v)$, respectively.

A *spanning circuit* in a graph G is defined as a closed trail that visits

(contains) each vertex of G . As an example, in the graph G depicted in Figure 1.3 (a), the closed trail $v_1v_2v_3v_1v_4v_5v_1$ forms a spanning circuit of G . A *Hamilton cycle* of a graph G can be regarded as a spanning circuit that visits each vertex of G exactly once, as the closed trail $v_1v_2v_3v_4v_5v_1$ in Figure 1.3 (a). An *Euler tour* of G can be regarded as a spanning circuit that traverses each edge of G exactly once. There does not exist an Euler tour in Figure 1.3 (a), because the graph contains vertices of odd degree. From the above definitions, it is clear that a spanning circuit can be considered as a common relaxation of a Hamilton cycle and an Euler tour. A graph is said to be *hamiltonian* if it contains a Hamilton cycle, and a graph is said to be *eulerian* if it admits an Euler tour. It is not difficult to see that each spanning circuit (if it exists) of a graph G corresponds to a spanning eulerian subgraphs of G . A graph is said to be *supereulerian* if it contains a spanning eulerian subgraph (spanning circuit).

Figure 1.3: Graphs illustrating key terminology and notations.

An *edge-coloring* of a graph G is defined as a mapping $c : E(G) \rightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. An *edge-colored graph* refers to a graph with a fixed edge-coloring. If the total number of different colors used in an edge-colored graph G is k , then we also say that G is a k -edge-colored graph. Two edges of a graph are said to be *consecutive* with respect to a trail (with a fixed orientation) if they are traversed consecutively along the trail. A *compatible spanning circuit* in an edge-colored graph refers to a spanning circuit in which any two consecutive edges have distinct colors, including the

final traversed edge and the first traversed edge. As an example, in the edge-colored graph G depicted in Figure 1.3 (b), the closed trail $v_1 v_2 v_3 v_1 v_4 v_5 v_1$ forms a compatible spanning circuit of G . An edge-colored graph is said to be *properly colored* if any two adjacent edges (i.e., two edges which are incident with a common end-vertex) of the graph have distinct colors, and it is said to be *rainbow* if any two edges of the graph have distinct colors. Thus, a compatible Hamilton cycle is properly colored, and a properly colored spanning circuit is compatible. Conversely, a compatible spanning circuit is obviously not necessarily properly colored. For example, in the edge-colored graph of Figure 1.3 (b), the compatible spanning circuit $v_1 v_2 v_3 v_1 v_4 v_5 v_1$ is not properly colored. Therefore, a compatible spanning circuit can be viewed as a generalization of a properly colored spanning circuit.

Let G be an edge-colored graph. We use $c(e)$ to denote the color appearing on the edge e of G , and we denote by $C(G)$ the set of colors appearing on the edges of G . For a vertex $v \in V(G)$ and a color $i \in C(G)$, we use $d_G^i(v)$ to denote the cardinality of the set $\{e \in E_G(v) \mid c(e) = i\}$. We let $\Delta_G^{mon}(v) = \max\{d_G^i(v) \mid i \in C(G)\}$ for a vertex $v \in V(G)$, and we let $\Delta^{mon}(G) = \max\{\Delta_G^{mon}(v) \mid v \in V(G)\}$; these two parameters are called the *maximum monochromatic degree* of a vertex v of G and the *maximum monochromatic degree* of a graph G , respectively. Similarly, we can also define the *minimum monochromatic degree* of a vertex of G or a graph G . In an edge-colored graph G , the *color degree* of a vertex v of G , denoted by $cd_G(v)$, is defined to be the number of different colors appearing on the edges of G incident with v . In particular, we write $\delta^c(G) = \min\{cd_G(v) \mid v \in V(G)\}$. As an example, in the edge-colored graph G depicted in Figure 1.3 (b), we have $\Delta_G^{mon}(v_1) = \Delta_G^{mon}(v_3) = \Delta_G^{mon}(v_4) = 2$ and $\Delta_G^{mon}(v_2) = \Delta_G^{mon}(v_5) = 1$, and hence $\Delta^{mon}(G) = 2$. For this graph, we also have $cd_G(v_1) = 2$ and $cd_G(v_i) = 3$ for $i \in \{2, 3, 4, 5\}$, and hence $\delta^c(G) = 2$. When no confusion can occur, we will use the notations $\Delta^{mon}(v)$ and $cd(v)$ instead of $\Delta_G^{mon}(v)$ and $cd_G(v)$, respectively.

For the sake of brevity, we just consider a digraph as a graph in which the directions of all the edges have been added, and we translate all the concepts defined above for graphs to the natural analogues for digraphs. In particular, an edge with a direction in a digraph is called an *arc*. Thus, an edge-coloring

of a digraph is accordingly called an arc-coloring of the digraph.

1.4 Research background and progress

In this section, we present a survey of the research background and progress on problems related to the existence of compatible spanning circuits in edge-colored graphs. In the first subsection, we focus on spanning circuits, and also in particular on Euler tours and Hamilton cycles. This is followed by a subsection on subgraphs with specific (edge) coloring patterns.

1.4.1 Euler tours, Hamilton cycles and spanning circuits

We start by shortly reviewing the research background and progress on problems related to Euler tours, Hamilton cycles and spanning circuits, respectively.

As we mentioned earlier, the origin of graph theory probably goes back to Euler [41] who solved the problem of the Seven Bridges of Königsberg (see [48]). This has led to the definition of *eulerian* graphs, i.e., graphs that admit an *Euler tour*, i.e., a closed trail that traverses every edge exactly once. Recreational mathematics problems related to this topic can be found in [117], whereas a translation of Euler's paper on a solution to the problem of the Seven Bridges of Königsberg and a discussion on related problems can be found in [10]. We refer to [15] for a proof of the well-known characterization that a connected graph G is eulerian if and only if the degree of each vertex of G is even. There one can also find a description of Fleury's polynomial algorithm for constructing an Euler tour in an eulerian graph. More details on the algorithmic aspects can be found in [81]. Various connections between eulerian graphs and hamiltonicity, nowhere-zero flows, the cycle-plus-triangles problem and other problems derived from it have been demonstrated in the survey [50]. For more details on eulerian graphs and related topics, we refer the interested reader to the monographs of Fleischner [48, 49].

We also introduced *hamiltonian* graphs, i.e., graphs that admit a *Hamilton cycle*, i.e., a closed walk visiting each vertex exactly once. The decision

problem of determining whether a given graph is hamiltonian is usually referred to as the *Hamilton Cycle Problem*, and is a special case of the *Traveling Salesman Problem* [117]. Whereas the counterpart for eulerian graphs is polynomially solvable, the Hamilton Cycle Problem is known to be NP-complete (see [60]). This is one of the reasons why researchers have mainly focussed on sufficient conditions for hamiltonicity. We refer the interested reader to the surveys of Gould for more details [62–64].

As variations on the above problems related to hamiltonian and eulerian graphs, we also shortly mentioned the *Chinese Postman Problem* and the *Supereulerian Problem*. The first of these problems is polynomially solvable by a combination of Fleury’s algorithm and an efficient algorithm due to Edmonds and Johnson [38] (see [14]), while determining whether a graph is supereulerian (i.e., contains a spanning circuit) is NP-complete (see [107]). In the past few decades, different groups of researchers studied variants of the Chinese Postman Problem for directed graphs, mixed graphs, edge-colored graphs, and graphs with wind, and also a hierarchical variant, and a variant for rural districts. More discussion on these variants can be found in the surveys [58, 89]. For more details on supereulerian graphs, we refer to Catlin’s survey [22] and its supplement due to Lai et al. [87].

1.4.2 Subgraphs with specific coloring patterns

In this subsection, we present a short review of the research background, progress and problems related to the existence of subgraphs with specific coloring patterns in edge-colored graphs.

The start of this research is from 1950 and marked by a result due to Erdős and Rado [39] who proved a counterpart of the well-known Ramsey’s Theorem, known as the *Canonical Ramsey Theorem*. Since then, problems on the existence of subgraphs with specific color patterns like monochromatic subgraphs, properly colored subgraphs and rainbow (also called heterochromatic, multicolored) subgraphs in edge-colored graphs have attracted considerable attention (see [6, 55, 83]). Apart from that, the problem on properly colored subgraphs of edge-colored graphs is closely related to other branches of graph theory, such as digraph theory and matching theory, and the problem

has been studied within algorithmic graph theory (see [6]). Besides some applications in graph theory and algorithm theory, properly colored subgraphs of edge-colored graphs have many applications in various other fields, such as in transportation and communication [56, 57, 65], in VLSI design [75], in social sciences [26] and in conflict analysis and resolution [43–45, 119, 120]. One can find more details and results on properly colored (alternating) cycles and paths of edge-colored graphs in the surveys [5, 70]. In particular, many practical problems arising from molecular biology can be modeled by means of edge-colored graphs (see [105, 106]). Given such an edge-colored graph, the original problems are equivalent to extracting subgraphs with specifically colored structures from the graph. In this context, the specific subgraphs often refer to compatible (i.e., properly colored or alternating) Hamilton cycles or paths (see [34–36]) and compatible Euler tours or trails (see [105, 106]).

As a generalization of a properly colored spanning circuit, the main research object in this thesis, i.e., a compatible spanning circuit is of interest in applications of graph theory, for example in genetic and molecular biology [106, 112, 113], in the design of printed circuit and wiring boards [114], and in channel assignment of wireless networks [3, 110].

Bang-Jensen and Gutin [5] indicated that Petersen's famous paper [104] published in 1891 seems to be the first place where one can find the discussion on compatible spanning circuits (cf. [99]). In particular, compatible Euler tours can be used to construct Hamilton cycles in certain digraphs that represent permutations (see [76]). In this context, the vertices of the digraph $P(n, k)$ correspond to k -permutations $(\sigma_1, \sigma_2, \dots, \sigma_k)$ of $[n] = \{1, \dots, n\}$ and its arcs correspond to $(k+1)$ -permutations of $[n]$, i.e., the arc corresponding to $\sigma_1\sigma_2\cdots\sigma_k\sigma_{k+1}$ is $(\sigma_1, \sigma_2, \dots, \sigma_k) \rightarrow (\sigma_2, \sigma_3, \dots, \sigma_k, \sigma_{k+1})$.

As an extremal case of compatible spanning circuits, the existence of compatible (i.e., properly colored or alternating) Hamilton cycles in specific edge-colored graphs has been studied extensively. Bánkfalvi et al. [7] gave a characterization of 2-edge-colored complete graphs containing compatible Hamilton cycles as early as 1968. For more results on the existence of compatible Hamilton cycles in 2-edge-colored (multi)graphs, we refer the reader to [25, 28, 29]. Throughout the rest of this section, we use K_n^c denote an edge-colored complete graph on $n \geq 3$ vertices (without any assumptions on the

number of colors). Daykin [31] asked whether there exists a constant μ such that every graph K_n^c with $\Delta^{mon}(K_n^c) \leq \mu n$ contains a compatible Hamilton cycle. This question was independently answered by Bollobás and Erdős [12] with $\Delta^{mon}(K_n^c) < n/69$, and Chen and Daykin [23] with $\Delta^{mon}(K_n^c) \leq n/17$. Moreover, Bollobás and Erdős [12] proposed the following conjecture.

Conjecture 1.4.1 (Bollobás and Erdős [12]). *If $\Delta^{mon}(K_n^c) < \lfloor n/2 \rfloor$, then K_n^c contains a compatible Hamilton cycle.*

Soon afterwards, Shearer [111] further showed that every graph K_n^c satisfying $\Delta^{mon}(K_n^c) < n/7$ contains a compatible Hamilton cycle. Alon and Gutin [4] in 1997 showed that for any $\epsilon > 0$ there exists an integer $N_0(\epsilon)$ such that for every integer n with $n \geq N_0(\epsilon)$, a graph K_n^c satisfying $\Delta^{mon}(K_n^c) \leq (1 - 1/\sqrt{2} - \epsilon)n$ contains a compatible Hamilton cycle. This topic has attracted new interest recently. Lo [92] in 2016 showed that for any $\epsilon > 0$ there exists an integer $N_0(\epsilon)$ such that for every integer n with $n \geq N_0(\epsilon)$, a graph K_n^c satisfying $\Delta^{mon}(K_n^c) \leq (1/2 - \epsilon)n$ contains a compatible Hamilton cycle, implying that Conjecture 1.4.1 is true asymptotically. At the same time, Lo [92] also obtained a corollary of the above result stating that for any $\epsilon > 0$ there exists an integer $N_0(\epsilon)$ such that for every integer n with $n \geq N_0(\epsilon)$, a graph K_n^c satisfying $\delta^c(K_n^c) \geq (1/2 + \epsilon)n$ contains a compatible Hamilton cycle. The condition on $\delta^c(K_n^c)$ in this result of Lo improved the previous condition that $\delta^c(K_n^c) \geq (7/8)n$ due to Bollobás and Erdős [12].

In this context, it is worth mentioning that Abouelaoualim et al. [1] established a sufficient condition in terms of the minimum monochromatic degree for the existence of compatible Hamilton cycles, when considering the existence of properly colored cycles of all possible lengths in edge-colored multigraphs. More results and problems (conjectures) related to the existence of compatible Hamilton cycles in specific edge-colored graphs can be found in [1, 2, 4, 23, 24, 54, 74, 92].

As another extremal case of compatible spanning circuits, the existence of compatible Euler tours in edge-colored eulerian graphs has also been considered in previous literature.

Kotzig [84] obtained a necessary and sufficient condition for the existence of compatible Euler tours in edge-colored eulerian graphs, as follows.

Theorem 1.4.1 (Kotzig [84]). *Let G be an edge-colored eulerian graph. Then a compatible Euler tour exists if and only if $\Delta^{mon}(v) \leq d(v)/2$ for each vertex v of G .*

Almost thirty years later, Benkoular et al. [9] considered the existence of compatible Euler tours in edge-colored eulerian graphs from an algorithmic perspective. In [9], they provided a polynomial-time algorithm for finding a compatible Euler tour in an edge-colored eulerian graph for which the above condition is satisfied. Independently, Pevzner [105] described a similar algorithm for solving the same problem.

By restricting the number of colors, Das and Rao [30] in 1983 considered the existence of more general compatible spanning circuits (i.e., not necessarily a compatible Hamilton cycle or Euler tour) in specific edge-colored graphs. In [30], they established necessary and sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly a specified number of times in 2-edge-colored complete graphs.

To the best of our knowledge, there are few results on the existence of more general compatible spanning circuits in edge-colored graphs without restrictions on the number of colors in existing literature. This is the main motivation for the contributions of this thesis, the main flavor of which will be described in the next section. This is followed by an outline of the thesis in the final section of this chapter.

1.5 Main contributions

The main aim of this thesis is to develop sufficient conditions for the existence of compatible spanning circuits with certain properties in various specific edge-colored graphs (without any assumptions on the number of colors), and to establish polynomial-time algorithms for finding such compatible spanning circuits. We also consider similar problems in (di)graphs with generalizations of (arc-) edge-colorings. Throughout the rest of this section, every edge-colored graph we refer to is always an edge-colored graph without restrictions on the number of colors, unless otherwise specified. The main flavor of our contributions can be summarized, as follows.

- We establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible positive integer k , in specific edge-colored graphs.
- We establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored graphs satisfying certain degree conditions, and in edge-colored graphs not containing certain forbidden induced subgraphs, respectively.
- We establish sufficient conditions for the asymptotical existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in edge-colored random graphs, and in edge-colored random geometric graphs, respectively.
- We analyze the complexity of the decision problem of determining whether an edge-colored connected graph contains a compatible spanning circuit, and we describe polynomial-time algorithms for finding compatible spanning circuits (with certain properties) in specific edge-colored complete graphs.
- We establish sufficient conditions for the existence of compatible Euler tours in eulerian (di)graphs for which certain generalizations of (arc-) edge-colorings have been defined.

1.6 Thesis outline

The main part of this thesis consists of seven chapters. For the convenience of the reader, some essential preliminaries that are frequently used in different chapters will appear repeatedly in each of these chapters. On the premise that the reader is familiar with the essentials of graph theory and the terminology and notations introduced in Section 1.3, each chapter can be read and understood independently. Throughout this section, every edge-colored graph we refer to is always an edge-colored graph without restrictions on

the number of colors, unless otherwise specified. Apart from this introductory chapter, the remainder of the main part of this thesis is organized in the following way.

In Chapter 2, we consider the existence of more general compatible spanning circuits (i.e., not necessarily a compatible Hamilton cycle or Euler tour) in edge-colored graphs. We establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible positive integer k , in edge-colored complete graphs, edge-colored complete equipartition r -partite graphs, and in specific edge-colored hamiltonian graphs with high order, respectively.

In Chapter 3, we consider the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored graphs satisfying certain degree conditions. We establish sufficient conditions for the existence of such compatible spanning circuits in edge-colored graphs satisfying Ore-type degree conditions, edge-colored graphs satisfying Fan-type degree conditions and edge-colored graphs with a high edge-connectivity, respectively. We also consider the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in edge-colored random graphs and edge-colored random geometric graphs.

In Chapter 4, we continue the research on sufficient conditions for the existence of compatible spanning circuits visiting each vertex at least a specified number of times. We consider edge-colored graphs that do not contain certain forbidden induced subgraphs. As applications, we also consider the existence of such compatible spanning circuits in edge-colored graphs G with $\kappa(G) \geq \alpha(G)$ and $\kappa(G) \geq \alpha(G) - 1$, respectively (where $\kappa(G)$ and $\alpha(G)$ denote the *connectivity* and the *independence number* of a graph G , respectively).

In Chapter 5, we consider the existence of compatible spanning circuits in edge-colored graphs from an algorithmic perspective. We first prove that determining whether an edge-colored connected graph contains a compatible spanning circuit is an NP-complete problem. Next, we describe two polynomial-time algorithms for finding compatible spanning circuits (with certain properties) in specific edge-colored complete graphs.

In Chapter 6, we consider the existing concept of a generalized transition system (over a graph), which can be viewed as a generalization of an edge-coloring of a graph. We further introduce the new concept of a weakly generalized transition system, which is an extension of a generalized transition system, and we establish Ore-type sufficient conditions for the existence of compatible (oriented) Euler tours in eulerian graphs with (weakly) generalized transition systems.

In Chapter 7, we further extend the results of Chapter 6 to the case of eulerian digraphs. We establish Ore-type sufficient conditions for the existence of compatible Euler tours in eulerian digraphs with generalized transition systems. As corollaries of some known results, we provide sufficient conditions for the existence of compatible Euler tours in arc-colored eulerian digraphs.

Remark

Note that we sometimes use the following abbreviations in order to avoid long headings of sections and titles throughout this thesis:

- We use k -CSCs as an abbreviation for compatible spanning k -circuits;
- We use E-C as an abbreviation for the term edge-colored;
- We use CSCs as an abbreviation for compatible spanning circuits;
- We use CETs as an abbreviation for compatible Euler tours.

Chapter 2

Compatible spanning k -circuits in edge-colored graphs

In this chapter, we consider the existence of compatible spanning circuits visiting each vertex exactly a specified number of times in edge-colored graphs (without any assumptions on the number of colors). We establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible positive integer k , in specific edge-colored graphs.

2.1 Introduction

As we mentioned in Chapter 1, the terminology and notations not defined but used in this chapter are standard and can be found in the most recent version of the textbook of Bondy and Murty [15]. Throughout this chapter, every edge-colored graph we refer to is always an edge-colored graph without restrictions on the number of colors, unless otherwise specified.

As an extremal case of compatible spanning circuits, the existence of compatible (i.e., properly colored or alternating) Hamilton cycles in specific edge-colored graphs has been extensively studied in previous literature. Bánkfalvi et al. [7] established a characterization of 2-edge-colored complete graphs

containing compatible Hamilton cycles as early as 1968. For more results on the existence of compatible Hamilton cycles in 2-edge-colored (multi)graphs, we refer the reader to [25, 28, 29]. The research on the existence of compatible Hamilton cycles in specific edge-colored graphs without restrictions on the number of colors dates back to the 1970s. Throughout the rest of this section, we use K_n^c to denote an edge-colored complete graph on $n \geq 3$ vertices. Daykin [31] asked whether there exists a constant μ such that every graph K_n^c with $\Delta^{mon}(K_n^c) \leq \mu n$ contains a compatible Hamilton cycle. This question was independently answered by Bollobás and Erdős [12] with $\Delta^{mon}(K_n^c) < n/69$, and Chen and Daykin [23] with $\Delta^{mon}(K_n^c) \leq n/17$. Moreover, Bollobás and Erdős [12] proposed the following conjecture.

Conjecture 2.1.1 (Bollobás and Erdős [12]). *If $\Delta^{mon}(K_n^c) < \lfloor n/2 \rfloor$, then K_n^c contains a compatible Hamilton cycle.*

Soon afterward, Shearer [111] showed that $\Delta^{mon}(K_n^c) < n/7$ is sufficient. Alon and Gutin [4] showed that $\Delta^{mon}(K_n^c) \leq (1 - 1/\sqrt{2} - o(1))n$ is sufficient. Recently, Lo [92] proved that Conjecture 2.1.1 is true asymptotically.

As another extremal case of compatible spanning circuits, the existence of compatible Euler tours in edge-colored eulerian graphs has also been considered. Kotzig [84] established a necessary and sufficient condition for the existence of compatible Euler tours in edge-colored eulerian graphs, as follows.

Theorem 2.1.1 (Kotzig [84]). *Let G be an edge-colored eulerian graph. Then a compatible Euler tour exists if and only if $\Delta^{mon}(v) \leq d(v)/2$ for each vertex v of G .*

From Theorem 2.1.1, we can obtain the following corollary.

Corollary 2.1.1. *If $\Delta^{mon}(K_n^c) \leq (n - 1)/2$, then K_n^c contains a compatible spanning circuit visiting each vertex exactly $\lfloor (n - 1)/2 \rfloor$ times.*

Proof. First we suppose that n is odd. Since $d_{K_n^c}(v) = n - 1$ for every vertex v of K_n^c , the graph K_n^c is an eulerian graph. If $\Delta^{mon}(K_n^c) \leq (n - 1)/2 = d_{K_n^c}(v)/2$, then the conclusion holds by Theorem 2.1.1.

Suppose now that n is even. Let $H = K_n^c - M$, where M is an arbitrary perfect matching of K_n^c . We have $d_H(v) = n - 2$ for every vertex v of K_n^c . Thus, the subgraph H is a spanning eulerian subgraph of K_n^c . If $\Delta^{mon}(K_n^c) \leq (n - 1)/2$, then we have $\Delta^{mon}(H) \leq \Delta^{mon}(K_n^c) \leq \lfloor (n - 1)/2 \rfloor = d_H(v)/2$. It follows that there exists a compatible Euler tour in H by Theorem 2.1.1. Therefore, the conclusion holds. $\square$

Corollary 2.1.1 implies that a graph K_n^c ($n \geq 4$) satisfying $\Delta^{mon}(K_n^c) < \lfloor n/2 \rfloor$ (i.e., the condition stated in Conjecture 2.1.1) contains a compatible spanning circuit visiting each vertex exactly $\lfloor (n - 1)/2 \rfloor$ times. Recently, Lo [92] asymptotically proved that a graph K_n^c satisfying the same condition (i.e., $\Delta^{mon}(K_n^c) < \lfloor n/2 \rfloor$) contains a compatible Hamilton cycle. Compared to a compatible Hamilton cycle, we intuitively feel that a weaker condition might imply that K_n^c contains a compatible spanning circuit. However, from the following construction given by Fujita and Magnant [54] (see Construction 2.1.1), we can show that the condition $\Delta^{mon}(K_n^c) < \lfloor n/2 \rfloor$ to guarantee the existence of compatible spanning circuits (even without restrictions on the number of times that it visits each vertex) in K_n^c is best possible for n even.

Construction 2.1.1 (Fujita and Magnant [54]). *For a given integer m with $m \geq 2$, let G be a complete graph on $2m$ vertices, and let u be one of the vertices of G . We label the remaining vertices with $v_1, \dots, v_{2m-1}$, respectively, and we color the edge uv_i with color i for each v_i , where $1 \leq i \leq 2m - 1$. Let $H = G - u$, and consider a decomposition of the edges of H into $m - 1$ Hamilton cycles (for the feasibility of the above decomposition, see [91]). We arbitrarily orient these Hamilton cycles such that they become directed cycles. We color the edge $v_i v_j$ with color j if the arc $\overrightarrow{v_i v_j}$ is an arc of one of these Hamilton cycles.*

The above construction defines an edge-coloring of a complete graph G on $2m$ ($m \geq 2$) vertices, thus a graph K_{2m}^c . The graph K_{2m}^c defined by Construction 2.1.1 satisfies $\Delta^{mon}(K_{2m}^c) = \lfloor 2m/2 \rfloor = m$, but it contains no compatible spanning circuit, because such a circuit cannot visit the vertex u compatibly.

Recall that a compatible Hamilton cycle can be regarded as a compatible spanning circuit visiting each vertex exactly once. Sufficient conditions for the existence of compatible Hamilton cycles and compatible spanning circuits visiting each vertex exactly $\lfloor (n-1)/2 \rfloor$ times in edge-colored complete graphs on n vertices have been established. It is natural to consider the following problem.

Problem 2.1.1. *Under what conditions does an edge-colored graph on n vertices contain a compatible spanning circuit visiting each vertex exactly k times for a given integer k with $1 \leq k \leq \lfloor (n-1)/2 \rfloor$?*

By restricting the number of colors, Das and Rao [30] established necessary and sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly a specified number of times in 2-edge-colored complete graphs back in the 1980s.

As far as we know, there are few results on the existence of compatible spanning circuits visiting each vertex exactly a specified number of times in edge-colored graphs without restrictions on the number of colors in existing literature.

In this chapter, we mainly consider Problem 2.1.1 in edge-colored graphs without restrictions on the number of colors, and we give some partial answers to the question of the problem.

2.2 k -CSCs in E-C complete graphs

In this section, we establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible positive integer k , in edge-colored complete graphs.

Before proceeding, we introduce the following basic result which will be used in the later proof of the main theorem of this section.

Theorem 2.2.1 (Laskar and Auerback [91]). *Let G be a complete graph on $n \geq 3$ vertices. Then G can be decomposed into $(n-1)/2$ edge-disjoint Hamilton cycles for n odd, and G can be decomposed into $(n-2)/2$ edge-disjoint Hamilton cycles and one perfect matching for n even.*

Next, we present the main result of this section, as follows.

Theorem 2.2.2. *Let K_n^c be an edge-colored complete graph on $n \geq 3$ vertices, and let k be an integer such that $1 \leq k \leq \lfloor (n-1)/2 \rfloor$. If $\Delta^{mon}(K_n^c) \leq k$, then K_n^c contains a compatible spanning circuit visiting each vertex exactly k times.*

Proof. Let K_n^c be an edge-colored complete graph on $n \geq 3$ vertices. Let k be a given integer such that $1 \leq k \leq \lfloor (n-1)/2 \rfloor$. It follows that there exist k edge-disjoint Hamilton cycles in K_n^c by Theorem 2.2.1. Let H be a $2k$ -regular spanning subgraph of K_n^c consisting of k edge-disjoint Hamilton cycles of K_n^c . Thus, H is a $2k$ -regular spanning eulerian subgraph of K_n^c .

If $\Delta^{mon}(K_n^c) \leq k$, then we have $\Delta^{mon}(H) \leq \Delta^{mon}(K_n^c) \leq k$. Since $d_H(v) = 2k$ for every vertex v of H , there exists a compatible Euler tour in H by Theorem 2.1.1. Therefore, the edge-colored complete graph K_n^c contains a compatible spanning circuit visiting each vertex exactly k times. This completes the proof. $\square$

Motivated by Conjecture 2.1.1 and Corollary 2.1.1, we intuitively feel that it is possible that an edge-colored complete graph on n vertices satisfying the condition stated in Conjecture 2.1.1 contains a compatible spanning circuit visiting each vertex exactly k times for any given integer k with $1 < k < \lfloor (n-1)/2 \rfloor$. We conclude this section with the following open problem.

Problem 2.2.1. *Let K_n^c be an edge-colored complete graph on n vertices such that $\Delta^{mon}(K_n^c) < \lfloor n/2 \rfloor$. Is it true that K_n^c contains a compatible spanning circuit visiting each vertex exactly k times for every integer k with $1 < k < \lfloor (n-1)/2 \rfloor$?*

2.3 k -CSCs in E-C complete multipartite graphs

In this section, we establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible positive integer k , in edge-colored complete equipartition r -partite graphs. In this context, a complete equipartition r -partite graph refers to a complete

r -partite graph in which any two partite sets have the same number of vertices.

Before proceeding, we introduce the following basic result which will be used in the later proof of the main theorem of this section.

Theorem 2.3.1 (Laskar and Auerback [91]). *Let r and n be integers such that $r \geq 2$ and $n \geq 2$, respectively, and let G be a complete equipartition r -partite graph on rn vertices. Then G can be decomposed into $n(r-1)/2$ edge-disjoint Hamilton cycles for even $n(r-1)$, and G can be decomposed into $(n(r-1)-1)/2$ edge-disjoint Hamilton cycles and one perfect matching for odd $n(r-1)$.*

Next, we present the main result of this section, as follows.

Theorem 2.3.2. *Let r and n be integers such that $r \geq 2$ and $n \geq 2$, respectively, and let $(K_n^r)^c$ be an edge-colored complete equipartition r -partite graph on rn vertices. Let k be an integer such that $1 \leq k \leq \lfloor n(r-1)/2 \rfloor$. If $\Delta^{mon}((K_n^r)^c) \leq k$, then $(K_n^r)^c$ contains a compatible spanning circuit visiting each vertex exactly k times.*

Proof. Let $(K_n^r)^c$ be an edge-colored complete equipartition r -partite graph on rn vertices, where $r \geq 2$ and $n \geq 2$. Let k be a given integer such that $1 \leq k \leq \lfloor n(r-1)/2 \rfloor$. It follows that there exist k edge-disjoint Hamilton cycles in $(K_n^r)^c$ by Theorem 2.3.1. Let H be a spanning subgraph of $(K_n^r)^c$ consisting of k edge-disjoint Hamilton cycles of $(K_n^r)^c$. Thus, H is a $2k$ -regular spanning eulerian subgraph of $(K_n^r)^c$.

If $\Delta^{mon}((K_n^r)^c) \leq k$, then we have $\Delta^{mon}(H) \leq \Delta^{mon}((K_n^r)^c) \leq k$. Since $d_H(v) = 2k$ for every vertex v of H , there exists a compatible Euler tour in H by Theorem 2.1.1. Therefore, the edge-colored complete equipartition r -partite graph $(K_n^r)^c$ contains a compatible spanning circuit visiting each vertex exactly k times. This completes the proof. $\square$

Remark 2.3.1. *The proofs of Theorems 2.2.2 and 2.3.2 are based on a decomposition of a regular graph into edge-disjoint Hamilton cycles (and one perfect matching if it is odd regular). Motivated by the proof technique of the two theorems, we can obtain a similar conclusion for any edge-colored regular graph G admitting a decomposition of G into edge-disjoint Hamilton cycles (and one perfect matching if G is odd regular).*

2.4 k -CSCs in E-C hamiltonian graphs

In this section, we continue to pay close attention to the existence of compatible spanning circuits visiting each vertex exactly k times for every feasible positive integer k . We consider specific edge-colored hamiltonian graphs with sufficiently large order.

Before proceeding, we introduce some basic results which will be used in the later proofs of the main theorems of this section. All the results that are listed below are from existing literature and due to different (groups of) researchers.

Let k be a positive integer. A graph G is said to be k -critical if G satisfies $\delta(G) \geq k$ and $\delta(G - e) < k$ for any edge e of G . A k -factor of a graph G refers to a k -regular (not necessarily connected) spanning subgraph of G .

Theorem 2.4.1 (Cai, Fang and Li [18]). *Let ℓ be an integer such that $\ell \geq 2$, and let G be an $n/2$ -critical graph on n vertices, where $n \geq \max\{8\ell - 14, 4\}$ and n is even. Then G has an ℓ -factor containing C for any given Hamilton cycle C of G , except if ℓ is odd and $G - E(C)$ consists of two components with an odd number of vertices.*

Theorem 2.4.2 (Matsuda [94]). *Let ℓ be an integer such that $\ell \geq 2$, and let G be a graph on n vertices with $\delta(G) \geq \ell$, where $n > 8\ell^2 - 2(\alpha + 12)\ell + 3\alpha + 16$ and $\alpha = 3$ for ℓ odd; $\alpha = 4$ for ℓ even. Suppose that ℓn is even. If $d(u) + d(v) \geq n + \alpha$ for every pair of nonadjacent vertices u, v of G , then G has an ℓ -factor containing C for any given Hamilton cycle C of G .*

Theorem 2.4.3 (Zhou [123]). *Let G be a 4-connected graph on n vertices such that $\max\{d(u), d(v)\} \geq n/2 + 2$ for every pair of vertices u, v of G with $\text{dist}(u, v) = 2$. Then G contains two edge-disjoint Hamilton cycles.*

Theorem 2.4.4 (Gao, Li and Li [59]). *Let ℓ be an integer such that $\ell \geq 2$, and let G be a graph on n vertices with $\delta(G) \geq \ell$, where $n > 12(\ell - 2)^2 + 2(5 - \alpha)(\ell - 2) - \alpha$ and $\alpha = 3$ for ℓ odd; $\alpha = 4$ for ℓ even. Suppose that ℓn is even. If $\max\{d(u), d(v)\} \geq (n + \alpha)/2$ for every pair of nonadjacent vertices u, v of G , then G has an ℓ -factor containing C for any given Hamilton cycle C of G such that $G - E(C)$ is connected.*

Next, we present the main results of this section. We first establish the following sufficient condition for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible positive integer k , in edge-colored $n/2$ -critical graphs on n vertices.

Theorem 2.4.5. *Let k be a positive integer, and let G be an edge-colored $n/2$ -critical graph on even n vertices, where $n \geq \max\{16k - 14, 4\}$. If $\Delta^{\text{mon}}(G) \leq k$, then G contains a compatible spanning circuit visiting each vertex exactly k times.*

Proof. Let k be a given positive integer, and let G be an edge-colored $n/2$ -critical graph on even n vertices, where $n \geq \max\{16k - 14, 4\}$. Set $\ell = 2k$. Thus, we have $n \geq \max\{16k - 14, 4\} = \max\{8\ell - 14, 4\}$. It follows from $\ell = 2k$ that G has a $2k$ -factor containing a given Hamilton cycle by Theorem 2.4.1. Hence, there exists a $2k$ -regular spanning eulerian subgraph H in G .

If $\Delta^{\text{mon}}(G) \leq k$, then we have $\Delta^{\text{mon}}(H) \leq \Delta^{\text{mon}}(G) \leq k$. Since $d_H(v) = 2k$ for every vertex v of H , there exists a compatible Euler tour in H by Theorem 2.1.1. Therefore, G contains a compatible spanning circuit visiting each vertex exactly k times. This completes the proof. $\square$

We also provide the following sufficient condition for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible positive integer k , in edge-colored graphs that satisfy an Ore-type degree condition.

Theorem 2.4.6. *Let k be a positive integer, and let G be an edge-colored graph on n vertices, where $n \geq 32(k-1)^2 - 3$, such that $\delta(G) \geq 2k$ and $d(u) + d(v) \geq n + 4$ for every pair of nonadjacent vertices u, v of G . If $\Delta^{\text{mon}}(G) \leq k$, then G contains a compatible spanning circuit visiting each vertex exactly k times.*

Proof. Let k be a given positive integer, and let G be an edge-colored graph on n vertices, where $n \geq 32(k-1)^2 - 3$, such that $\delta(G) \geq 2k$ and $d(u) + d(v) \geq n + 4$ for every pair of nonadjacent vertices u, v of G . Set $\ell = 2k$. Thus, we have $n \geq 32(k-1)^2 - 3 = 32k^2 - 64k + 29 = 8\ell^2 - 32\ell + 29 > 8\ell^2 - 2(4 + 12)\ell + 12 + 16$. It follows from $\ell = 2k$ that G has a $2k$ -factor

containing a given Hamilton cycle by Theorem 2.4.2. Hence, there exists a $2k$ -regular spanning eulerian subgraph H in G .

If $\Delta^{mon}(G) \leq k$, then we have $\Delta^{mon}(H) \leq \Delta^{mon}(G) \leq k$. Since $d_H(v) = 2k$ for every vertex v of H , there exists a compatible Euler tour in H by Theorem 2.1.1. Therefore, G contains a compatible spanning circuit visiting each vertex exactly k times. This completes the proof. $\square$

By slightly weakening the condition stated in Theorem 2.4.6, we provide another sufficient condition for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible positive integer k , as follows.

Theorem 2.4.7. *Let k be a positive integer, and let G be an edge-colored 4-connected graph on n vertices, where $n \geq 4(3k - 2)(4k - 5) + 1$, such that $\delta(G) \geq 2k$ and $\max\{d(u), d(v)\} \geq n/2 + 2$ for every pair of nonadjacent vertices u, v of G . If $\Delta^{mon}(G) \leq k$, then G contains a compatible spanning circuit visiting each vertex exactly k times.*

Proof. Let k be a given positive integer, and let G be an edge-colored 4-connected graph on n vertices, where $n \geq 4(3k - 2)(4k - 5) + 1$, such that $\delta(G) \geq 2k$ and $\max\{d(u), d(v)\} \geq n/2 + 2$ for every pair of nonadjacent vertices u, v of G . Since G is 4-connected, it has two edge-disjoint Hamilton cycles by Theorem 2.4.3. Let C be one of the two edge-disjoint Hamilton cycles of G . Thus, the graph $G - E(C)$ is connected. Set $\ell = 2k$. Thus, we have $n \geq 4(3k - 2)(4k - 5) + 1 = 48k^2 - 92k + 41 = 12\ell^2 - 46\ell + 41 > 12(\ell - 2)^2 + 2(\ell - 2) - 4$. It follows from $\ell = 2k$ and $G - E(C)$ is connected that G has a $2k$ -factor containing C by Theorem 2.4.4. Hence, there exists a $2k$ -regular spanning eulerian subgraph H in G .

If $\Delta^{mon}(G) \leq k$, then we have $\Delta^{mon}(H) \leq \Delta^{mon}(G) \leq k$. Since $d_H(v) = 2k$ for every vertex v of H , there exists a compatible Euler tour in H by Theorem 2.1.1. Therefore, G contains a compatible spanning circuit visiting each vertex exactly k times. This completes the proof. $\square$

Remark 2.4.1. *The proofs of Theorems 2.4.5, 2.4.6 and 2.4.7 are based on the existence of even k -factors containing Hamilton cycles (i.e., hamiltonian k -factors). Motivated by the proof technique applied in the proofs of these three*

theorems, we can obtain a similar conclusion for any edge-colored graph admitting such a hamiltonian k -factor.

2.5 Conclusions and future work

In this chapter, we established sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible integer k , in some specific edge-colored graphs.

In future work, we look forward to establishing sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly k times for a given positive integer k in other classes of edge-colored graphs.

For a graph G , let f denote a positive integer-valued function on $V(G)$. Motivated by the early work due to Das and Rao [30] and the concept of an f -factor, a more challenging and interesting problem is to develop sufficient conditions for the existence of compatible spanning circuits visiting each vertex v exactly $f(v)$ times in specific classes of edge-colored graphs.

Chapter 3

Compatible spanning circuits in edge-colored graphs satisfying certain degree conditions

In this chapter, we consider the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored graphs satisfying certain degree conditions (without any assumptions on the number of colors). We establish sufficient conditions for the existence of such compatible spanning circuits in edge-colored graphs satisfying Ore-type degree conditions, and in edge-colored graphs satisfying Fan-type degree conditions, respectively. Moreover, we confirm the existence of such compatible spanning circuits in edge-colored graphs with a high edge-connectivity, and we confirm the asymptotical existence of such compatible spanning circuits in specific edge-colored random graphs.

3.1 Introduction

As we mentioned in Chapter 1, the terminology and notations not defined but used in this chapter are standard and can be found in the most recent version of the textbook of Bondy and Murty [15]. Throughout this chapter, every edge-colored graph we refer to is always an edge-colored graph without restrictions on the number of colors, unless otherwise specified.

As the two extremal cases of compatible spanning circuits, the existence of compatible (i.e., properly colored or alternating) Hamilton cycles and compatible Euler tours in specific edge-colored graphs have been extensively studied. Back in 1968, Bánkfalvi et al. [7] established a characterization of 2-edge-colored complete graphs containing compatible Hamilton cycles. We refer the reader to [25, 28, 29] for more results on the existence of compatible Hamilton cycles in 2-edge-colored (multi)graphs. The research on the existence of compatible Hamilton cycles in specific edge-colored graphs without restrictions on the number of colors dates back to the 1970s (see [12, 23, 31, 111]), and this topic has also attracted new attention recently (see [1, 2, 4, 92]). On the other hand, Kotzig [84] in 1968 obtained the following necessary and sufficient condition for the existence of compatible Euler tours in edge-colored eulerian graphs (see Theorem 3.1.1). For more details on the existence of compatible Euler tours, we refer the reader to [9, 48, 51, 105].

Theorem 3.1.1 (Kotzig [84]). *Let G be an edge-colored eulerian graph. Then a compatible Euler tour exists if and only if $\Delta^{mon}(v) \leq d(v)/2$ for each vertex v of G .*

By restricting the number of colors, Das and Rao [30] established necessary and sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly a specified number of times in 2-edge-colored complete graphs back in the 1980s.

More recently, Guo et al. [67] provided sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly k times, for every feasible k , in specific edge-colored graphs without restrictions on the number of colors.

In this chapter, we consider the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored graphs satisfying Ore-type degree conditions, and in edge-colored graphs satisfying Fan-type degree conditions, respectively. Moreover, we consider the existence of such compatible spanning circuits in edge-colored graphs with a high edge-connectivity and edge-colored random graphs.

3.2 Ore-type degree conditions

In this section, we establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in specific edge-colored graphs satisfying Ore-type degree conditions.

Before proceeding, we list the following two lemmas on supereulerian graphs that are essential for our proofs of the main theorems of this section. Our lengthy proofs of these two key lemmas are postponed to Section 3.6.

Lemma 3.2.1. *Let G be a connected graph on $n \geq 3$ vertices such that $d(u) + d(v) \geq n$ for every pair of vertices u, v of G with $\text{dist}(u, v) = 2$. Then G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - 2$ for each vertex v of H .*

Lemma 3.2.2. *Let G be a 2-connected graph on n vertices such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of nonadjacent vertices u, v of G . Then G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - 2$ for each vertex v of H .*

Next, we present the main results of this section. We first establish the following sufficient condition for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in edge-colored graphs that satisfy an Ore-type degree condition.

Theorem 3.2.1. *Let G be an edge-colored connected graph on $n \geq 3$ vertices such that $d(u) + d(v) \geq n$ for every pair of vertices u, v of G with $\text{dist}(u, v) = 2$. If $\Delta^{\text{mon}}(v) \leq (d(v) - 1)/2$ for each vertex v with $d(v) \geq 3$, and $\Delta^{\text{mon}}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.*

Proof. The proof is based on Lemma 3.2.1, which will be proved in Section 3.6. Let G be an edge-colored connected graph on $n \geq 3$ vertices such that $d(u) + d(v) \geq n$ for every pair of vertices u, v of G with $\text{dist}(u, v) = 2$. By Lemma 3.2.1, G contains a spanning eulerian subgraph H such that $d_H(v) = d_G(v) - 1$ for each vertex v of G of odd degree and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even degree. If $\Delta_G^{\text{mon}}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G with $d_G(v) \geq 3$, then we have $\Delta_H^{\text{mon}}(v) \leq \Delta_G^{\text{mon}}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each vertex v of G with $d_G(v) \geq 3$. Since H is a spanning eulerian subgraph of G , we have $d_H(v) = d_G(v) = 2$ for each vertex v of G with $d_G(v) = 2$. Thus, we have $\Delta_H^{\text{mon}}(v) = 1 = d_H(v)/2$ for each vertex v of G with $d_G(v) = 2$. Based on the above argument, there exists a compatible Euler tour in H by Theorem 3.1.1. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

From Theorem 3.2.1, we immediately obtain the following corollaries.

Corollary 3.2.1. *Let G be an edge-colored graph on $n \geq 3$ vertices such that $d(u) + d(v) \geq n$ for every pair of nonadjacent vertices u, v of G . If $\Delta^{\text{mon}}(v) \leq (d(v) - 1)/2$ for each vertex v with $d(v) \geq 3$, and $\Delta^{\text{mon}}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.*

Corollary 3.2.2. *Let G be an edge-colored graph on $n \geq 3$ vertices such that $d(v) \geq n/2$ for every vertex v of G . If $\Delta^{\text{mon}}(v) \leq (n - 2)/4$ for each vertex v with $d(v) \geq 3$, and $\Delta^{\text{mon}}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex at least $\lfloor (n - 2)/4 \rfloor$ times.*

We also provide another sufficient condition for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times, as follows.

Theorem 3.2.2. *Let G be an edge-colored 2-connected graph on n vertices such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of nonadjacent vertices u, v of G . If $\Delta^{\text{mon}}(v) \leq (d(v) - 1)/2$ for each vertex v with $d(v) \geq 3$, and $\Delta^{\text{mon}}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.*

Proof. The proof is based on Lemma 3.2.2, which will be proved in Section 3.6. Let G be an edge-colored 2-connected graph on n vertices such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of nonadjacent vertices u, v of G . By Lemma 3.2.2, G contains a spanning eulerian subgraph H such that $d_H(v) = d_G(v) - 1$ for each vertex v of G of odd degree and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even degree. If $\Delta_G^{mon}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G with $d_G(v) \geq 3$, then we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each vertex v of G with $d_G(v) \geq 3$. Since H is a spanning eulerian subgraph of G , we have $d_H(v) = d_G(v) = 2$ for each vertex v of G with $d_G(v) = 2$. Thus, we have $\Delta_H^{mon}(v) = 1 = d_H(v)/2$ for each vertex v of G with $d_G(v) = 2$. Based on the above argument, there exists a compatible Euler tour in H by Theorem 3.1.1. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

Remark 3.2.1. *The following example shows that the bounds on $\Delta^{mon}(v)$ in Theorems 3.2.1 and 3.2.2 are both tight for a vertex v of odd degree.*

Example 3.2.1. *Let m be an odd integer such that $m \geq 3$, and let G be a balanced complete bipartite graph on $2m$ vertices with partite sets X and Y . Let $X' \subset X$ and $Y' \subset Y$ such that $|X'| = |Y'| = (m + 1)/2$, and let u be a vertex in $X \setminus X'$. We color all the edges between X' and Y' with color i , and we color all the edges between u and Y' with color j ($j \neq i$). Finally, we color the remaining edges of G with pairwise distinct new colors.*

One can check that the graph G of Example 3.2.1 satisfies the property requested in Theorems 3.2.1 and 3.2.2, respectively, and that it satisfies $\Delta^{mon}(u) = (m + 1)/2 = (d(u) - 1)/2 + 1$ for the above mentioned vertex u . However, the graph G contains no compatible spanning circuit visiting every vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.

As shown above, sufficient conditions for the existence of compatible spanning circuits visiting every vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in specific edge-colored graphs satisfying Ore-type degree conditions have been established. It is natural to consider this problem in edge-colored graphs satisfying Fan-type degree conditions. In hamiltonian graph theory, the sufficient Ore-type condition “ $d(u) + d(v) \geq n$ for all pairs of nonadjacent

vertices u, v ” can be replaced by the weaker sufficient Fan-type condition “ $\max\{d(u), d(v)\} \geq n/2$ for all pairs of vertices u, v with distance 2” in order to guarantee the existence of a Hamilton cycle in a graph on $n \geq 3$ vertices, or in a 2-connected graph on n vertices, respectively. In the light of Theorems 3.2.1 and 3.2.2 it would be natural to expect that the conclusions of these two results would still hold if one replaces “ $d(u) + d(v) \geq n$ ” by “ $\max\{d(u), d(v)\} \geq n/2$ ” in Theorem 3.2.1, or “nonadjacent pairs” by “distance 2 pairs” in Theorem 3.2.2. However, there seems no obvious way to prove this, and we are not sure that such a result in fact holds. In the next section, we show that such a result holds if we impose the mild additional minimum degree condition that $\delta(G) \geq 3$.

3.3 Fan-type degree conditions

In this section, we concentrate our efforts to consider the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored graphs satisfying Fan-type degree conditions. We establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in specific edge-colored graphs satisfying Fan-type degree conditions.

Before proceeding, we list the following two lemmas on supereulerian graphs and a basic result from existing literature, which will be used in the later proofs of the main results of this section.

Lemma 3.3.1. *Let G be a 2-connected graph on n vertices with $\delta(G) \geq 3$ such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v of G with $\text{dist}(u, v) = 2$. Then G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - 2$ for each vertex v of H .*

Our lengthy proof of this key lemma that is essential for the later proof of Theorem 3.3.2 is postponed to Section 3.6. The following lemma on supereulerian graphs will be used in the later proof of Corollary 3.3.1.

Lemma 3.3.2. *Let G be a graph admitting two edge-disjoint Hamilton paths. Then G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - 2$ for each vertex v of H .*

Proof. Let G be a graph admitting two edge-disjoint Hamilton paths. We suppose that G has 2ℓ vertices of odd degree, where ℓ is some fixed nonnegative integer. If $\ell = 0$, then G itself is eulerian, and hence it is a desired spanning eulerian subgraph of G . Next, we assume that $\ell \geq 1$.

We use the notation $O(G) = \bigcup_{i=1}^{\ell} \{x_i, y_i\}$ to denote the set of vertices of G of odd degree. Let P be one of the two edge-disjoint Hamilton paths of G . It follows that for each integer i with $1 \leq i \leq \ell$, there exists a unique path Q_i connecting x_i and y_i in P . Let F be the set of edges of P which appear an odd number of times on all paths Q_i ($1 \leq i \leq \ell$). Let $H = G - F$. Hence, we have $O(H) = \emptyset$. Since G has another Hamilton path that is edge-disjoint from P , the graph H is connected. Moreover, H satisfies that $d_H(v) \geq d_G(v) - 2$ for each vertex v of H . Thus, H is a desired spanning eulerian subgraph of G . This completes the proof. $\square$

The following theorem deals with the existence of edge-disjoint Hamilton cycles.

Theorem 3.3.1 (Zhou [123]). *Let G be a 4-connected graph on n vertices such that $\max\{d(u), d(v)\} \geq n/2 + 2$ for every pair of vertices u, v of G with $\text{dist}(u, v) = 2$. Then G contains two edge-disjoint Hamilton cycles.*

Next, we present the main results of this section. We first establish the following sufficient condition for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in edge-colored graphs that satisfy a Fan-type degree condition. This is the result we anticipated in the discussion at the end of Section 3.2.

Theorem 3.3.2. *Let G be an edge-colored 2-connected graph on n vertices with $\delta(G) \geq 3$ such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v of G with $\text{dist}(u, v) = 2$. If $\Delta^{\text{mon}}(v) \leq (d(v) - 1)/2$ for each vertex v of G , then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.*

Proof. The proof is based on Lemma 3.3.1, which will be proved in Section 3.6. Let G be an edge-colored 2-connected graph on n vertices with $\delta(G) \geq 3$ such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v

of G with $\text{dist}(u, v) = 2$. By Lemma 3.3.1, G contains a spanning eul-
 rian subgraph H such that $d_H(v) = d_G(v) - 1$ for each vertex v of G of
 odd degree and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even de-
 gree. If $\Delta_G^{\text{mon}}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G , then we have
 $\Delta_H^{\text{mon}}(v) \leq \Delta_G^{\text{mon}}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each vertex v of H .
 Based on the above argument, there exists a compatible Euler tour in H by
 Theorem 3.1.1. Therefore, G contains a compatible spanning circuit visiting
 each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

Remark 3.3.1. *The following example (i.e., Example 3.2.1 presented in Sec-
 tion 3.2) also shows that the bound on $\Delta^{\text{mon}}(v)$ in Theorem 3.3.2 is tight for a
 vertex v of odd degree.*

Example 3.3.1. *Let m be an odd integer such that $m \geq 3$, and let G be a
 balanced complete bipartite graph on $2m$ vertices with partite sets X and Y . Let
 $X' \subset X$ and $Y' \subset Y$ such that $|X'| = |Y'| = (m + 1)/2$, and let u be a vertex in
 $X \setminus X'$. We color all the edges between X' and Y' with color i , and we color all
 the edges between u and Y' with color j ($j \neq i$). Finally, we color the remaining
 edges of G with pairwise distinct new colors.*

One can check that the graph G of Example 3.3.1 satisfies the property
 requested in Theorem 3.3.2, and that it satisfies $\Delta^{\text{mon}}(u) = (m + 1)/2 =$
 $(d(u) - 1)/2 + 1$ for the above mentioned vertex u . However, the graph
 G contains no compatible spanning circuit visiting each vertex v at least
 $\lfloor (d(v) - 1)/2 \rfloor$ times.

From Theorem 3.3.2, the following corollary is immediate. Nevertheless,
 we will provide an alternative proof of the corollary, which does not depend
 on the proof of Theorem 3.3.2.

Corollary 3.3.1. *Let G be an edge-colored 4-connected graph on n vertices
 such that $\max\{d(u), d(v)\} \geq n/2 + 2$ for every pair of vertices u, v of G with
 $\text{dist}(u, v) = 2$. If $\Delta^{\text{mon}}(v) \leq (d(v) - 1)/2$ for each vertex v of G , then G
 contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) -$
 $1)/2 \rfloor$ times.*

Proof. Let G be an edge-colored 4-connected graph on n vertices such that
 $\max\{d(u), d(v)\} \geq n/2 + 2$ for every pair of vertices u, v of G with $\text{dist}(u, v)$

$= 2$. By Theorem 3.3.1, G contains two edge-disjoint Hamilton cycles. We can obtain two edge-disjoint Hamilton paths of G by deleting one edge from each of these two edge-disjoint Hamilton cycles. By Lemma 3.3.2, G contains a spanning eulerian subgraph H such that $d_H(v) = d_G(v) - 1$ for each vertex v of G of odd degree and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even degree. If $\Delta_G^{mon}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G , then we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each vertex v of H . Based on the above argument, there exists a compatible Euler tour in H by Theorem 3.1.1. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

Remark 3.3.2. *The proof of Corollary 3.3.1 is based on Lemma 3.3.2. In the proof of Lemma 3.3.2, we adopt a key technique of deleting those edges of a Hamilton path P of a graph G which appear an odd number of times on all sub-paths of P connecting a pair of vertices of G of odd degree. In addition, the proof of Corollary 3.3.1 is also depended on the existence of two edge-disjoint Hamilton cycles. Using the proof technique applied in the proof of Corollary 3.3.1, we can prove a similar conclusion for any edge-colored graph admitting two edge-disjoint Hamilton cycles.*

Remark 3.3.3. *The following example shows that the bound on $\Delta^{mon}(v)$ in Corollary 3.3.1 is tight for a vertex v of odd degree.*

Example 3.3.2. *Let m be an odd integer such that $m \geq 7$, and let $G = \overline{K}_m \vee K_{m+4}$. The graph G on $2m + 4$ vertices is obtained from two disjoint graphs $\overline{K}_m$ and K_{m+4} by joining each vertex of $\overline{K}_m$ to each vertex of K_{m+4} . Let $X \subset V(\overline{K}_m)$ and $Y \subset V(K_{m+4})$ such that $|X| = |Y| = (m + 5)/2$, and let u be a vertex in $V(\overline{K}_m) \setminus X$ (the condition $m \geq 7$ guarantees the existence of such a vertex u). We color all the edges between X and Y with color i , and we color all the edges between u and Y with color $j \neq i$. Finally, we color the remaining edges of G with pairwise distinct new colors.*

One can check that the graph G of Example 3.3.2 satisfies the properties required in Corollary 3.3.1, and that it satisfies $\Delta^{mon}(u) = (m + 5)/2 = (d(u) - 1)/2 + 1$ for the above mentioned vertex u . However, the graph

G contains no compatible spanning circuit visiting every vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.

3.4 CSCs in E-C $2(k + 1)$ -edge-connected graphs

In this section, we establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored $2(k + 1)$ -edge-connected graphs.

Before proceeding, we introduce the following two basic results which will be used in the later proof of the main theorem of this section.

Theorem 3.4.1 (Gusfield [69]). *Let k be a positive integer, and let G be a $2k$ -edge-connected graph. Then G contains k edge-disjoint spanning trees.*

Theorem 3.4.2 (Hasanvand [73]). *Let k be a positive integer, and let G be a graph with k edge-disjoint spanning trees. Then G contain a spanning tree T such that $d_T(v) \leq \lceil (d_G(v) - 1)/k \rceil + 1$ for each vertex v of G .*

Next, we present the main result of this section, as follows.

Theorem 3.4.3. *Let k be an integer such that $k \geq 2$, and let G be an edge-colored $2(k+1)$ -edge-connected graph. If for each vertex v of G , $\Delta^{mon}(v) \leq r/2$, where $r = \lfloor (k-1)(d(v)-1)/k \rfloor$, then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor r/2 \rfloor$ times.*

Proof. Let G be an edge-colored $2(k+1)$ -edge-connected graph, where k is an integer and $k \geq 2$. We suppose that G has 2ℓ vertices of odd degree, where ℓ is some fixed nonnegative integer. If $\ell = 0$, then G itself is an eulerian graph. If for each vertex v of G , $\Delta^{mon}(v) \leq r/2$, where $r = \lfloor (k-1)(d(v)-1)/k \rfloor$, then we have $\Delta^{mon}(v) < d(v)/2$ for each vertex v of G . It follows that G contains a compatible Euler tour by Theorem 3.1.1. This implies that G contains a compatible spanning circuit visiting each vertex v at least $\lfloor r/2 \rfloor$ times.

Next, we assume that $\ell \geq 1$. We use the notation $O(G) = \bigcup_{i=1}^{\ell} \{x_i, y_i\}$ to denote the set of vertices of G of odd degree. Since G is $2(k+1)$ -edge-connected, it contains $k+1$ edge-disjoint spanning trees by Theorem 3.4.1.

Let G' be a spanning subgraph of G obtained from G by deleting all edges of one of the $k + 1$ edge-disjoint spanning trees of G . Thus, the subgraph G' contains the remaining k edge-disjoint spanning trees of G . By Theorem 3.4.2, the subgraph G' contains a spanning tree T such that for each vertex v of G' , $d_T(v) \leq \lceil (d_{G'}(v) - 1)/k \rceil + 1$. It follows that for each integer i with $1 \leq i \leq \ell$, there exists a unique path Q_i between x_i and y_i in the above mentioned spanning tree T . Let F be the set of edges of T which appear an odd number of times on all paths Q_i ($1 \leq i \leq \ell$). Let $H = G - F$. Hence, we have $O(H) = \emptyset$. Note that T is a spanning tree of G' . Thus, G has another spanning tree that is edge-disjoint from T by the construction of G' . We conclude that the graph H is connected. Thus, H is a spanning eulerian subgraph of G . Moreover, H satisfies that for each vertex v of H , $d_H(v) \geq d_G(v) - d_T(v) \geq d_G(v) - (\lceil (d_{G'}(v) - 1)/k \rceil + 1) \geq d_G(v) - 1 - \lceil (d_G(v) - 1)/k \rceil \geq \lfloor (k - 1)(d_G(v) - 1)/k \rfloor = r$. If $\Delta_G^{mon}(v) \leq r/2$ for each vertex v of G , then we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq r/2 \leq d_H(v)/2$ for each vertex v of H . Based on the above argument, there exists a compatible Euler tour in H by Theorem 3.1.1. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor r/2 \rfloor$ times. This completes the proof. $\square$

Remark 3.4.1. *The edge-connectivity condition of Theorem 3.4.3 implies that the graphs in Theorem 3.4.3 need to have a large minimum degree.*

3.5 CSCs in E-C random graphs

In this section, we continue to pay close attention to the existence of compatible spanning circuits visiting each vertex at least a specified number of times. We consider edge-colored random graphs, and we establish sufficient conditions for the asymptotical existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in specific edge-colored random graphs.

A random graph process $\tilde{\mathcal{G}}(n) = (G_0, G_1, \dots, G_m, \dots)$ on the vertex set $V_n = \{v_1, \dots, v_n\}$, introduced by Bollobás and Frieze [13], is defined as a Markov process (a random process in which the future state is independent

of the past state and is only dependent on the present state) in which G_m is a graph with $V(G_m) = V_n$ and $e(G_m) = m$ for each integer m with $m \geq 0$. The initial graph G_0 is an empty graph. For $m \geq 1$, the graph G_m is obtained from G_{m-1} by choosing an edge $e \in \binom{V_n}{2} \setminus E(G_{m-1})$ uniformly at random (u.a.r. for short), where $\binom{V_n}{2}$ denotes the set of all pairs of vertices of V_n , and putting $E(G_m) = E(G_{m-1}) \cup \{e\}$. In fact, the random graph G_m constructed by the random graph process $\tilde{\mathcal{G}}(n)$ is distributed exactly as the well-known random graph $G_{n,m}$ constructed by the original random graph process introduced by Erdős and Rényi [40] (i.e., a graph that is chosen u.a.r. from the set of all graphs with the vertex set V_n and exactly m edges).

We say that a property $\mathcal{P}$ of a graph on n vertices holds *asymptotically almost surely* (a.a.s. for short) if the probability that $\mathcal{P}$ holds tends to 1 as n tends to infinity ($n \rightarrow \infty$ for short).

Before proceeding, we introduce the following basic result due to Bollobás and Frieze [13], which will be used in the later proof of Theorem 3.5.2.

Theorem 3.5.1 (Bollobás and Frieze [13]). *Let k be an integer such that $k \geq 2$, and let G be a random graph constructed by the random graph process $\tilde{\mathcal{G}}(n)$ such that $\delta(G) \geq k$. Then G contains a.a.s. $\lfloor k/2 \rfloor$ edge-disjoint Hamilton cycles.*

Next, we present the main results of this section.

We first establish the following sufficient condition for the asymptotical existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in edge-colored random graphs that are constructed by the random graph process due to Bollobás and Frieze [13].

Theorem 3.5.2. *Let G be an edge-colored random graph constructed by the random graph process $\tilde{\mathcal{G}}(n)$ such that $\delta(G) \geq 4$. If $\Delta^{\text{mon}}(v) \leq (d(v) - 1)/2$ for each vertex v of G , then G contains a.a.s. a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.*

Proof. Let G be an edge-colored random graph constructed by the random graph process $\tilde{\mathcal{G}}(n)$ such that $\delta(G) \geq 4$. By Theorem 3.5.1, the random graph

G contains a.a.s. at least two edge-disjoint Hamilton cycles. Now suppose that G is a random graph constructed by the random graph process $\tilde{\mathcal{G}}(n)$, and that it contains at least two edge-disjoint Hamilton cycles. We suppose that G has 2ℓ vertices of odd degree, where ℓ is some fixed nonnegative integer. If $\ell = 0$, then G itself is an eulerian graph. If $\Delta^{mon}(v) \leq (d(v) - 1)/2$ for each vertex v of G , then we have $\Delta^{mon}(v) < d(v)/2$ for each vertex v of G . It follows that G contains a compatible Euler tour by Theorem 3.1.1. This implies that G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.

Next, we assume that $\ell \geq 1$. We use the notation $O(G) = \bigcup_{i=1}^{\ell} \{x_i, y_i\}$ to denote the set of vertices of G of odd degree. Let P be a Hamilton path of G obtained from one of the two edge-disjoint Hamilton cycles of G by deleting one edge. It follows that for each integer i with $1 \leq i \leq \ell$, there exists a unique path Q_i connecting x_i and y_i in P . Let F be the set of edges of P which appear an odd number of times on all paths Q_i ($1 \leq i \leq \ell$). Let $H = G - F$. Hence, we have $O(H) = \emptyset$. Since G has another Hamilton path that is edge-disjoint from P , the graph H is connected. Thus, H is a spanning eulerian subgraph of G . Moreover, H satisfies that $d_H(v) = d_G(v) - 1$ for each vertex v of G of odd degree and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even degree. If $\Delta_G^{mon}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G , then we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each vertex v of H . Based on the above argument, there exists a compatible Euler tour in H by Theorem 3.1.1. Thus, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times. Therefore, an edge-colored random graph G constructed by the random graph process $\tilde{\mathcal{G}}(n)$ satisfying $\delta(G) \geq 4$ contains a.a.s. a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

We further consider the asymptotical existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in another class of edge-colored random graphs.

Let d be a positive integer, and let r be a real number. A *random d -dimensional geometric graph process* $\tilde{\mathcal{G}}^d(n) = (G^d(n, r))_{0 \leq r < \infty}$ on the vertex set $V_n = \{v_1, \dots, v_n\}$, generalized from Gilbert's definition [61], is defined as

a continuous process as r ranges from 0 to ∞ in which $G^d(n, r)$ is a random d -dimensional geometric graph defined by placing independently and uniformly at random all vertices of V_n in the d -dimensional unit square $[0, 1]^d$, and joining two vertices v_i and v_j whenever $\|v_i - v_j\|_p \leq r$ for the given real number r with $0 \leq r < \infty$, where $\|\cdot\|_p$ denotes the standard ℓ_p -norm for some fixed p with $1 < p \leq \infty$ (see [100]). For more details on random geometric graph processes and related applications, we refer the reader to [115].

Before presenting our result, we introduce a basic result due to Müller et al. [100] which will be used in the later proof of Theorem 3.5.4.

Theorem 3.5.3 (Müller, Pérez-Giménez and Wormald [100]). *Let k and d be two positive integers such that $k \geq 2$, and let G be a k -connected random d -dimensional geometric graph constructed by the random d -dimensional geometric graph process $\tilde{\mathcal{G}}^d(n)$. Then G contains a.a.s. $\lfloor k/2 \rfloor$ edge-disjoint Hamilton cycles.*

We establish a sufficient condition for the asymptotical existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in edge-colored random geometric graphs, as follows.

Theorem 3.5.4. *Let d be a positive integer, and let G be an edge-colored 4-connected random d -dimensional geometric graph constructed by the random d -dimensional geometric graph process $\tilde{\mathcal{G}}^d(n)$. If $\Delta^{\text{mon}}(v) \leq (d(v) - 1)/2$ for each vertex v of G , then G contains a.a.s. a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.*

Proof. Let G be an edge-colored 4-connected random d -dimensional geometric graph constructed by the random d -dimensional geometric graph process $\tilde{\mathcal{G}}^d(n)$, where d is a positive integer. By Theorem 3.5.3, the random d -dimensional geometric graph G contains a.a.s. at least two edge-disjoint Hamilton cycles. Now suppose that G is a random d -dimensional geometric graph constructed by the random d -dimensional geometric graph process $\tilde{\mathcal{G}}^d(n)$, and that it contains at least two edge-disjoint Hamilton cycles. We suppose that G has 2ℓ vertices of odd degree, where ℓ is some fixed non-negative integer. If $\ell = 0$, then G itself is an eulerian graph. If $\Delta^{\text{mon}}(v) \leq$

$(d(v) - 1)/2$ for each vertex v of G , then we have $\Delta^{mon}(v) < d(v)/2$ for each vertex v of G . It follows that G contains a compatible Euler tour by Theorem 3.1.1. This implies that G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.

Next, we assume that $\ell \geq 1$. We use the notation $O(G) = \bigcup_{i=1}^{\ell} \{x_i, y_i\}$ to denote the set of vertices of G of odd degree. Let P be a Hamilton path of G obtained from one of the two edge-disjoint Hamilton cycles of G by deleting one edge. It follows that for each integer i with $1 \leq i \leq \ell$, there exists a unique path Q_i connecting x_i and y_i in P . Let F be the set of edges of P which appear an odd number of times on all paths Q_i ($1 \leq i \leq \ell$). Let $H = G - F$. Hence, we have $O(H) = \emptyset$. Since G has another Hamilton path that is edge-disjoint from P , the graph H is connected. Thus, H is a spanning eulerian subgraph of G . Moreover, H satisfies that $d_H(v) = d_G(v) - 1$ for each vertex v of G of odd degree and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even degree. If $\Delta_G^{mon}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G , then we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each vertex v of H . Based on the above argument, there exists a compatible Euler tour in H by Theorem 3.1.1. Thus, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. Therefore, an edge-colored 4-connected random d -dimensional geometric graph constructed by the random d -dimensional geometric graph process $\tilde{\mathcal{G}}^d(n)$ contains a.s. a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

Remark 3.5.1. *The connectivity condition of Theorem 3.5.4 implies that the random d -dimensional geometric graphs in Theorem 3.5.4 need to have minimum degree at least 4. Moreover, Penrose [103] showed that the first added edge that makes the random d -dimensional geometric graph have minimum degree k is a.s. the first one that makes it k -connected for each integer d with $d \geq 2$ and some positive integer k .*

3.6 Proofs of key lemmas

In this section, we present the lengthy proofs of the three key lemmas on supereulerian graphs that have not been proven but have been used in our proofs of the main theorems of this chapter.

Before proceeding, we first introduce some additional terminology and notations, and we also present some auxiliary results that will be used in the later proofs of these key lemmas.

A *linear forest* of a graph G is defined as a forest of G in which every component is a path. For a vertex v and a subgraph A of a graph G , we denote by $N_G(v, A)$ the set of neighbors of v in G contained in A . For two disjoint subgraphs A and B of a graph G , we let $N_G(B, A) = \bigcup_{v \in V(B)} N_G(v, A)$ and $E_G(B, A) = \{uv \in E(G) \mid u \in V(B), v \in V(A)\}$. We write $d_G(v, A) = |N_G(v, A)|$ and $e_G(B, A) = |E_G(B, A)|$. For two disjoint subsets S_1 and S_2 of $V(G)$, we can similarly define $N_G(S_1, S_2)$ and $E_G(S_1, S_2)$. In particular, if $S_1 = \{v\}$, then we use $N_G(v, S_2)$ and $E_G(v, S_2)$ instead of $N_G(\{v\}, S_2)$ and $E_G(\{v\}, S_2)$, respectively. For a subset S of $V(G)$ or $E(G)$, we use $G[S]$ to denote the subgraph of G induced by S .

For a subgraph H of a graph G and two disjoint subsets E_1, E_2 of $E(G)$ such that $E_1 \subseteq E(G) \setminus E(H)$, we use the notation $H + E_1 \oplus E_2$ to denote the operation of adding these edges of E_1 to H , and for each edge $e \in E_2$, removing the edge e from H in the case $e \in E(H)$ and otherwise adding the edge e to H .

The following theorem due to Fan [42] is well-known in the hamiltonian graph theory community.

Theorem 3.6.1 (Fan [42]). *Let G be a 2-connected graph on $n \geq 3$ vertices. If $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v of G with $\text{dist}(u, v) = 2$, then G is hamiltonian.*

We present an easy but useful lemma, as follows.

Lemma 3.6.1. *Let G be a hamiltonian graph on n vertices. Then there exists a linear forest F of G such that $N_1(F) = O(G)$ and $e(F) \leq n/2$.*

Proof. If $O(G) = \emptyset$, then an empty (edgeless) linear forest of G is a desired linear forest. Now, we assume that $O(G) \neq \emptyset$. Let C be a Hamilton cycle of G with a given orientation $\vec{C}$. Set $O(G) = \{v_1, \dots, v_k\}$. For convenience, we assume that these vertices $v_1, \dots, v_k$ appear in this order along $\vec{C}$. We use $\vec{C}[u, v]$ to denote the (oriented) uv -path along $\vec{C}$. Let $F_1 = \bigcup_{i=1}^{k/2} \vec{C}[v_{2i-1}, v_{2i}]$ and $F_2 = \bigcup_{i=1}^{k/2} \vec{C}[v_{2i}, v_{2i+1}]$, where $v_{k+1} = v_1$. Clearly, the subgraph F_i is a linear forest of G with $N_1(F_i) = O(G)$ for $i \in \{1, 2\}$. Since $e(F_1) + e(F_2) = n$, either $e(F_1) \leq n/2$ or $e(F_2) \leq n/2$. $\square$

We now have all the ingredients for the proofs of the key lemmas. Next, we present our proofs.

3.6.1 Proof of Lemma 3.2.1

Let G be a connected graph on $n \geq 3$ vertices such that $d(u) + d(v) \geq n$ for every pair of vertices u, v with $\text{dist}(u, v) = 2$. It is not difficult to see that G is 2-connected. It follows that G is a hamiltonian graph by Theorem 3.6.1. By Lemma 3.6.1, there exists a linear forest F of G such that $N_1(F) = O(G)$ and $e(F) \leq n/2$. Let $G' = G - E(F)$. Thus, we can conclude that $d_{G'}(v) \geq d_G(v) - 2$ and $d_{G'}(v)$ is even for each vertex $v \in V(G)$. If the graph G' is connected, then it is a desired spanning eulerian subgraph of G . Next, we assume that G' is disconnected. We divide the remainder of the proof into two cases that might occur.

We first consider the case that the graph $G[V(A_0)]$ is not a complete graph for some component A_0 of G' with $v(A_0) \leq n/2$.

Case 1. *There exists a component A_0 of G' with $v(A_0) \leq n/2$ such that the graph $G[V(A_0)]$ is not a complete graph.*

Let $B = G' - A_0$. Note that $O(A_0) = \emptyset$ and the component A_0 is connected and not a complete graph. This implies that $v(A_0) \geq 4$ and hence $n \geq 8$, and that there exist two vertices $u_1, u_2 \in V(A_0)$ with $\text{dist}_G(u_1, u_2) = 2$. It follows that $n \leq d_G(u_1) + d_G(u_2) \leq v(A_0) - 2 + 2 + v(A_0) - 2 + 2 = 2v(A_0)$. Recall that $v(A_0) \leq n/2$. Therefore, we have $v(A_0) = v(B) = n/2$, $d_G(u_1, A_0) = d_G(u_2, A_0) = v(A_0) - 2$ and $d_G(u_1, B) = d_G(u_2, B) = 2$. Hence, we have $N_G(u_1, A_0) = N_G(u_2, A_0) = V(A_0) \setminus \{u_1, u_2\}$.

Figure 3.1: The graph illustrating Case 1.1.

Depending on whether or not $N_G(u_1, B) \cap N_G(u_2, B) = \emptyset$, we divide Case 1 into two subcases (i.e., Cases 1.1 and 1.2). We first consider the case that $N_G(u_1, B) \cap N_G(u_2, B) \neq \emptyset$.

Case 1.1. $N_G(u_1, B) \cap N_G(u_2, B) \neq \emptyset$ (see Figure 3.1).

Let $v_0 \in N_G(u_1, B) \cap N_G(u_2, B)$. It is easy to see that $N_G(u_1, B) \cap N_G(u_2, B) = \{v_0\}$; otherwise, there would be a cycle in F . Recall that $N_G(u_1, A_0) = V(A_0) \setminus \{u_1, u_2\}$. Let u be an arbitrary vertex in $V(A_0) \setminus \{u_1, u_2\}$. Since $u_1 u \in E(G)$ and $v_0 u \notin E(G)$, we have $\text{dist}_G(u, v_0) = 2$. It follows that $n \leq d_G(u) + d_G(v_0) \leq v(A_0) - 1 + 2 + d_G(v_0, B) + 2$, implying that $d_G(v_0, B) \geq v(B) - 3$.

If $d_G(v_0, B) = v(B) - 3$, then we have $d_G(u, B) = 2$ for every vertex $u \in V(A_0)$. We conclude that $e(F) \geq 2(n/2) > n/2$, a contradiction. If $d_G(v_0, B) = v(B) - 2$, then we have $d_G(u, B) \geq 1$ for every vertex $u \in V(A_0)$. Recall that $d_G(u_1, B) = d_G(u_2, B) = 2$. We conclude that $e(F) \geq 4 + (n/2 - 2) > n/2$, a contradiction. Now, we assume that $d_G(v_0, B) = v(B) - 1$. Thus, the vertex v_0 is adjacent to all vertices in $V(B) \setminus \{v_0\}$. Let v be an arbitrary vertex in $V(B) \setminus \{v_0\}$. Therefore, there exists a vertex $u_i \in \{u_1, u_2\}$ such that $\text{dist}_G(u_i, v) = 2$. It follows that $n \leq d_G(u_i) + d_G(v) \leq v(A_0) - 2 + 2 + v(B) - 1 + d_G(v, A_0)$, implying that $d_G(v, A_0) \geq 1$. Recall that $d_G(v_0, A_0) = 2$. We have $e(F) \geq 2 + (n/2 - 1) > n/2$, a contradiction.

This concludes the proof for Case 1.1. The remaining subcase of Case 1 is that $N_G(u_1, B) \cap N_G(u_2, B) = \emptyset$.

Case 1.2. $N_G(u_1, B) \cap N_G(u_2, B) = \emptyset$.

Let $N_G(u_1, B) = \{v_1, v_2\}$. We first suppose that $N_G(v_1, A_0) = \{u_1\}$ (see Figure 3.2 (a)). Recall that $N_G(u_1, A_0) = V(A_0) \setminus \{u_1, u_2\}$. Let u be an arbitrary vertex in $V(A_0) \setminus \{u_1, u_2\}$. We have $\text{dist}_G(u, v_1) = 2$. It follows that $n \leq d_G(u) + d_G(v_1) \leq v(A_0) - 1 + d_G(u, B) + v(B) - 1 + 1$, implying that $d_G(u, B) \geq 1$. Recall that $d_G(u_1, B) = d_G(u_2, B) = 2$. We have $e(F) \geq 4 + (n/2 - 2) > n/2$, a contradiction.

Figure 3.2: Graphs illustrating Case 1.2.

Suppose now that $N_G(v_1, A_0) = \{u_1, u_3\}$, where $u_3 \in V(A_0) \setminus \{u_1, u_2\}$ (see Figure 3.2 (b)). Let u be an arbitrary vertex in $V(A_0) \setminus \{u_1, u_2, u_3\}$. We have $\text{dist}_G(u, v_1) = 2$. It follows that $n \leq d_G(u) + d_G(v_1) \leq v(A_0) - 1 + 2 + d_G(v_1, B) + 2$, implying that $d_G(v_1, B) \geq v(B) - 3$.

If $d_G(v_1, B) = v(B) - 3$, then we have $d_G(u, B) = 2$ for every vertex $u \in V(A_0) \setminus \{u_3\}$. Recall that $d_G(u_3, B) \geq 1$. We have $e(F) \geq 1 + 2(n/2 - 1) > n/2$, a contradiction. If $d_G(v_1, B) = v(B) - 2$, then we have $d_G(u, B) \geq 1$ for every vertex $u \in V(A_0)$. Recall that $d_G(u_1, B) = d_G(u_2, B) = 2$. We conclude that $e(F) \geq 4 + (n/2 - 2) > n/2$, a contradiction. Now, we assume that $d_G(v_1, B) = v(B) - 1$. Let v be an arbitrary vertex in $V(B) \setminus \{v_1, v_2\}$. Thus, we have $\text{dist}_G(u_1, v) = 2$. It follows that $n \leq d_G(u_1) + d_G(v) \leq v(A_0) - 2 + 2 + v(B) - 1 + d_G(v, A_0)$, implying that $d_G(v, A_0) \geq 1$. Recall that $d_G(v_1, A_0) = 2$ and $d_G(v_2, A_0) \geq 1$. We conclude that $e(F) \geq 2 + (n/2 - 1) > n/2$, a contradiction.

This concludes the proof for Case 1. The remaining case is that the graph $G[V(A)]$ is a complete graph for every component A of G' with $v(A) \leq n/2$.

Case 2. The graph $G[V(A)]$ is a complete graph for every component A of G' with $v(A) \leq n/2$.

Depending on whether or not the graph G' has an isolated vertex, we divide Case 2 into two subcases (i.e., Cases 2.1 and 2.2). We first consider the case that G' has at least one isolated vertex.

Case 2.1. *The graph G' has at least one isolated vertex.*

We prove several claims in order to deal with Case 2.1.

Claim 1. *The graph G' has exactly one isolated vertex.*

Proof. Let S be the set of isolated vertices of G' . Set $B = G' - S$ and $S' = N_G(S, B)$. If $V(B) \setminus S' = \emptyset$ (i.e., $V(G) = S \cup S'$), then we have $|S'| \leq e_G(S, S')$ and $e(G[S]) < |S|$; otherwise, there would be a cycle in F . If $|S| \geq |S'|$, then we have $e(F) \geq 2|S| - e(G[S]) > |S| \geq n/2$, a contradiction; if $|S'| > |S|$, then we have $e(F) \geq e_G(S, S') > n/2$, also a contradiction. Therefore, we conclude that $V(B) \setminus S' \neq \emptyset$. Thus there exist two vertices $u_1 \in S$ and $v_1 \in V(B) \setminus S'$ such that $\text{dist}_G(u_1, v_1) = 2$. It follows that $n \leq d_G(u_1) + d_G(v_1) \leq 2 + v(B) - 1$, implying that $v(B) \geq n - 1$ and $|S| \leq 1$. Recall that the graph G' has at least one isolated vertex. Hence, $|S| = 1$. This confirms our claim. $\square$

Let u_1 be the isolated vertex of G' . We prove another claim to show that the subgraph $B = G' - u_1$ is connected.

Claim 2. *The subgraph $B = G' - u_1$ is connected.*

Proof. Suppose, to the contrary, that B is disconnected. Let B_1 be a component of B such that $N_G(u_1, B_1) \neq \emptyset$. Note that the graph G' has no component with two vertices. Thus, every component of B has at least three vertices. We can see that $3 \leq v(B_1) \leq n - 4$, and that there is a vertex $v \in V(B_1)$ with $\text{dist}_G(u_1, v) = 2$. Thus, we have $n \leq d_G(u_1) + d_G(v) \leq 2 + v(B_1) - 1 + 2 \leq n - 1$, a contradiction. $\square$

Clearly, we have $d_G(u_1) = 2$. Let $N_G(u_1, B) = \{v_1, v_2\}$. If $v_1 v_2 \in E(G)$, then we have $v_1 v_2 \in E(G')$; otherwise, the sequence $v_1 u_1 v_2 v_1$ would form a cycle in F . Thus, the subgraph $H = G' + \{u_1 v_1, u_1 v_2\} - \{v_1 v_2\}$ is a desired spanning eulerian subgraph of G . Next, we assume that $v_1 v_2 \notin E(G)$.

We claim that $N_G(v_1, B) = N_G(v_2, B)$. Suppose, to the contrary, that there exists a vertex $v_0 \in N_G(v_1, B) \setminus N_G(v_2, B)$. Since $\text{dist}_G(u_1, v_0) = 2$, we have $n \leq d_G(u_1) + d_G(v_0) \leq 2 + n - 1 - 2 < n$, a contradiction. Thus as we claimed, we have $N_G(v_1, B) = N_G(v_2, B)$.

Let $N_G(v_1, B) = N_G(v_2, B) = R$. Since $\text{dist}_G(v_1, v_2) = 2$, we have $|R| \geq (n - 2)/2$. Note that $O(B) = \emptyset$, and that the component B is connected and not a complete graph. This implies that $v(B) \geq 4$ and hence $n \geq 5$.

We claim that $N_{G'}(v_1, B) \cap N_{G'}(v_2, B) \neq \emptyset$. Suppose, to the contrary, that $N_{G'}(v_1, B) \cap N_{G'}(v_2, B) = \emptyset$. Based on the fact that $N_G(v_1, B) = N_G(v_2, B)$ and the component B is connected, we have $d_{G'}(v, B) = 1$ for every vertex $v \in \{v_1, v_2\}$; otherwise, there would be a vertex $v' \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$. This is a contradiction with the fact that $O(B) = \emptyset$. Thus as we claimed, we have $N_{G'}(v_1, B) \cap N_{G'}(v_2, B) \neq \emptyset$. Let $N_{G'}(v_1, B) \cap N_{G'}(v_2, B) = S$.

If there exists a vertex $v_3 \in S$ such that $E_G(v_3) \cap E(F) = \emptyset$, then let $H = G' + \{u_1 v_1, u_1 v_2\} - \{v_1 v_3, v_2 v_3\}$. Since $\text{dist}_G(u_1, v_3) = 2$ and $d_G(u_1) = 2$, we have $d_G(v_3) \geq n - 2$. Recall that $n \geq 5$ and $v(B) = n - 1$. One can check that $N_{G'}(v_3, B) \cap N_{G'}(v, B) \neq \emptyset$ for some vertex $v \in \{v_1, v_2\}$. Thus, the subgraph H is a desired spanning eulerian subgraph of G . Next, we assume that $E_G(v) \cap E(F) \neq \emptyset$ for every vertex $v \in S$. We prove the following claim to show how we can construct a desired spanning eulerian subgraph of G , for the case that $E_G(v) \cap E(F) \neq \emptyset$ for every vertex $v \in S$.

Claim 3. *There exist two vertices $v_3, v_4 \in S$ such that $v_3 v_4 \in E(F)$.*

Proof. First we suppose that $S = R$. Let $F' = F - \{u_1 v_1, u_1 v_2\}$. We have $e(F') \leq n/2 - 2$. Recall that $|S| = |R| \geq n/2 - 1$. Hence, there exist two vertices $v_3, v_4 \in S$ such that $v_3 v_4 \in E(F')$.

Suppose now that $|R \setminus S| = 1$. Let $v_5 \in R \setminus S$. Note that it is impossible that $\{v_1 v_5, v_2 v_5\} \subset E(F)$; otherwise, the sequence $u_1 v_1 v_5 v_2 u_1$ would form a cycle in F . Without loss of generality, we suppose that $v_1 v_5 \in E(F)$. Let $F' = F - \{u_1 v_1, u_1 v_2, v_1 v_5\}$. We have $e(F') \leq n/2 - 3$. Recall that $|S| = |R| - 1 \geq n/2 - 2$. Hence, there exist two vertices $v_3, v_4 \in S$ such that $v_3 v_4 \in E(F')$.

Finally, we suppose that $|R \setminus S| = 2$. Set $\{v_1 v_5, v_2 v_6\} \subset E(F)$. Let $F' = F - \{u_1 v_1, u_1 v_2, v_1 v_5, v_2 v_6\}$. We have $e(F') \leq n/2 - 4$. Recall that $|S| = |R| - 2 \geq$

$n/2 - 3$. Hence, there exist two vertices $v_3, v_4 \in S$ such that $v_3v_4 \in E(F')$. This confirms our claim. $\square$

By Claim 3, the subgraph $H = G' + \{u_1v_1, u_1v_2, v_3v_4\} - \{v_1v_3, v_2v_4\}$ is a desired spanning eulerian subgraph of G (see Figure 3.3).

Figure 3.3: The graph illustrating Case 2.1.

This concludes the proof for Case 2.1. The remaining subcase of Case 2 is that the graph G' has no isolated vertex.

Case 2.2. *The graph G' has no isolated vertex.*

Note that there is no component A with $v(A) = 2$ in G' ; otherwise, the component A would be a complete graph on two vertices, contradicting the fact that $O(A) = \emptyset$. Thus, we can conclude that every component of G' has at least three vertices. The following claim shows that the graph G' has exactly two components.

Claim 4. *The graph G' has exactly two components.*

Proof. Let A_0 be a component of G' with $v(A_0) \leq n/2$, and let B be another component of G' such that $N_G(A_0, B) \neq \emptyset$. Note that the graph $G[V(A_0)]$ is complete, and that the component A_0 has at least three vertices. Thus, there exist two vertices $u \in V(A_0)$ and $v \in V(B)$ such that $\text{dist}_G(u, v) = 2$. It follows that $n \leq d_G(u) + d_G(v) \leq v(A_0) - 1 + 2 + v(B) - 1 + 2$, implying that $v(A_0) + v(B) \geq n - 2$. If G' has the third component C , then we have $v(C) \leq 2$, a contradiction. $\square$

Let A and B be the two components of G' . We suppose that $v(A) \leq v(B)$, and that $e_{G'}(A) \leq e_{G'}(B)$ if $v(A) = v(B)$. Depending on whether or not the component A is a complete graph in G' , we divide Case 2.2 into two sub-cases (i.e., Cases 2.2.1 and 2.2.2). We first consider the case that A is not a complete graph in G' .

Case 2.2.1. *The component A is not a complete graph in G' (see Figure 3.4).*

Figure 3.4: The graph illustrating Case 2.2.1.

Recall that we assume throughout Case 2 that the graph $G[V(A)]$ is a complete graph. The next claim characterizes the structure of $G'[V(A)]$.

Claim 5. *The component A of G' satisfies $A = G[V(A)] - M$ for some perfect matching M of $G[V(A)]$.*

Proof. Recall that $G' = G - E(F)$ and the graph $G[V(A)]$ is a complete graph. Hence, the graph $G[V(A)]$ is regular. Note that the component A of G' is not a complete graph. If $d_G(u, A) - d_{G'}(u, A) \equiv 0 \pmod{2}$ for each vertex $u \in V(A)$, then the graph induced by $E(F) \cap E(G[V(A)])$ contains a cycle, contradicting the fact that F is a linear forest. Thus, there exists a vertex u of A such that $d_{G'}(u, A) = d_G(u, A) - 1$. Since the graph $G[V(A)]$ is regular and every vertex of G' has an even degree, we have $d_{G'}(u, A) = d_G(u, A) - 1$ for each vertex $u \in V(A)$. This confirms our claim. $\square$

Clearly, there exists an edge $u_1 v_1 \in E(F)$ with $u_1 \in V(A)$ and $v_1 \in V(B)$ (see Figure 3.4). Let $u_1 u_2 \in M$ (by Claim 5). Thus, we have $\text{dist}_G(u_2, v_1) = 2$; otherwise, the sequence $u_1 u_2 v_1 u_1$ would form a cycle in F . It follows that

$n \leq d_G(u_2) + d_G(v_1) \leq v(A) - 1 + 1 + d_G(v_1, B) + 2$, implying that $d_G(v_1, B) \geq v(B) - 2$. Let v be an arbitrary vertex in $N_G(v_1, B)$. Thus, we have $\text{dist}_G(u_1, v) = 2$. It follows that $n \leq d_G(u_1) + d_G(v) \leq v(A) - 1 + 1 + v(B) - 1 + d_G(v, A)$, implying that $d_G(v, A) \geq 1$. By Claim 5, we have $v(A) \geq 4$ and $e(M) \geq 2$. Thus, we have $e(F) \geq v(B) - 2 + 1 + 2 > n/2$, a contradiction.

This concludes the proof for Case 2.2.1. The remaining subcase of Case 2.2 is that the component A is a complete graph in G' .

Case 2.2.2. *The component A is a complete graph in G' .*

Let $N_G(A, B) = S$. Since G is 2-connected, we have $|S| \geq 2$.

We first suppose that there exist two vertices v_1, v_2 of S such that $v_1 v_2 \in E(G)$. If $N_G(v_1, A) \cap N_G(v_2, A) \neq \emptyset$, say $u_1 \in N_G(v_1, A) \cap N_G(v_2, A)$, then we have $v_1 v_2 \in E(G')$; otherwise, the sequence $v_1 u_1 v_2 v_1$ would form a cycle in F . Thus, the subgraph $H = G' + \{u_1 v_1, u_1 v_2\} - \{v_1 v_2\}$ is a desired spanning eulerian subgraph of G (see Figure 3.5).

Figure 3.5: The graph illustrating the case that $N_G(v_1, A) \cap N_G(v_2, A) \neq \emptyset$.

Now, let us consider the case that $N_G(v_1, A) \cap N_G(v_2, A) = \emptyset$. Let $u_1 \in N_G(v_1, A)$ and $u_2 \in N_G(v_2, A)$. We define a graph H such that

$$H = \begin{cases} G' + \{u_1 v_1, u_2 v_2\} - \{u_1 u_2, v_1 v_2\}, & \text{if } v_1 v_2 \notin E(F); \\ G' + \{u_1 v_1, u_2 v_2, v_1 v_2\} - \{u_1 u_2\}, & \text{otherwise.} \end{cases}$$

Thus, the subgraph H is a desired spanning eulerian subgraph of G (see Figure 3.6 (a) and (b)).

Figure 3.6: Graphs illustrating the case that $N_G(v_1, A) \cap N_G(v_2, A) = \emptyset$.

Suppose now that the set S is an independent set in G . If $d_G(v, A) = 2$ for some vertex $v \in S$, say $N_G(v, A) = \{u_1, u_2\}$, then the subgraph $H = G' + \{u_1v, u_2v\} - \{u_1u_2\}$ is a desired spanning eulerian subgraph of G . Next, we assume that $d_G(v, A) = 1$ for every vertex $v \in S$. Recall that the component A of G' is a complete graph. Thus, there exist two vertices $u \in V(A)$ and $v \in S$ such that $\text{dist}_G(u, v) = 2$. It follows that $n \leq d_G(u) + d_G(v) \leq v(A) - 1 + 2 + v(B) - |S| + 1$, implying that $|S| \leq 2$. Recall that $|S| \geq 2$, so we have $|S| = 2$. Let u be an arbitrary vertex in A . Clearly, there is a vertex $v \in V(B)$ such that $\text{dist}_G(u, v) = 2$. It follows that $n \leq d_G(u) + d_G(v) \leq v(A) - 1 + d_G(u, B) + v(B) - 2 + 1$, implying that $d_G(u, B) \geq 2$. Thus, we have $N_G(u, B) = S$ for every vertex $u \in A$. Recall that the component A has at least three vertices. We conclude that there exists a cycle in F , contradicting the fact that F is a linear forest. This completes the proof of Lemma 3.2.1. $\square$

3.6.2 Proof of Lemma 3.2.2

The proof of this lemma is modelled along similar lines as the proof of Lemma 3.2.1. Let G be a 2-connected graph on n vertices such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of nonadjacent vertices u, v of G . The assertion is trivial if $n \leq 4$. Therefore, we assume that $n \geq 5$. It follows that G is a hamiltonian graph by Theorem 3.6.1. By Lemma 3.6.1, there exists a linear forest F of G such that $N_1(F) = O(G)$ and $e(F) \leq n/2$. Let $G' = G - E(F)$. Thus, we can conclude that $d_{G'}(v) \geq d_G(v) - 2$ and $d_{G'}(v)$ is even for each vertex $v \in V(G)$. If the graph G' is connected, then it is a desired spanning

eulerian subgraph of G . Next, we assume that G' is disconnected. We divide the remainder of the proof into two cases that might occur.

We first consider the case that the graph $G[V(A_0)]$ is not a complete graph for some component A_0 of G' with $v(A_0) \leq n/2$.

Case 1. *There exists a component A_0 of G' with $v(A_0) \leq n/2$ such that the graph $G[V(A_0)]$ is not a complete graph.*

Let $B = G' - A_0$. Note that $O(A_0) = \emptyset$ and the component A_0 is connected and not a complete graph. This implies that $v(A_0) \geq 4$ and hence $n \geq 8$, and that there exist two nonadjacent vertices $u_1, u_2 \in V(A_0)$. Without loss of generality, we suppose that $d_G(u_1) \geq n/2$. It follows that $n/2 \leq d_G(u_1) \leq v(A_0) - 2 + 2 = v(A_0)$. Recall that $v(A_0) \leq n/2$. Therefore, we have $v(A_0) = v(B) = n/2$, $d_G(u_1, A_0) = v(A_0) - 2$ and $d_G(u_1, B) = 2$. Hence, we have $N_G(u_1, A_0) = V(A_0) \setminus \{u_1, u_2\}$.

We claim that the subgraph B is connected. Suppose, to the contrary, that B is disconnected. Since $v(B) = n/2$, there exists a component B_1 of B with $v(B_1) \leq n/4$. Thus, we have $d_G(v) \leq n/4 - 1 + 2 < n/2$ for every vertex $v \in V(B_1)$. Let u be an arbitrary vertex in A_0 . If the vertex $u \in N_G(B_1, A_0)$, then we have $d_G(u, B) \geq 1$. If the vertex $u \notin N_G(B_1, A_0)$, then we can conclude that $n/2 \leq d_G(u) \leq v(A_0) - 1 + d_G(u, B)$, implying that $d_G(u, B) \geq 1$. Recall that $d_G(u_1, B) = 2$. We have $e(F) \geq 2 + (n/2 - 1) > n/2$, a contradiction. Thus as we claimed, the subgraph B is connected.

Let $N_G(u_1, B) = \{v_1, v_2\}$. If $v_1 v_2 \in E(G)$, then we have $v_1 v_2 \in E(G')$; otherwise, the sequence $v_1 u_1 v_2 v_1$ would form a cycle in F . Thus, the subgraph $H = G' + \{u_1 v_1, u_1 v_2\} - \{v_1 v_2\}$ is a desired spanning eulerian subgraph of G . Next, we assume that $v_1 v_2 \notin E(G)$.

We claim that $d_G(v) \geq n/2$ for every vertex $v \in V(B)$. Suppose, to the contrary, that $d_G(v) < n/2$ for some vertex $v \in V(B)$. Let u be an arbitrary vertex in A_0 . If $u \in N_G(v, A_0)$, then we have $d_G(u, B) \geq 1$. If $u \notin N_G(v, A_0)$, then we can conclude that $n/2 \leq d_G(u) \leq v(A_0) - 1 + d_G(u, B)$, implying that $d_G(u, B) \geq 1$. Recall that $d_G(u_1, B) = 2$. We have $e(F) \geq 2 + (n/2 - 1) > n/2$, a contradiction. Thus as we claimed, we have $d_G(v) \geq n/2$ for every vertex $v \in V(B)$.

We have $d_G(v, A_0) \geq 1$ for every vertex $v \in V(B)$ by the fact that $v(B) = n/2$. Moreover, we can conclude that $d_G(v_1, A_0) = d_G(v_2, A_0) = 2$ by the assumption that $v_1 v_2 \notin E(G)$. Hence, we have $e(F) \geq 4 + (n/2 - 2) > n/2$, a contradiction.

This concludes the proof for Case 1. The remaining case is that the graph $G[V(A)]$ is a complete graph for every component A of G' with $v(A) \leq n/2$.

Case 2. *The graph $G[V(A)]$ is a complete graph for every component A of G' with $v(A) \leq n/2$.*

Depending on whether or not every component A of G' with $v(A) \leq n/2$ is a complete graph, we divide Case 2 into two subcases (i.e., Cases 2.1 and 2.2). We first consider the case that there exists a component A_0 of G' with $v(A_0) \leq n/2$ that is not a complete graph.

Case 2.1. *There exists a component A_0 of G' with $v(A_0) \leq n/2$ that is not a complete graph.*

We prove the following claim to show the structure of the component A_0 , for the case that A_0 is not complete.

Claim 1. *The component A_0 satisfies $A_0 = G[V(A_0)] - M$ for some perfect matching M of $G[V(A_0)]$.*

Proof. Recall that $G' = G - E(F)$ and the graph $G[V(A_0)]$ is a complete graph. Hence, $G[V(A_0)]$ is regular. Note that the component A_0 is not a complete graph. If $d_G(u, A_0) - d_{G'}(u, A_0) \equiv 0 \pmod{2}$ for each vertex $u \in V(A_0)$, then the graph induced by $E(F) \cap E(G[V(A_0)])$ contains a cycle, contradicting the fact that F is a linear forest. Thus, there exists a vertex u of A_0 such that $d_{G'}(u, A_0) = d_G(u, A_0) - 1$. Since the graph $G[V(A_0)]$ is regular and every vertex of G' has an even degree, we have $d_{G'}(u, A_0) = d_G(u, A_0) - 1$ for each vertex $u \in V(A_0)$. This confirms our claim. $\square$

Let $B = G' - A_0$. If $v(A_0) = n/2$, then there exists a vertex $u \in V(A_0)$ with $d_G(u, B) = 0$; otherwise, we would have $e(F) \geq n/4 + n/2 > n/2$ by Claim 1. Let v be an arbitrary vertex in $V(B)$. Note that $uv \notin E(G)$ and $d_G(u) \leq v(A_0) - 1 < n/2$. It follows that $n/2 \leq d_G(v) \leq v(B) - 1 + d_G(v, A_0)$,

implying that $d_G(v, A_0) \geq 1$. Thus, we have $e(F) \geq n/4 + n/2 > n/2$, a contradiction. Therefore, we conclude that $v(A_0) < n/2$.

By Claim 1, we have $v(A_0) \geq 4$ and $d_G(u) \leq v(A_0) - 1 + 1 = v(A_0) < n/2$ for every vertex $u \in V(A_0)$. For each vertex $v \in V(B)$, there exists a vertex $u \in V(A_0)$ such that $uv \notin E(G)$. This implies that $d_G(v) \geq n/2$ for every vertex $v \in V(B)$.

If the subgraph B is disconnected, then there exists a component B_1 of B with $v(B_1) \leq v(B)/2$. Recall that $v(A_0) \geq 4$. We have $d_G(v) \leq v(B_1) - 1 + 2 \leq v(B)/2 + 1 = (n - v(A_0))/2 + 1 < n/2$ for every vertex $v \in V(B_1)$, a contradiction. Therefore, we conclude that the subgraph B is connected.

Let $N_G(A_0, B) = S$. Since G is 2-connected, we have $|S| \geq 2$. For two vertices v_1, v_2 of S , let $u_1 \in N_G(v_1, A_0)$ and $u_2 \in N_G(v_2, A_0)$ (note that $u_1 \neq u_2$ by Claim 1). We define a graph $H_1(v_1, v_2)$ such that

$$H_1(v_1, v_2) = \begin{cases} G' + \{u_1v_1, u_2v_2\} - \{u_1u_2\}, & \text{if } u_1u_2 \notin M; \\ G' + \{u_1v_1, u_2v_2, u_1u_2\}, & \text{otherwise.} \end{cases}$$

If there exist two vertices v_1, v_2 of S such that $v_1v_2 \in E(G')$, then the subgraph $H = H_1(v_1, v_2) - \{v_1v_2\}$ is a desired spanning eulerian subgraph of G (see Figure 3.7 (a) and (b)). Next, we assume that the set S is an independent set in G' .

Figure 3.7: Graphs illustrating the case that $v_1v_2 \in E(G')$.

We prove the following claim to show how we can construct a desired spanning eulerian subgraph of G , for the case that the set S is an independent set in G' .

Claim 2. For any two vertices v_1, v_2 of S , one of the following holds:

- (1) there exists a vertex $v_3 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ such that $E_G(v_3) \cap E(F) = \emptyset$;
- (2) there exist two vertices $v_3 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ and $v_4 \in N_{G'}(v_1, B) \cup N_{G'}(v_2, B)$ such that $v_3 v_4 \in E(F)$; or
- (3) there exist two vertices $v_3, v_4 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ such that $v_3 v_5, v_4 v_5 \in E(F)$ for some vertex $v_5 \in V(B) \setminus (N_{G'}(v_1, B) \cup N_{G'}(v_2, B) \cup \{v_1, v_2\})$.

Proof. Let v_1 and v_2 be two vertices of S , and let $S_0 = N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$. Let $S_1 = N_{G'}(v_1, B) \setminus S_0$, $S_2 = N_{G'}(v_2, B) \setminus S_0$ and $S_3 = V(B) \setminus (N_{G'}(v_1, B) \cup N_{G'}(v_2, B) \cup \{v_1, v_2\})$.

If there exists a vertex $v_3 \in S_0$ such that $E_G(v_3) \cap E(F) = \emptyset$, then the assertion (1) holds. Now, we assume that $E_G(v) \cap E(F) \neq \emptyset$ for every vertex $v \in S_0$. If there exists a vertex $v_3 \in S_0$ such that $E_G(v_3, S_3) \cap E(F) = \emptyset$, then the assertion (2) holds. Next, we assume that $E_G(v, S_3) \cap E(F) \neq \emptyset$ for every vertex $v \in S_0$. Recall that $d_G(v) \geq n/2$ for every vertex $v \in V(B)$. We have $n/2 \leq d_G(v_1) \leq |S_1| + |S_0| + 2$ and $n/2 \leq d_G(v_2) \leq |S_2| + |S_0| + 2$. It follows that $|S_1| + |S_0| + |S_2| + |S_0| \geq n - 4$. However, we have $|S_1| + |S_0| + |S_2| + |S_3| \leq n - 6$ by the fact that $v(A_0) \geq 4$. Therefore, we have $|S_3| < |S_0|$. Thus, the assertion (3) holds. $\square$

Figure 3.8: Graphs illustrating (1) and (2) of Claim 2.

Let v_1 and v_2 be two given vertices of S . We first suppose that the assertion (1) of Claim 2 holds. Recall that $d_G(v_i) \geq n/2$ for $i \in \{1, 2, 3\}$.

One can check that $N_{G'}(v_1, B) \cap N_{G'}(v_3, B) \neq \emptyset$. Thus, the subgraph $H = H_1(v_1, v_2) - \{v_1 v_3, v_2 v_3\}$ is a desired spanning eulerian subgraph of G (see Figure 3.8 (a)).

Suppose now that the assertion (2) of Claim 2 holds. Without loss of generality, we suppose that $v_4 \in N_{G'}(v_1, B)$. In that case, the subgraph $H = H_1(v_1, v_2) + \{v_3 v_4\} - \{v_1 v_4, v_2 v_3\}$ is a desired spanning eulerian subgraph of G (see Figure 3.8 (b)).

Finally, we suppose that the assertion (3) of Claim 2 holds. In that case, the subgraph $H = H_1(v_1, v_2) + \{v_3 v_5, v_4 v_5\} - \{v_1 v_4, v_2 v_3\}$ is a desired spanning eulerian subgraph of G (see Figure 3.9).

Figure 3.9: The graph illustrating (3) of Claim 2.

This concludes the proof for Case 2.1. The remaining subcase of Case 2 is that every component A of G' with $v(A) \leq n/2$ is a complete graph.

Case 2.2. Every component A of G' with $v(A) \leq n/2$ is a complete graph.

Depending on whether or not the graph G' has an isolated vertex, we divide Case 2.2 into two subcases (i.e., Cases 2.2.1 and 2.2.2). We first consider the case that G' has at least one isolated vertex.

Case 2.2.1. The graph G' has at least one isolated vertex.

Let u_1 be an isolated vertex of G' , and let $B = G' - u_1$. Clearly, we have $d_G(u_1) = 2$. Let $N_G(u_1, B) = \{u_2, u_3\}$.

We first suppose that $u_2 u_3 \in E(G)$. It follows that $u_2 u_3 \in E(G')$; otherwise, the sequence $u_1 u_2 u_3 u_1$ would form a cycle in F . If the subgraph B is connected, then the subgraph $H = G' + \{u_1 u_2, u_1 u_3\} - \{u_2 u_3\}$ is a desired

spanning eulerian subgraph of G . Next, we assume that the subgraph B is disconnected.

We claim that the subgraph B has exactly two components. Suppose, to the contrary, that the subgraph B has at least three components. Note that neither the vertex u_2 nor the vertex u_3 is an isolated vertex of B by the supposition that $u_2u_3 \in E(G)$. Recall that $n \geq 5$. We conclude that there is no isolated vertex in B ; otherwise, there would be an isolated vertex u of B such that $u_1u \notin E(G)$ and $\max\{d(u_1), d(u)\} \leq 2 < n/2$. Note that there is no component on two vertices in B ; otherwise, there would be a vertex of odd degree in B . Thus, there exists a component B_1 of B with $3 \leq v(B_1) \leq n/2 - 2$. We have $d_G(v) \leq n/2 - 3 + 2 < n/2$ for every vertex $v \in V(B_1)$. Therefore, there is a vertex $v \in V(B_1)$ such that $u_1v \notin E(G)$ and $\max\{d_G(u_1), d_G(v)\} < n/2$, a contradiction. Thus as we claimed, the subgraph B has exactly two components.

Let B_1 and B_2 be the two components of B such that $3 \leq v(B_1) \leq v(B_2)$. Note that $d_G(u_1) < n/2$, and that there exists a vertex $v \in V(B_1)$ such that $u_1v \notin E(G)$. It follows that $n/2 \leq d_G(v) \leq v(B_1) - 1 + 2$, implying that $v(B_1) \geq n/2 - 1$. Therefore, we have $v(B_2) \leq n - 1 - (n/2 - 1) = n/2$. We conclude that the two components B_1 and B_2 are both complete graphs by the supposition that $v(B_1) \leq v(B_2)$. Without loss of generality, we suppose that u_2 and u_3 are in B_1 . Since G is 2-connected, there exist two nonadjacent edges $x_1y_1, x_2y_2 \in E(F)$ with $x_1, x_2 \in V(B_1)$ and $y_1, y_2 \in V(B_2)$. If $\{x_1, x_2\} \cap \{u_2, u_3\} = \emptyset$, then the subgraph $H = G' + \{u_1u_2, u_1u_3, x_1y_1, x_2y_2\} - \{u_2u_3, x_1x_2, y_1y_2\}$ is a desired spanning eulerian subgraph of G (see Figure 3.10). Next, we assume that $\{x_1, x_2\} \cap \{u_2, u_3\} \neq \emptyset$.

If $|\{x_1, x_2\} \cap \{u_2, u_3\}| = 1$, then without loss of generality suppose that $x_1 = u_2$. In that case the subgraph $H = G' + \{u_1u_2, u_1u_3, x_1y_1, x_2y_2\} - \{u_3x_2, y_1y_2\}$ is a desired spanning eulerian subgraph of G (see Figure 3.10). If $\{x_1, x_2\} = \{u_2, u_3\}$, say $x_1 = u_2$ and $x_2 = u_3$, then the subgraph $H = G' + \{u_1u_2, u_1u_3, x_1y_1, x_2y_2\} - \{y_1y_2\}$ is a desired spanning eulerian subgraph of G (see Figure 3.10).

Suppose now that $u_2u_3 \notin E(G)$. Without loss of generality, we suppose that $d_G(u_3) \geq n/2$. Let $G^* = G - \{u_1, u_2\}$, and let u_4 be a fixed vertex of

Figure 3.10: The graph illustrating the case that B is disconnected.

$N_G(u_2, G^*)$. Let S be a subset of $V(G^*)$ such that

$$S = \begin{cases} O(G), & \text{if } d_G(u_2) \equiv 0 \pmod{2}; \\ O(G - \{u_2 u_4\}), & \text{otherwise.} \end{cases}$$

Let v be an arbitrary vertex in $V(G^*) \setminus \{u_3\}$. Since $u_1 v \notin E(G)$, we have $d_G(v) \geq n/2$. Recall that $d_G(u_3) \geq n/2$. Thus, we have $d_G(v) \geq n/2$ for every vertex $v \in V(G) \setminus \{u_1, u_2\}$ and hence $d_{G^*}(v) \geq (n-2)/2 = v(G^*)/2$ for every vertex $v \in V(G^*)$. By Dirac's theorem, the graph G^* has a Hamilton cycle C . We fix an orientation $\vec{C}$ on C . Let $S = \{v_1, \dots, v_k\}$, where these vertices appear in this order along $\vec{C}$. Clearly, we can see that k is even. Let $F_1^* = \bigcup_{i=1}^{k/2} \vec{C}[v_{2i-1}, v_{2i}]$ and $F_2^* = \bigcup_{i=1}^{k/2} \vec{C}[v_{2i}, v_{2i+1}]$, where $v_{k+1} = v_1$. Thus, the subgraph F_i^* is a linear forest of G^* with $N_1(F_i^*) = S$ for $i \in \{1, 2\}$. Therefore, there exists a linear forest F^* of G^* such that $N_1(F^*) = S$ and $u_4 \notin N_2(F^*)$ for the case that $d_G(u_4) \equiv 1 \pmod{2}$.

Now, we let

$$G'' = \begin{cases} G - E(F^*), & \text{if } d_G(u_2) \equiv 0 \pmod{2}; \\ G - (E(F^*) \cup \{u_2 u_4\}), & \text{otherwise.} \end{cases}$$

One can check that $d_{G''}(v) \geq d_G(v) - 2$ and $d_{G''}(v)$ is even for each vertex $v \in V(G)$. If the graph G'' is connected, then it is a desired spanning eulerian subgraph of G . Next, we assume that G'' is disconnected.

If the graph G'' has an isolated vertex v , then we have $v \notin \{u_1, u_2\}$ by

the fact that $u_1u_2 \in E(G'')$. Recall that $n \geq 5$. We have $d_G(v) = 2 < n/2$, contradicting the fact that $d_G(v) \geq n/2$. Therefore, we conclude that G'' has no isolated vertex. We prove another claim to show that G'' has exactly two components.

Claim 3. *The graph G'' has exactly two components.*

Proof. Suppose, to the contrary, that G'' has at least three components. Note that there is no component on two vertices in G'' ; otherwise, there would be a vertex of odd degree in G'' . Thus, there exists a component B_1 of G'' with $3 \leq v(B_1) \leq n/3$, implying that $n \geq 9$. Let v be an arbitrary vertex in $V(B_1) \setminus \{u_1, u_2\}$. Recall that $d_G(v) \geq n/2$ for every vertex $v \in V(G) \setminus \{u_1, u_2\}$. Thus, we have $n/2 \leq d_G(v) \leq n/3 - 1 + 2 < n/2$, a contradiction. $\square$

Let B_1 and B_2 be the two components of G'' such that $\{u_1, u_2\} \subseteq V(B_1)$. Recall that $v(B_1) \geq 3$. Let $B_1^* = B_1 - \{u_1, u_2\}$. If $v(B_2) < n/2 - 1$, then for every vertex $v \in V(B_2)$, we have $d_G(v) \leq v(B_2) - 1 + 2 < n/2$, a contradiction. If $v(B_2) > n/2$, then we have $v(B_1^*) < n/2 - 2$. For every vertex $v \in V(B_1^*)$, we have $d_{G^*}(v) \leq v(B_1^*) - 1 + 2 < n/2 - 1$, a contradiction. Therefore, we conclude that $n/2 - 1 \leq v(B_2) \leq n/2$ and $n/2 - 2 \leq v(B_1^*) \leq n/2 - 1$.

We first suppose that the component B_2 is a complete graph. If there exists a vertex $v \in V(B_1^*)$ such that $d_{G^*}(v, B_2) = 2$, say $N_{G^*}(v, B_2) = \{v', v''\}$, then the subgraph $H = G'' + \{vv', vv''\} - \{v'v''\}$ is a desired spanning eulerian subgraph of G . Next, we assume that $d_{G^*}(v, B_2) \leq 1$ for every vertex $v \in V(B_1^*)$. If the subgraph $G[V(B_1^*)]$ is not complete, then there exists a vertex $v \in V(B_1^*)$ such that $d_{G^*}(v) \leq v(B_1^*) - 2 + 1 \leq n/2 - 1 - 2 + 1 < n/2 - 1$, a contradiction. Therefore, we conclude that the subgraph $G[V(B_1^*)]$ is a complete graph.

If $v(B_1^*) = 1$, then the component B_1 is a triangle and $V(B_1) = \{u_1, u_2, u_3\}$, contradicting the supposition that $u_2u_3 \notin E(G)$. Now, we assume that $v(B_1^*) \geq 2$. For every vertex $v \in V(B_1^*)$, we have $n/2 - 1 \leq d_{G^*}(v) \leq v(B_1^*) - 1 + d_{G^*}(v, B_2) \leq n/2 - 1 - 1 + d_{G^*}(v, B_2)$. This implies that $d_{G^*}(v, B_2) \geq 1$ for every vertex $v \in V(B_1^*)$. Recall that $d_{G^*}(v, B_2) \leq 1$ for every vertex $v \in V(B_1^*)$. Thus, we have $d_{G^*}(v, B_2) = 1$ for every vertex $v \in V(B_1^*)$. Let v_1 and v_2 be two vertices in $V(B_1^*)$, and let $v'_i \in N_{G^*}(v_i, B_2)$ for $i \in \{1, 2\}$. We define a

graph $H(v_1, v_2)$ such that

$$H(v_1, v_2) = \begin{cases} G'' + \{v'_1 v_1, v'_1 v_2\}, & \text{if } v'_1 = v'_2; \\ G'' + \{v'_1 v_1, v'_2 v_2\} - \{v'_1 v'_2\}, & \text{otherwise.} \end{cases}$$

Now, we let

$$H = \begin{cases} H(v_1, v_2) + \{v_1 v_2\}, & \text{if } v_1 v_2 \in E(F^*); \\ H(v_1, v_2) - \{v_1 v_2\}, & \text{otherwise.} \end{cases}$$

Thus, the subgraph H is a desired spanning eulerian subgraph of G .

Suppose now that the component B_2 is not a complete graph. Let v_0 be a vertex of B_2 such that $d_{G''}(v_0, B_2) \leq v(B_2) - 2$. We have $n/2 \leq d_G(v_0) \leq d_{G''}(v_0, B_2) + d_G(v_0, F^*) \leq v(B_2) - 2 + 2 \leq n/2 - 2 + 2 = n/2$. This implies that $v(B_2) = n/2$ and hence $v(B_1^*) = n/2 - 2$.

We claim that the subgraph B_1^* is a complete graph. Suppose, to the contrary, that the subgraph B_1^* is not complete. Let v' be a vertex of B_1^* such that $d_{G''}(v', B_1^*) \leq v(B_1^*) - 2$. Recall that $v(B_1^*) = n/2 - 2$ and $d_{G^*}(v') \geq n/2 - 1$. We have $n/2 - 1 \leq d_{G^*}(v') \leq d_{G''}(v', B_1^*) + d_G(v', F^*) \leq v(B_1^*) - 2 + 2 \leq n/2 - 2 - 2 + 2 < n/2 - 1$, a contradiction. Thus as we claimed, the subgraph B_1^* is a complete graph.

Let v be an arbitrary vertex in $V(B_1^*)$. Based on the fact that $d_{G^*}(v) \geq n/2 - 1$, we have $n/2 - 1 \leq d_{G^*}(v) \leq v(B_1^*) - 1 + d_{G^*}(v, B_2) = n/2 - 3 + d_{G^*}(v, B_2)$, implying that $d_{G^*}(v, B_2) = 2$. Let $N_{G^*}(v_0, B_2) = \{v_1, v_2\}$ for a given vertex $v_0 \in V(B_1^*)$. If $v_1 v_2 \in E(G)$, then we have $v_1 v_2 \in E(B_2)$; otherwise, the sequence $v_0 v_1 v_2 v_0$ would form a cycle in F^* . Thus, the graph $H = G'' + \{v_0 v_1, v_0 v_2\} - \{v_1 v_2\}$ is a desired spanning eulerian subgraph of G . Next, we assume that $v_1 v_2 \notin E(G)$.

Recall that $v(B_2) = n/2$ and $d_G(v_i) \geq n/2$ for $i \in \{1, 2\}$. We have $n/2 \leq d_G(v_i) \leq d_{G''}(v_i, B_2) + d_G(v_i, F^*) \leq v(B_2) - 2 + 2 = n/2 - 2 + 2 = n/2$. This implies that $d_{G''}(v_i, B_2) = n/2 - 2$ and $d_G(v_i, F^*) = 2$ for $i \in \{1, 2\}$. Note that $v_1 v_2 \notin E(G)$ and $v(B_2) = n/2$. Thus, we have $N_{G''}(v_i, B_2) = V(B_2) \setminus \{v_1, v_2\}$ and $d_G(v_i, B_1^*) = 2$ for $i \in \{1, 2\}$. Let $N_G(v_1, B_1^*) = \{v_0, v'_0\}$. Recall that the subgraph B_1^* is a complete graph. Thus, the graph $H = G'' + \{v_1 v_0, v_1 v'_0\} -$

$\{v_0v'_0\}$ is a desired spanning eulerian subgraph of G .

This concludes the proof for Case 2.2.1. The remaining subcase of Case 2.2 is that the graph G' has no isolated vertex.

Case 2.2.2. *The graph G' has no isolated vertex.*

We prove several claims in order to deal with Case 2.2.2.

Claim 4. *The graph G' has exactly two components.*

Proof. Suppose, to the contrary, that G' has at least three components. Let A and B be the smallest two components of G' such that $v(A) \leq v(B)$, and let $C = G' - (A \cup B)$. Thus, we have $v(B) \leq (n - v(A))/2$. Since every vertex of G' has an even degree, we have $3 \leq v(A) \leq n/3$, implying that $n \geq 9$. Let v be a given vertex of B . Clearly, there exists a vertex $u \in V(A)$ such that $uv \notin E(G)$ and $d_G(u) \leq n/3 - 1 + 2 < n/2$. It follows that $n/2 \leq d_G(v) \leq v(B) - 1 + 2 \leq (n - v(A))/2 + 1 \leq (n - 3)/2 + 1 < n/2$, a contradiction. $\square$

Let A and B be the two components of G' such that $v(A) \leq v(B)$. If $v(A) = n/2$, then we have $v(B) = n/2$. In that case, the two components A and B are both complete graphs in G' . Since G is 2-connected, there exist two nonadjacent edges $u_1v_1, u_2v_2 \in E(F)$ with $u_1, u_2 \in V(A)$ and $v_1, v_2 \in V(B)$. Thus, the subgraph $H = G' + \{u_1v_1, u_2v_2\} - \{u_1u_2, v_1v_2\}$ is a desired spanning eulerian subgraph of G (see Figure 3.11).

Figure 3.11: The graph illustrating the case that $v(A) = v(B) = n/2$.

Now, we assume that $3 \leq v(A) < n/2$, implying that $n \geq 7$. Recall that the component A is a complete graph. If there exist a vertex $v \in V(B)$ such that

$d_G(v, A) = 2$, say $N_G(v, A) = \{u_1, u_2\}$, then the subgraph $H = G' + \{vu_1, vu_2\} - \{u_1u_2\}$ is a desired spanning eulerian subgraph of G . Next, we assume that $d_G(v, A) \leq 1$ for every vertex $v \in V(B)$.

Let $N_G(A, B) = S$. Since G is 2-connected, we have $|S| \geq 2$. Let v_1 and v_2 be two vertices of S such that $u_1 \in N_G(v_1, A)$ and $u_2 \in N_G(v_2, A)$. We define a graph $H_2(v_1, v_2)$ such that

$$H_2(v_1, v_2) = \begin{cases} G' + \{u_1v_1, u_1v_2\}, & \text{if } u_1 = u_2; \\ G' + \{u_1v_1, u_2v_2\} - \{u_1u_2\}, & \text{otherwise.} \end{cases}$$

If there exist two vertices v_1, v_2 of S such that $v_1v_2 \in E(G')$, then the subgraph $H = H_2(v_1, v_2) - \{v_1v_2\}$ is a desired spanning eulerian subgraph of G (see Figure 3.12 (a) and (b)). Next, let us consider the case that the set S is an independent set in G' .

Figure 3.12: Graphs illustrating the case that $v_1v_2 \in E(G')$.

We prove the following claim to show that every vertex $v \in V(B)$ has a large degree in G .

Claim 5. *We have $d_G(v) \geq n/2$ for every vertex $v \in V(B)$.*

Proof. We first suppose that $(n-2)/2 \leq v(A) \leq (n-1)/2$. If there exists a vertex $u \in V(A)$ with $d_G(u, B) = 0$, then we have $d_G(u) \leq v(A) - 1 < n/2$. This implies that $d_G(v) \geq n/2$ for every vertex $v \in V(B)$. If $d_G(u, B) \geq 1$ for every $u \in V(A)$, then there exists at most one vertex $u_1 \in V(A)$ such that $d_G(u, B) = 1$ for every vertex $u \in V(A) \setminus \{u_1\}$; otherwise, we would have $e(F) > n/2$. This implies that $d_G(u) < n/2$ for every vertex $u \in V(A) \setminus \{u_1\}$. Based on the fact that $v(A) \geq 3$ and the assumption that $d_G(v, A) \leq 1$, we

can conclude that there exists a vertex $u \in V(A)$ such that $d_G(u) < n/2$ and $uv \notin E(G)$ for every vertex $v \in V(B)$. Thus, we have $d_G(v) \geq n/2$ for every vertex $v \in V(B)$.

Suppose now that $3 \leq v(A) \leq (n-3)/2$. We have $d_G(u) < n/2$ for every vertex $u \in V(A)$. We can conclude that there exists a vertex $u \in V(A)$ such that $d_G(u) < n/2$ and $uv \notin E(G)$ for every vertex $v \in V(B)$. Thus, we have $d(v) \geq n/2$ for every vertex $v \in V(B)$. This confirms our claim. $\square$

We prove the following claim to show how we can construct a desired spanning eulerian subgraph of G , for the case that the set S is an independent set in G' .

Claim 6. *For any two vertices v_1, v_2 of S , one of the following holds:*

- (1) *there exists a vertex $v_3 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ such that $E_G(v_3) \cap E(F) = \emptyset$;*
- (2) *there exist two vertices $v_3 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ and $v_4 \in N_{G'}(v_1, B) \cup N_{G'}(v_2, B)$ such that $v_3 v_4 \in E(F)$; or*
- (3) *there exist two vertices $v_3, v_4 \in N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$ such that $v_3 v_5, v_4 v_5 \in E(F)$ for some vertex $v_5 \in V(B) \setminus (N_{G'}(v_1, B) \cup N_{G'}(v_2, B) \cup \{v_1, v_2\})$.*

Proof. Let v_1 and v_2 be two vertices of S , and let $S_0 = N_{G'}(v_1, B) \cap N_{G'}(v_2, B)$. Let $S_1 = N_{G'}(v_1, B) \setminus S_0$, $S_2 = N_{G'}(v_2, B) \setminus S_0$ and $S_3 = V(B) \setminus (N_{G'}(v_1, B) \cup N_{G'}(v_2, B) \cup \{v_1, v_2\})$.

If there exists a vertex $v_3 \in S_0$ such that $E_G(v_3) \cap E(F) = \emptyset$, then the assertion (1) holds. Now, we assume that $E_G(v) \cap E(F) \neq \emptyset$ for every vertex $v \in S_0$. If there exists a vertex $v_3 \in S_0$ such that $E_G(v_3, S_3) \cap E(F) = \emptyset$, then the assertion (2) holds. Next, we assume that $E_G(v, S_3) \cap E(F) \neq \emptyset$ for every vertex $v \in S_0$. By Claim 5, we have $n/2 \leq d_G(v_1) \leq |S_1| + |S_0| + 2$, and $n/2 \leq d_G(v_2) \leq |S_2| + |S_0| + 2$. It follows that $|S_1| + |S_0| + |S_2| + |S_0| \geq n - 4$. However, we have $|S_1| + |S_0| + |S_2| + |S_3| \leq n - 5$ by the fact of $v(A) \geq 3$. Therefore, we have $|S_3| < |S_0|$. Thus, the assertion (3) holds. $\square$

Let v_1 and v_2 be two given vertices of S . We first suppose that the assertion (1) of Claim 6 holds. Recall that $d_G(v_i) \geq n/2$ for $i \in \{1, 2, 3\}$.

Figure 3.13: The graph illustrating (1) of Claim 6.

One can check that $N_{G'}(v_1, B) \cap N_{G'}(v_3, B) \neq \emptyset$. In that case, the subgraph $H = H_2(v_1, v_2) - \{v_1 v_3, v_2 v_3\}$ is a desired spanning eulerian subgraph of G (see Figure 3.13).

Suppose now that the assertion (2) of Claim 6 holds. Without loss of generality, we suppose that $v_4 \in N_{G'}(v_1, B)$. In that case, the subgraph $H = H_2(v_1, v_2) + \{v_3 v_4\} - \{v_1 v_4, v_2 v_3\}$ is a desired spanning eulerian subgraph of G (see Figure 3.14 (a)).

Figure 3.14: Graphs illustrating (2) and (3) of Claim 6.

Finally, we suppose that the assertion (3) of Claim 6 holds. In that case, the subgraph $H = H_2(v_1, v_2) + \{v_3 v_5, v_4 v_5\} - \{v_1 v_4, v_2 v_3\}$ is a desired spanning eulerian subgraph of G (see Figure 3.14 (b)). This completes the proof of Lemma 3.2.2. $\square$

3.6.3 Proof of Lemma 3.3.1

The proof of this lemma is modelled along similar lines as the proofs of the above two lemmas. Let G be a 2-connected graph on n vertices with $\delta(G) \geq 3$ such that $\max\{d(u), d(v)\} \geq n/2$ for every pair of vertices u, v of G with $\text{dist}(u, v) = 2$. It follows that G is a hamiltonian graph by Theorem 3.6.1. If $n \leq 5$, then a Hamilton cycle of G is a desired spanning eulerian subgraph of G . Next, we assume that $n \geq 6$. By Lemma 3.6.1, there exists a linear forest F of G such that $N_1(F) = O(G)$ and $e(F) \leq n/2$. Let $G' = G - E(F)$. Thus, we can conclude that $d_{G'}(v) \geq d_G(v) - 2$ and $d_{G'}(v)$ is even for each vertex $v \in V(G)$. If the graph G' is connected, then it is a desired spanning eulerian subgraph of G . Next, we assume that G' is disconnected. Note that there is no component A with $v(A) = 2$ in G' ; otherwise, the component A would be a complete graph on two vertices, contradicting the fact that $O(A) = \emptyset$. Since $\delta(G) \geq 3$, there is no isolated vertex in G' . Therefore, we have $v(A) \geq 3$ for every component A of G' . We divide the remainder of the proof into two cases that might occur.

We first consider the case that the graph $G[V(A_0)]$ is not a complete graph for some component A_0 of G' with $v(A_0) \leq n/2$.

Case 1. *There exists a component A_0 of G' with $v(A_0) \leq n/2$ such that the graph $G[V(A_0)]$ is not a complete graph (see Figure 3.15).*

Figure 3.15: The graph illustrating Case 1.

Let $B = G' - A_0$. Note that $O(A_0) = \emptyset$ and the component A_0 is connected and not a complete graph. This implies that $v(A_0) \geq 4$ and hence $n \geq 8$, and that there exist two vertices $u_1, u_2 \in V(A_0)$ with $\text{dist}_G(u_1, u_2) = 2$. Without

loss of generality, we suppose that $d_G(u_1) \geq n/2$. It follows that $n/2 \leq d_G(u_1) \leq v(A_0) - 2 + 2 = v(A_0)$. Recall that $v(A_0) \leq n/2$. Therefore, we have $v(A_0) = v(B) = n/2$, $d_G(u_1, A_0) = v(A_0) - 2$ and $d_G(u_1, B) = 2$. Hence, we have $N_{G'}(u_1, A_0) = V(A_0) \setminus \{u_1, u_2\}$, and we let $N_G(u_1, B) = \{v_1, v_2\}$.

We first prove the following claim to show the subgraph B is connected.

Claim 1. *The subgraph B is connected.*

Proof. Suppose, to the contrary, that the subgraph B is disconnected. Recall that $v(B) = n/2$ and $v(A) \geq 3$ for every component A of G' . We have $3 \leq v(D) \leq n/2 - 3$ for every component D of B . This implies that $d_G(v_i) \leq n/2 - 3 - 1 + 2 < n/2$ for $i \in \{1, 2\}$. Note that there exists a vertex $v \in \{v_1, v_2\}$ such that $uv \notin E(G)$ for every vertex $u \in V(A_0) \setminus \{u_1, u_2\}$; otherwise, the sequence $u_1 v_1 u v_2 u_1$ would form a cycle in F . Recall that $N_{G'}(u_1, A_0) = V(A_0) \setminus \{u_1, u_2\}$. We can conclude that there exists a vertex $v \in \{v_1, v_2\}$ such that $\text{dist}_G(u, v) = 2$ for every vertex $u \in V(A_0) \setminus \{u_1, u_2\}$. It follows from $d_G(v) < n/2$ that $d_G(u) \geq n/2$ for every vertex $u \in V(A_0) \setminus \{u_1, u_2\}$. Recall that $e(F) \leq n/2$ and $d_G(u_1, B) = 2$. It follows from $v(A_0) = n/2$ that $d_G(u, B) = 1$ and $N_{G'}(u, A) = V(A_0) \setminus \{u\}$ for every vertex $u \in V(A_0) \setminus \{u_1, u_2\}$. This implies that $uu_2 \in E(G')$ for every vertex $u \in V(A_0) \setminus \{u_1, u_2\}$.

Let $N_G(u, B) = \{v'\}$ for a given vertex $u \in V(A_0) \setminus \{u_1, u_2\}$. If $v'u_2 \in E(G)$, then we have $e(F) \geq n/2 - 2 + 2 + 1 > n/2$, a contradiction. Now, we assume that $v'u_2 \notin E(G)$. We have $\text{dist}_G(v', u_2) = 2$. Note that $d_G(v') \leq n/2 - 3 - 1 + 2 < n/2$. It follows that $n/2 \leq d_G(u_2) \leq v(A_0) - 2 + d_G(u_2, B)$, implying that $d_G(u_2, B) \geq 2$. Therefore, we have $e(F) \geq n/2 - 2 + 2 + 2 > n/2$, a contradiction. $\square$

If $v_1 v_2 \in E(G)$, then we have $v_1 v_2 \in E(G')$; otherwise, the sequence $u_1 v_1 v_2 u_1$ would form a cycle in F . Thus, the graph $H = G' + \{u_1 v_1, u_1 v_2\} - \{v_1 v_2\}$ is a desired spanning eulerian subgraph of G (see Figure 3.15). Next, we assume that $v_1 v_2 \notin E(G)$.

If $d_G(v_1) \geq n/2$ and $d_G(v_2) \geq n/2$, then we have $d_G(v_1, A_0) = d_G(v_2, A_0) = 2$. Let $N_G(v_1, A_0) = \{u_1, u_3\}$ and $N_G(v_2, A_0) = \{u_1, u_4\}$. We have $u_3 \neq u_4$; otherwise the sequence $u_1 v_1 u_3 v_2 u_1$ would form a cycle in F . Without loss of generality, we suppose that $u_3 \neq u_2$. Note that $u_1 u_3 \in E(G')$. Thus, the graph

$H = G' + \{v_1u_1, v_1u_3\} - \{u_1u_3\}$ is a desired spanning eulerian subgraph of G (see Figure 3.16). Next, we assume that there exists a vertex $v \in \{v_1, v_2\}$ such that $d_G(v) < n/2$. Without loss of generality, we suppose that $d_G(v_1) < n/2$.

Figure 3.16: The graph illustrating the case that $d_G(v_i) \geq n/2$ for $i \in \{1, 2\}$.

If there exists a vertex $u \in V(A_0) \setminus \{u_1, u_2\}$ such that $uv_1 \in E(G)$ or $uv_2 \in E(G)$, then the graph $H = G' + \{v_1u_1, v_1u\} - \{u_1u\}$ or $H = G' + \{v_2u_1, v_2u\} - \{u_1u\}$ is a desired spanning eulerian subgraph of G (see Figure 3.16).

Next, we assume that $uv_1 \notin E(G)$ and $uv_2 \notin E(G)$ for every vertex $u \in V(A_0) \setminus \{u_1, u_2\}$. It follows from $\text{dist}_G(v_1, v_2) = 2$ and $d_G(v_1) < n/2$ that $n/2 \leq d_G(v_2) \leq d_{G'}(v_2, B) + 2$, implying that $d_{G'}(v_2, B) = n/2 - 2$ and hence $N_{G'}(v_2, B) = V(B) \setminus \{v_1, v_2\}$. Let u be an arbitrary vertex in $V(A_0) \setminus \{u_1, u_2\}$. It follows from $\text{dist}_G(v_1, u) = 2$ and $d_G(v_1) < n/2$ that $n/2 \leq d_G(u) \leq v(A_0) - 1 + d_G(u, B)$, implying that $d_G(u, B) \geq 1$. Let $v' \in N_G(u, B)$ for a given vertex $u \in V(A_0) \setminus \{u_1, u_2\}$. We have $v_2v' \in E(G')$ by the fact that $N_{G'}(v_2, B) = V(B) \setminus \{v_1, v_2\}$. For any two vertices $u_3, u_4 \in V(A_0) \setminus \{u_1, u_2\}$ such that $u_3u_4 \notin E(G')$, one can check that $|N_{G'}(u_3, A_0) \cap N_{G'}(u_4, A_0)| \geq 2$ by the facts that $d_G(u_i) \geq n/2$ for $i \in \{3, 4\}$ and $n \geq 8$. Therefore, the subgraph $H = G' + \{u_1v_2, uv'\} - \{u_1u, v_2v'\}$ is a desired spanning eulerian subgraph of G (see Figure 3.17).

This concludes the proof for Case 1. The remaining case is that the graph $G[V(A)]$ is a complete graph for every component A of G' with $v(A) \leq n/2$.

Case 2. The graph $G[V(A)]$ is a complete graph for every component A of G' with $v(A) \leq n/2$.

Figure 3.17: The graph illustrating the case that $d_G(v_1) < n/2$.

We prove the following claim to show that there exists a large component in G' .

Claim 2. *There exists a component A with $v(A) \geq n/2$ in G' .*

Proof. Suppose, to the contrary, that every component A of G' satisfies that $3 \leq v(A) < n/2$. This implies that $n \geq 7$. It follows that there exists a component A' of G' with $v(A') = \lfloor (n-1)/2 \rfloor$; otherwise, we would have $d_G(v) \leq \lfloor (n-1)/2 \rfloor - 1 - 1 + 2 < n/2$ for every vertex $v \in V(G)$. Based on the fact that $v(A) \geq 3$ for every component A of G' , we have $v(A) \leq n/2 - 2$ for every component A of G' , except for A' . Let $B = G' - A'$, and let u be an arbitrary vertex of A' . Recall that the graph $G[V(A')]$ is a complete graph. Thus, there exists a vertex v of B such that $\text{dist}_G(u, v) = 2$. Note that $d_G(v) \leq n/2 - 2 - 1 + 2 < n/2$. It follows that $n/2 \leq d_G(u) \leq \lfloor (n-1)/2 \rfloor - 1 + d_G(u, B)$, implying that $d_G(u, B) = 2$. Therefore, we have $e(F) \geq 2\lfloor (n-1)/2 \rfloor > n/2$, a contradiction. $\square$

Let A_0 be a component of G' such that $v(A_0) \geq n/2$.

The next claim deals with the case that some smaller component of G' is not complete.

Claim 3. *If a component A of G' with $v(A) \leq n/2$ is not a complete graph, then the component A satisfies $A = G[V(A)] - M$ for some perfect matching M of $G[V(A)]$.*

Proof. Recall that $G' = G - E(F)$ and the graph $G[V(A)]$ is a complete graph. Hence, $G[V(A)]$ is regular. Note that the component A is not a complete graph. If $d_G(u, A) - d_{G'}(u, A) \equiv 0 \pmod{2}$ for each vertex $u \in V(A)$, then the graph induced by $E(F) \cap E(G[V(A)])$ contains a cycle, contradicting the fact that F is a linear forest. Thus, there exists a vertex u of A such that $d_{G'}(u, A) = d_G(u, A) - 1$. Since $G[V(A)]$ is regular and every vertex of G' has an even degree, we have $d_{G'}(u, A) = d_G(u, A) - 1$ for each vertex $u \in V(A)$. This confirms our claim. $\square$

If there exists another component A_1 of G' with $v(A_1) \geq n/2$, then we have $v(A_0) = v(A_1) = n/2$. In that case, the graphs $G[V(A_0)]$ and $G[V(A_1)]$ are both complete graphs. Let $N_G(A_1, A_0) = U$. Since G is 2-connected, we have $|U| \geq 2$. For two different vertices $u_1, u_2 \in U$, let $v_1 \in N_G(u_1, A_1)$ and $v_2 \in N_G(u_2, A_1)$. For this case, we perform the operation: $G' + \{u_1 v_1, u_2 v_2\} \oplus \{u_1 u_2, v_1 v_2\}$ if $v_1 \neq v_2$, and we perform the operation: $G' + \{u_1 v_1, u_2 v_1\} \oplus \{u_1 u_2\}$ otherwise. Clearly, the resulting graph is a desired spanning eulerian subgraph of G (see Figure 3.18 (a) and (b)).

Figure 3.18: Graphs illustrating the case that $v(A_0) = v(A_1) = n/2$.

Next, we assume that every component $A \neq A_0$ of G' satisfies $v(A) < n/2$. Throughout the remainder of the proof, we suppose that the graph G' has k ($k \geq 1$) components of order less than $n/2$. The following two additional claims deal with the case that there exist at least two components of order less than $n/2$ in G' .

Claim 4. *If there exist two components A_1, A_2 of G' with $v(A_i) < n/2$ for $i \in \{1, 2\}$, then we have $E_G(A_1, A_2) = \emptyset$.*

Proof. Suppose, to the contrary, that there exist two components A_1, A_2 with the requested properties such that $E_G(A_1, A_2) \neq \emptyset$. By Claim 2 and the fact that $v(A) \geq 3$ for every component A of G' , we have $3 \leq v(A_i) \leq n/2 - 3$ for $i \in \{1, 2\}$. Clearly, there exist two vertices u_1 and u_2 with $u_i \in V(A_i)$ ($i \in \{1, 2\}$) and $\text{dist}_G(u_1, u_2) = 2$. However, we have $d_G(u_1) \leq v(A_1) - 1 + 2 \leq n/2 - 3 - 1 + 2 < n/2$ and $d_G(u_2) \leq v(A_2) - 1 + 2 \leq n/2 - 3 - 1 + 2 < n/2$, a contradiction. $\square$

Claim 5. *If there exist two components A_1, A_2 of G' with $v(A_i) < n/2$ and $N_G(A_i, A_0) = U_i$ for $i \in \{1, 2\}$, then we have $U_1 \cap U_2 = \emptyset$.*

Proof. Suppose, to the contrary, that there exist two components A_1, A_2 with the requested properties such that $U_1 \cap U_2 \neq \emptyset$. By Claim 2 and the fact that $v(A) \geq 3$ for every component A of G' , we have $3 \leq v(A_i) \leq n/2 - 3$ for $i \in \{1, 2\}$. Clearly, by Claim 4, there exist two vertices u_1 and u_2 with $u_i \in V(A_i)$ ($i \in \{1, 2\}$) and $\text{dist}_G(u_1, u_2) = 2$. However, we have $d_G(u_1) \leq v(A_1) - 1 + 2 \leq n/2 - 3 - 1 + 2 < n/2$ and $d_G(u_2) \leq v(A_2) - 1 + 2 \leq n/2 - 3 - 1 + 2 < n/2$, a contradiction. $\square$

Figure 3.19: The structure of a spanning subgraph of G .

Based on Claims 4 and 5 and the condition that G is 2-connected, we can conclude that the graph G has a spanning subgraph as depicted in Figure 3.19.

Next, for each component of G' of order less than $n/2$, we will perform one operation of the type $H + E_1 \oplus E_2$ for some subgraph H of G and two given subsets E_1, E_2 of $E(G)$ such that the graph $G[E_2 \cap E(G[V(A_0)])]$ is a path.

For each component A_i ($i \in \{1, \dots, k\}$) of G' of order less than $n/2$, let $N_G(A_i, A_0) = U_i$. Initially, we let $H = G'$ and $R_0 = \emptyset$. We first consider the following discussion on the different situations that might occur for the component A_1 of G' of order less than $n/2$.

Since G is 2-connected, we have $|U_1| \geq 2$. We first suppose that there exist two vertices $u_1^1, u_2^1 \in U_1$ such that $u_1^1 u_2^1 \in E(G)$. Let $v_1^1 \in N_G(u_1^1, A_1)$ and $v_2^1 \in N_G(u_2^1, A_1)$ (where possibly $v_1^1 = v_2^1$). For this case, we perform the operation $H + \{u_1^1 v_1^1, u_2^1 v_2^1\} \oplus \{u_1^1 u_2^1, v_1^1 v_2^1\}$ if $v_1^1 \neq v_2^1$; otherwise, we perform the operation $H + \{u_1^1 v_1^1, u_2^1 v_1^1\} \oplus \{u_1^1 u_2^1\}$.

Suppose now that the set U_1 is an independent set in G . If there exists a vertex $u^1 \in U_1$ such that $d_G(u^1, A_1) = 2$, say $N_G(u^1, A_1) = \{v_3^1, v_4^1\}$, then we perform the operation $H + \{u^1 v_3^1, u^1 v_4^1\} \oplus \{v_3^1 v_4^1\}$. Next, we assume that $d_G(u^1, A_1) = 1$ for every vertex $u^1 \in U_1$. We prove the following claim to show that every vertex $u^1 \in U_1$ has a large degree in G .

Claim 6. *We have $d_G(u^1) \geq n/2$ for every vertex $u^1 \in U_1$.*

Proof. Let u^1 be an arbitrary vertex of U_1 . Recall that the component A_1 of G' satisfies $3 \leq v(A_1) < n/2$. This implies that $n \geq 7$.

We first suppose that the component A_1 of G' satisfies $v(A_1) = \lfloor (n-1)/2 \rfloor$. We claim that there exists a vertex $v^1 \in V(A_1)$ with $\text{dist}_G(u^1, v^1) = 2$ such that $d_G(v^1) < n/2$. Suppose, to the contrary, that we have $d_G(v^1) \geq n/2$ for every vertex $v^1 \in V(A_1)$ with $\text{dist}_G(u^1, v^1) = 2$. Recall that the graph $G[V(A_1)]$ is complete and $d_G(u^1, A_1) = 1$. We conclude that $\text{dist}_G(u^1, v^1) = 2$ for every vertex $v^1 \in V(A_1) \setminus N_G(u^1, A_1)$. It follows that $n/2 \leq d_G(v^1) \leq \lfloor (n-1)/2 \rfloor - 1 + d_G(v^1, A_0)$, implying that $d_G(v^1, A_0) = 2$ for every vertex $v^1 \in V(A_1) \setminus N_G(u^1, A_1)$. Thus, we have $e(F) \geq 2(\lfloor (n-1)/2 \rfloor - 1) + 1 > n/2$,

a contradiction. Thus as we claimed, there exists a vertex $v^1 \in V(A_1)$ with $\text{dist}_G(u^1, v^1) = 2$ such that $d_G(v^1) < n/2$. Therefore, we have $d_G(u^1) \geq n/2$ for every vertex $u^1 \in U_1$.

Suppose now that the component A_1 of G' satisfies $3 \leq v(A_1) \leq \lfloor (n-1)/2 \rfloor - 1$. This implies that $d_G(v^1) < n/2$ for every vertex $v^1 \in V(A_1)$. It follows that there exists a vertex v^1 of A_1 such that $\text{dist}_G(u^1, v^1) = 2$ for each vertex $u^1 \in U_1$. Therefore, we have $d_G(u^1) \geq n/2$ for every vertex $u^1 \in U_1$. This confirms our claim. $\square$

We prove another claim to show how we can perform one effective operation for the component A_1 , for the case that the set U_1 is an independent set in G and $d_G(u^1, A_1) = 1$ for every vertex $u^1 \in U_1$.

Claim 7. *For any two vertices u_1^1, u_2^1 of U_1 , one of the following holds:*

- (1) *there exists a vertex $u_3^1 \in N_{G'}(u_1^1, A_0) \cap N_{G'}(u_2^1, A_0)$ such that $E_G(u_3^1) \cap E(F) = \emptyset$;*
- (2) *there exist two vertices $u_3^1 \in N_{G'}(u_1^1, A_0) \cap N_{G'}(u_2^1, A_0)$ and $u_4^1 \in N_{G'}(u_1^1, A_0) \cup N_{G'}(u_2^1, A_0)$ such that $u_3^1 u_4^1 \in E(F)$; or*
- (3) *there exist two vertices $u_3^1, u_4^1 \in N_{G'}(u_1^1, A_0) \cap N_{G'}(u_2^1, A_0)$ such that $u_3^1 u_5^1, u_4^1 u_5^1 \in E(F)$ for some vertex $u_5^1 \in V(A_0) \setminus (N_{G'}(u_1^1, A_0) \cup N_{G'}(u_2^1, A_0) \cup \{u_1^1, u_2^1\})$.*

Proof. Let u_1^1 and u_2^1 be two vertices of U_1 , and let $S_0 = N_{G'}(u_1^1, A_0) \cap N_{G'}(u_2^1, A_0)$. Let $S_1 = N_{G'}(u_1^1, A_0) \setminus S_0$, $S_2 = N_{G'}(u_2^1, A_0) \setminus S_0$ and $S_3 = V(A_0) \setminus (N_{G'}(u_1^1, A_0) \cup N_{G'}(u_2^1, A_0) \cup \{u_1^1, u_2^1\})$.

If there exists a vertex $u_3^1 \in S_0$ such that $E_G(u_3^1) \cap E(F) = \emptyset$, then the assertion (1) holds. Now, we assume that $E_G(u^1) \cap E(F) \neq \emptyset$ for every vertex $u^1 \in S_0$. If there exists a vertex $u_3^1 \in S_0$ such that $E_G(u_3^1, S_3) \cap E(F) = \emptyset$, then the assertion (2) holds. Next, we assume that $E_G(u^1, S_3) \cap E(F) \neq \emptyset$ for every vertex $u^1 \in S_0$. By Claim 6 and the facts that $d_F(u_j^1) \leq 2$ for $j \in \{1, 2\}$, we have $n/2 \leq d_G(u_1^1) \leq |S_1| + |S_0| + 2$ and $n/2 \leq d_G(u_2^1) \leq |S_2| + |S_0| + 2$. It follows that $|S_1| + |S_0| + |S_2| + |S_0| \geq n - 4$. However, we have $|S_1| + |S_0| + |S_2| + |S_3| \leq n - 3k - 2$ by the fact that the graph G' has k

($k \geq 1$) components of order less than $n/2$ and $v(A) \geq 3$ for every component A of G' . Therefore, we have $|S_3| < |S_0|$. Thus, the assertion (3) holds. This confirms our claim. $\square$

Recall that the graph G is 2-connected, and that $d_G(u^1, A_1) = 1$ for every vertex $u^1 \in U_1$. This implies that $|U_1| \geq 2$ and $|N_G(U_1, A_1)| \geq 2$, and that there exist two different vertices $u_1^1, u_2^1 \in U_1$ such that $v_1^1 \in N_G(u_1^1, A_1)$ and $v_2^1 \in N_G(u_2^1, A_1)$ for two different vertices $v_1^1, v_2^1 \in V(A_1)$. Let u_1^1 and u_2^1 be such two different vertices of U_1 , and let $v_1^1 \in N_G(u_1^1, A_1)$ and $v_2^1 \in N_G(u_2^1, A_1)$ and $v_1^1 \neq v_2^1$. Our next purpose is to perform one effective operation for the component A_1 , subject to the condition that one of the three assertions of Claim 7 holds.

We first consider the case that the assertion (1) of Claim 7 holds for the above mentioned two vertices u_1^1, u_2^1 . We claim that $d_G(u^1) \geq n/2$ for every vertex $u^1 \in N_{G'}(u_1^1, A_0) \cap N_{G'}(u_2^1, A_0)$ with $E_G(u^1) \cap E(F) = \emptyset$. Suppose, to the contrary, that there exists a vertex $u_0^1 \in N_{G'}(u_1^1, A_0) \cap N_{G'}(u_2^1, A_0)$ with $E_G(u_0^1) \cap E(F) = \emptyset$ such that $d_G(u_0^1) < n/2$. The condition that $E_G(u_0^1) \cap E(F) = \emptyset$ implies $N_G(u_0^1, A_1) = \emptyset$. It follows from $\text{dist}_G(u_0^1, v_i^1) = 2$ ($i \in \{1, 2\}$) that $n/2 \leq d_G(v_i^1) \leq v(A_1) - 1 + d_G(v_i^1, A_0)$, implying that $v(A_1) = \lfloor (n-1)/2 \rfloor$ and $d_G(v_i^1, A_0) = 2$ for $i \in \{1, 2\}$. Based on the assumption that $d_G(u^1, A_1) = 1$ for every vertex $u^1 \in U_1$, we have $|N_G(v_1^1, A_0) \cup N_G(v_2^1, A_0)| = 4$. Recall that the set $N_G(v_1^1, A_0) \cup N_G(v_2^1, A_0) \subset U_1$ is an independent set in G . We have $d_G(u^*) \leq n - \lfloor (n-1)/2 \rfloor - 4 + 1 < n/2$ for every vertex $u^* \in N_G(v_1^1, A_0) \cup N_G(v_2^1, A_0)$, contradicting the assertion of Claim 6. Thus as we claimed, we have $d_G(u^1) \geq n/2$ for every vertex $u^1 \in N_{G'}(u_1^1, A_0) \cap N_{G'}(u_2^1, A_0)$ with $E_G(u^1) \cap E(F) = \emptyset$. It is not difficult to check that $N_{G'}(u_1^1, A_0) \cap N_{G'}(u_3^1, A_0) \neq \emptyset$. Now, we perform the operation $H + \{u_1^1 v_1^1, u_2^1 v_2^1\} \oplus \{u_1^1 u_3^1, u_2^1 u_3^1, v_1^1 v_2^1\}$, and let $R_1 = R_0 \cup \{u_3^1\}$. For the case that the assertion (1) of Claim 7 does not hold for the above mentioned two vertices u_1^1, u_2^1 , let $R_1 = R_0$.

We now consider the case that the assertion (2) of Claim 7 holds for the above mentioned two vertices u_1^1, u_2^1 . Without loss of generality, we suppose that $u_4^1 \in N_{G'}(u_1^1, A_0)$. Now, we perform the operation $H + \{u_1^1 v_1^1, u_2^1 v_2^1\} \oplus \{u_1^1 u_4^1, u_4^1 u_3^1, u_3^1 u_2^1, v_1^1 v_2^1\}$ (see Figure 3.20 (a)). Note that this operation keeps

the degree of u in the resulting graph at least $d_{G'}(u)$ for each vertex $u \in V(A_0)$.

Figure 3.20: Graphs illustrating the case that U_1 is an independent set of G .

Finally, we consider the case that the assertion (3) of Claim 7 holds for the above mentioned two vertices u_1^1, u_2^1 . For this case, we perform the operation $H + \{u_1^1 v_1^1, u_2^1 v_2^1\} \oplus \{u_1^1 u_4^1, u_2^1 u_3^1, u_3^1 u_5^1, u_4^1 u_5^1, v_1^1 v_2^1\}$ (see Figure 3.20 (b)). Note that this operation also keeps the degree of u in the resulting graph at least $d_{G'}(u)$ for each vertex $u \in V(A_0)$.

If $k = 1$, then the resulting graph is a desired spanning eulerian subgraph of G . Next, we assume that $k \geq 2$. For each other component A_i ($i \in \{2, \dots, k\}$) of G' of order less than $n/2$, we will perform a similar operation as we did for the component A_1 . For simplicity of the notation, we again use H to denote the resulting graph after performing each operation of the type $H + E_1 \oplus E_2$.

Whenever the case that the set U_i ($2 \leq i \leq k$) is an independent set in G and $d_G(u^i, A_i) = 1$ for every vertex $u^i \in U_i$ occurs, we will need more discussion, as follows.

Let i be an integer such that $i \geq 2$ and the set U_i is an independent set in G as well as $d_G(u^i, A_i) = 1$ for every vertex $u^i \in U_i$. Recall that G is 2-connected. We have $|U_i| \geq 2$. Note that the component A_i of G' satisfies $3 \leq v(A_i) \leq n/2 - 3$ and the graph $G[V(A_i)]$ is complete. This implies that $d_G(v^i) < n/2$ for every vertex $v^i \in V(A_i)$, and that there exists a vertex v^i of A_i such that $\text{dist}_G(u^i, v^i) = 2$ for each vertex $u^i \in U_i$. Therefore, we have $d_G(u^i) \geq n/2$ for every vertex $u^i \in U_i$.

Similarly to Claim 7, we prove the following claim to show how we can perform one effective operation for the component A_i , in case the set U_i is an independent set in G and $d_G(u^i, A_i) = 1$ for every vertex $u^i \in U_i$.

Claim 8. *For any two vertices u_1^i, u_2^i of U_i , one of the following holds:*

- (1) *there exists a vertex $u_3^i \in (N_{G'}(u_1^i, A_0) \cap N_{G'}(u_2^i, A_0)) \setminus R_{i-1}$ such that $E_G(u_3^i) \cap E(F) = \emptyset$;*
- (2) *there exist two vertices $u_3^i \in (N_{G'}(u_1^i, A_0) \cap N_{G'}(u_2^i, A_0)) \setminus R_{i-1}$ and $u_4^i \in N_{G'}(u_1^i, A_0) \cup N_{G'}(u_2^i, A_0)$ such that $u_3^i u_4^i \in E(F)$; or*
- (3) *there exist two vertices $u_3^i, u_4^i \in (N_{G'}(u_1^i, A_0) \cap N_{G'}(u_2^i, A_0)) \setminus R_{i-1}$ such that $u_3^i u_5^i, u_4^i u_5^i \in E(F)$ for some vertex $u_5^i \in V(A_0) \setminus (N_{G'}(u_1^i, A_0) \cup N_{G'}(u_2^i, A_0) \cup \{u_1^i, u_2^i\})$.*

Proof. Let u_1^i and u_2^i be two vertices of U_i , and let $S_0 = N_{G'}(u_1^i, A_0) \cap N_{G'}(u_2^i, A_0)$ and $S'_0 = S_0 \setminus R_{i-1}$. Let $S_1 = N_{G'}(u_1^i, A_0) \setminus S_0$, $S_2 = N_{G'}(u_2^i, A_0) \setminus S_0$ and $S_3 = V(A_0) \setminus (N_{G'}(u_1^i, A_0) \cup N_{G'}(u_2^i, A_0) \cup \{u_1^i, u_2^i\})$.

If there exists a vertex $u_3^i \in S'_0$ such that $E_G(u_3^i) \cap E(F) = \emptyset$, then the assertion (1) holds. Now, we assume that $E_G(u^i) \cap E(F) \neq \emptyset$ for every $u^i \in S'_0$. If there exists a vertex $u_3^i \in S'_0$ such that $E_G(u_3^i, S_3) \cap E(F) = \emptyset$, then the assertion (2) holds. Next, we assume that $E_G(u^i, S_3) \cap E(F) \neq \emptyset$ for every $u^i \in S'_0$. By the facts that $d_G(u_j^i) \geq n/2$ and $d_F(u_j^i) \leq 2$ for $j \in \{1, 2\}$, we have $n/2 \leq d_G(u_1^i) \leq |S_1| + |S_0| + 2$ and $n/2 \leq d_G(u_2^i) \leq |S_2| + |S_0| + 2$. It follows that $|S_1| + |S_0| + |S_2| + |S_0| \geq n - 4$. However, we have $|S_1| + |S_0| + |S_2| + |S_3| \leq n - 3k - 2$ by the fact that the graph G' has k ($k \geq 1$) components of order less than $n/2$ and $v(A) \geq 3$ for every component A of G' . Therefore, we have $|S_0| - |S_3| \geq 3k - 2$. Note that $|R_{i-1}| \leq k - 1$. It follows that $|S'_0| - |S_3| \geq |S_0| - (k - 1) - |S_3| \geq 2k - 1$, implying that $|S'_0| > |S_3|$. Thus, the assertion (3) holds. This confirms our claim. $\square$

For the case that the assertion (1) of Claim 8 holds, we have $d_G(u_3^i) \geq n/2$ by the fact that $d_G(v^i) < n/2$ for every vertex $v^i \in V(A_i)$. Based on the facts that $d_G(u_j^i) \geq n/2$ for $j \in \{1, 2\}$, we can conclude that $|N_{G'}(u_1^i, A_0) \cap N_{G'}(u_3^i, A_0)| \geq 3k - 2$. Let $v_1^i \in N_G(u_1^i, A_i)$ and $v_2^i \in N_G(u_2^i, A_i)$. Now, we

perform the operation $H + \{u_1^i v_1^i, u_2^i v_2^i\} \oplus \{u_1^i u_3^i, u_2^i u_3^i, v_1^i v_2^i\}$ if $v_1^i \neq v_2^i$; otherwise, we perform the operation $H + \{u_1^i v_1^i, u_2^i v_1^i\} \oplus \{u_1^i u_3^i, u_2^i u_3^i\}$, and let $R_i = R_{i-1} \cup \{u_3^i\}$. For the case that the other assertion of Claim 8 holds (i.e., the assertion (1) of Claim 8 does not hold for any two vertices u_1^i, u_2^i of U_i), we let $R_i = R_{i-1}$, and we perform a similar operation as we did for the component A_1 .

We can see that each operation reduces the number of components of H of order less than $n/2$, as required. After k such operations, one can check that the resulting graph H is connected. Moreover, the resulting graph H satisfies that all the degrees of its vertices are even, and that $d_H(v) \geq d_G(v) - 1$ for each vertex v of G of odd degree and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even degree. Therefore, the resulting graph H is a desired spanning eulerian subgraph of G . This completes the proof of Lemma 3.3.1. $\square$

3.7 Conclusions and future work

In this chapter, we established sufficient conditions for the existence of compatible spanning circuits visiting each vertex at least a specified large number of times (i.e., dense or almost eulerian compatible spanning circuits) in specific edge-colored graphs satisfying certain degree conditions.

In future work, we look forward to establishing sufficient conditions for the existence of compatible spanning circuits visiting each vertex at least k times for a given small positive integer k (such as compatible Hamilton cycles or sparse compatible spanning circuits) in specific classes of edge-colored graphs.

For a graph G , let f denote a positive integer-valued function on $V(G)$. A more challenging and interesting problem is to develop sufficient conditions for the existence of compatible spanning circuits visiting each vertex v at least $f(v)$ times in specific classes of edge-colored graphs.

Chapter 4

Compatible spanning circuits in edge-colored $\mathcal{H}$ -free graphs

In this chapter, we continue to consider the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored graphs (without any assumptions on the number of colors). We establish sufficient conditions for the existence of such compatible spanning circuits in specific edge-colored graphs that do not contain one or a small set of fixed graphs as induced subgraphs. As applications, we also establish sufficient conditions for the existence of such compatible spanning circuits in edge-colored graphs G with $\kappa(G) \geq \alpha(G)$ and $\kappa(G) \geq \alpha(G) - 1$, respectively.

4.1 Introduction

As we mentioned in Chapter 1, the terminology and notations not defined but used in this chapter are standard and can be found in the most recent version of the textbook of Bondy and Murty [15]. Throughout this chapter, every edge-colored graph we refer to is always an edge-colored graph without restrictions on the number of colors, unless otherwise specified.

Let $\mathcal{H}$ be a set of connected graphs. If $|\mathcal{H}| \geq 2$, then for any two graphs $H_1, H_2 \in \mathcal{H}$, we assume that H_i is not an induced subgraph of H_{3-i} ($i \in$

$\{1, 2\}$). A graph G is said to be $\mathcal{H}$ -free if G contains no graph $H \in \mathcal{H}$ as an induced subgraph. In this context, each graph $H \in \mathcal{H}$ is called a *forbidden induced subgraph* of G . If G is $\{H\}$ -free, then we simply say that G is H -free, and we say that G is *claw-free* if $H = K_{1,3}$. Throughout this chapter, when we consider a $\mathcal{H}$ -free graph, we always assume that any member of $\mathcal{H}$ is not a path P_2 or P_3 , because a P_2 -free graph is an empty (edgeless) graph and a connected P_3 -free graph is a complete graph (the problem we consider in this chapter has been solved for complete graphs in [67]). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two sets of connected graphs. We write $\mathcal{H}_1 \preceq \mathcal{H}_2$ if there exists a graph $H_1 \in \mathcal{H}_1$ such that H_1 is an induced subgraph of H_2 for each graph $H_2 \in \mathcal{H}_2$. Clearly, if $\mathcal{H}_1 \preceq \mathcal{H}_2$, then every $\mathcal{H}_1$ -free graph is also $\mathcal{H}_2$ -free.

As introduced in the previous chapters, the existence of compatible (i.e., properly colored or alternating) Hamilton cycles, as an extremal case of compatible spanning circuits, in specific edge-colored (multi)graphs has been extensively studied in existing literature (see [1, 2, 4, 7, 12, 23, 25, 28, 29, 31, 74, 92, 111]).

On the other hand, Kotzig [84] in 1968 provided a necessary and sufficient condition for the existence of compatible Euler tours in edge-colored eulerian graphs. Almost thirty years later, Benkour et al. [9] described a polynomial-time algorithm for finding a compatible Euler tour in an edge-colored eulerian graph G in which $\Delta^{mon}(v) \leq d(v)/2$ for each vertex v of G . Independently, Pevzner [105] presented a similar algorithm for solving the same problem.

By restricting the number of colors, Das and Rao [30] considered the existence of more general compatible spanning circuits (i.e., not necessarily a compatible Hamilton cycle or Euler tour) in specific edge-colored graphs back in the 1980s. They [30] established necessary and sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly a specified number of times in 2-edge-colored complete graphs.

More recently, sufficient conditions for the existence of compatible spanning circuits visiting each vertex at least (or exactly) a specified number of times in specific edge-colored graphs satisfying certain degree conditions

(without restrictions on the number of colors) have been established in existing literature [66, 67].

In this chapter, we consider the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored graphs not containing certain forbidden induced subgraphs. As applications, we also consider the existence of such compatible spanning circuits in edge-colored graphs G with $\kappa(G) \geq \alpha(G)$ and $\kappa(G) \geq \alpha(G) - 1$, respectively (where $\kappa(G)$ and $\alpha(G)$ denote the *connectivity* and the *independence number* of the graph G , respectively).

4.2 Preliminaries

In this section, we present the following key theorem and its corollaries that are essential for our proofs of the main results of this chapter, and we list a basic known result on the existence of compatible Euler tours that will be frequently used in the later proofs of our results. In addition, we also introduce some essential graphs used as forbidden induced subgraphs in the later sections.

Theorem 4.2.1. *Let G be a claw-free hamiltonian graph. Then G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - 2$ for each vertex v of H .*

Proof. Let H be a given Hamilton cycle of the graph G . Note that the Hamilton cycle H itself is a spanning eulerian subgraph of G . We start with the Hamilton cycle H . Using a greedy idea, we extend the Hamilton cycle H to become a desired spanning eulerian subgraph of G by repeatedly performing the following two operations (in any order) as long as possible:

Operation 1. *If there exists a triangle in the subgraph $G' = G - E(H)$, then add the edges of the triangle to H ;*

Operation 2. *If there exists a 2-path uwv in the subgraph $G' = G - E(H)$ for an edge uv of H , then replace the edge uv of H by the edges uw and wv .*

For simplicity of the notation, we still use the notation H to denote the resulting spanning eulerian subgraph of G after each of the above operations.

Next, we prove by contradiction that the eventual resulting spanning eulerian subgraph H of G satisfies $d_H(v) \geq d_G(v) - 2$ for each vertex v of H . Suppose, to the contrary, that there exists a vertex x of H such that $d_H(x) \leq d_G(x) - 3$. It follows that there exist three neighbors x_1, x_2 and x_3 of x in G such that $xx_i \notin E(H)$ for $i \in \{1, 2, 3\}$. Since the graph G is claw-free, we have $\alpha(G[\{x_1, x_2, x_3\}]) \leq 2$. Without loss of generality, we suppose that $x_1x_2 \in E(G)$. If $x_1x_2 \notin E(H)$, then we can continue to perform Operation 1 (i.e., add these edges xx_1, x_1x_2 and xx_2 to H). If $x_1x_2 \in E(H)$, then we can continue to perform Operation 2 (i.e., replace the edge x_1x_2 of H by the two edges x_1x and xx_2). In both cases we obtain a contradiction with the assumption that H is the resulting graph after the termination of the above process. This completes the proof. $\square$

Using similar arguments as in the proof of Theorem 4.2.1, we can obtain the following more general result in a straightforward way.

Corollary 4.2.1. *Let k be an integer such that $k \geq 3$, and let G be a $K_{1,k}$ -free hamiltonian graph. Then G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - k + 1$ for each vertex v of H .*

Remark 4.2.1. *The following example shows that the bound on $d_H(v)$ in Corollary 4.2.1 is tight. In particular, the case with $h = 4$ in the example also shows that the bound on $d_H(v)$ in Theorem 4.2.1 is tight.*

Example 4.2.1. *Let h be an integer such that $h \geq 4$, and let G be a graph obtained from a Hamilton cycle $H = uv_1v_2u_1v_3u_2v_4 \cdots u_{h-3}v_{h-1}v_hu$ on $2h - 2$ vertices by joining vertices u and v_i by an edge for each integer i with $2 \leq i \leq h - 1$ (see Figure 4.1).*

One can check that $d_G(u) = h$ and $\alpha(G[N(u)]) = h - 2$ for the graph G of Example 4.2.1. Hence, for $h \geq 5$ it follows that the graph G is $K_{1,h-1}$ -free, since all other vertices ($\neq u$) have degree at most 3 in G . For $h = 4$, it is straightforward to check that G is claw-free. Observing the vertices with degree 2, it is easy to check that for all cases that $h \geq 4$, the graph G has a

Figure 4.1: The graph illustrating Example 4.2.1.

unique spanning eulerian subgraph, namely the Hamilton cycle H of G . We have $d_H(u) = 2 = h - (h - 1) + 1 = d_G(u) - (h - 1) + 1$.

Using similar arguments as in the proof of Theorem 4.2.1, we immediately obtain the following counterpart for supereulerian graphs, by starting with a spanning eulerian subgraph of the supereulerian graph, which is not necessarily a Hamilton cycle.

Corollary 4.2.2. *Let k be an integer such that $k \geq 3$, and let G be a $K_{1,k}$ -free supereulerian graph. Then G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - k + 1$ for each vertex v of H .*

Kotzig [84] established a necessary and sufficient condition for the existence of compatible Euler tours in edge-colored eulerian graphs, as follows.

Theorem 4.2.2 (Kotzig [84]). *Let G be an edge-colored eulerian graph. Then a compatible Euler tour exists if and only if $\Delta^{\text{mon}}(v) \leq d(v)/2$ for each vertex v of G .*

Next, we introduce some essential graphs that are used as forbidden induced subgraphs in the later sections. For integers i, j, k with $0 \leq i \leq j \leq k$, let $N_{i,j,k}$ be a graph obtained from a triangle and three disjoint paths of lengths i, j, k , respectively by identifying three vertices of the triangle with

three end vertices of the paths, one for each of the paths. (As an example, the graph $N_{1,2,3}$ is depicted in Figure 4.2 (a).) Let P_n be a path on n vertices, and let $\mathcal{P}$ denote the class of all graphs obtained from two disjoint triangles $a_1 a_2 a_3 a_1$, $b_1 b_2 b_3 b_1$ by joining each pair of vertices a_i, b_i by a path $P_{k_i} = a_i c_i^1 c_i^2 \cdots c_i^{k_i-2} b_i$ with $k_i \geq 3$ or by a triangle $a_i b_i c_i a_i$ for $i \in \{1, 2, 3\}$. We denote a graph from $\mathcal{P}$ by P_{x_1, x_2, x_3} , where $x_i = k_i$ if the two vertices a_i and b_i are joined by a path P_{k_i} , and $x_i = T$ if the two vertices a_i and b_i are joined by a triangle $a_i b_i c_i a_i$ for $i \in \{1, 2, 3\}$. (As an example, the graph $P_{3,4,T}$ is depicted in Figure 4.2 (b).) Let R be a graph obtained by removing one vertex of degree 4 from $P_{T,T,T}$ (see Figure 4.2 (c)). Note that the labels in these graphs are only used to illustrate how these graphs are constructed from triangles and paths, but that we do consider them as unlabeled graphs, so R is the unique unlabeled graph isomorphic to the labeled graph of Figure 4.2 (c).

Figure 4.2: The graphs $N_{1,2,3}$, $P_{3,4,T}$ and R .

4.3 CSCs in E-C hamiltonian graphs

In this section, we establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in specific edge-colored claw-free hamiltonian graphs. As an application, we

provide a sufficient condition for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - \kappa(G))/2 \rfloor$ times in edge-colored connected graphs G with $\kappa(G) \geq \alpha(G)$.

Before proceeding, we list the following basic results on hamiltonian graphs which will be used in the later proofs of the main theorems of this section. All the results that are listed below are from existing literature and due to different (groups of) researchers. The forbidden induced subgraphs involved in the following theorem have been introduced in Section 4.2.

Theorem 4.3.1. *Let G be a 2-connected claw-free graph on $n \geq 3$ vertices. Then G is hamiltonian if one of the following holds:*

- (1) (Matthews and Sumner [95]) $\delta(G) \geq (n - 2)/3$;
- (2) (Broersma [16], and Zhang [121]) $\sigma_3(G) \geq n - 2$;
- (3) (Favaron and Fraise [47]) G is 3-connected and $\delta(G) \geq (n + 38)/10$;
- (4) (Kaiser and Vrána [82]) G is 5-connected and $\delta(G) \geq 6$;
- (5) (Ryjáček [109]) G is 7-connected;
- (6) (Bedrossian [8]) G is H -free, where H is an induced subgraph of P_6 , $N_{0,1,2}$ or $N_{1,1,1}$;
- (7) (Faudree et al. [46]) G is H -free, where H is an induced subgraph of $N_{0,0,3}$, and $n \geq 10$;
- (8) (Brousek [17]) G is $\mathcal{H}$ -free, where $\mathcal{H} \preceq \{P_7, P_{T,T,T}\}$, $\mathcal{H} \preceq \{N_{1,1,2}, P_{T,T,T}\}$, $\mathcal{H} \preceq \{N_{0,1,2}, P_{3,3,3}\}$ or $\mathcal{H} \preceq \{N_{0,1,3}, R\}$.

Chvátal and Erdős [27] proved the following well-known theorem which combines the independence number, connectivity and hamiltonicity of a graph.

Theorem 4.3.2 (Chvátal and Erdős [27]). *Let G be a connected graph on $n \geq 3$ vertices. If $\kappa(G) \geq \alpha(G)$, then G is hamiltonian.*

Next, we present the main results of this section. We first provide sufficient conditions for the existence of compatible spanning circuits visiting

each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in edge-colored 2-connected claw-free graphs, as follows. The forbidden induced subgraphs involved in the following theorem have been introduced before stating Theorem 4.3.1.

Theorem 4.3.3. *Let G be an edge-colored 2-connected claw-free graph on $n \geq 3$ vertices such that $\Delta^{mon}(v) \leq (d(v) - 1)/2$ for each vertex v with $d(v) \geq 3$, and $\Delta^{mon}(v) = 1$ otherwise. Then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times, if one of the following holds:*

- (1) $\delta(G) \geq (n - 2)/3$;
- (2) $\sigma_3(G) \geq n - 2$;
- (3) G is 3-connected and $\delta(G) \geq (n + 38)/10$;
- (4) G is 5-connected and $\delta(G) \geq 6$;
- (5) G is 7-connected;
- (6) G is H -free, where H is an induced subgraph of P_6 , $N_{0,1,2}$ or $N_{1,1,1}$;
- (7) G is H -free, where H is an induced subgraph of $N_{0,0,3}$, and $n \geq 10$;
- (8) G is $\mathcal{H}$ -free, where $\mathcal{H} \preceq \{P_7, P_{T,T,T}\}$, $\mathcal{H} \preceq \{N_{1,1,2}, P_{T,T,T}\}$, $\mathcal{H} \preceq \{N_{0,1,2}, P_{3,3,3}\}$ or $\mathcal{H} \preceq \{N_{0,1,3}, R\}$.

Proof. Let G be an edge-colored 2-connected claw-free graph on $n \geq 3$ vertices satisfying one of Conditions (1) – (8) of Theorem 4.3.3. It follows that G is a claw-free hamiltonian graph by Theorem 4.3.1. By Theorem 4.2.1, G contains a spanning eulerian subgraph H such that $d_H(v) = d_G(v) - 1$ for each vertex v of G of odd degree and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even degree. If $\Delta_G^{mon}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G with $d_G(v) \geq 3$, then we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each vertex v of G with $d_G(v) \geq 3$. Since H is a spanning eulerian subgraph of G , we have $d_H(v) \geq 2$ for each vertex v of G . Thus, we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) = 1 \leq d_H(v)/2$ for each vertex v of G with $d_G(v) = 2$. Based on the above argument, there exists a compatible Euler tour in H by Theorem 4.2.2. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

Remark 4.3.1. In Theorem 4.3.3, it is worth noting that Condition (1) implies Condition (2), and Condition (5) implies Condition (4).

Using the proof technique we applied in our proof of Theorem 4.3.3, we can also confirm the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored connected graphs G with $\kappa(G) \geq \alpha(G)$, as follows.

Theorem 4.3.4. Let G be an edge-colored connected graph on $n \geq 3$ vertices such that $\kappa(G) \geq \alpha(G)$. If $\Delta^{mon}(v) \leq (d(v) - \kappa(G))/2$ for each vertex v with $d(v) \geq \kappa(G) + 2$, and $\Delta^{mon}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - \kappa(G))/2 \rfloor$ times.

Proof. Let G be an edge-colored connected graph on $n \geq 3$ vertices such that $\kappa(G) \geq \alpha(G)$. It follows that G is hamiltonian by Theorem 4.3.2. Since we have $\alpha(G) \leq \kappa(G)$, G is a $K_{1, \kappa(G)+1}$ -free graph. If $\kappa(G) = 1$, then we have $\alpha(G) = 1$, implying that G is a complete graph. It is not difficult to check that the complete graph G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - \kappa(G)$ for each vertex v of H (in fact $H = G$, if n is odd; otherwise, $H = G - M$, where M is an arbitrary perfect matching of G). Now, we consider the case that $\kappa(G) \geq 2$. By Corollary 4.2.1, G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - \kappa(G)$ for each vertex v of H . If $\Delta_G^{mon}(v) \leq (d_G(v) - \kappa(G))/2$ for each vertex v of G with $d_G(v) \geq \kappa(G) + 2$, then we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq (d_G(v) - \kappa(G))/2 \leq d_H(v)/2$ for each vertex v of G with $d_G(v) \geq \kappa(G) + 2$. Since H is a spanning eulerian subgraph of G , we have $d_H(v) \geq 2$ for each vertex v of G . Thus, we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) = 1 \leq d_H(v)/2$ for each vertex v of G with $d_G(v) \leq \kappa(G) + 1$. Based on the above argument, there exists a compatible Euler tour in H by Theorem 4.2.2. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - \kappa(G))/2 \rfloor$ times. This completes the proof. $\square$

4.4 CSCs in E-C supereulerian graphs

In this section, we continue to pay close attention to the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored graphs which do not contain certain forbidden induced subgraphs. We establish sufficient conditions for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in edge-colored 2-connected claw-free supereulerian graphs, and we also confirm the existence of such compatible spanning circuits in specific edge-colored 2-edge-connected claw-free graphs. As an application, we provide a sufficient condition for the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - \kappa(G) - 1)/2 \rfloor$ times in edge-colored 2-connected graphs G with $\kappa(G) \geq \alpha(G) - 1$.

Before proceeding, we list a number of basic results on supereulerian graphs which will be used in the later proofs of the main theorems of this section. All the results that are listed below are from existing literature and due to different (groups of) researchers.

Figure 4.3: Three induced subgraphs of $P_{3,3,3}$.

Before presenting these known results, we introduce three induced subgraphs B_i ($i \in \{1, 2, 3\}$) of the graph $P_{3,3,3}$, which are depicted in Figure 4.3. We further define three families of pairs of graphs: $\mathbb{H}_1 = \{\{P_{3,3,3}, Y\} \mid Y \in \{P_8, N_{0,1,4}, N_{1,1,3}, N_{1,2,2}\}\}$, $\mathbb{H}_2 = \{\{X, Y\} \mid X \in \{B_1, B_2\}, Y \in \{P_9, N_{0,1,5}, N_{1,1,4}, N_{1,2,3}, N_{2,2,2}\}\}$ and $\mathbb{H}_3 = \{\{B_3, Y\} \mid Y \in \{P_{10}, N_{0,2,5}, N_{1,2,4}, N_{2,2,3}\}\}$, which

serve as forbidden induced subgraphs in the following results. Other forbidden induced subgraphs involved in the following results have been introduced in Section 4.2.

Theorem 4.4.1. *Let G be a 2-connected claw-free graph. Then G is supereulerian if one of the following holds:*

- (1) (Catlin [21], and Jaeger [80]) G is 4-edge-connected;
- (2) (Lv and Xiong [93]) G is H -free, where H is an induced subgraph of P_7 , $N_{0,1,3}$ or $N_{1,1,2}$;
- (3) (Wang and Xiong [116]) G is $\mathcal{H}$ -free, where $\mathcal{H} \preceq \mathcal{H}'$ and $\mathcal{H}' \in \mathbb{H}_1 \cup \mathbb{H}_2 \cup \mathbb{H}_3$;
- (4) (Wang and Xiong [116]) G is $\mathcal{P}^*$ -free, where $\mathcal{P}^* = \{P_{x_1, x_2, x_3} \in \mathcal{P} \mid x_1, x_2, x_3 \neq T \text{ and } 3 \leq x_1 \leq x_2 \leq x_3\}$;
- (5) (Wang and Xiong [116]) G is a graph with the longest induced cycle of length at most 5.

We will also use the following result on supereulerian graphs in the later proof of Theorem 4.4.5. Before stating this result, we introduce the graphs $K_{2,3}$, $K_{2,3}(1)$, $K_{2,3}(2)$ and $K'_{2,3}$ that are used as exceptional graphs in the following result, and that are depicted in Figure 4.4. In the following theorem, when we say that a graph G' is obtained from a graph G by replacing a vertex v of G by a graph H disjoint with G , we mean that the number of edges in G' joining H to $G - v$ exactly equals $d_G(v)$ (we do not need to care which vertices of H are incident with the edges of G' joining H to $G - v$).

Figure 4.4: The graphs $K_{2,3}$, $K_{2,3}(1)$, $K_{2,3}(2)$ and $K'_{2,3}$.

Theorem 4.4.2 (Han et al. [72]). *Let G be a 2-connected graph. If $\kappa(G) \geq \alpha(G) - 1$, then G is supereulerian, unless*

- (1) $G \in \{P, K_{2,3}, K_{2,3}(1), K_{2,3}(2), K'_{2,3}\}$, where P is the Petersen graph; or
- (2) G is one of 2-connected graphs obtained from $K_{2,3}$ or $K_{2,3}(1)$ by replacing a vertex all neighbors of which have degree 3 in $K_{2,3}$ or $K_{2,3}(1)$ by a complete graph on at least three vertices.

Next, we present the main results of this section. By replacing the conditions of Theorem 4.3.3 by the ones that are listed above in Theorem 4.4.1, we can still guarantee the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in edge-colored 2-connected claw-free graphs, as shown in the following theorem. The forbidden induced subgraphs involved in the following theorem have been introduced before stating Theorem 4.4.1.

Theorem 4.4.3. *Let G be an edge-colored 2-connected claw-free graph such that $\Delta^{\text{mon}}(v) \leq (d(v) - 1)/2$ for each vertex v with $d(v) \geq 3$, and $\Delta^{\text{mon}}(v) = 1$ otherwise. Then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times, if one of the following holds:*

- (1) G is 4-edge-connected;
- (2) G is H -free, where H is an induced subgraph of P_7 , $N_{0,1,3}$ or $N_{1,1,2}$;
- (3) G is $\mathcal{H}$ -free, where $\mathcal{H} \preceq \mathcal{H}'$ and $\mathcal{H}' \in \mathbb{H}_1 \cup \mathbb{H}_2 \cup \mathbb{H}_3$;
- (4) G is $\mathcal{P}^*$ -free, where $\mathcal{P}^* = \{P_{x_1, x_2, x_3} \in \mathcal{P} \mid x_1, x_2, x_3 \neq T \text{ and } 3 \leq x_1 \leq x_2 \leq x_3\}$;
- (5) G is a graph with the longest induced cycle of length at most 5.

Proof. Let G be an edge-colored 2-connected claw-free graph satisfying one of Conditions (1) – (5) of Theorem 4.4.3. It follows that G is a claw-free supereulerian graph by Theorem 4.4.1. By Corollary 4.2.2, G contains a spanning eulerian subgraph H such that $d_H(v) = d_G(v) - 1$ for each vertex v of G of odd degree and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even

degree. If $\Delta_G^{mon}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G with $d_G(v) \geq 3$, then we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each vertex v of G with $d_G(v) \geq 3$. Since H is a spanning eulerian subgraph of G , we have $d_H(v) \geq 2$ for each vertex v of G . Thus, we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) = 1 \leq d_H(v)/2$ for each vertex v of G with $d_G(v) = 2$. Based on the above argument, there exists a compatible Euler tour in H by Theorem 4.2.2. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

We also show that the sufficient conditions of Theorem 4.4.3 can still guarantee the existence of compatible spanning circuits visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times in edge-colored 2-edge-connected claw-free graphs, as follows. So we can relax 2-connectivity to 2-edge-connectivity.

Theorem 4.4.4. *Let G be an edge-colored 2-edge-connected claw-free graph satisfying one of Conditions (1) – (5) of Theorem 4.4.3. If $\Delta^{mon}(v) \leq (d(v) - 1)/2$ for each vertex v with $d(v) \geq 3$, and $\Delta^{mon}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - 1)/2 \rfloor$ times.*

Proof. Let G be an edge-colored 2-edge-connected claw-free graph satisfying one of Conditions (1) – (5) of Theorem 4.4.3. If G is 2-connected, then it is a claw-free supereulerian graph by Theorem 4.4.1. Next, we assume that G is a graph with connectivity 1. Now, we consider the nontrivial blocks of G . Note that every nontrivial block of G is 2-connected. Moreover, we conclude that every nontrivial block of G satisfies one of Conditions (1) – (5) in Theorem 4.4.3; otherwise, G would not satisfy any one of Conditions (1) – (5) in Theorem 4.4.3. It follows that every nontrivial block of G is a supereulerian graph by Theorem 4.4.1. We can show that G is supereulerian by taking one spanning eulerian subgraph from every nontrivial block of G and then combining these spanning eulerian subgraphs to obtain a spanning eulerian subgraph of G . Recall that G is a claw-free graph. By Corollary 4.2.2, G contains a spanning eulerian subgraph H such that $d_H(v) = d_G(v) - 1$ for each vertex v of G of odd degree and $d_H(v) \geq d_G(v) - 2$ for each vertex v of G of even degree. If $\Delta_G^{mon}(v) \leq (d_G(v) - 1)/2$ for each vertex v of G with

$d_G(v) \geq 3$, then we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq \lfloor (d_G(v) - 1)/2 \rfloor \leq d_H(v)/2$ for each vertex v of G with $d_G(v) \geq 3$. Since the graph H is a spanning eulerian subgraph of G , we have $d_H(v) \geq 2$ for each vertex v of G . Thus, we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) = 1 \leq d_H(v)/2$ for each vertex v of G with $d_G(v) = 2$. Based on the above argument, there exists a compatible Euler tour in H by Theorem 4.2.2. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - 1)/2 \rfloor$ times. This completes the proof. $\square$

Using the proof technique we applied in our proof of Theorem 4.4.3, we can also confirm the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored 2-connected graphs G with $\kappa(G) \geq \alpha(G) - 1$, as shown in the following theorem. The exceptional graphs involved in the following theorem have been introduced before stating Theorem 4.4.2.

Theorem 4.4.5. *Let G be an edge-colored 2-connected graph such that $\kappa(G) \geq \alpha(G) - 1$. If $\Delta^{mon}(v) \leq (d(v) - \kappa(G) - 1)/2$ for each vertex v with $d(v) \geq \kappa(G) + 3$, and $\Delta^{mon}(v) = 1$ otherwise, then G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d(v) - \kappa(G) - 1)/2 \rfloor$ times, unless*

- (1) $G \in \{P, K_{2,3}, K_{2,3}(1), K_{2,3}(2), K'_{2,3}\}$, where P is the Petersen graph; or
- (2) G is one of 2-connected graphs obtained from $K_{2,3}$ or $K_{2,3}(1)$ by replacing a vertex all neighbors of which have degree 3 in $K_{2,3}$ or $K_{2,3}(1)$ by a complete graph on at least three vertices.

Proof. Let G be an edge-colored 2-connected graph such that $\kappa(G) \geq \alpha(G) - 1$ and $G \notin \{P, K_{2,3}, K_{2,3}(1), K_{2,3}(2), K'_{2,3}\}$, where P is the Petersen graph and the other graphs have been depicted in Figure 4.4. We further assume that G is not any one of 2-connected graphs obtained from $K_{2,3}$ or $K_{2,3}(1)$ by replacing a vertex all neighbors of which have degree 3 in $K_{2,3}$ or $K_{2,3}(1)$ by a complete graph on at least three vertices. It follows that G is a supereulerian graph by Theorem 4.4.2. Since we have $\alpha(G) \leq \kappa(G) + 1$, G is a $K_{1, \kappa(G)+2}$ -free graph. By Corollary 4.2.2, G contains a spanning eulerian subgraph H such that $d_H(v) \geq d_G(v) - \kappa(G) - 1$ for each vertex v of H . If $\Delta_G^{mon}(v) \leq$

$(d_G(v) - \kappa(G) - 1)/2$ for each vertex v of G with $d_G(v) \geq \kappa(G) + 3$, then we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) \leq (d_G(v) - \kappa(G) - 1)/2 \leq d_H(v)/2$ for each vertex v of G with $d_G(v) \geq \kappa(G) + 3$. Since H is a spanning eulerian subgraph of G , we have $d_H(v) \geq 2$ for each vertex v of G . Thus, we have $\Delta_H^{mon}(v) \leq \Delta_G^{mon}(v) = 1 \leq d_H(v)/2$ for each vertex v of G with $d_G(v) \leq \kappa(G) + 2$. Based on the above argument, there exists a compatible Euler tour in H by Theorem 4.2.2. Therefore, G contains a compatible spanning circuit visiting each vertex v at least $\lfloor (d_G(v) - \kappa(G) - 1)/2 \rfloor$ times. This completes the proof. $\square$

4.5 Conclusions and an open problem

In this chapter, we first proved a key theorem (i.e., Theorem 4.2.1) that deals with the existence of spanning eulerian subgraphs H of G satisfying $d_H(v) \geq d_G(v) - 2$ for each vertex v of G , and we obtained some related corollaries. We then established sufficient conditions for the existence of compatible spanning circuits visiting each vertex at least a specified number of times in specific edge-colored graphs that do not contain certain forbidden induced subgraphs, by combining the conditions of Theorem 4.2.1 (or its corollaries) with the conditions of Theorem 4.2.2. As applications, we also considered the existence of such compatible spanning circuits in edge-colored graphs G with $\kappa(G) \geq \alpha(G)$ and $\kappa(G) \geq \alpha(G) - 1$, respectively.

Although we did not consider any counterparts of our results for digraphs, we realize that counterparts for digraphs analogous to Theorem 4.2.2 have been discussed (see [51, 68, 76]). Motivated by the ideas we explored in this chapter, it is natural to consider the existence of compatible spanning circuits visiting each vertex at least a specified number of times in arc-colored digraphs. However, it looks difficult or even impossible to obtain a straightforward analogue of Theorem 4.2.1 for digraphs, indicating that it is very likely that different approaches are necessary. We leave this as an open problem.

Problem 4.5.1. *Let D be an arc-colored digraph. Can D contain a compatible spanning circuit visiting each vertex at least a specified number of times? If so, under what conditions does D contain such a compatible spanning circuit?*

Chapter 5

Algorithmic results for compatible spanning circuits in edge-colored graphs

In this chapter, we consider the existence of compatible spanning circuits in edge-colored graphs (without any assumptions on the number of colors) from an algorithmic perspective. We first show that determining whether an edge-colored connected graph contains a compatible spanning circuit is an NP-complete problem. Next, we describe two polynomial-time algorithms for finding compatible spanning circuits (with certain properties) in specific edge-colored complete graphs. These results in some sense give partial support to an as-yet-unproven conjecture on the existence of compatible Hamilton cycles in edge-colored complete graphs due to Bollobás and Erdős from the 1970s.

5.1 Introduction

As we mentioned in Chapter 1, the terminology and notations not defined but used in this chapter are standard and can be found in the most recent

version of the textbook of Bondy and Murty [15]. Throughout this chapter, every edge-colored graph we refer to is always an edge-colored graph without restrictions on the number of colors, unless otherwise specified.

From a sufficient condition perspective, the existence of a kind of extremal compatible spanning circuits called compatible (i.e., properly colored or alternating) Hamilton cycles in specific edge-colored (multi)graphs has been extensively studied in previous literature (see [1, 2, 4, 7, 12, 23, 25, 28, 29, 31, 74, 92, 111]).

As another extremal case of compatible spanning circuits, Kotzig [84] in 1968 provided a necessary and sufficient condition for the existence of compatible Euler tours in edge-colored eulerian graphs. Almost thirty years later, Benkour et al. [9], from an algorithmic perspective, considered the existence of compatible Euler tours in edge-colored eulerian graphs. They [9] established a polynomial-time algorithm for finding a compatible Euler tour in an edge-colored eulerian graph G in which $\Delta^{mon}(v) \leq d(v)/2$ for each vertex v of G . Independently, Pevzner [105] described a similar algorithm for solving the same problem.

By restricting the number of colors, Das and Rao [30] established necessary and sufficient conditions for the existence of compatible spanning circuits visiting each vertex exactly a specified number of times in 2-edge-colored complete graphs back in the 1980s.

In recent work [66, 67], sufficient conditions for the existence of compatible spanning circuits visiting each vertex at least (or exactly) a specified number of times in specific edge-colored graphs (without restrictions on the number of colors) have been established.

In this chapter, we consider the existence of compatible spanning circuits in edge-colored graphs from an algorithmic perspective.

5.2 NP-completeness results

In this section, we deal with the complexity of finding compatible spanning circuits in edge-colored graphs. Based on an early NP-completeness result on the Hamilton problem, we show that determining whether an edge-colored

connected graph contains a compatible spanning circuit is an NP-complete problem.

In the later proof of our result, we will use the NP-completeness of the following special case of the Hamilton problem, an early complexity result due to Garey et al. [60].

Problem 5.2.1. (*Garey, Johnson and Stockmeyer [60]*)

Instance: A connected graph G with $\Delta(G) = 3$.

Question: Does G contain a Hamilton cycle?

Next, we present the main result of this section, as follows.

Theorem 5.2.1. *The decision problem of determining whether an edge-colored connected graph contains a compatible spanning circuit is NP-complete.*

Proof. It is clear that Problem 5.2.1 can easily be reduced to the following special case of the decision problem we stated in Theorem 5.2.1.

Problem 5.2.2.

Instance: An edge-colored connected graph G^c with $\Delta(G^c) = 3$.

Question: Does G^c contain a compatible spanning circuit?

First of all, Problem 5.2.2 clearly belongs to the class NP: for any candidate subgraph H corresponding to a compatible spanning circuit in G^c , it can be verified in polynomial time whether the subgraph H contains all vertices of G^c , $d_H(v) = 2$ and $\Delta_H^{mon}(v) = 1$ for each vertex v of H .

For any instance G of Problem 5.2.1, we construct a rainbow edge-colored graph by coloring all edges of G with pairwise distinct colors, to obtain an instance G^c of Problem 5.2.2. It is obvious that the graph G contains a Hamilton cycle if and only if the edge-colored graph G^c corresponding to G contains a compatible spanning circuit.

It follows directly from our construction that the above reduction is polynomial. This proves that Problem 5.2.2 is NP-complete. Since Problem 5.2.2 is a special case of the decision problem we stated in Theorem 5.2.1, the result is immediate. $\square$

5.3 Polynomial-time algorithms

In this section, we deal with the existence of compatible spanning circuits in specific classes of edge-colored graphs from an algorithmic perspective. We describe two polynomial-time algorithms for finding compatible spanning circuits (with certain properties) in specific edge-colored complete graphs.

Motivated by the NP-completeness result in Section 5.2, we consider the existence of compatible spanning circuits in specific classes of edge-colored graphs from an algorithmic perspective, and we analyze the complexity of the associated algorithms.

Several sufficient conditions for the existence of compatible Hamilton cycles in edge-colored complete graphs have been obtained (see [4, 12, 23, 31, 92, 111]). In particular, Bollobás and Erdős [12] considered the problem and proposed the following conjecture on the existence of compatible Hamilton cycles in edge-colored complete graphs back in the 1970s (see Conjecture 5.3.1). Recently, Lo [92] proved that this conjecture is true asymptotically. Throughout the rest of this section, we use K_n^c to denote an edge-colored complete graph on n vertices, where $n \geq 3$.

Conjecture 5.3.1 (Bollobás and Erdős [12]). *If $\Delta^{mon}(K_n^c) < \lfloor n/2 \rfloor$, then K_n^c contains a compatible Hamilton cycle.*

We first deal with the existence of compatible spanning circuits (with no restrictions) in graphs K_n^c with $\Delta^{mon}(K_n^c) \leq \lfloor (n-1)/2 \rfloor$, as follows.

Theorem 5.3.1. *If $\Delta^{mon}(K_n^c) \leq \lfloor (n-1)/2 \rfloor$, then K_n^c contains a compatible spanning circuit. Moreover, such a compatible spanning circuit can be found by an $O(n^4)$ algorithm.*

We postpone the lengthy proof of this theorem to Section 5.4.

Remark 5.3.1. *The following example, extended from a construction given by Fujita and Magnan [54], shows that the bound on $\Delta^{mon}(K_n^c)$ in Theorem 5.3.1 is tight.*

Example 5.3.1. *Let G be a complete graph on $n \geq 3$ vertices, and let u be one of the vertices of G . We label the remaining vertices with $v_1, \dots, v_{n-1}$, respectively,*

and we color the edge uv_i with color i for each v_i , where $1 \leq i \leq n-1$. Let $H = G - u$, and consider a decomposition of the edges of H into $\lfloor (n-2)/2 \rfloor$ Hamilton cycles (together with one perfect matching M , if n is odd) (for the feasibility of the above decomposition, see [91]). We arbitrarily orient these Hamilton cycles (and M , if n is odd) such that they become directed cycles (a directed perfect matching). We color the edge $v_i v_j$ with color j if the arc $\overrightarrow{v_i v_j}$ is an arc of one of these Hamilton cycles (perfect matching). This defines an edge-coloring of G , thus a K_n^c .

One can check that the edge-colored complete graph K_n^c of Example 5.3.1 satisfies $\Delta^{mon}(K_n^c) = \lfloor (n-1)/2 \rfloor + 1$, but it contains no compatible spanning circuit, because such a circuit cannot visit the vertex u compatibly.

We next consider the existence of compatible spanning circuits visiting every vertex, except for one specific vertex, exactly $(n-2)/2$ times in graphs K_n^c , and we obtain the following result.

Theorem 5.3.2. *Let n be an even integer such that $n \geq 4$. If $\Delta^{mon}(K_n^c) \leq (n-2)/2$ and $cd(v_0) \geq n - \lfloor (\sqrt{4n-3} + 1)/2 \rfloor$ for some vertex v_0 of K_n^c , then K_n^c contains a compatible spanning circuit visiting every vertex of K_n^c , except for v_0 , exactly $(n-2)/2$ times. Moreover, such a compatible spanning circuit can be found by an $O(n^4)$ algorithm.*

The lengthy proof of this theorem will be postponed to Section 5.4.

Remark 5.3.2. *The edge-colored complete graphs on even $n \geq 4$ vertices of Example 5.3.1 also show that the bound on $\Delta^{mon}(K_n^c)$ in Theorem 5.3.2 is tight. However, we do not know whether the bound on $cd(v_0)$ in Theorem 5.3.2 is tight.*

5.4 Proofs of the two main theorems

In this section, we present the lengthy proofs of Theorems 5.3.1 and 5.3.2 of Section 5.3.

Before proceeding, we first introduce some additional terminology.

For a given trail $T_i = x_1 x_2 \cdots x_i$ ($i \geq 2$) of a graph H , we use H_i to denote the (spanning) subgraph of H obtained from H by deleting all the

edges of T_i . For a given (compatible) trail $T_i = x_1x_2\cdots x_i$ ($i \geq 2$) of an edge-colored graph H , an edge x_ix_{i+1} of H is said to be *suitable* for T_i in H if $x_ix_{i+1} \in E_{H_i}(x_i)$ and $c(x_ix_{i+1})$ satisfies that $c(x_ix_{i+1}) \neq c(x_{i-1}x_i)$ and $d_{H_i}^{c_0}(x_i) = \max_{c \neq c(x_{i-1}x_i)} \{d_{H_i}^c(x_i)\}$, where $c_0 = c(x_ix_{i+1})$.

5.4.1 Proof of Theorem 5.3.1

We prove Theorem 5.3.1 by considering the following polynomial algorithm (see Algorithm 1 on the next page), and proving its correctness. We use CSC as shorthand for compatible spanning circuit. The ideas behind Algorithm 1 were inspired by similar ideas due to Pevzner [105] for an efficient algorithm to construct a compatible Euler tour in an edge-colored eulerian graph G in which $\Delta^{mon}(v) \leq d(v)/2$ for each vertex v of G .

Next, we show the correctness of Algorithm 1 by proving the following lemmas, with the notations H_i , T_i , H and T defined as above.

Lemma 5.4.1. *We have $\Delta_{H_i}^{mon}(v) \leq d_{H_i}(v)/2$ for each integer i with $i \geq 2$ such that $T_i \neq T$, and each vertex v of H_i , excluding possibly x_1 and x_i .*

Proof. Suppose, to the contrary, that the statement of Lemma 5.4.1 does not hold. Let i_0 be the minimum integer such that Lemma 5.4.1 fails. Clearly, we have $i_0 > 2$. Thus, for some color c and some vertex v distinct from x_1 and x_{i_0} , we have $d_{H_{i_0}}^c(v) > d_{H_{i_0}}(v)/2$. It is not difficult to see that $v = x_{i_0-1}$; otherwise, Lemma 5.4.1 would already fail for the integer $i_0 - 1$. It follows from $d_{H_{i_0}}^c(x_{i_0-1}) > d_{H_{i_0}}(x_{i_0-1})/2$ and $d_{H_{i_0}}(x_{i_0-1})$ is even that $d_{H_{i_0}}^c(x_{i_0-1}) \geq d_{H_{i_0}}(x_{i_0-1})/2 + 1$. Clearly, we have $d_{H_{i_0-1}}^c(x_{i_0-1}) \geq d_{H_{i_0}}^c(x_{i_0-1}) \geq d_{H_{i_0}}(x_{i_0-1})/2 + 1 = (d_{H_{i_0-1}}(x_{i_0-1}) - 1)/2 + 1 = (d_{H_{i_0-1}}(x_{i_0-1}) + 1)/2$. We first prove the following claim in order to complete the proof of Lemma 5.4.1.

Claim 1. $c(x_{i_0-1}x_{i_0}) = c$.

Proof. Suppose, to the contrary, that $c(x_{i_0-1}x_{i_0}) \neq c$. Recall that $d_{H_{i_0-1}}^c(x_{i_0-1}) \geq (d_{H_{i_0-1}}(x_{i_0-1}) + 1)/2$ and $T_{i_0} \neq T$. It follows that $c(x_{i_0-2}x_{i_0-1}) = c$ by the rules of Step 3 of Algorithm 1. Thus, we have $d_{H_{i_0-2}}^c(x_{i_0-1}) = d_{H_{i_0-1}}^c(x_{i_0-1}) + 1 \geq (d_{H_{i_0-1}}(x_{i_0-1}) + 1)/2 + 1 = d_{H_{i_0-2}}(x_{i_0-1})/2 + 1$, contradicting the minimality of i_0 . This confirms our claim. $\square$

Algorithm 1 Finding a CSC in K_n^c with $\Delta^{mon}(K_n^c) \leq \lfloor (n-1)/2 \rfloor$.

Input: A graph K_n^c with $\Delta^{mon}(K_n^c) \leq \lfloor (n-1)/2 \rfloor$;

Output: A CSC T of K_n^c ;

Step 1. If n is odd, then let $H = K_n^c$; otherwise, choose an arbitrary perfect matching M of K_n^c , and let $H = K_n^c - M$;

Step 2. Choose an arbitrary vertex x_1 of H , and put $T_1 = x_1$;
Choose the next vertex x_2 such that $c(x_1x_2)$ is (one of) the least frequent colors among the edges of $E_H(x_1)$, and put $T_2 = x_1x_2$;

Step 3. Based on $T_i = x_1x_2 \cdots x_i$ ($i \geq 2$), build up $T_{i+1} = x_1x_2 \cdots x_ix_{i+1}$ according to the following **rules**:

if $V(H) \setminus V(T_i) = \emptyset$ and $x_ix_1 \in E_{H_i}(x_i)$, as well as $c(x_{i-1}x_i) \neq c(x_ix_1)$ and $c(x_ix_1) \neq c(x_1x_2)$ **then**
 put $T = T_{i+1} = x_1x_2 \cdots x_ix_1$, and go to **Step 5**;

else

if there exists a vertex $x_{i+1} \in V(H) \setminus V(T_i)$ such that x_ix_{i+1} is suitable for T_i in H **then**

 choose such a vertex x_{i+1} and preferentially choose the vertex x_{i+1} such that $c(x_ix_{i+1}) = c(x_1x_2)$ when $x_i = x_1$, and
 put $T_{i+1} = x_1x_2 \cdots x_ix_{i+1}$;

else

 choose a vertex $x_{i+1} \in V(T_i)$ such that x_ix_{i+1} is suitable for T_i in H , and preferentially choose the vertex x_{i+1} such that $c(x_ix_{i+1}) = c(x_1x_2)$ when $x_i = x_1$, and put $T_{i+1} = x_1x_2 \cdots x_ix_{i+1}$;

end if

end if

Step 4. $i \leftarrow i + 1$, and go to **Step 3**;

Step 5. T is a CSC of K_n^c ; terminate the process;

return T .

By Claim 1, we have $c(x_{i_0-2}x_{i_0-1}) \neq c$ and $c(x_{i_0-1}x_{i_0}) = c$. Hence, we have $d_{H_{i_0-2}}^c(x_{i_0-1}) = d_{H_{i_0}}^c(x_{i_0-1}) + 1 \geq d_{H_{i_0}}(x_{i_0-1})/2 + 1 + 1 = (d_{H_{i_0-2}}(x_{i_0-1}) - 2)/2 + 1 + 1 = d_{H_{i_0-2}}(x_{i_0-1})/2 + 1$, contradicting the minimality of i_0 . This completes the proof of Lemma 5.4.1. $\square$

Lemma 5.4.2. *We have $\Delta_{H_i}^{mon}(x_1) \leq \lceil d_{H_i}(x_1)/2 \rceil$ for each integer i with $i \geq 2$ such that $x_i \neq x_1$.*

Proof. By Step 2 of Algorithm 1, we have $\Delta_{H_2}^{mon}(x_1) = \Delta_{H_1}^{mon}(x_1) = \Delta_H^{mon}(x_1)$. For the case that n is odd, we have $\Delta_H^{mon}(x_1) = \Delta_{K_n^c}^{mon}(x_1) \leq (n-1)/2 = \lceil (n-2)/2 \rceil = \lceil d_{H_2}(x_1)/2 \rceil$. For the case that n is even, we have $\Delta_H^{mon}(x_1) \leq \Delta_{K_n^c}^{mon}(x_1) \leq (n-2)/2 = \lceil (n-3)/2 \rceil = \lceil d_{H_2}(x_1)/2 \rceil$. Thus, Lemma 5.4.2 holds for $i = 2$.

Next, we assume that $i \geq 3$. Suppose, to the contrary, that the statement of Lemma 5.4.2 does not hold. Let i_0 be the minimum integer such that Lemma 5.4.2 fails. Since $x_{i_0} \neq x_1$, we conclude that $x_{i_0-1} = x_1$; otherwise Lemma 5.4.2 would already fail for some integer less than i_0 . Thus, for some color c , we have $d_{H_{i_0}}^c(x_1) \geq \lceil d_{H_{i_0}}(x_1)/2 \rceil + 1$. Note that $d_{H_{i_0}}(x_1)$ is odd. Let $d_{H_{i_0}}(x_1) = 2k - 1$ ($k \geq 1$). Obviously, we have $d_{H_{i_0-1}}^c(x_1) \geq d_{H_{i_0}}^c(x_1) \geq \lceil d_{H_{i_0}}(x_1)/2 \rceil + 1 = \lceil (2k-1)/2 \rceil + 1 = k + 1 = d_{H_{i_0-1}}(x_1)/2 + 1$. We first prove the following claim in order to complete the proof of Lemma 5.4.2.

Claim 2. $c(x_1x_{i_0}) = c$.

Proof. Suppose, to the contrary, that $c(x_1x_{i_0}) \neq c$. Recall that $x_{i_0-1} = x_1$ and $d_{H_{i_0-1}}^c(x_1) \geq d_{H_{i_0-1}}(x_1)/2 + 1$. It follows that $c(x_{i_0-2}x_1) = c$ by the rules of Step 3 of Algorithm 1. Thus, we have $d_{H_{i_0-2}}^c(x_1) = d_{H_{i_0-1}}^c(x_1) + 1 \geq d_{H_{i_0-1}}(x_1)/2 + 1 + 1 = k + 1 + 1 = \lceil d_{H_{i_0-2}}(x_1)/2 \rceil + 1$, contradicting the minimality of i_0 . This confirms our claim. $\square$

By Claim 2, we have $c(x_{i_0-2}x_1) \neq c$ and $c(x_1x_{i_0}) = c$. Hence, we have $d_{H_{i_0-2}}^c(x_1) = d_{H_{i_0}}^c(x_1) + 1 \geq \lceil d_{H_{i_0}}(x_1)/2 \rceil + 1 + 1 = k + 1 + 1 = \lceil d_{H_{i_0-2}}(x_1)/2 \rceil + 1$, contradicting the minimality of i_0 . This completes the proof of Lemma 5.4.2. $\square$

From Lemmas 5.4.1 and 5.4.2, we can obtain the following lemma immediately.

Lemma 5.4.3. *For each integer i with $i \geq 2$ such that $T_i \neq T$, we have*

$$\Delta_{H_{i-1}}^{mon}(x_i) \leq \begin{cases} d_{H_{i-1}}(x_i)/2, & \text{if } x_i \neq x_1; \\ (d_{H_{i-1}}(x_i) + 1)/2, & \text{if } x_i = x_1. \end{cases}$$

Lemma 5.4.3 implies that for each trail $T_i \neq T$, there always exists an edge $x_i x_{i+1}$ that is suitable for T_i in H .

Next, we show that Algorithm 1 will terminate, by proving the following lemma.

Lemma 5.4.4. *For an integer i with $i \geq 3$ such that $x_i \neq x_1$ and $E_{H_i}(x_i) = E_{H_i}(x_1) = \{x_i x_1\}$, we have $V(T_i) = V(H)$, as well as $c(x_{i-1} x_i) \neq c(x_i x_1)$ and $c(x_i x_1) \neq c(x_1 x_2)$.*

Proof. Let i be an integer with $i \geq 3$ such that $x_i \neq x_1$ and $E_{H_i}(x_i) = E_{H_i}(x_1) = \{x_i x_1\}$.

We first claim that $V(T_i) = V(H)$. Suppose, to the contrary, that there exists a vertex $v \in V(H) \setminus V(T_i)$. Obviously, we have $v \notin \{x_1, x_i\}$. Recall that the graph K_n^c is a complete graph on n vertices, where $n \geq 3$. It follows from the construction of H that at least one of vx_i and vx_1 is an edge of H_i , contradicting the fact that $E_{H_i}(x_i) = E_{H_i}(x_1) = \{x_i x_1\}$. Thus as we claimed, we have $V(T_i) = V(H)$.

Recall that $E_{H_i}(x_i) = \{x_i x_1\}$. It follows that $\Delta_{H_{i-1}}^{mon}(x_i) = 1$ by Lemma 5.4.3. Therefore, we conclude that $c(x_{i-1} x_i) \neq c(x_i x_1)$.

Next, we prove the assertion that $c(x_i x_1) \neq c(x_1 x_2)$. Suppose, to the contrary, that $c(x_i x_1) = c(x_1 x_2) = c_0$. Let us consider T_i as the oriented trail in the direction from x_1 to x_2 (see Figure 5.1). Note that $d_H(x_1) = n - 1$, if n is odd, and $d_H(x_1) = n - 2$, otherwise. We use $y_1, y_2, \dots, y_{n-3}, y_{n-2}$ (and y_{n-1} , if n is odd) to denote the vertices of H adjacent to the vertex x_1 according to the order in which they are visited by the oriented trail T_i (see Figure 5.1 (a) and (b)). We prove the following claim in order to complete the proof of Lemma 5.4.4.

Figure 5.1: (a) the oriented trail T_i for n odd;
(b) the oriented trail T_i for n even.

Claim 3. *There exists an integer j with $2 \leq j \leq n-4$ ($2 \leq j \leq n-3$, if n is odd) such that $c(\overrightarrow{y_j x_1}) \neq c_0$ and $c(\overrightarrow{x_1 y_{j+1}}) \neq c_0$.*

Proof. Suppose, to the contrary, that at least one of $c(\overrightarrow{y_j x_1})$ and $c(\overrightarrow{x_1 y_{j+1}})$ is c_0 for every integer j with $2 \leq j \leq n-4$ ($2 \leq j \leq n-3$, if n is odd). It follows that $d_H^{c_0}(x_1) \geq (n-4)/2+2 > (n-2)/2$ ($d_H^{c_0}(x_1) \geq (n-3)/2+2 > (n-1)/2$, if n is odd), contradicting the fact that $\Delta^{mon}(K_n^c) \leq \lfloor (n-1)/2 \rfloor$. This confirms our claim. $\square$

Let $j_0 = \max\{j \mid c(\overrightarrow{y_j x_1}) \neq c_0 \text{ and } c(\overrightarrow{x_1 y_{j+1}}) \neq c_0\}$. We suppose that $y_{j_0} = x_k$. Thus, we have $x_{k+1} = x_1$. We conclude that $d_{H_{k+1}}^{c_0}(x_1) \geq (n-3-j_0-1)/2+1$ and $d_{H_{k+1}}(x_1) = n-2-j_0$ ($d_{H_{k+1}}^{c_0}(x_1) \geq (n-2-j_0-1)/2+1$ and $d_{H_{k+1}}(x_1) = n-1-j_0$, if n is odd), implying that $d_{H_{k+1}}^{c_0}(x_1) \geq d_{H_{k+1}}(x_1)/2$. Recall that $c(\overrightarrow{y_{j_0} x_1}) \neq c_0$. By definition, the edge of $E_{H_{k+1}}(x_1)$ with color c_0 is suitable for T_{k+1} in H . It follows that $c(\overrightarrow{x_1 y_{j_0+1}}) = c_0$ by the rules of Step 3 of Algorithm 1. However, as supposed, we have $c(\overrightarrow{x_1 y_{j_0+1}}) \neq c_0$, a contradiction. This completes the proof of Lemma 5.4.4. $\square$

Lemma 5.4.4 implies that Algorithm 1 will terminate in the case that $E_{H_i}(x_i) = E_{H_i}(x_1) = \{x_i x_1\}$ for some integer i with $i \geq 3$ such that $x_i \neq x_1$. However, it is possible that Algorithm 1 terminates earlier. It is not difficult to check that in all cases the output T of Algorithm 1 is a compatible spanning circuit of K_n^c .

Now, we analyze the time complexity of Algorithm 1. It is obvious from the structure of the algorithm that the combination of Step 4 and Step 3 dominates and determines its time complexity. Since each edge of H is traversed at most once, Step 3 is performed at most $|E(H)| = O(n^2)$ times according to Step 4 of Algorithm 1. In Step 3 of Algorithm 1, it requires at most $O(n^2)$ time to choose a vertex x_{i+1} such that $x_i x_{i+1}$ is suitable for T_i in H : this requires checking and comparing these colors that appear on $x_{i-1} x_i$ and the at most $O(n^2)$ edges of $E_H(x_i)$. Thus, Step 3 takes at most $O(n^2)$ time, yielding an overall time complexity $O(n^4)$. This completes the proof of Theorem 5.3.1.

5.4.2 Proof of Theorem 5.3.2

We prove Theorem 5.3.2 by using a known algorithm due to Pevzner [105] as a subroutine to construct an $O(n^4)$ algorithm, and proving its correctness.

Pevzner [105] provided a polynomial algorithm for constructing a compatible Euler tour (CET for short) in an edge-colored eulerian graph G in which $\Delta^{mon}(v) \leq d(v)/2$ for each vertex v of G (see Algorithm 2). Benkoular et al. [9] independently described a different algorithm for solving the same problem, requiring solving a perfect matching problem for a specific class of complete k -partite graphs.

We use Algorithm 2 as a subroutine to construct an $O(n^4)$ algorithm for finding a compatible spanning circuit visiting every vertex of K_n^c , except for one specific vertex, exactly $(n-2)/2$ times (SCSC for short), subject to the conditions that n is an even integer and $n \geq 4$, as well as the graph K_n^c satisfies $\Delta^{mon}(K_n^c) \leq (n-2)/2$ and $cd(v_0) \geq n - \lfloor (\sqrt{4n-3} + 1)/2 \rfloor$ for some vertex v_0 of K_n^c (see Algorithm 3).

In Algorithm 3, we denote $G' = K_n^c - v_0$, where v_0 is a specific vertex of K_n^c with $cd(v_0) \geq n - \lfloor (\sqrt{4n-3} + 1)/2 \rfloor$. It is not difficult to see that $\Delta_{G'}^{mon}(v) \leq \Delta^{mon}(K_n^c) \leq (n-2)/2 = d_{G'}(v)/2$ for each vertex v of G' . Thus, the graph G' satisfies the conditions of Algorithm 2. This implies that we can use Algorithm 2 as a subroutine in Step 2 of Algorithm 3.

Algorithm 2 [105] Finding a CET in an edge-colored eulerian graph G with $\Delta^{mon}(v) \leq d(v)/2$ for each vertex v .

Input: An edge-colored eulerian graph G with $\Delta^{mon}(v) \leq d(v)/2$ for each vertex v ;

Output: A CET T of G ;

Step 1. $i \leftarrow 1$;

$\mathcal{C} \leftarrow \emptyset$;

Step 2. Choose an arbitrary vertex x_1 of G , and put $T_1 = x_1$;
Choose the next vertex x_2 such that $c(x_1x_2)$ is the most frequent color among the edges of $E(x_1)$, and put $T_2 = x_1x_2$;

while there exists an edge x_jx_{j+1} suitable for T_j in G **do**

Based on $T_j = x_1x_2 \cdots x_j$ ($j \geq 2$), build up $T_{j+1} = x_1x_2 \cdots x_jx_{j+1}$ by choosing a vertex x_{j+1} such that x_jx_{j+1} is suitable for T_j in G , and preferentially choosing the vertex x_{j+1} such that $c(x_jx_{j+1}) = c(x_1x_2)$ when $x_j = x_1$;
 $j \leftarrow j + 1$;

end while

$C_i \leftarrow T_j$ (note that T_j is a closed trail);

$\mathcal{C} \leftarrow \mathcal{C} \cup \{C_i\}$;

$G \leftarrow G - E(C_i)$;

Step 3. **if** $E(G) \neq \emptyset$ **then**

$i \leftarrow i + 1$, and go to **Step 2**;

end if

Step 4. To construct T , **if** $\mathcal{C} \setminus \{C_1\} \neq \emptyset$, then start walking along C_1 , **until** an intersection vertex with another closed trail C_p of $\mathcal{C}$ is found;

Continue walking along C_p while preserving the compatibility on the intersection vertex in the case of walking into C_p and walking out of C_p , **until** C_p is entirely walked out;

Then continue walking along the remaining part of C_1 , **until** C_1 is entirely walked out;

We use C_1 to denote the new closed trail that is the combination of C_1 and C_p , and continue to combine the remaining elements of $\mathcal{C}$, if any, in this way, **until** all elements of $\mathcal{C}$ have been combined into the closed trail T ;

return T .

Algorithm 3 Finding a SCSC in K_n^c (even $n \geq 4$) with $\Delta^{mon}(K_n^c) \leq (n-2)/2$ and $cd(v_0) \geq n - \lfloor (\sqrt{4n-3} + 1)/2 \rfloor$ for some vertex v_0 .

Input: An edge-colored graph K_n^c (even $n \geq 4$) with $\Delta^{mon}(K_n^c) \leq (n-2)/2$ and $cd(v_0) \geq n - \lfloor (\sqrt{4n-3} + 1)/2 \rfloor$ for some vertex v_0 ;

Output: A compatible spanning circuit T of K_n^c visiting every vertex of K_n^c , except for one specific vertex, exactly $(n-2)/2$ times;

Step 1. Choose a specific vertex v_0 of K_n^c with $cd(v_0) \geq n - \lfloor (\sqrt{4n-3} + 1)/2 \rfloor$, and let $G' = K_n^c - v_0$;

Step 2. Perform Algorithm 2 on G' to produce a compatible Euler tour of G' , denoted by $T' = x_1 x_2 \cdots x_1$;

Step 3. Choose an edge $x_i x_{i+1}$ of T' such that $c(x_{i-1} x_i) \neq c(x_i v_0)$ and $c(x_i v_0) \neq c(v_0 x_{i+1})$, as well as $c(v_0 x_{i+1}) \neq c(x_{i+1} x_{i+2})$, where the subscripts are taken modulo $\binom{n-1}{2}$;

Step 4. Let $T = T' \cup \{x_i v_0, v_0 x_{i+1}\} \setminus \{x_i x_{i+1}\}$;

return T .

Next, we show the correctness of Algorithm 3 by stating and proving the following lemma.

Lemma 5.4.5. *There exists an edge $x_i x_{i+1}$ of T' such that $c(x_{i-1} x_i) \neq c(x_i v_0)$ and $c(x_i v_0) \neq c(v_0 x_{i+1})$, as well as $c(v_0 x_{i+1}) \neq c(x_{i+1} x_{i+2})$, where the subscripts are taken modulo $\binom{n-1}{2}$.*

Proof. Suppose, to the contrary, that for each integer i such that $c(x_i v_0) \neq c(v_0 x_{i+1})$, either $c(x_{i-1} x_i) = c(x_i v_0)$, or $c(v_0 x_{i+1}) = c(x_{i+1} x_{i+2})$.

Let $cd(v_0) = n - \ell \geq n - \lfloor (\sqrt{4n-3} + 1)/2 \rfloor$. Thus, we have $\ell \leq (\sqrt{4n-3} + 1)/2$. Let $P = \{\{v_0 x_i, v_0 x_{i+1}\} \subset E_{K_n^c}(v_0) \mid c(v_0 x_i) \neq c(v_0 x_{i+1})\}$. We can conclude that $|P| \geq \binom{n-1}{2} - \binom{\ell}{2} = ((n-1)(n-2))/2 - (\ell(\ell-1))/2 = (n^2 - 3n + 2 - \ell^2 + \ell)/2$.

Based on the above supposition, we have either $c(x_{i-1} x_i) = c(x_i v_0)$, or $c(v_0 x_{i+1}) = c(x_{i+1} x_{i+2})$ for each pair $\{v_0 x_i, v_0 x_{i+1}\}$ of P . Note that the graph G' is a complete graph on $n-1$ vertices. It follows from $\ell \leq (\sqrt{4n-3} + 1)/2$ that $\frac{|P|}{n-1} \geq \frac{n^2 - 3n + 2 - \ell^2 + \ell}{2(n-1)} \geq \frac{n-3}{2} > \frac{n-4}{2}$. Therefore, there exists a vertex v of G' such that $d_{K_n^c}^{c_0}(v) \geq (n-4)/2 + 1 + 1 > (n-2)/2$, where

$c_0 = c(v_0v)$, contradicting that $\Delta^{mon}(K_n^c) \leq (n-2)/2$. This completes the proof of Lemma 5.4.5. $\square$

Lemma 5.4.5 clearly shows that we can always find an edge satisfying the requested conditions at Step 3 of Algorithm 3. The correctness of Algorithm 3 follows directly from Lemma 5.4.5 (and the correctness of Algorithm 2 due to Pevzner [105]). It is not difficult to check that the closed trail returned by Algorithm 3 is a desired compatible spanning circuit of K_n^c .

Now, we analyze the time complexity of Algorithm 3. In order to analyze the time complexity of Algorithm 3, we first need to analyze the time complexity of Algorithm 2 due to Pevzner, since we use it as a subroutine. In fact, it is clear that Algorithm 2 is the dominating factor regarding the time complexity of Algorithm 3. Due to the similarity with Algorithm 1, it is not difficult to see that Algorithm 2 has time complexity $O(n^4)$ (in case the graph G is a complete graph on n vertices). Therefore, the whole time complexity of Algorithm 3 is $O(n^4)$. This completes the proof of Theorem 5.3.2. $\square$

5.5 Conclusions and future work

In this chapter, we considered the existence of compatible spanning circuits in edge-colored graphs from an algorithmic perspective. We first proved that the decision problem of determining whether an edge-colored connected graph contains a compatible spanning circuit is NP-complete, even within graphs with maximum degree 3. We then developed two polynomial-time algorithms for finding compatible spanning circuits (with certain properties) in specific edge-colored complete graphs. In particular, our Algorithm 1 returns a compatible spanning circuit (with no restrictions) directly. In previous work from literature, this was done in two steps. We also presented Algorithm 3 for finding compatible spanning circuits visiting every vertex, except for one specific vertex, exactly $(n-2)/2$ times in edge-colored complete graphs G on even n ($n \geq 4$) vertices with $\Delta^{mon}(G) \leq (n-2)/2$ and $cd(v_0) \geq n - \lfloor (\sqrt{4n-3} + 1)/2 \rfloor$ for some vertex v_0 of G .

In future work, we look forward to establishing polynomial-time algorithms for finding compatible spanning circuits in other classes of edge-colored

graphs. As another future direction, a more challenging problem is to develop polynomial-time algorithms for finding compatible spanning circuits visiting every vertex exactly (or at least) a specified number of times in some specific classes of edge-colored graphs.

Chapter 6

Compatible Euler tours in eulerian graphs with generalized transition systems

In this chapter, we consider the existence of compatible Euler tours in eulerian graphs for which certain generalizations of edge-colorings have been defined. Based on the existing concept of a generalized transition system that can be viewed as a generalization of an edge-coloring, we introduce the new concept of a weakly generalized transition system over a graph, which is an extension of an generalized transition system. We also establish sufficient conditions for the existence of compatible (oriented) Euler tours in eulerian graphs with (weakly) generalized transition systems.

6.1 Introduction

Throughout this chapter, we consider only finite undirected graphs without loops, but possibly containing multiple edges. As we mentioned in Chapter 1, the terminology and notations not defined but used in this chapter are standard and can be found in the most recent version of the textbook of Bondy and Murty [15].

For a vertex v of a graph G , we denote by $E(v)$ the set of edges of G incident with v , and we denote by $P(v)$ a partition of $E(v)$. If a partition $P(v)$ has been specified for each vertex v of G , we will say that G has a *partition system*, denoted by $P(G) = \{P(v) \mid v \in V(G)\}$. An Euler tour X of an eulerian graph G with a partition system $P(G)$ is said to be *compatible* with $P(G)$ if any two consecutive edges of X through v are in distinct classes of $P(v)$ for each vertex v of G . In an eulerian graph G with a partition system $P(G)$, a *compatible Euler tour* of G refers to an Euler tour compatible with $P(G)$. In particular, a compatible Euler tour of an edge-colored eulerian graph refers to an Euler tour in which any two consecutive edges have distinct colors. In this sense, partition systems can be considered as generalizations of edge-colorings in the context of compatible Euler tours.

Kotzig [84] in 1968 established a necessary and sufficient condition for the existence of compatible Euler tours in eulerian graphs with partition systems, as follows.

Theorem 6.1.1 (Kotzig [84]). *Let G be an eulerian graph with a partition system $P(G)$. Then there exists a compatible Euler tour in G if and only if the cardinality of every class of $P(v)$ is at most $d(v)/2$ for each vertex $v \in V(G)$.*

Almost thirty years later, Benkour et al. [9] established a necessary and sufficient condition for the existence of compatible Euler tours in edge-colored eulerian graphs, as follows.

Theorem 6.1.2 (Benkour, Manoussakis, Paschos and Saad [9]). *Let G be an edge-colored eulerian graph. Then there exists a compatible Euler tour in G if and only if $d^i(v) \leq \sum_{j \neq i} d^j(v)$ for each vertex $v \in V(G)$ and colors $i, j \in C(G)$.*

In [9], the authors also described a polynomial-time algorithm for finding a compatible Euler tour in an edge-colored eulerian graph that satisfies the above mentioned conditions.

As a special case of a partition system of a graph, Jackson [77] introduced the concept of a transition system. For a vertex v of a graph G with a partition system $P(G)$, a class of $P(v)$ with two edges is called a *transition*, and a *transition decomposition* $T(v)$ at v is defined as a partition of $E(v)$ into transitions. If a transition decomposition $T(v)$ has been specified for each

vertex v of an eulerian graph G , we will say that the eulerian graph G has a *transition system*, denoted by $T(G) = \{T(v) \mid v \in V(G)\}$. Two transition systems $T_1(G) = \{T_1(v) \mid v \in V(G)\}$ and $T_2(G) = \{T_2(v) \mid v \in V(G)\}$ of an eulerian graph G are said to be *compatible* if $T_1(v) \cap T_2(v) = \emptyset$ for each vertex $v \in V_4(G)$, where $V_4(G)$ denotes the set of vertices of G of degree at least 4. There is a natural bijection between transition systems of an eulerian graph and closed trail decompositions of the eulerian graph (i.e., partitions of the edge set of the graph into closed trails): two edges form a transition if and only if they are consecutive in a closed trail. We use $T(\mathcal{C})$ to denote the transition system associated with the closed trail decomposition $\mathcal{C}$. In particular, each Euler tour X defines a transition system $T(X)$ associated with X . Thus, the compatibility of two Euler tours X_1 and X_2 can be defined by the compatibility of the two associated transition systems $T(X_1)$ and $T(X_2)$. Jackson [77] considered the problem of deciding when a given eulerian graph contains several pairwise compatible Euler tours (see also [48, 52, 78, 79]).

Jackson [78] studied a variation of the problem considered in [77] by giving a set of transitions over an eulerian graph, and finding sufficient conditions for the existence of Euler tours in which no two consecutive edges form a transition in the given set. In order to deal with this problem, Jackson [78] introduced the concept of a generalized transition system. A *generalized transition system* over a graph G is defined as a set of mappings $F(G) = \{F_v \mid v \in V(G)\}$ such that for each vertex $v \in V(G)$, $F_v : E(v) \rightarrow 2^{E(v)}$, where $2^{E(v)}$ denotes the power set of $E(v)$, and whenever $h \in F_v(g)$, we have $g \in F_v(h)$. It is clear that each generalized transition system of a graph G defines a set of transitions over G . An Euler tour of an eulerian graph G with a generalized transition system $F(G)$ is said to be *compatible* with $F(G)$ if no two consecutive edges of the Euler tour form a transition defined by $F(G)$. In an eulerian graph G with a generalized transition system $F(G)$, a *compatible Euler tour* of G refers to an Euler tour compatible with $F(G)$. We denote by $f_v(g)$ the cardinality of $F_v(g)$ for a vertex $v \in V(G)$ and an edge $g \in E(v)$. Jackson [78] established a Dirac-type sufficient condition for the existence of compatible Euler tours in eulerian graphs with generalized transition systems, as follows.

Theorem 6.1.3 (Jackson [78]). *Let G be an eulerian graph with a generalized transition system $F(G)$. If $f_v(g) \leq d(v)/2 - 1$ for each vertex v of G with $d(v) \equiv 0 \pmod{4}$ or $d(v) = 2$ and each incident pair (v, g) , and $f_v(g) \leq d(v)/2 - 2$ otherwise, then there exists a compatible Euler tour in G .*

Dvořák [37] proved that determining whether we can find a compatible Euler tour in an eulerian graph with a generalized transition system is an NP-complete problem. Recently, Krivelevich, Lee and Sudakov [86] presented a different description for the concept of a generalized transition system. A generalized transition system $F(G)$ is said to be k -bounded if $f_v(g) \leq k$ for each vertex $v \in V(G)$ and each edge $g \in E(v)$. The authors [86] pointed out that a k -bounded generalized transition system over a graph can be viewed as a quantitative measure of robustness of graph properties, and that further study of how various extremal results can be strengthened using this concept appears to be a promising direction of research.

Figure 6.1: The graph K_3 with vertex and edge labels.

It is not difficult to see that for each edge-coloring of a graph G , there always exists a trivial partition system $P(G)$ of G such that for each vertex v of G , each class of $P(v)$ consists of all edges of $E(v)$ with the same color. But the converse is not necessarily true. For some partition system of a graph, there may be no such edge-coloring of the graph corresponding to it. For example, let us consider the graph K_3 depicted in Figure 6.1. For the partition system $P(K_3) = \{P(v_1), P(v_2), P(v_3)\}$, where $P(v_1) = \{\{e_1, e_3\}\}$, $P(v_2) = \{\{e_1, e_2\}\}$ and $P(v_3) = \{\{e_2\}, \{e_3\}\}$, we cannot find an edge-coloring of K_3 corresponding to it. Thus, a partition system of a graph G can be viewed as a generalization of an edge-coloring of G . It is trivial that each partition system defines a generalized transition system. But the converse is not necessarily true. For

example, let us consider the graph $K_{1,3}$ depicted in Figure 6.2. For the generalized transition system $F(K_{1,3}) = \{F_u, F_{v_1}, F_{v_2}, F_{v_3}\}$, where $F_u(e_1) = \{e_2, e_3\}$, $F_u(e_2) = \{e_1\}$ and $F_u(e_3) = \{e_1\}$, and $F_{v_1}(e_1) = F_{v_2}(e_2) = F_{v_3}(e_3) = \emptyset$, we cannot find a partition system of $K_{1,3}$ corresponding to it. Thus, a generalized transition system over a graph G can be viewed as a generalization of a partition system of G . Therefore, a generalized transition system over a graph G can also be viewed as a generalization of an edge-coloring of G .

Figure 6.2: The graph $K_{1,3}$ with vertex and edge labels.

There are some interesting open problems and conjectures on compatible Euler tours (see [78]). Moreover, compatible Euler tours have many important applications in various areas. Pevzner [105] used compatible Euler tours of edge-colored eulerian graphs to reconstruct DNA from its segments. Compatible Euler tours also have applications in routing problems. In practice, there are often additional restrictions on the allowed turns in a route. For instance, we could restrict the edges of a road network such that a compatible Euler tour is a route on which a mail truck avoids turning left [97]. UPS and other delivery services [96] used such routes to reduce the time of deliveries and the number of accidents.

In this chapter, we first consider the existence of compatible Euler tours in eulerian graphs with generalized transition systems. We further introduce the new concept of a weakly generalized transition system, which is an extension of a generalized transition system, and we also consider the existence of compatible oriented Euler tours in eulerian graphs with weakly generalized transition systems.

6.2 CETs for generalized transition systems

In this section, we establish Ore-type sufficient conditions for the existence of compatible Euler tours in eulerian graphs with generalized transition systems, which slightly improve some previous results due to other researchers.

Before proceeding, we introduce some essential terminology, and we list some basic results which will be used in the later proof of the main theorem of this section. All the results that are listed below are from existing literature and due to different (groups of) researchers.

A subset U of $V(G)$ is called an *E-separator* of a graph G if there exist two nonempty subgraphs G_1 and G_2 of G such that:

- (a) at least one of G_1 and G_2 is eulerian;
- (b) $V(G_1) \cap V(G_2) = U$; and
- (c) $E(G_1) \cap E(G_2) = \emptyset$ and $E(G_1) \cup E(G_2) = E(G)$.

We use $L(G)$ to denote the line graph of a graph G . Let G be an eulerian graph with a generalized transition system $F(G)$. For $v \in V(G)$, let $L'_v = L_v - F(v)$, where $L_v = L(G[E(v)])$ and $F(v) = \{gh \in E(L_v) \mid h \in F_v(g)\}$. A vertex v of G is called a *bad vertex* if there exists a perfect matching M of L_v such that M is not contained in any Hamilton cycle of $L_v^* = L'_v \cup M$, under the condition that $d_{L_v^*}(g) + d_{L_v^*}(h) \geq |V(L_v^*)|$ for every pair of nonadjacent vertices g, h of L_v^* ; otherwise, the vertex v is called a *good vertex* of G .

In 1960, Ore [101] obtained the following classic sufficient condition for a graph to be hamiltonian.

Theorem 6.2.1 (Ore [101]). *Let G be a graph on $n \geq 3$ vertices. If $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices x, y of G , then G is hamiltonian.*

Kotzig [84] presented the following observation.

Observation 6.2.1 (Kotzig [84]). *Let G be an eulerian graph. Then each closed trail decomposition of G forms exactly one transition system of G , and for each transition system $T(G)$ of G , there exists exactly one closed trail decomposition $\mathcal{C}$ of G such that $T(\mathcal{C}) = T(G)$.*

Jackson and Wormald [79] provided a sufficient condition guaranteeing that a perfect matching of a graph is contained in a Hamilton cycle of the graph, as follows.

Lemma 6.2.1 (Jackson and Wormald [79]). *Let G be a graph on n vertices, where $n \equiv 0 \pmod{2}$, and let M be a perfect matching of G . If $d(x) + d(y) \geq n$ for every pair of nonadjacent vertices x, y of G , then M is contained in a Hamilton cycle of G , unless at least one of the following holds:*

- (a) $G = K_2 + (K_{2s} \cup K_{2t})$, where $K_2 \in M$;
- (b) $G - M = 2K_{2t+1}$;
- (c) $K_{2t, 2t+2} \subseteq G - M \subset K_{2t} + \overline{K_{2t+2}}$.

Lemma 6.2.1 is used to characterize the bad vertices of an eulerian graph with a generalized transition system. From Lemma 6.2.1, the following corollary is immediate.

Corollary 6.2.1 (Jackson and Wormald [79]). *Let G be a graph on n vertices, where $n \equiv 0 \pmod{4}$. If $d(x) \geq n/2$ for every vertex x of G , then every perfect matching of G is contained in a Hamilton cycle of G .*

Häggkvist [71] also provided a sufficient condition guaranteeing that a perfect matching of a graph is contained in a Hamilton cycle of the graph, as follows.

Theorem 6.2.2 (Häggkvist [71]). *Let G be a graph on n vertices, where $n \equiv 0 \pmod{2}$, and let M be a perfect matching of G . If $d(x) + d(y) \geq n + 1$ for every pair of nonadjacent vertices x, y of G , then M is contained in a Hamilton cycle of G .*

Next, we present the main results of this section. We establish the following Ore-type sufficient condition for the existence of compatible Euler tours in eulerian graphs with generalized transition systems, which is weaker than Jackson's condition in Theorem 6.1.3.

Theorem 6.2.3. *Let G be an eulerian graph with a generalized transition system $F(G)$. If $f_v(g) + f_v(h) \leq d(v) - 2$ for each vertex $v \in V(G)$ and every pair of edges g, h with $h \in F_v(g)$, and G has no E -separator consisting of bad vertices, then there exists a compatible Euler tour in G .*

Proof. Let G be an eulerian graph with a generalized transition system $F(G)$. Let v be a vertex of G , and let $L'_v = L_v - F(v)$, where $L_v = L(G[E(v)])$ and $F(v) = \{gh \in E(L_v) \mid h \in F_v(g)\}$. It follows from $f_v(g) + f_v(h) \leq d(v) - 2$ for every pair of edges g, h with $h \in F_v(g)$ that $d_{L'_v}(g) + d_{L'_v}(h) \geq d(v) = |V(L'_v)|$ for every pair of nonadjacent vertices g, h of L'_v . By Theorem 6.2.1, the graph L'_v is hamiltonian. Since the graph G is an eulerian graph, we conclude that $d(v)$ is even. Thus, there exists a perfect matching M in L'_v .

Based on the perfect matching M , we can construct a transition decomposition $T(v)$ at v : each two edges in $E(v)$ corresponding to an edge of M forms a transition in $T(v)$. By making such a construction for each vertex v of G , we can get a transition system of G , denoted by $T(G) = \{T(v) \mid v \in V(G)\}$. By Observation 6.2.1, there exists a closed trail decomposition $\mathcal{C} = \{C_1, \dots, C_m\}$ of G such that $T(\mathcal{C}) = T(G)$. By the construction of $T(G)$, it can be checked that each closed trail in $\mathcal{C}$ is compatible with $F(G)$. If $m = 1$, then the closed trail decomposition $\mathcal{C}$ gives a desired Euler tour.

Now, let us consider the case that $m \geq 2$. For a closed trail in the closed trail decomposition $\mathcal{C}$, we can form an E-separator by considering the closed trail as a subgraph. Let U be the E-separator associated with a given closed trail in $\mathcal{C}$. It follows that there exists at least one good vertex $u \in U$ such that u is a common vertex of at least two closed trails in $\mathcal{C}$. Let u_0 be such a good vertex in U . By the following discussion, we can combine all closed trails through u_0 in $\mathcal{C}$ into one closed trail that is compatible with $F(G)$.

We use $T(u_0) = \{\{e_1, e_2\}, \{e_3, e_4\}, \dots, \{e_{d(u_0)-1}, e_{d(u_0)}\}\}$ to denote the transition decomposition at u_0 , according to the order in which these transitions through u_0 appear, when we travel along closed trails in $\mathcal{C}$ one by one. Without loss of generality, we use C_1 to denote the first closed trail of $\mathcal{C}$ in which a transition of $T(u_0)$ appears. We denote by t_1 ($t_1 \leq d(u_0)/2 - 1$) the number of transitions through u_0 that appear in C_1 . Hence, we have $C_1 = e_1 e_2 T_{2,3} e_3 \cdots e_{2t_1-1} e_{2t_1} T_{2t_1,1} e_1$, where $T_{i,j}$ denotes the subtrail of C_1 connecting e_i and e_j for $1 \leq i, j \leq 2t_1$. We can denote all closed trails through u_0 in $\mathcal{C}$, in such a similar way. It is not difficult to see that we can obtain a new perfect matching M' of L_{u_0} , which corresponds to all pairs of edges of $E(u_0)$ connected by the subtrails $T_{i,j}$ of all closed trails through u_0 in $\mathcal{C}$.

Let $L'_{u_0} = L_{u_0} - F(u_0)$, where $L_{u_0} = L(G[E(u_0)])$ and $F(u_0) = \{gh \in E(L_{u_0}) \mid h \in F_{u_0}(g)\}$, and let $L^*_{u_0} = L'_{u_0} \cup M'$. It follows from $f_{u_0}(g) + f_{u_0}(h) \leq d(u_0) - 2$ for every pair of edges g, h with $h \in F_{u_0}(g)$ that $d_{L'_{u_0}}(g) + d_{L'_{u_0}}(h) \geq |V(L'_{u_0})|$ for every pair of nonadjacent vertices g, h of L'_{u_0} . Thus, we conclude that $d_{L^*_{u_0}}(g) + d_{L^*_{u_0}}(h) \geq |V(L^*_{u_0})|$ for every pair of nonadjacent vertices g, h of $L^*_{u_0}$. Recall that u_0 is a good vertex (good vertices were characterized by Lemma 6.2.1). It follows that M' is contained in a Hamilton cycle C of $L^*_{u_0}$. Let $M'' = C - M'$. Clearly, M'' is another perfect matching of L'_{u_0} . Based on the perfect matching M'' , we can construct another transition decomposition $T'(u_0)$ at u_0 : each two edges in $E(u_0)$ corresponding to an edge of M'' forms a transition in $T'(u_0)$. Replacing $T(u_0)$ by $T'(u_0)$ in $T(G)$, we can combine all closed trails through u_0 in $\mathcal{C}$ into one closed trail compatible with $F(G)$. This operation reduces the number of closed trails in $\mathcal{C}$, as required. After a finite number of such combinations, we can obtain a compatible Euler tour. This completes the proof. $\square$

Figure 6.3: The graph illustrating Example 6.2.1.

Remark 6.2.1. *The following example shows that the bound on $f_v(g) + f_v(h)$ in Theorem 6.2.3 is tight for some vertex $v \in V(G)$ and some pair of edges g, h with $h \in F_v(g)$.*

Example 6.2.1. *Let G be an eulerian graph as depicted in Figure 6.3, where all subtrails T_i connecting g_i and h_i ($1 \leq i \leq 2t + 1$) are disjoint from each other and each T_i along with g_i and h_i forms a compatible subtrail. Let $F(G)$ be a generalized transition system of G such that for the vertex v with $d(v) = 4t + 2$, $F_v(g_i) = \{h_j \mid j \neq i\}$ for $1 \leq i \leq 2t + 1$, and $F_v(h_1) = \{g_i \mid i \neq 1\} \cup \{h_2\}$, $F_v(h_2) = \{g_i \mid i \neq 2\} \cup \{h_1\}$ and $F_v(h_j) = \{g_i \mid i \neq j\}$ for $3 \leq j \leq 2t + 1$.*

One can check that the eulerian graph G of Example 6.2.1 satisfies that $f_v(g_1) + f_v(h_2) = f_v(g_2) + f_v(h_1) = 4t + 1 = d(v) - 1$ for the vertex v with $d(v) = 4t + 2$. However, the graph G has no Euler tour compatible with $F(G)$.

From Theorem 6.2.3, we can obtain the following corollary.

Corollary 6.2.2. *Let G be an eulerian graph with a generalized transition system $F(G)$. If $f_v(g) \leq d(v)/2 - 1$ for each vertex $v \in V(G)$ and each incident pair (v, g) , and G has no E -separator consisting of bad vertices, then there exists a compatible Euler tour in G .*

Using similar arguments as in the proof of Theorem 6.2.3, we can provide an alternative proof for Theorem 6.1.3 due to Jackson [78] by replacing Lemma 6.2.1 with Corollary 6.2.1 for a vertex v with $d(v) \equiv 0 \pmod{4}$ and Theorem 6.2.2 for a vertex v with $d(v) \equiv 2 \pmod{4}$ and $d(v) \neq 2$, respectively.

6.3 CETs for weakly generalized transition systems

In this section, we first introduce the new concept of a weakly generalized transition system, which is an extension of a generalized transition system. We further establish sufficient conditions for the existence of compatible oriented Euler tours in eulerian graphs with weakly generalized transition systems.

Based on the concept of a generalized transition system over a graph due to Jackson [78], we can more naturally introduce the concept of a weakly

generalized transition system. A *weakly generalized transition system* over a graph G is defined as a set of mappings $\vec{F}(G) = \{F_v^+ \mid v \in V(G)\}$ such that for each vertex $v \in V(G)$, $F_v^+ : E(v) \rightarrow 2^{E(v)}$, where $2^{E(v)}$ denotes the power set of $E(v)$, and when $h \in F_v^+(g)$, it is not necessary that $g \in F_v^+(h)$. It is clear that each weakly generalized transition system of a graph G defines a set of oriented transitions over G . An oriented Euler tour of an eulerian graph G with a weakly generalized transition system $\vec{F}(G)$ is said to be *compatible* with $\vec{F}(G)$ if no two consecutive edges in the oriented Euler tour form an oriented transition defined by $\vec{F}(G)$. In an eulerian graph G with a weakly generalized transition system $\vec{F}(G)$, a *compatible oriented Euler tour* of G refers to an oriented Euler tour compatible with $\vec{F}(G)$. Moreover, for a vertex $v \in V(G)$ and an edge $g \in E(v)$, we write $F_v^-(g) = \{h \mid g \in F_v^+(h)\}$, and we use $f_v^+(g)$ and $f_v^-(g)$ to denote the cardinality of $F_v^+(g)$ and the cardinality of $F_v^-(g)$, respectively.

Replacing each transition of a generalized transition system by two oriented transitions with different orientations, we can get a weakly generalized transition system. Thus, a weakly generalized transition system is a more general form.

Before proceeding, we introduce some essential terminology, and we list the following basic results which will be used in the later proof of the main theorem of this section. Woodall [118] obtained the following classic sufficient condition for a digraph to be hamiltonian.

Theorem 6.3.1 (Woodall [118]). *Let D be a digraph on n vertices. If $d^+(x) + d^-(y) \geq n$ for every pair of vertices x, y with $xy \notin A(D)$, then D is hamiltonian.*

A *directed matching* of a digraph is defined as a set of arcs in which no two arcs share a common vertex. Let M be a directed perfect matching of a digraph D . A balanced partition (B, W) of $V(D)$ is said to be a *partition induced by M* if B is the set of all tails of arcs in M and W is the set of all heads of arcs in M .

We provide the following sufficient condition that guarantees every directed perfect matching of a digraph is contained in a directed Hamilton cycle of the digraph.

Lemma 6.3.1. *Let D be a digraph on n vertices, where $n \equiv 0 \pmod{2}$. If $d^+(x) + d^-(y) \geq 3n/2$ for every pair of vertices x, y with $xy \notin A(D)$, then every directed perfect matching of D is contained in a directed Hamilton cycle of D .*

Proof. By Theorem 6.3.1, the digraph D contains a directed Hamilton cycle. Since the digraph D has even n vertices, it contains a directed perfect matching. Let M be a given directed perfect matching of D , and let D' be a new digraph such that $V(D') = V(D)$ and $uv \in A(D')$ if and only if there is an arc from the head of u to the tail of v in D . Clearly, there exists a directed Hamilton cycle in D' if and only if there exists a directed Hamilton cycle containing M in D .

Let (B, W) be the balanced partition of D induced by M . It follows from $d^+(x) + d^-(y) \geq 3n/2$ for every pair of vertices x, y with $xy \notin A(D)$ that $d_B^+(w) + d_W^-(b) \geq n/2 + 2$ for every pair of vertices $w \in W, b \in B$ with $wb \notin A(D)$. Note that there may be a reversed arc in D for each arc of M . Thus, we can conclude that $d_{D'}^+(u) + d_{D'}^-(v) \geq n/2$ for every pair of vertices u, v with $uv \notin A(D')$. By Theorem 6.3.1, there exists a directed Hamilton cycle in D' . Therefore, there exists a directed Hamilton cycle containing M in D . $\square$

Next, we present the main results of this section. We establish an Ore-type sufficient condition for the existence of compatible oriented Euler tours in eulerian graphs with weakly generalized transition systems, as follows. The following theorem will be proved by using an approach similar to that taken by Jackson [78]. Throughout the rest of this section, we use $V_4(G)$ to denote the set of vertices of degree at least 4 of a graph G .

Theorem 6.3.2. *Let G be an eulerian graph with a weakly generalized transition system $\vec{F}(G)$. If $f_v^+(g) + f_v^-(h) \leq d(v)/2 - 2$ for each vertex $v \in V_4(G)$ and every pair of edges g, h with $h \in F_v^+(g)$, and $f_v^+(g) = f_v^-(g) = 0$ for every incident pair (v, g) with $d(v) = 2$, then there exists a compatible oriented Euler tour in G .*

Proof. We prove the theorem by induction on $|V_4(G)|$. If $|V_4(G)| = 0$, then the theorem is trivial. Now, we assume that $|V_4(G)| \geq 1$. Choose a vertex

$v \in V_4(G)$. Define a new digraph D_v with $V(D_v) = E(v)$ in which $gh \in A(D_v)$ if and only if $h \notin F_v^+(g)$. Given a subset S of $E(v)$, define the detachment of G at v along S to be the graph G^S with $V(G^S) = (V(G) - \{v\}) \cup \{v', v''\}$, $E(G^S) = E(G)$, $E_{G^S}(u) = E_G(u)$ for all $u \in V(G) - \{v\}$, $E_{G^S}(v') = S$ and $E_{G^S}(v'') = E_G(v) - S$.

Let R be the set of all transitions S at v in the graph G such that G^S is disconnected. By giving an orientation for each transition in R , we obtain a set $\vec{R}$ of edge-disjoint oriented transitions. Let $T(v)$ be an oriented transition decomposition at v which contains $\vec{R}$. Based on the fact that each oriented transition in $T(v)$ corresponds to an arc of D_v or $\overline{D_v}$, let $D'_v = D_v \cup T(v)$. It is clear that $T(v)$ corresponds to a directed perfect matching M of D'_v . It follows from $f_v^+(g) + f_v^-(h) \leq d(v)/2 - 2$ for every pair of edges g, h with $h \in F_v^+(g)$ that $d_{D'_v}^+(g) + d_{D'_v}^-(h) \geq 3d(v)/2$ for every pair of vertices g, h with $gh \notin A(D'_v)$. Thus, we conclude that the directed perfect matching M is contained in a directed Hamilton cycle C of D'_v by Lemma 6.3.1. Let $M' = C - M$. Clearly, M' is another directed perfect matching of D_v . We can obtain another oriented transition decomposition $T'(v)$ at v corresponding to M' . Let G' be the graph obtained by recursively detaching G along each oriented transition in $T'(v)$. We may deduce from the choice of $T'(v)$ that the graph G' is connected and hence eulerian. We may now apply induction to G' to obtain an oriented Euler tour of G' compatible with the weakly generalized transition system $\vec{F}'(G')$ obtained by putting $F_u'^+(g) = F_u^+(g)$ for each vertex $u \in V(G) - \{v\}$ and each edge $g \in E(u)$, and putting $F_u'^+(g) = \emptyset$, otherwise. This gives rise to an oriented Euler tour of G compatible with the weakly generalized transition system $\vec{F}(G)$. This completes the proof. $\square$

Remark 6.3.1. We do not know whether the bound on $f_v^+(g) + f_v^-(h)$ in Theorem 6.3.2 is tight. The following example shows that the bound on $f_v^+(g) + f_v^-(h)$ is at most $d(v) - 1$.

Example 6.3.1. let G be an eulerian graph with a weakly generalized transition system $\vec{F}(G)$. For a given vertex $v \in V(G)$ and a given pair of edges $g, h \in E(v)$, let $E'(v) = \{e_i \in E(v) \mid 1 \leq i \leq d(v)/2 - 1 \text{ and } e_i \neq g, h\}$. We suppose that $\vec{F}(G)$ satisfies that $F_v^+(g) = E'(v) \cup \{h\}$, $F_v^+(h) = E'(v) \cup \{g\}$ and $F_v^+(e_i) = (E'(v) \setminus \{e_i\}) \cup \{g, h\}$ for each integer i with $1 \leq i \leq d(v)/2 - 1$.

One can check that the eulerian graph G of Example 6.3.1 satisfies $f_v^+(g) + f_v^-(h) = d(v)$, $f_v^+(h) + f_v^-(g) = d(v)$ and $f_v^+(e_i) + f_v^-(e_j) = d(v)$ for $1 \leq i, j \leq d(v)/2 - 1$, and that there is no oriented Euler tour compatible with $\vec{F}(G)$ in G .

From Theorem 6.3.2, we can obtain the following corollary. In order to state the corollary, we let $f_v(g) = \max\{f_v^+(g), f_v^-(g)\}$ for a vertex $v \in V(G)$ and an incident pair (v, g) .

Corollary 6.3.1. *Let G be an eulerian graph with a weakly generalized transition system $\vec{F}(G)$. If $f_v(g) \leq d(v)/4 - 1$ for each vertex $v \in V_4(G)$ and each incident pair (v, g) , and $f_v(g) = 0$ otherwise, then there exists a compatible oriented Euler tour in G .*

6.4 Conclusions and future work

In this chapter, as an extremal case of compatible spanning circuits, we established sufficient conditions for the existence of compatible (oriented) Euler tours in eulerian graphs with (weakly) generalized transition systems. Our results slightly improve some previous results due to other researchers. As we analyzed earlier, a (weakly) generalized transition system over a graph G can be viewed as a generalization of an edge-coloring of G . As a consequence, our results provide sufficient conditions for the existence of compatible (oriented) Euler tours in eulerian graphs for which some generalizations of edge-colorings have been defined.

As another extremal case of compatible spanning circuits, sufficient conditions for the existence of compatible Hamilton cycles in Dirac graphs with generalized transition systems and random graphs with generalized transition systems have also been provided in [85, 86].

In future work, we look forward to establishing sufficient conditions for the existence of more general compatible (oriented) spanning circuits (i.e., not necessarily a compatible Hamilton cycle or Euler tour) in graphs with (weakly) generalized transition systems.

Chapter 7

Compatible Euler tours in eulerian digraphs with generalized transition systems

In this chapter, we consider the existence of compatible Euler tours in eulerian digraphs with generalizations of arc-colorings. We establish sufficient conditions for the existence of compatible Euler tours in eulerian digraphs with generalized transition systems. As corollaries of some known results, we also provide sufficient conditions for the existence of compatible Euler tours in specific arc-colored eulerian digraphs.

7.1 Introduction

Throughout this chapter, we consider only finite digraphs without loops, but possibly containing multiple arcs. As we mentioned in Chapter 1, the terminology and notations not defined but used in this chapter are standard and can be found in the most recent version of the textbook of Bondy and Murty [15].

Let D be a digraph. We denote by $V(D)$ and $A(D)$ the set of vertices of D and the set of arcs of D , respectively. For a vertex $v \in V(D)$, we use $A_D^-(v)$

and $A_D^+(v)$ to denote the set of arcs of D oriented into v and the set of arcs of D oriented away from v , respectively. The *indegree* $d_D^-(v)$ of a vertex v in D is defined to be the cardinality of $A_D^-(v)$, and the *outdegree* $d_D^+(v)$ of a vertex v in D is defined to be the cardinality of $A_D^+(v)$. Let $A_D(v) = A_D^-(v) \cup A_D^+(v)$. The *degree* $d_D(v)$ of a vertex v in D is defined to be the cardinality of $A_D(v)$, i.e., $d_D(v) = d_D^-(v) + d_D^+(v)$. When no confusion can occur, we will use the notations $A^-(v)$, $A^+(v)$, $A(v)$, $d^-(v)$, $d^+(v)$ and $d(v)$ instead of $A_D^-(v)$, $A_D^+(v)$, $A_D(v)$, $d_D^-(v)$, $d_D^+(v)$ and $d_D(v)$, respectively.

For a vertex v of a digraph D , we denote by $P(v)$ a partition of the set $A(v)$. If a partition $P(v)$ has been specified for every vertex v of D , we will say that D has a *partition system*, denoted by $P(D) = \{P(v) \mid v \in V(D)\}$. An Euler tour X of an eulerian digraph D with a partition system $P(D)$ is said to be *compatible* with $P(D)$ if any two consecutive arcs of X through v are in distinct classes of $P(v)$ for each vertex $v \in V(D)$. In an eulerian digraph D with a partition system $P(D)$, a *compatible Euler tour* of D refers to an Euler tour compatible with $P(D)$. Fleischner and Fulmek [51] showed that the necessary and sufficient condition for the existence of compatible Euler tours in eulerian graphs with partition systems, provided by Kotzig [84], cannot be reformulated in a straightforward way in order to obtain a necessary and sufficient condition for the existence of compatible Euler tours in eulerian digraphs with partition systems, by constructing the following small example.

Example 7.1.1 (Fleischner and Fulmek [51]). *Let D be an eulerian digraph such that $V(D) = \{v_1, v_2\}$, $A^+(v_1) = A^-(v_2) = \{a_1, a_3, a_5\}$ and $A^-(v_1) = A^+(v_2) = \{a_2, a_4, a_6\}$. Let $P(D)$ be a partition system of D such that $P(v_1) = \{\{a_1, a_2\}, \{a_3, a_4\}, \{a_5, a_6\}\}$ and $P(v_2) = \{\{a_1, a_2\}, \{a_3, a_6\}, \{a_4, a_5\}\}$.*

It is easy to verify that the partition system $P(D)$ of D of Example 7.1.1 satisfies that the cardinality of every class of $P(v_i)$ is at most $d(v_i)/2$ for $i \in \{1, 2\}$. However, the eulerian digraph D has no Euler tour compatible with $P(D)$. Thus, the condition stated by Kotzig [84] is not sufficient for the existence of compatible Euler tours in eulerian digraphs with partition systems. The problem of determining necessary and sufficient conditions for the existence of compatible Euler tours in eulerian digraphs with partition systems seems much harder.

Afterward, Fleischner and Fulmek [51] established the following sufficient condition for the existence of compatible Euler tours in eulerian digraphs with partition systems.

Theorem 7.1.1 (Fleischner and Fulmek [51]). *Let D be an eulerian digraph with a partition system $P(D)$. Then there exists a compatible Euler tour in D , if the cardinality of every class of $P(v)$ is at most $d(v)/2$ for each vertex v of D , and $|P(v)| \geq d(v)/2 + 2$ for each vertex v of D with $d(v) \geq 4$.*

More recently, Carraher and Hartke [19] provided necessary and sufficient conditions for the existence of compatible Euler tours in arc-colored eulerian digraphs avoiding certain vertices of outdegree 3. In [20], the same authors further established necessary and sufficient conditions for the existence of compatible Euler tours in several specific families of arc-colored eulerian digraphs (the first half and the latter half of an arc can receive different colors), whose vertex set consists of vertices of outdegree and indegree 3.

For a vertex v of a digraph D with a partition system $P(D)$, a class of $P(v)$ consisting of one arc oriented into v and one arc oriented away from v is called a *transition*, and a *transition decomposition* $T(v)$ at v is defined as a partition of $A(v)$ into transitions. A *transition system* of an eulerian digraph D , denoted by $T(D) = \{T(v) \mid v \in V(D)\}$, refers to a partition system of D in which each class is a transition. Two transition systems $T_1(D) = \{T_1(v) \mid v \in V(D)\}$ and $T_2(D) = \{T_2(v) \mid v \in V(D)\}$ are said to be *compatible* if $T_1(v) \cap T_2(v) = \emptyset$ for each vertex $v \in V_2(D)$, where $V_2(D)$ denotes the set of vertices of D of indegree and outdegree at least 2. There is a natural bijection between transition systems of an eulerian digraph and closed trail decompositions of the digraph (a partition of the arc set into closed trails): two arcs form a transition if and only if they are consecutive in a closed trail. We use $T(\mathcal{C})$ to denote the transition system associated with the closed trail decomposition $\mathcal{C}$. In particular, each Euler tour X defines a transition system $T(X)$ associated with X . Thus, the compatibility of two Euler tours X_1 and X_2 can be defined by the compatibility of the two associated transition systems $T(X_1)$ and $T(X_2)$. The problem of deciding when a given eulerian digraph contains several pairwise compatible Euler tours was considered in [53].

Similarly to the case of undirected graphs, a *generalized transition system* over a digraph D is defined as a set of mappings $F(D) = \{F_v^+ \mid v \in V(D)\}$ such that for each vertex $v \in V(D)$, $F_v^+ : A^-(v) \rightarrow 2^{A^+(v)}$, where $2^{A^+(v)}$ denotes the power set of $A^+(v)$. Moreover, for a vertex $v \in V(D)$ and an arc $b \in A^+(v)$, we write $F_v^-(b) = \{a \in A^-(v) \mid b \in F_v^+(a)\}$, and we use $f_v^+(a)$ and $f_v^-(b)$ to denote the cardinality of $F_v^+(a)$ and the cardinality of $F_v^-(b)$, respectively. Considering only an incident pair (v, a) , we use $f_v(a)$ to denote $f_v^+(a)$ or $f_v^-(a)$.

It is not difficult to see that for each arc-coloring of a digraph D , there always exists a generalized transition system $F(D)$ of D such that for each vertex v of D and each arc $a \in A^-(v)$, $F_v^+(a) = \{b \in A^+(v) \mid c(b) = c(a)\}$, where $c(a)$ and $c(b)$ denote the color appearing on the arc a and the color appearing on the arc b , respectively. But the converse is not necessarily true. For some generalized transition system of a digraph, there may be no such arc-coloring of the digraph corresponding to it. For example, let us consider the digraph $\vec{K}_3$ depicted in Figure 7.1. Let $F(\vec{K}_3)$ be a generalized transition system over $\vec{K}_3$ such that $F_{v_1}^+(a_1) = \{a_2\}$, $F_{v_2}^+(a_2) = \{a_3\}$ and $F_{v_3}^+(a_3) = \emptyset$. We cannot find an arc-coloring of $\vec{K}_3$ corresponding to $F(\vec{K}_3)$. Thus, a generalized transition system over a digraph D can be viewed as a generalization of an arc-coloring of D .

Figure 7.1: The digraph $\vec{K}_3$ with vertex and arc labels.

Isaak [76] provided a Dirac-type sufficient condition for the existence of compatible Euler tours in eulerian digraphs with generalized transition systems, as follows.

Theorem 7.1.2 (Isaak [76]). *Let D be an eulerian digraph with a generalized transition system $F(D)$. If $f_v(a) \leq d^+(v)/2 - 1$ for each incident pair (v, a) , then there exists a compatible Euler tour in D .*

In this chapter, we continue the research on sufficient conditions for the existence of compatible Euler tours in eulerian digraphs with generalized transition systems. We also consider the existence of compatible Euler tours in specific arc-colored eulerian digraphs.

7.2 CETs for generalized transition systems

In this section, we establish Ore-type sufficient conditions for the existence of compatible Euler tours in eulerian digraphs with generalized transition systems, which slightly improve some previous results due to other researchers.

Before proceeding, we introduce some essential terminology, and we list some basic results which will be used in the later proof of the main theorem of this section. All the results that are listed below are from existing literature and due to different (groups of) researchers.

A subset U of $V(D)$ is called an *E-separator* of a digraph D if there exist two nonempty subgraphs D_1 and D_2 of D such that:

- (a) at least one of D_1 and D_2 is eulerian;
- (b) $V(D_1) \cap V(D_2) = U$; and
- (c) $A(D_1) \cap A(D_2) = \emptyset$ and $A(D_1) \cup A(D_2) = A(D)$.

For a vertex v of an eulerian digraph D with a generalized transition system $F(D)$, let L_v be a balanced complete bipartite graph with partite sets $A^-(v)$ and $A^+(v)$. Let $L'_v = L_v - F(v)$, where $F(v) = \{ab \in E(L_v) \mid b \in F_v^+(a)\}$. A vertex v of D is called a *bad vertex* if there exists a perfect matching M of L_v such that M is not contained in any Hamilton cycle of $L_v^* = L'_v \cup M$, under the condition that $d_{L_v^*}(a) + d_{L_v^*}(b) \geq |V(L_v^*)|/2 + 1$ for every pair of vertices a, b of L_v^* with $a \in A^-(v)$, $b \in A^+(v)$ and $ab \notin E(L_v^*)$; otherwise, the vertex v is called a *good vertex* of D .

In 1963, Moon and Moser [98] established the following sufficient condition for a balanced bipartite graph to be hamiltonian.

Theorem 7.2.1 (Moon and Moser [98]). *Let G be a balanced bipartite graph on $2n$ vertices with partite sets X and Y . If $d(x) + d(y) \geq n + 1$ for every pair of vertices x, y of G with $x \in X$, $y \in Y$ and $xy \notin E(G)$, then G is hamiltonian.*

For the case of eulerian digraphs, the following statement similar to Observation 6.2.1 holds immediately.

Observation 7.2.1. *Let D be an eulerian digraph. Then each closed trail decomposition of D forms exactly one transition system of D , and for each transition system $T(D)$ of D , there exists exactly one closed trail decomposition $\mathcal{C}$ of D such that $T(\mathcal{C}) = T(D)$.*

Figure 7.2: Exceptional graphs for Theorem 7.2.2.

Zhang et al. [122] introduced the following classes of graphs. Let $\mathcal{G}_1$ denote the class of graphs constructed by identifying an edge of a complete

bipartite graph $K_{m+1,m+1}$ with an edge of another complete bipartite graph $K_{n+1,n+1}$, and let $\mathcal{M}_1$ denote the set of perfect matchings of $G \in \mathcal{G}_1$ containing the identified edge. Let $\mathcal{G}_2$ denote the class of graphs constructed from a balanced bipartite graph G_1 consisting of $n+1$ independent edges, whose partite sets are X and Y , and an arbitrary balanced bipartite graph G_2 on $2n$ vertices containing at least one perfect matching, whose partite sets are X_1 and Y_1 , by connecting each vertex of X to each vertex of Y_1 , and by connecting each vertex of Y to each vertex of X_1 . Furthermore, let $\mathcal{M}_2$ denote the set of perfect matchings of $G \in \mathcal{G}_2$ containing all the edges of G_1 (see Figure 7.2 (a)). Let G_3 be the graph shown in Figure 7.2 (b), and let $\mathcal{G}_3$ denote the class of graphs constructed from a complete bipartite graph $K_{n,n}$ with partite sets X and Y and the graph G_3 , by connecting each vertex of X to y_0 and y_1 , and by connecting each vertex of Y to x_0 and x_1 , and possibly adding any of the edges y_0x_1 , y_1x_0 , or both. Furthermore, let $\mathcal{M}_3$ denote the set of perfect matchings of $G \in \mathcal{G}_3$ containing the thick edges of G_3 (see Figure 7.2 (c)). Finally, let G_4 be the graph shown in Figure 7.2 (d), and let M_4 be a perfect matching of G_4 consisting of the thick edges.

Zhang et al. [122] provided a sufficient condition guaranteeing that a perfect matching of a balanced bipartite graph is contained in a Hamilton cycle of the graph, as follows.

Theorem 7.2.2 (Zhang, Zhang and Wen [122]). *Let G be a balanced bipartite graph on $2n$ vertices with partite sets X and Y , and let M be a perfect matching of G . If $d(x) + d(y) \geq n+1$ for every pair of vertices x, y of G with $x \in X$, $y \in Y$ and $xy \notin E(G)$, then M is contained in a Hamilton cycle of G unless one of the following holds:*

- (a) $G \in \mathcal{G}_1$ and $M \in \mathcal{M}_1$;
- (b) $G \in \mathcal{G}_2$ and $M \in \mathcal{M}_2$;
- (c) $G \in \mathcal{G}_3$ and $M \in \mathcal{M}_3$;
- (d) $G = G_4$ and $M = M_4$.

Theorem 7.2.2 is used to characterize the bad vertices of an eulerian digraph with a generalized transition system.

Las Vergnas [90] also provided a sufficient condition guaranteeing that a perfect matching of a balanced bipartite graph is contained in a Hamilton

cycle of the graph, as follows.

Theorem 7.2.3 (Las Vergnas [90]). *Let G be a balanced bipartite graph on $2n$ vertices with partite sets X and Y , and let M be a perfect matching of G . If $d(x) + d(y) \geq n + 2$ for every pair of vertices x, y of G with $x \in X$, $y \in Y$ and $xy \notin E(G)$, then M is contained in a Hamilton cycle of G .*

Next, we present the main results of this section. We establish the following Ore-type sufficient condition for the existence of compatible Euler tours in eulerian digraphs with generalized transition systems, which is weaker than Isaak's condition in Theorem 7.1.2.

Theorem 7.2.4. *Let D be an eulerian digraph with a generalized transition system $F(D)$. If $f_v^+(a) + f_v^-(b) \leq d^+(v) - 1$ for each vertex $v \in V(D)$ and every pair of arcs a, b with $b \in F_v^+(a)$, and D has no E -separator consisting of bad vertices, then there exists a compatible Euler tour in D .*

Proof. Let D be an eulerian digraph with a generalized transition system $F(D)$. Let v be a vertex of D , and let L_v be a balanced complete bipartite graph with partite sets $A^-(v)$ and $A^+(v)$. Let $L'_v = L_v - F(v)$, where $F(v) = \{ab \in E(L_v) \mid b \in F_v^+(a)\}$. It follows from $f_v^+(a) + f_v^-(b) \leq d^+(v) - 1$ for every pair of arcs a, b with $b \in F_v^+(a)$ that $d_{L'_v}(a) + d_{L'_v}(b) \geq d^+(v) + 1 = |V(L'_v)|/2 + 1$ for every pair of vertices a, b of L'_v with $a \in A^-(v)$, $b \in A^+(v)$ and $ab \notin E(L'_v)$. Since D is an eulerian digraph, the graph L'_v is a balanced bipartite graph. By Theorem 7.2.1, the graph L'_v is hamiltonian. Therefore, there exists a perfect matching M in L'_v .

Based on the perfect matching M , we can construct a transition decomposition $T(v)$ at v : each two arcs in $A(v)$ corresponding to an edge of M form a transition in $T(v)$. By making such a construction for each vertex v of D , we can get a transition system of D , denoted by $T(D) = \{T(v) \mid v \in V(D)\}$. By Observation 7.2.1, there exists a closed trail decomposition $\mathcal{C} = \{C_1, \dots, C_m\}$ such that $T(\mathcal{C}) = T(D)$. By the construction of $T(D)$, it can be checked that every closed trail in $\mathcal{C}$ is compatible with $F(D)$. If $m = 1$ then the closed trail decomposition $\mathcal{C}$ gives a desired Euler tour.

Now let us consider the case that $m \geq 2$. For a closed trail in the closed trail decomposition $\mathcal{C}$, we can form an E -separator by considering the closed

trail as a subgraph. Let U be the E-separator associated with a given closed trail in $\mathcal{C}$. It follows that there exists at least one good vertex $u \in U$ such that u is a common vertex of at least two closed trails of $\mathcal{C}$. Let u_0 be such a good vertex in U . By the following discussion, we can combine all closed trails in $\mathcal{C}$ through u_0 into one closed trail that is compatible with $F(D)$.

We use $T(u_0) = \{\{a_1, a_2\}, \{a_3, a_4\}, \dots, \{a_{d(u_0)-1}, a_{d(u_0)}\}\}$ to denote the transition decomposition at u_0 , according to the order in which these transitions through u_0 appear, when we travel along closed trails in $\mathcal{C}$ one by one. Without loss of generality, we use C_1 to denote the first closed trail of $\mathcal{C}$ in which a transition of $T(u_0)$ appears. We denote by t_1 ($t_1 \leq d(u_0)/2 - 1$) the number of transitions through u_0 that appear in C_1 . Hence, we have $C_1 = a_1 a_2 T_{2,3} a_3 \cdots a_{2t_1-1} a_{2t_1} T_{2t_1,1} a_1$, where $T_{i,j}$ denotes the subtrail of C_1 connecting a_i and a_j for $1 \leq i, j \leq 2t_1$. We can denote all closed trails through u_0 in $\mathcal{C}$, in a similar way. It is not difficult to see that we can obtain a new perfect matching M' of L_{u_0} , which corresponds to all pairs of arcs of $A(u_0)$ connected by the subtrails $T_{i,j}$'s of all closed trails through u_0 in $\mathcal{C}$.

Let L_{u_0} be a balanced complete bipartite graph with partite sets $A^-(u_0)$ and $A^+(u_0)$, and let $L'_{u_0} = L_{u_0} - F(u_0)$, where $F(u_0) = \{ab \in E(L_{u_0}) \mid b \in F_{u_0}^+(a)\}$. Furthermore, let $L_{u_0}^* = L'_{u_0} \cup M'$. It follows from $f_{u_0}^+(a) + f_{u_0}^-(b) \leq d^+(u_0) - 1$ for every pair of arcs a, b with $b \in F_{u_0}^+(a)$ that $d_{L'_{u_0}}(a) + d_{L'_{u_0}}(b) \geq |V(L'_{u_0})|/2 + 1$ for every pair of vertices a, b of L'_{u_0} with $a \in A^-(u_0)$, $b \in A^+(u_0)$ and $ab \notin E(L_{u_0})$. Thus, we conclude that $d_{L_{u_0}^*}(a) + d_{L_{u_0}^*}(b) \geq |V(L_{u_0}^*)|/2 + 1$ for every pair of vertices a, b of $L_{u_0}^*$ with $a \in A^-(u_0)$, $b \in A^+(u_0)$ and $ab \notin E(L_{u_0}^*)$. Recall that u_0 is a good vertex (good vertices were characterized by Theorem 7.2.2). It follows that M' is contained in a Hamilton cycle C of $L_{u_0}^*$. Let $M'' = C - M'$. Clearly, M'' is another perfect matching of L'_{u_0} . Based on the perfect matching M'' , we can construct another transition decomposition $T'(u_0)$ at u_0 : each two arcs in $A(u_0)$ corresponding to an edge of M'' form a transition in $T'(u_0)$. Replacing $T(u_0)$ by $T'(u_0)$ in $T(D)$, we can combine all closed trails through u_0 in $\mathcal{C}$ into one closed trail compatible with $F(D)$. This operation reduces the number of closed trails in $\mathcal{C}$ as required. After a finite number of such combinations, we can obtain a compatible Euler tour. This completes the proof. $\square$

Remark 7.2.1. The following example shows that the bound on $f_v^+(a) + f_v^-(b)$ in Theorem 7.2.4 is tight for some vertex $v \in V(D)$ and some pair of arcs a, b with $b \in F_v^+(a)$.

Example 7.2.1. Let D be an eulerian digraph as depicted in Figure 7.3, where all subtrails T_i connecting b_i and a_i ($1 \leq i \leq 2s$) are disjoint from each other and each T_i along with b_i and a_i forms a compatible subtrail. Let $F(D)$ be a generalized transition system of D such that for the vertex v with $d^+(v) = d^-(v) = 2s$, $F_v^+(a_i) = \{b_j \mid s+1 \leq j \leq 2s\}$ for $1 \leq i \leq s$ and $F_v^+(a_i) = \{b_j \mid 1 \leq j \leq s\}$ for $s+1 \leq i \leq 2s$.

Figure 7.3: The digraph illustrating Example 7.2.1.

One can check that the eulerian digraph D of Example 7.2.1 satisfies that for the vertex v with $d^+(v) = d^-(v) = 2s$, $f_v^+(a_k) + f_v^-(b_\ell) = 2s = d^+(v)$ for $1 \leq k \leq s$ and $s+1 \leq \ell \leq 2s$, and $f_v^+(a_k) + f_v^-(b_\ell) = 2s = d^+(v)$ for $s+1 \leq k \leq 2s$ and $1 \leq \ell \leq s$. However, the digraph D has no Euler tour compatible with $F(D)$.

From Theorem 7.2.4, we can obtain the following corollary. In the next corollary, we use $f_v(a)$ to denote $f_v^+(a)$ or $f_v^-(a)$ for an incident pair (v, a) .

Corollary 7.2.1. *Let D be an eulerian digraph with a generalized transition system $F(D)$. If $f_v(a) \leq (d^+(v) - 1)/2$ for each vertex $v \in V(D)$ and each incident pair (v, a) , and D has no E -separator consisting of bad vertices, then there exists a compatible Euler tour in D .*

Using similar arguments as in the proof of Theorem 7.2.4, we can also provide an alternative proof for Theorem 7.1.2 due to Isaak [76] by replacing Theorem 7.2.2 with Theorem 7.2.3.

7.3 CETs in arc-colored eulerian digraphs

In this section, we provide sufficient conditions for the existence of compatible Euler tours in arc-colored eulerian digraphs, based on some known results due to different (groups of) researchers.

Benkoular et al. [9] considered the existence of compatible Euler tours in edge-colored eulerian graphs, and provided a polynomial-time algorithm for finding a compatible Euler tour in a specific edge-colored eulerian graph. In [9], the authors also claimed that a similar algorithm can be given for arc-colored eulerian digraphs, and they stated a necessary and sufficient condition for the existence of compatible Euler tours in arc-colored eulerian digraphs. However, Carraher and Hartke [19] noticed that the sufficient condition stated by Benkoular et al. [9] is in fact not sufficient for the existence of compatible Euler tours in arc-colored eulerian digraphs, by giving the following small counterexample (see Figure 7.4). Moreover, so far, there is still no result on necessary and sufficient condition for the existence of compatible Euler tours in arc-colored eulerian digraphs (without any assumptions on the class of digraphs).

Figure 7.4: The digraph illustrating a counterexample.

Let D be an arc-colored eulerian digraph. We use $c(a)$ to denote the color appearing on the arc a of D , and we denote by $C(D)$ the set of colors appearing on the arcs of D . For a color $i \in C(D)$, we use D^i to denote the subgraph of D induced by $\{a \in A(D) \mid c(a) = i\}$. For a vertex v of D , we let $\Delta^{-mon}(v) = \max\{d_{D^i}^-(v) \mid i \in C(D)\}$ and $\Delta^{+mon}(v) = \max\{d_{D^i}^+(v) \mid i \in C(D)\}$; these two parameters are called the *maximum monochromatic indegree* and the *maximum monochromatic outdegree* of the vertex v , respectively. Furthermore, we define the *refined maximum monochromatic indegree* of a vertex v of D as $\Delta_r^{-mon}(v) = \max\{d_{D^i}^-(v) \mid i \in C(D) \text{ and } d_{D^i}^+(v) \neq 0\}$, and we define the *refined maximum monochromatic outdegree* of a vertex v of D as $\Delta_r^{+mon}(v) = \max\{d_{D^i}^+(v) \mid i \in C(D) \text{ and } d_{D^i}^-(v) \neq 0\}$. It is clear that $\Delta_r^{-mon}(v) \leq \Delta^{-mon}(v)$ and $\Delta_r^{+mon}(v) \leq \Delta^{+mon}(v)$ for each vertex v of D .

Isaak [76] provided the following sufficient condition for the existence of compatible Euler tours in arc-colored eulerian digraphs, when considering the existence of Hamilton cycles in digraphs.

Theorem 7.3.1 (Isaak [76]). *Let D be an arc-colored eulerian digraph. If $\Delta^{-mon}(v) \leq d^+(v)/2 - 1$ and $\Delta^{+mon}(v) \leq d^+(v)/2 - 1$ for each vertex $v \in V(D)$, then there exists a compatible Euler tour in D .*

Motivated by Theorem 7.2.4, we provide the following sufficient condition for the existence of compatible Euler tours in arc-colored eulerian digraphs. This condition is weaker than Isaak's condition in Theorem 7.3.1.

Proposition 7.3.1. *Let D be an arc-colored eulerian digraph. If $\Delta_r^{-mon}(v) \leq d^+(v)/2 - 1$ and $\Delta_r^{+mon}(v) \leq d^+(v)/2 - 1$ for each vertex v of D , then there exists a compatible Euler tour in D .*

Proof. The proof of Proposition 7.3.1 directly follows from the conclusion of Theorem 7.1.2. For this proof, we just need to consider an arc-coloring of an eulerian digraph as a special case of a generalized transition system of the eulerian digraph. Therefore, we omitted the details. $\square$

Based on Theorem 7.1.1, we also obtain another sufficient condition for the existence of compatible Euler tours in arc-colored eulerian digraphs, as follows. In the next proposition, we use $cd(v)$ to denote the color degree of

a vertex v of D , i.e., the number of different colors appearing on the arcs of D incident with v .

Proposition 7.3.2. *Let D be an arc-colored eulerian digraph. If $cd(v) \geq d(v)/2 + 2$ for each vertex v of D with $d(v) \geq 4$, and $cd(v) = 2$ otherwise, then there exists a compatible Euler tour in D .*

Proof. The proof of Proposition 7.3.2 directly follows from the conclusion of Theorem 7.1.1. For this proof, we just need to consider an arc-coloring of an eulerian digraph as a special case of a partition system of the eulerian digraph. The condition that $cd(v) \geq d(v)/2 + 2$ for each vertex v of D with $d(v) \geq 4$ directly implies that $|P(v)| \geq d(v)/2 + 2$ for each vertex v of D with $d(v) \geq 4$. It is easy to check that the cardinality of every class of $P(v)$ is at most $d(v)/2$ for each vertex v of D ; otherwise, the assertion that $|P(v)| \geq d(v)/2 + 2$ for some vertex v of D with $d(v) \geq 4$ would not hold, or this would contradict the condition that $cd(v) = 2$ for some vertex v of D with $d(v) = 2$. Therefore, this conclusion holds by Theorem 7.1.1. $\square$

7.4 Conclusions and future work

In this chapter, we established sufficient conditions for the existence of compatible Euler tours in eulerian digraphs with generalized transition systems. Our results slightly improve some previous results due to other researchers. As we discussed earlier, a generalized transition system over a digraph D can be viewed as a generalization of an arc-coloring of D . As a consequence, our results provide sufficient conditions for the existence of compatible Euler tours in eulerian digraphs for which some generalizations of arc-colorings have been defined. In addition, we also provided sufficient conditions for the existence of compatible Euler tours in arc-colored eulerian digraphs, based on some known results due to different (groups of) researchers.

As far as we know, there are very few results on the existence of compatible spanning circuits (or even for the special cases of compatible Hamilton cycles or compatible Euler tours) in arc-colored digraphs or digraphs with generalized transition systems. For instance, so far, there has been no analogue of the necessary and sufficient condition for the existence of compatible

Euler tours in eulerian graphs with partition systems (or edge-colored eulerian graphs) due to Kotzig [84], for the case of eulerian digraphs. Moreover, the problem of determining sufficient conditions for the existence of compatible spanning circuits (or related structures) in arc-colored digraphs or digraphs with generalized transition systems looks also much more difficult.

In future work, we look forward to establishing more sufficient conditions for the existence of compatible spanning circuits (or related structures) in arc-colored digraphs and digraphs with generalized transition systems.

Summary

This thesis contains contributions to the research on the existence of compatible spanning circuits in edge-colored graphs, as well as some sufficient conditions for the existence of compatible Euler tours in eulerian (di)graphs with generalizations of (arc-) edge-colorings.

Compatible spanning circuits are common generalizations of Euler tours and Hamilton cycles, two popular and well-studied concepts within graph theory. Whereas determining whether a graph admits an Euler tour is easy (this can be decided in polynomial time), the analogous decision problem for a Hamilton cycle is difficult (it is one of the most well-known NP-complete problems). Similarly, it has been proved that determining whether a graph contains a spanning circuit (corresponding to a spanning eulerian subgraph) is NP-complete (see [107]). These complexity results have spurred the research into sufficient conditions and necessary conditions for the existence of these objects and their generalizations. As an example, the research on the existence of compatible (i.e., properly colored or alternating) Hamilton cycles in specific edge-colored graphs has been attracting considerable attention (see [92]). It is clear that a Hamilton cycle of a graph can be regarded as a spanning circuit visiting each vertex of the graph exactly once. Based on the above results (facts), it is tempting to consider under what conditions an edge-colored graph contains a compatible spanning circuit that is not necessarily a Hamilton cycle. This problem is exactly what this thesis mainly focuses on.

It is a reasonable approach to consider the existence of compatible spanning circuits in specific edge-colored hamiltonian graphs. In Chapters 2

and 3, we first considered some classic hamiltonian graphs, such as graphs satisfying (Dirac-) Ore-type degree conditions and graphs satisfying Fan-type degree conditions, and we established several sufficient conditions for the existence of compatible spanning circuits visiting each vertex at least (or exactly) a specified number of times in these specific edge-colored graphs. In Chapter 3, using a similar proof technique, we also confirmed the existence of such compatible spanning circuits in edge-colored graphs with a high edge-connectivity, as well as the asymptotical existence of such compatible spanning circuits in edge-colored random graphs. Based on some known results on sufficient conditions involving forbidden induced subgraphs for a graph to be hamiltonian (or supereulerian), we established sufficient conditions for the existence of compatible spanning circuits visiting each vertex at least a specified number of times in edge-colored graphs not containing certain forbidden induced subgraphs, in Chapter 4.

As an extremal case of compatible spanning circuits, Kotzig [84] provided a necessary and sufficient condition for the existence of compatible Euler tours in edge-colored eulerian graphs in 1968. Almost thirty years later, Benkoular et al. [9] considered the existence of compatible Euler tours from an algorithmic perspective, and they described a polynomial-time algorithm for finding a compatible Euler tour in a specific edge-colored eulerian graph. Inspired by the research approach Benkoular et al. [9] used, we analyzed the complexity of the decision problem of determining whether an edge-colored connected graph contains a compatible spanning circuit, and we described two polynomial-time algorithms for finding compatible spanning circuits in edge-colored complete graphs satisfying certain conditions, in Chapter 5. The last part of this thesis (i.e., Chapters 6 and 7) mainly presents several new sufficient conditions for the existence of compatible Euler tours in eulerian (di)graphs with generalizations of (arc-) edge-colorings.

Most of the work presented in this thesis is focused on edge-colored (undirected) graph. It is usually more difficult to deal with a problem on the existence of subgraphs with specific properties in edge-colored graphs than to deal with an analogue of the problem in uncolored graphs. For example, unlike in the case of uncolored complete graphs, it is far from easy to determine whether an edge-colored complete graph contains a compatible (i.e.,

properly colored or alternating) Hamilton cycle. Moreover, perhaps surprisingly there has been no results on the complexity of this decision problem, for the case that the number of colors used in the complete graph is at least 3. This thesis only deals with the existence of compatible spanning circuits, not with the existence of properly colored spanning eulerian subgraphs. It is not difficult to see that the latter concept is stronger than the former. As a matter of fact, the problem on the existence of properly colored spanning eulerian subgraphs in edge-colored graphs looks far more difficult to solve.

As presented in this thesis, there are some interesting problems and conjectures that remain unresolved. At the same time, we also present several new open problems. We hope that these problems and conjectures attract more attention from other researchers.

Samenvatting

Dit proefschrift bevat theoretische resultaten op het gebied van de existentie van bepaalde compatibele opspannende circuits in grafen met een lijnkleuring. Daarnaast bevat het proefschrift ook voldoende voorwaarden voor de existentie van compatibele Eulertours in (gerichte) eulerse grafen met (pijl-) lijnkleuringen.

Een compatibel opspannend circuit is een gemeenschappelijke generalisatie van een Eulertour en een Hamiltoncykel, twee populaire en veel bestudeerde concepten binnen de grafentheorie. Terwijl het een eenvoudig probleem is om te beslissen of een graaf een Eulertour bezit (dit kan in polynomiale tijd beslist worden), is het analoge beslissingsprobleem voor een Hamiltoncykel moeilijk (dit is één van de bekendste NP-volledige problemen). Tevens is bewezen dat het bepalen of een graaf een opspannend circuit heeft ook een NP-volledig probleem is (zie [107]). Deze complexiteitsresultaten hebben geleid tot veel onderzoek met betrekking tot voldoende voorwaarden en noodzakelijke voorwaarden voor de existentie van deze objecten en hun generalisaties. Een voorbeeld hiervan is het onderzoek naar de existentie van compatibele (ofwel correct gekleurde of alternerende) Hamiltoncyclen in specifieke grafen met een lijnkleuring, dat aanzienlijk veel aandacht van onderzoekers heeft getrokken (zie [92]). Het is eenvoudig in te zien dat een Hamiltoncykel kan worden opgevat als een opspannend circuit dat elk punt van de graaf precies één keer bezoekt. Op basis van de bovenstaande resultaten (feiten), is het een natuurlijke uitdaging om te beschouwen onder welke voorwaarden een lijngekleurde graaf compatibele opspannende circuits bevat die niet per se een Hamiltoncykel zijn. Dit is precies het hoofdonderwerp waar dit proefschrift zich op richt.

Allereerst wordt de existentie van compatibele opspannende circuits in specifieke lijngekleurde hamiltonse grafen onder de loep genomen. In Hoofdstuk 2 en 3 beschouwen we enkele klassieke hamiltonse grafen, zoals grafen die aan bepaalde graadvoorwaarden van Ore en Fan voldoen. Voor dit type lijngekleurde grafen leiden we verscheidene voldoende voorwaarden af voor de existentie van compatibele opspannende circuits die elk punt minstens (of precies) een gespecificeerd aantal keren bezoeken. In Hoofdstuk 3 doen we iets soortgelijks voor grafen met een hoge lijnsamenhang en ook beschouwen we daarin asymptotische resultaten voor random grafen. In Hoofdstuk 4 richten we ons op analoge resultaten voor grafen die bepaalde vaste kleine grafen niet bevatten als geïnduceerde deelgraaf.

In 1968 heeft Kotzig [84] een noodzakelijke en voldoende voorwaarde afgeleid voor het extreme geval van compatibele opspannende circuits, namelijk voor compatibele Eulertours. Bijna dertig jaar later hebben Benkouar et al. [9] deze compatibele Eulertours bestudeerd vanuit een algoritmisch perspectief, en een polynomiaal algoritme beschreven voor het vinden van dergelijke tours in specifieke lijngekleurde eulerse grafen. Geïnspireerd door deze benadering van Benkouar et al. [9], hebben we de complexiteit van het overeenkomstige probleem voor het bestaan van compatibele opspannende circuits geanalyseerd. In Hoofdstuk 5 beschrijven we twee polynomiale algoritmen voor het vinden van compatibele opspannende circuits in lijngekleurde grafen die aan bepaalde voorwaarden voldoen. In het laatste gedeelte van dit proefschrift, de Hoofdstukken 6 en 7, presenteren we verscheidene nieuwe voldoende voorwaarden voor de existentie van compatibele Eulertours in eulerse (gerichte) grafen die eigenschappen hebben die generalisaties zijn van (pijl-) lijnkleuringen.

Het merendeel van het werk dat ten grondslag licht aan dit proefschrift is gericht op lijngekleurde (ongerichte) grafen. In het algemeen zijn de problemen die betrekking hebben op de existentie van deelgrafen met specifieke eigenschappen in lijngekleurde grafen moeilijker dan de analoge problemen in ongekleurde grafen. Als voorbeeld noemen we de existentie van een Hamiltoncykel in een ongekleurde volledige graaf versus de existentie van een compatibele Hamiltoncykel in een lijngekleurde volledige graaf. Het mag de lezer wellicht verrassen dat er geen enkel complexiteitsresultaat voor

het laatstgenoemde beslissingsprobleem bekend is voor het geval dat het aantal gebruikte kleuren in de volledige graaf minstens 3 is.

Dit proefschrift behandelt de existentie van compatibele opspannende circuits, maar bevat geen resultaten op het gebied van de existentie van correct gekleurde opspannende circuits. Het is niet moeilijk om in te zien dat deze laatstgenoemde circuits aan striktere voorwaarden moeten voldoen, en dus een sterker concept vormen. Feitelijk lijkt het probleem met betrekking tot de existentie van correct gekleurde opspannende circuits in lijngekleurde grafen veel moeilijker op te lossen.

Zoals reeds aangegeven in het proefschrift blijven een aantal interessante problemen en vermoedens tot op heden onopgelost. Bovendien hebben we een aantal nieuwe open problemen gepresenteerd. We hopen dat deze problemen en vermoedens de aandacht zullen trekken van andere onderzoekers.

Bibliography

- [1] A. Abouelaoualim, K.Ch. Das, W. Fernandez de la Vega, M. Karpinski, Y. Manoussakis, C.A. Martinhon and R. Saad, Cycles and paths in edge-colored graphs with given degrees, *J. Graph Theory* **64** (2010) 63–86.
- [2] R. Águeda, V. Borozan, R. Díaz, Y. Manoussakis and L. Montero, Proper hamiltonian cycles in edge-colored multigraphs, *Discrete Math.* **340** (2017) 1897–1902.
- [3] S.K. Ahuja, Algorithms for routing and channel assignment in wireless infrastructure networks, Ph.D. thesis, *Univ. Arizona*, 2010.
- [4] N. Alon and G. Gutin, Properly colored Hamilton cycles in edge-colored complete graphs, *Random Struct. Algor.* **11** (1997) 179–186.
- [5] J. Bang-Jensen and G. Gutin, Alternating cycles and paths in edge-coloured multigraphs: a survey, *Discrete Math.* **165/166** (1997) 39–60.
- [6] J. Bang-Jensen and G. Gutin, Digraphs: theory, algorithms and applications, second edition, Springer Monographs in Mathematics, Springer-Verlag London Ltd., London, 2009.
- [7] M. Bánkfalvi and Z. Bánkfalvi, Alternating hamiltonian circuit in two-coloured complete graphs, in: Theory of Graphs (Proc. Colloq. Tihany 1968), Academic Press, New York, (1968) 11–18.
- [8] P. Bedrossian, Forbidden subgraph and minimum degree conditions for hamiltonicity, Ph.D. thesis, *Memphis State University*, 1991.

- [9] A. Benkour, Y. Manoussakis, V.Th. Paschos and R. Saad, Hamiltonian problems in edge-colored complete graphs and eulerian cycles in edge-colored graphs: some complexity results, *RAIRO Oper. Res.* **30** (1996) 417–438.
- [10] N.L. Biggs, E.K. Lloyd and R.J. Wilson, Graph Theory 1736–1936, second edition, *Clarendon Press*, New York, 1986.
- [11] F.T. Boesch, C. Suffel and R. Tindell, The spanning subgraphs of eulerian graphs, *J. Graph Theory* **1** (1977) 79–84.
- [12] B. Bollobás and P. Erdős, Alternating hamiltonian cycles, *Israel J. Math.* **23** (1976) 126–131.
- [13] B. Bollobás and A. Frieze, On matchings and hamiltonian cycles in random graphs, *Ann. Discrete Math.* **28** (1985) 23–46.
- [14] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications, Macmillan London and Elsevier, New York, 1976.
- [15] J.A. Bondy and U.S.R. Murty, Graph Theory, Graduate Texts in Mathematics, Volume 244, Springer, New York, 2008.
- [16] H.J. Broersma, Hamilton cycles in graphs and related topics, Ph.D. thesis, *University of Twente*, 1988.
- [17] J. Brousek, Forbidden triples for hamiltonicity, *Discrete Math.* **251** (2002) 71–76.
- [18] M. Cai, Q. Fang and Y. Li, Existence of hamiltonial k -factors, *J. Sys. Sci. Complexity* **17** (4) (2004) 464–471.
- [19] J.M. Carraher and S.G. Hartke, Eulerian circuits with no monochromatic transitions in edge-colored digraphs, *SIAM J. Discrete Math.* **27** (4) (2013) 1924–1939.
- [20] J.M. Carraher and S.G. Hartke, Eulerian circuits with no monochromatic transitions in edge-colored digraphs with all vertices of outdegree three, *SIAM J. Discrete Math.* **31** (1) (2017) 190–209.

- [21] P.A. Catlin, A reduction method to find spanning eulerian subgraphs, *J. Graph Theory* **12** (1988) 29–45.
- [22] P.A. Catlin, Supereulerian graphs: a survey, *J. Graph Theory* **16** (1992) 177–196.
- [23] C.C. Chen and D.E. Daykin, Graphs with hamiltonian cycles having adjacent lines different colors, *J. Combin. Theory Ser. B* **21** (1976) 135–139.
- [24] X. Chen, F. Huang and J. Yuan, Proper vertex-pancyclicity of edge-colored complete graphs without monochromatic triangles, *Discrete Appl. Math.* **265** (2019) 199–203.
- [25] A.G. Chetwynd and A.J.W. Hilton, Alternating hamiltonian cycles in two colored complete bipartite graphs, *J. Graph Theory* **16** (1992) 153–158.
- [26] W.S. Chou, Y. Manoussakis, O. Megalakaki, M. Spyratos and Zs. Tuza, Paths through fixed vertices in edge-colored graphs, *Math. Inform. Sci. Humaines* **127** (1994) 49–58.
- [27] V. Chvátal and P. Erdős, A note on hamiltonian circuits, *Discrete Math.* **2** (1972) 111–113.
- [28] A. Contreras-Balbuena, H. Galeana-Sánchez and I.A. Goldfeder, A new sufficient condition for the existence of alternating hamiltonian cycles in 2-edge-colored multigraphs, *Discrete Appl. Math.* **229** (2017) 55–63.
- [29] A. Contreras-Balbuena, H. Galeana-Sánchez and I.A. Goldfeder, Alternating hamiltonian cycles in 2-edge-colored multigraphs, *Discrete Math. Theoret. Comput. Sci.* **21** (2019) 1–15.
- [30] P. Das and S.B. Rao, Alternating eulerian trails with prescribed degrees in two edge-colored complete graphs, *Discrete Math.* **43** (1983) 9–20.
- [31] D.E. Daykin, Graphs with cycles having adjacent lines of different colors, *J. Combin. Theory Ser. B* **20** (1976) 149–152.

- [32] R. Diestel, Graph Theory, fifth edition, Springer, New York, 2017.
- [33] R. Diestel and D. Kühn, On infinite cycles I, *Combinatorica* **24** (2004) 68–89.
- [34] D. Dorninger, On permutations of chromosomes, in: *Contributions to General Algebra, 5 (Salzburg, 1986)*, Hölder-Pichler-Tempsky, Vienna (1987) 95–103.
- [35] D. Dorninger, Hamiltonian circuits determining the order of chromosomes, *Discrete Appl. Math.* **50** (1994) 159–168.
- [36] D. Dorninger and W. Timischl, Geometrical constraints on Bennett’s predictions of chromosome order, *Heredity* **58** (1987) 321–325.
- [37] Z. Dvořák, Eulerian tours in graphs with forbidden transitions and bounded degree, *KAM-DIMATIA* **669** (2004) 1–12.
- [38] J. Edmonds and E.L. Johnson, Matching, Euler tours and the Chinese postman, *Math. Programming* **5** (1973) 88–124.
- [39] P. Erdős and R. Rado, A combinatorial theorem, *J. London Math. Soc.* **25** (1950) 249–255.
- [40] P. Erdős and A. Rényi, On the evolution of random graphs, *Publ. Math. Inst. Hung. Acad. Sci.* **5** (1960) 17–61.
- [41] L. Euler, Solvtio problematis ad geometriam sitvs pertinentis, *Comment. Academiae Sci. Imp. Petropolitanae* **8** (1736) 128–140.
- [42] G.H. Fan, New sufficient conditions for cycles in graphs, *J. Combin. Theory Ser. B* **37** (1984) 221–227.
- [43] L. Fang, K.W. Hipel and D.M. Kilgour, Interactive decision making: the graph model for conflict resolution, Wiley, New York, 1993.
- [44] L. Fang, K.W. Hipel, D.M. Kilgour and X. Peng, A decision support system for interactive decision making, part I: model formulation, *IEEE Transactions on Systems, Man, and Cybernetics— Part C: Applications and Reviews* **33** (2003) 42–55.

- [45] L. Fang, K.W. Hipel, D.M. Kilgour and X. Peng, A decision support system for interactive decision making, part II: analysis and output interpretation, *IEEE Transactions on Systems, Man, and Cybernetics— Part C: Applications and Reviews* **33** (2003) 56–66.
- [46] R.J. Faudree, R.J. Gould, Z. Ryjáček and I. Schiermeyer, Forbidden subgraphs and pancyclicity, *Congress Numer.* **109** (1995) 13–32.
- [47] O. Favaron and P. Fraisse, Hamiltonicity and minimum degree in 3-connected claw-free graphs, *J. Combin. Theory Ser. B* **82** (2001) 297–305.
- [48] H. Fleischner, Eulerian Graphs and Related Topics, Part 1. Vol. 1, Ann. Discrete Math. 45, North-Holland, Amsterdam 1990.
- [49] H. Fleischner, Eulerian Graphs and Related Topics, Part 1. Vol. 2, Ann. Discrete Math. 50, North-Holland, Amsterdam 1991.
- [50] H. Fleischner, (Some of) the many uses of eulerian graphs in graph theory (plus some applications), *Discrete Math.* **230** (2001) 23–43.
- [51] H. Fleischner and M. Fulmek, P(D)-compatible eulerian trails in digraphs and a new splitting lemma, in Contemporary Methods in Graph Theory, Bibliographisches Institut, Mannheim, Germany (1990) 291–303.
- [52] H. Fleischner, A.J.W. Hilton and B. Jackson, On the maximum number of pairwise compatible Euler cycles, *J. Graph Theory* **14** (1) (1990) 51–63.
- [53] H. Fleischner and B. Jackson, Compatible Euler tours in eulerian digraphs, in Cycles and Rays (Montreal, PQ, 1987), *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* 30, Kluwer, Dordrecht (1990) 95–100.
- [54] S. Fujita and C. Magnant, Properly colored paths and cycles, *Discrete Appl. Math.* **159** (2011) 1391–1397.
- [55] S. Fujita, C. Magnant and K. Ozeki, Rainbow generalizations of Ramsey theory: a survey, *Graphs Combin.* **26** (2010) 1–30.

- [56] G. Galbiati, S. Gualandi and F. Maffioli, On minimum reload cost cycle cover, *Electronic Notes in Discrete Math.* **36** (2010) 81–88.
- [57] I. Gamvros, Satellite network design, optimization and management, Ph.D. Thesis, *University of Maryland*, 2006.
- [58] Jingzhen Gao and Bo Gao, Chinese postman problem over 50 years, *Operations Research Transactions* **17** (1) (2013) 17–28.
- [59] Y. Gao, G. Li and X. Li, Degree condition for the existence of a k -factor containing a given hamiltonian cycle, *Discrete Math.* **309** (2009) 2373–2381.
- [60] M.R. Garey, D.S. Johnson and L. Stockmeyer, Some simplified NP-complete problems, In *Proceedings of Sixth Annual ACM Symposium on Theory of Computing*. ACS, Washington, D.C. (1974) 47–63.
- [61] E.N. Gilbert, Random plane networks, *J. Soc. Indust. Appl. Math.* **9** (1961) 533–543.
- [62] R.J. Gould, Updating the hamiltonian problem—a survey, *J. Graph Theory* **15** (1991) 121–157.
- [63] R.J. Gould, Advances on the hamiltonian problem—a survey, *Graphs Combin.* **19** (2003) 7–52.
- [64] R.J. Gould, Recent Advances on the hamiltonian Problem: Survey III, *Graphs Combin.* **30** (2014) 1–46.
- [65] L. Gourvès, A. Lyra, C. Martinhon and J. Monnot, The minimum reload $s - t$ path, trail, walk problems, *Discrete Appl. Math.* **158** (13) (2010) 1404–1417.
- [66] Z. Guo, H.J. Broersma, B. Li and S. Zhang, Almost eulerian compatible spanning circuits in edge-colored graphs, submitted.
- [67] Z. Guo, B. Li, X. Li and S. Zhang, Compatible spanning circuits in edge-colored graphs, *Discrete Math.* **343** (2020) 111908.

- [68] Z. Guo, X. Li, C. Xu and S. Zhang, Compatible eulerian circuits in eulerian (di)graphs with generalized transition systems, *Discrete Math.* **341** (2018) 2104–2112.
- [69] D. Gusfield, Connectivity and edge-disjoint spanning trees, *Inform. Process. Lett.* **16** (1983) 87–89.
- [70] G. Gutin and E.J. Kim, Properly coloured cycles and paths: results and open problems, *Lect. Notes in Comp. Science* **5420** (2009) 200–208.
- [71] R. Häggkvist, On F-hamiltonian graphs, *Graph Theory and Related Topics*, Academic Press, New York and London, (1979) 219–231.
- [72] L. Han, H.-J. Lai, L. Xiong and H. Yan, The Chvátal-Erdős condition for supereulerian graphs and the hamiltonian index, *Discrete Math.* **310** (2010) 2082–2092.
- [73] M. Hasanvand, Spanning trees and spanning eulerian subgraphs with small degrees, *Discrete Math.* **338** (2015) 1317–1321.
- [74] A.J.W. Hilton, Alternating hamiltonian circuits in edge-coloured bipartite graphs, *Discrete Appl. Math.* **35** (1992) 271–273.
- [75] T.C. Hu and Y.S. Kuo, Graph folding and programmable logical arrays, *Networks* **17** (1987) 19–37.
- [76] G. Isaak, Hamiltonicity of digraphs for universal cycles of permutations, *European J. Combin.* **27** (6) (2006) 801–805.
- [77] B. Jackson, Compatible Euler tours for transition systems in eulerian graphs, *Discrete Math.* **66** (1987) 127–131.
- [78] B. Jackson, On circuit covers, circuit decompositions and Euler tours of graphs, In *Surveys in Combinatorics, 1993* (Keele), *London Math. Soc. Lecture Note Ser.* 187, Cambridge University Press, Cambridge, UK (1993) 191–210.
- [79] B. Jackson and N.C. Wormald, Cycles containing matchings and pairwise compatible Euler tours, *J. Graph Theory* **14** (1) (1990) 127–138.

- [80] F. Jaeger, A note on sub-eulerian graphs, *J. Graph Theory* **3** (1979) 91–93.
- [81] D. Jungnickel, *Graphs, Networks and Algorithms*, fourth edition, Springer, Berlin, 2013.
- [82] T. Kaiser and P. Vrána, Hamilton cycles in 5-connected line graphs, *Europ. J. Comb.* **33** (2012) 924–947.
- [83] M. Kano and X. Li, Monochromatic and heterochromatic subgraphs in edge-colored graphs—a survey, *Graphs Combin.* **24** (2008) 237–263.
- [84] A. Kotzig, Moves without forbidden transitions in a graph, *Math. Fyz. Časopis* **18** (1968) 76–80.
- [85] M. Krivelevich, C. Lee and B. Sudakov, Compatible Hamilton cycles in random graphs, *Random Struct. Algor.* **49** (3) (2016) 533–557.
- [86] M. Krivelevich, C. Lee and B. Sudakov, Compatible Hamilton cycles in Dirac graphs, *Combinatorica* **37** (4) (2017) 697–732.
- [87] H.-J. Lai, Y. Shao and H. Yan, An update on supereulerian graphs, *WSEAS Trans Math.* **12** (2013) 926–940.
- [88] T.H. Lai and S.S. Wei, The edge hamiltonian path problem is NP-complete for bipartite graphs, *Inform. Process. Lett.* **46** (5) (1993) 21–26.
- [89] G. Laporte and I.H. Osman, Routing problems: a bibliography, *Ann. Oper. Res.* **61** (1995) 227–262.
- [90] M. Las Vergnas, Problèmes de couplages et problèmes hamiltoniens en théorie des graphes, Ph.D. Thesis, *Université Paris XI*, 1972.
- [91] R. Laskar and B. Auerback, On decomposition of r -partite graphs into edge-disjoint Hamilton circuits, *Discrete Math.* **14** (1976) 265–268.
- [92] A. Lo, Properly coloured hamiltonian cycles in edge-coloured complete graphs, *Combinatorica* **36** (2016) 471–492.

- [93] S. Lv and L. Xiong, Forbidden pairs for spanning (closed) trails, *Discrete Math.* **340** (2017) 1012–1018.
- [94] H. Matsuda, Regular factors containing a given hamiltonian cycle, *Lecture Notes in Computer Science, Combinatorial Geometry and Graph Theory* (2005) 123–132.
- [95] M.M. Matthews and D.P. Sumner, Hamiltonian results in $K_{1,3}$ -free graphs, *J. Graph Theory* **8** (1984) 139–146.
- [96] A. Mayyasi, Why UPS trucks don't turn left, <http://priceconomics.com/why-ups-trucks-dont-turn-left> (2014).
- [97] R. McBride, Controlling left and u-turns in the routing of refuse collection vehicles, *Comput. Oper. Res.* **9** (1982) 145–152.
- [98] J. Moon and M. Moser, On hamiltonian bipartite graphs, *Israel J. Math.* **1** (1963) 357–369.
- [99] H.M. Mulder, Julius Petersen's theory of regular graphs, *Discrete Math.* **100** (1992) 157–175.
- [100] T. Müller, X. Pérez-Giménez and N. Wormald, Disjoint Hamilton cycles in the random geometric graph, *J. Graph Theory* **68** (2011) 299–322.
- [101] O. Ore, Note on hamiltonian circuits, *Amer. Math. Monthly* **67** (1960) 55–55.
- [102] T. Paoletti, Leonard Euler's solution to the Königsberg bridge problem, <https://www.maa.org/press/periodicals/convergence/leonard-eulers-solution-to-the-konigsberg-bridge-problem>.
- [103] M.D. Penrose, On k -connectivity for a geometric random graph, *Random Struct. Algor.* **15** (2) (1999) 145–164.
- [104] J. Petersen, Die Theorie der regulären graphs, *Acta Math.* **15** (1891) 193–220.
- [105] P.A. Pevzner, DNA physical mapping and properly edge-colored eulerian cycles in colored graphs, *Algorithmica* **13** (1995) 77–105.

- [106] P.A. Pevzner, Computational molecular biology: an algorithmic approach, MIT Press, Cambridge, MA, 2000.
- [107] W.R. Pulleyblank, A note on graphs spanned by eulerian graphs, *J. Graph Theory* **3** (1979) 309–310.
- [108] F.S. Roberts, Graph theory and its applications to problems of society, CBMS-NSF Monograph No. 29, SIAM, Philadelphia, 1978.
- [109] Z. Ryjáček, On a closure concept in claw-free graphs, *J. Combin. Theory Ser. B* **70** (1997) 217–224.
- [110] S. Sankararaman, A. Efrat and S. Ramasubramanian, On channel discontinuity-constraint routing in wireless networks, *Ad Hoc Networks* **13** (2014) 153–169.
- [111] J. Shearer, A property of the colored complete graph, *Discrete Math.* **25** (1979) 175–178.
- [112] M. Szachniuk, M.C. De Cola, G. Felici and J. Blazewicz, The orderly colored longest path problem – a survey of applications and new algorithms, *RAIRO-Oper. Res.* **48** (2014) 25–51.
- [113] M. Szachniuk, M. Popenda, R.W. Adamiak and J. Blazewicz, An assignment walk through 3D NMR spectrum, in: *Proc. 2009 IEEE Symposium on Computational Intelligence in Bioinformatics and Computational Biology* (2009) 215–219.
- [114] I.L. Tseng, H.W. Chen and C.I. Lee, Obstacle-aware longest path routing with parallel MILP solvers, in: *Proc WCECS-ICCS* **2** (2010) 827–831.
- [115] M. Walters, Random geometric graphs, in: *Surveys in Combinatorics 2011, London Math. Soc. Lecture Note Ser.* **392** (2011) 365–401.
- [116] S. Wang and L. Xiong, Forbidden set of induced subgraphs for 2-connected supereulerian graphs, *Discrete Math.* **340** (2017) 2792–2797.

- [117] R.J. Wilson, Introduction to graph theory, fourth edition, Longman, Harlow, England, 1996.
- [118] D.R. Woodall, Sufficient conditions for cycles in digraphs, *Proc. London Math. Soc.* **24** (1972) 739–755.
- [119] H. Xu, K.W. Li, D.M. Kilgour and K.W. Hipel, A matrix-based approach to searching colored paths in a weighted colored multidigraph, *Appl. Math. and Comp.* **215** (2009) 353–366.
- [120] H. Xu, M. Kilgour, K. Hipel and G. Kemkes, Using matrices to link conflict evolution and resolution in a graph model, *European J. Oper. Res.* **207** (2010) 318–329.
- [121] C.-Q. Zhang, Hamilton cycles in claw-free graphs, *J. Graph Theory* **12** (1988) 209–216.
- [122] Z. Zhang, X. Zhang and X. Wen, Directed Hamilton cycles in digraphs and matching alternating Hamilton cycles in bipartite graphs, *SIAM J. Discrete Math.* **27** (1) (2013) 274–289.
- [123] S. Zhou, Disjoint hamiltonian cycles in Fan $2k$ -type graphs, *J. Graph Theory* **17** (1993) 673–678.

Acknowledgements

After countless revisions and refinements, this thesis finally came out. This thesis could not have been successfully completed without the collaboration, support, encouragement and help from a lot of people. Looking back on the past four years of my doctoral study, I feel that my doctoral experience is like a long trip with bends and turns. Luckily, I was not alone in this unforgettable trip of my life, and I received a lot of support, encouragement and help from many people along the way. Now, I would like to express my sincere gratitude to all the people who helped me in this journey in various different ways.

First of all, I would like to express my sincere gratitude to my supervisors Professor Xueliang Li and Professor Shenggui Zhang at the Northwestern Polytechnical University (NPU). I am particularly grateful to them for giving me the opportunity to become a PhD candidate and to carry out this research, and for their support, encouragement and positive supervision. Professor Li provided me with scientific guidance and useful suggestions to enlarge my insights. His extensive professional knowledge and excellent academic taste deeply inspired and influenced me. Professor Zhang patiently taught me a lot of details that I could not learn from the books but are very useful for me, and he also showed me effective ways to think about scientific problems. Professor Zhang also supported me to participate in a lot of important academic conferences, and he supported me to visit the University of Twente (UT) in order to expand my horizons and enrich my life experience. Professor Zhang always encouraged me to discuss problems with different scholars outside the group at the NPU, and he created many opportunities for me to communicate with famous scholars in the field of graph theory. I feel very lucky and happy to study and carry out research under the supervision of

Professor Zhang.

My sincere thanks also go to Professor Hajo Broersma, my supervisor at the UT. I am very grateful to him for giving me the opportunity to carry out my research under his supervision, and for his support, concern and encouragement. In each discussion, he was always very patient to hear what we wanted to express, and he was smart to give inspiring suggestions. He always had enough patience to revise our papers and theses in a detailed manner, and he patiently marked every comment to make us find it easily. I learned a lot of writing skills from him. He also supported me to visit the Technische Universität Bergakademie Freiberg, Germany in order to improve my academic ability. Apart from his profound professional knowledge and scrupulous scientific spirit, many of his noble qualities, such as modesty, forgiveness and kindness influenced me a lot.

I would also like to express my deep gratitude to all my coauthors. Chuan-dong Xu and Binlong Li, who are like brothers to me, were always generous in sharing their valuable experience, helpful discussions and kind advices with me. Ruonan Li also selflessly shared related research results and creative ideas with me, and she provided me with a lot of practical guide and helpful advices for surviving in the Netherlands. I also particularly thank Professor Ingo Schiermeyer, Christoph Brause and Maximilian Geißer for their creative ideas and valuable discussions, and for their warmhearted and thoughtful help for my personal life in Germany. In addition, I am particularly grateful to Professor Cun-Quan Zhang and Professor Hong-Jian Lai for their helpful discussions and inspiring suggestions.

I also wish to thank Professor Haixing Zhao for bringing me to the interesting field of graph theory and creating the valuable opportunity for me to pursue my Doctor degree. At the same time, I would like to thank Professor Yaping Mao for his positive guidance and generous help in my research and life.

My deep thanks also go to all my colleagues at the NPU for creating the pleasant atmosphere on discussing, research, playing, etc. In particular, I am very grateful to Pengfei Wan and Xinzhuang Chen for their a lot of help, especially in computer problems, and I also wish to thank Tingting Han for

her selfless help, especially in handling the affairs of the NPU for me during my visit to the Netherlands. I would also like to express my deep thanks to all FMT members at the UT for the interesting activities they have organized and their warmhearted help in daily life. In addition, my sincere thanks go to my English teacher in Enschede, Mrs. Phing Choo, who provided me with a lot of help in my study and life.

I would also like to thank all my friends and relatives for their generous support, understanding and encouragement. In particular, I am very grateful to all my friends I got to know in Enschede for making my life in the Netherlands so enjoyable.

Last but not least, my deepest thanks go to my family members: my parents, my girlfriend, my younger sister and my brother-in-law, who are the most important part of my life. They always provided me with spiritual and financial support, encouragement and deep concern, even though they did not always understand what I was doing. I really appreciate everything they have done for me.

Zhiwei Guo

August 2020, Enschede

About the Author

The author Zhiwei Guo was born on December 23, 1989 in Lingshi County, Shanxi Province, P.R. China. After completing his elementary and secondary education, he was admitted as a student by the Lingshi No.1 Middle School after the high school entrance examination in 2005.

In September 2009, he started to study at the Department of Mathematics (now School of Mathematics and Statistics), Qinghai Normal University, Xining, Qinghai. He was awarded the National Endeavor Scholarship in 2011. After obtaining his Bachelor degree in July 2013, he was recommended to continue his study for a Master degree without the entrance examination at the same university. In 2015, he was awarded the Master Graduate National Scholarship. Under the supervision of Professor Haixing Zhao, he obtained his Master degree in July 2016.

In September 2016, he became a PhD candidate at the Department of Applied Mathematics (now School of Mathematics and Statistics), Northwestern Polytechnical University, Xi'an, Shaanxi, under the supervision of Professor Xueliang Li and Professor Shenggui Zhang.

Starting from October 2018, as a joint PhD student, he visited the Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, The Netherlands to carry out the research on the existence of compatible spanning circuits in edge-colored graphs, under the supervision of Professor Hajo Broersma. This research has been sponsored by the China Scholarship Council. The main results obtained from the research that was carried out since September 2016 have been collected in this thesis.