Mean-field analysis of the convergence time of message-passing computation of harmonic influence in social networks

W. S. Rossi * **P. Frasca** **

* *University of Twente, 7500 AE Enschede, The Netherlands (e-mail: w.s.rossi@utwente.nl).*
** *Univ. Grenoble Alpes, CNRS, Inria, GIPSA-lab, F-38000 Grenoble, France (e-mail: paolo.frasca@gipsa-lab.fr).*

Abstract: The concept of harmonic influence has been recently proposed as a metric for the importance of nodes in a social network. A distributed message passing algorithm for its computation has been proposed by Vassio et al. (2014) and proved to converge on general graphs by Rossi and Frasca (2016a). In this paper, we want to evaluate the convergence time of this algorithm by using a mean-field approach. The mean-field dynamics is first introduced in a "homogeneous" setting, where it is exact, then heuristically extended to a non-homogeneous setting. The rigorous analysis of the mean-field dynamics is complemented by numerical examples and simulations that demonstrate the validity of the approach.

Keywords: Distributed algorithm, Message passing, Opinion dynamics, Social networks, Convergence analysis, Nonlinear recursion.

1. INTRODUCTION

In the study of social networks and dynamical processes therein, the identification of the most influential leaders is an important issue. In this work, we assume that a leader has to compete against an external field of influence in order to win the opinions of the other individuals. Following a consolidated research line, we shall postulate that the opinions of the nodes follow a linear dynamics with fixed confidence weights (see for instance the survey by Proskurnikov and Tempo (2017)). More precisely, the leader node has a fixed opinion, whereas the remaining "regular" agents update their opinions to weighted averages of the opinions of their neighbors, their own, and the external field. The opinions converge asymptotically to values that depend on the confidence weights and on the position of the leader, but not on the initial opinions of the regular agents.

The influence of the leader is defined as its effectiveness in moving the average opinion of the social network. More precisely, we define as *harmonic influence* of the leader the sum of the asymptotic opinions that it induces in the other agents. The influence of a node is the influence obtained if that node was the leader. This definition is equivalent to the *Harmonic Influence Centrality* introduced in Vassio et al. (2014) and implicitly used in Acemoglu et al. (2013); Yildiz et al. (2013). Also other definitions have bee used to evaluate nodes as potential leaders, see for instance Lin et al. (2014); Fitch and Leonard (2016).

In principle, the computation of the harmonic influence of a set of n nodes requires the solution of n linear systems. This approach requires global knowledge of the graph and does not exploit apparent redundancies in the computations. For this reason, Vassio et al. (2014) have proposed a *Message Passing Algorithm* (MPA) that aims to compute the nodes' influence in a distributed and concurrent way. If the graph is a tree, then the algorithm computes the nodes' influence in a number of steps equal to the diameter of the graph. On general graphs, the algorithm converges asymptotically to a meaningful approximation of the nodes' influence, as proved in Rossi and Frasca (2016a,b).

In this paper, we want to have a closer look at the convergence of the MPA by a *mean-field* approach. The basic idea behind mean-field approaches is to reduce the complexity of dynamical systems with large state space by neglecting correlations between state variables and mainly caring for mean values. Typically, this leads to simpler low-dimensional dynamical systems which satisfy self-consistent equations. A related example is discussed by Massar and Massar (2013).

In our case, the mean-field analysis brings us to study a pair of coupled scalar recursions, which is undertaken in Section 3 and which provides us insights on convergence time of the MPA. This two-dimensional system is exactly equivalent to the original $2n$-dimensional system under some "homogeneity" assumptions, namely homogeneity in the degrees of the nodes, the confidence weights, and the influence of the field. This homogeneity is partially lifted in Section 4 with the support of simulations, by considering non-uniform external opinion fields.

Paper Structure. Section 2 recalls the Message Passing Algorithm as described by Rossi and Frasca (2016a). The mean-field recursion is studied in Sections 3 under the homogeneity assumptions, before discussing its extension in Section 4.

Notation. The set of real and non-negative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. Vectors are denoted with boldface letters and matrices with capital letters. The all-zero and all-one vectors are denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively. A matrix Q is termed "Schur stable" if the largest absolute value of its eigenvalues is strictly smaller than one. The cardinality of the set S is denoted by $|S|$. Let $\mathcal{G} = (I, E)$ be a graph where I is the set of vertices (also called nodes) and E is the set of edges, that are unordered pairs of vertices. A graph is *connected* if for any pair of nodes i, j there exists a sequence of adjacent edges that joins them. The set $N_i = \{j \in I : \{i, j\} \in E\}$ contains the neighbors of i in $\mathcal{G}$; the degree of i is $d_i = |N_i|$. A graph is d-regular if every node has degree d.

2. THE MESSAGE PASSING ALGORITHM

Given a connected graph $\mathcal{G} = (I, E)$, consider the matrix $Q \in [0, 1]^{I \times I}$ and the vector $\mathbf{q} \in [0, 1]^I$ defined by:

$$Q_{ij} = \begin{cases} \dfrac{1}{d_i + \gamma_i} & \text{if } \{i, j\} \in E \\ 0 & \text{else}\,, \end{cases} \tag{1}$$

$$q_i = \frac{\gamma_i}{d_i + \gamma_i}\,, \tag{2}$$

where the vector $\boldsymbol{\gamma} \in \mathbb{R}_+^I$ is non-identically zero. The matrix Q is adapted to $\mathcal{G}$, the vector $\mathbf{q}$ is non-identically zero and $q_i + \sum_j Q_{ij} = 1$ for every $i \in I$. Moreover, this choice of Q and $\mathbf{q}$ satisfies Assumptions 0 and 1 in Rossi and Frasca (2016a).

The MPA works as follows. Let $t \in \{0, 1, \ldots\}$ be the iteration counter. For any ordered pair of nodes (j, i) such that $\{i, j\} \in E$, the node i sends to its neighbour j two messages, $W^{i\to j}(t) \in [0, 1]$ and $H^{i\to j}(t) \in \mathbb{R}_+$. All the messages are initialised by:

$$W^{i\to j}(0) = 1 \qquad H^{i\to j}(0) = 1\,,$$

and updated synchronously following the rules:

$$W^{i\to j}(t+1) = \frac{1}{1 + \gamma_i + \sum_{k \in N_i^j} (1 - W^{k\to i}(t))} \tag{3}$$

$$H^{i\to j}(t+1) = 1 + \textstyle\sum_{k \in N_i^j} W^{k\to i}(t)\, H^{k\to i}(t)\,, \tag{4}$$

where $N_i^j := N_i \setminus \{j\}$. The rules above are obtained from equations (6) and (7) in Rossi and Frasca (2016a), with the substitution of the matrix Q and the vector $\mathbf{q}$ defined in (1) and (2). At any time t, the node $\ell \in I$ can compute an approximation of its harmonic influence measure $H(\ell)$ using:

$$H^\ell(t) = 1 + \textstyle\sum_{i \in N_\ell} W^{i\to \ell}(t)\, H^{i\to \ell}(t)\,.$$

Theorem 1. (Convergence). Consider a connected graph $\mathcal{G} = (I, E)$ and a non-identically zero vector $\boldsymbol{\gamma} \in \mathbb{R}_+^I$. Then, the MPA described above converges.

The convergence theorem is proved in Rossi and Frasca (2016a) –Theorem 3 therein– for a wider class of matrices Q and vector $\mathbf{q}$. If the graph $\mathcal{G} = (I, E)$ is a tree, the MPA converges in a number of steps equal to the diameter of the graph and is exact, i.e. $\lim_{t\to\infty} H^\ell(t) = H(\ell)$. In general, however, the convergence is asymptotical (not in finite time) and the limit values do not coincide with the exact values of the harmonic influence (that is, $H^\ell(\infty) \neq H(\ell)$).

Figure 1 shows the convergence of the MPA on a connected Erdős-Rényi random graph $\mathcal{G}_{\mathrm{ER}} = (I, E_{\mathrm{ER}})$ with $n = |I| =$

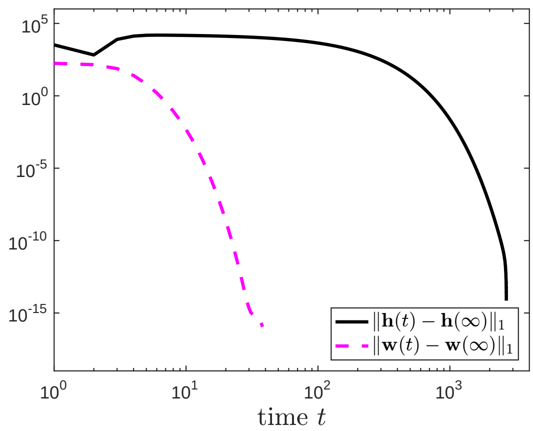

Fig. 1. The convergence pattern of the MPA on $\mathcal{G}_{\mathrm{ER}}$. The solid black line is the error between the vector $\mathbf{h}(t) \in \mathbb{R}_+^I$ that stacks the values $H^\ell(t)$ and its limit $\mathbf{h}(\infty)$. The dashed magenta line is the error between the vector $\mathbf{w}(t) \in [0, 1]^{2|E_{\mathrm{ER}}|}$ which stacks all the messages $W^{i\to j}(t)$ and its corresponding limit $\mathbf{w}(\infty)$.

50 nodes using the uniform vector $\boldsymbol{\gamma} = \frac{1}{25}\mathbf{1}$. The graph $\mathcal{G}_{\mathrm{ER}}$ contains $|E_{\mathrm{ER}}| = 123$ edges (the link probability to generate it was 0.1), has many cycles and its diameter is 5. The distance between the messages $W^{i\to j}(t)$ and their limit values (dashed magenta line) becomes negligible within 30 iterations, while the distance between the harmonic influence values computed by the MPA, i.e. $H^\ell(t)$, and the corresponding limits (solid black line) requires about 2500 iterations to become negligible. The messages $W^{i\to j}(t)$ seem to always converge faster –with a significant time scale separation– than the harmonic influence estimate $H^\ell(t)$, which basically tracks the messages $H^{i\to j}(t)$. The proof of the convergence theorem suggests a qualitative reason for the different convergence times. The messages $W^{i\to j}(t)$ form an independent system (see their update law (3)), are decreasing and converge –typically in about ten steps– to the their limit. The update law of the messages $H^{i\to j}(t)$, in eq. (4), can be rewritten in a matrix form, with the update matrix depending on $W^{i\to j}(t)$. Such matrix is not initially Schur stable and during the transient the messages $H^{i\to j}(t)$ typically accumulate a large overshoot. As the messages $W^{i\to j}(t)$ converge, eventually the update matrix becomes Schur stable and the messages $H^{i\to j}(t)$ start to converge.

In the remainder of the paper, we quantify the different convergence times of the messages $W^{i\to j}(t)$ and $H^{i\to j}(t)$, first in a homogenous setting and then in a less special case.

3. FULLY HOMOGENEOUS CASE

If the graph $\mathcal{G} = (I, E)$ is d-regular, i.e. every node has a degree d, and the vector $\boldsymbol{\gamma}$ is uniform, i.e. $\boldsymbol{\gamma} = \gamma\mathbf{1}$, with $\gamma > 0$, the equations of the MPA become fully homogeneous. The messages $W^{i\to j}(t)$ coincide for every ordered pair (j, i) such that $\{i, j\} \in E$, and so do the messages $H^{i\to j}$. Hence, all the values of $H^\ell(t)$ are equal in the fully homogeneous scenario: the MPA computes the same influence estimate for every node, regardless of the topology of the interconnections, and hence cannot provide a ranking.

Given $d \geq 2$ and $\gamma > 0$, we define the function $f(x) : [0,1] \to [0,1]$ as:

$$f(x) := \frac{1}{1+\gamma+(d-1)(1-x)}\,, \tag{5}$$

and consider the sequences $\omega(t)$ and $\eta(t)$ defined by the recursions:

$$\omega(t+1) = f(\omega(t))$$
$$\eta(t+1) = 1 + (d-1)\omega(t)\eta(t)$$

initialized with $\omega(0) = 1$ and $\eta(0) = 1$. In the fully homogeneous scenario, as proved in the following lemma, the above sequences coincide with the messages of the MPA; the influence estimates coincide with the sequence:

$$\theta(t) = 1 + d\omega(t)\eta(t)\,.$$

Lemma 2. Consider a connected, d-regular graph $\mathcal{G} = (I, E)$ and a uniform vector $\boldsymbol{\gamma} = \gamma\mathbf{1}$, where $\gamma > 0$. For every $t \geq 0$, every (i,j) such that $\{i,j\} \in E$ and every $\ell \in I$, the MPA messages and influence estimates satisfy:

$$W^{i\to j}(t) = \omega(t)\,, \qquad H^{i\to j}(t) = \eta(t)\,, \qquad H^{\ell}(t) = \theta(t)\,.$$

Proof. $W^{i\to j}(0) = \omega(0) = 1$ and $W^{i\to j}(1) = \omega(1) = \frac{1}{1+\gamma}$, for every ordered pair (j,i) such that $\{i,j\} \in E$. Assume $W^{i\to j}(t) = \omega(t)$ for every (j,i) and observe that this implies $W^{i\to j}(t+1) = f(\omega(t)) = \omega(t+1)$. Hence, by induction, $W^{i\to j}(t) = \omega(t)$ for every ordered pair (j,i) and for every $t \geq 0$. The proof proceeds similarly for the other sequences. □

Since the scalar sequences $\omega(t)$, $\eta(t)$ and $\theta(t)$ fully reproduce the MPA under these assumptions, we proceed to study their behaviour and convergence time. We first show that the sequence $\omega(t)$ decreases monotonically to a limit $\bar{\omega}$ such that $(d-1)\bar{\omega} < 1$. Then, we estimate the time required by $\omega(t)$ to become close to its limit. Finally, we use the result to discuss the convergence of the sequences $\eta(t)$ and study its convergence time. The sequence $\theta(t)$ simply tracks the behaviour of $\eta(t)$.

Lemma 3. Let $d \geq 2$ and $\gamma > 0$. The sequence $\omega(t)$ is strictly decreasing and admits the limit:

$$\bar{\omega} = \frac{1}{2} + \frac{1+\gamma}{2(d-1)} - \frac{1}{2}\sqrt{1 + \frac{2\gamma-2}{d-1} + \frac{(1+\gamma)^2}{(d-1)^2}}\,.$$

The product $(d-1)\bar{\omega}$ belongs to $(0,1)$ and decreases in γ.

Proof. First, observe that the function $f(x)$ defined in (5) is continuous, strictly increasing and convex in $[0,1]$. Moreover, the image corresponding to $[0,1]$ is $[\frac{1}{d+\gamma}, \frac{1}{1+\gamma}]$ where $\gamma > 0$, so $f(x)$ admits a unique fixed point in $[0,1]$. Assuming $\omega(t) < \omega(t-1)$, we have that:

$$\omega(t+1) = f(\omega(t)) < f(\omega(t-1)) = \omega(t)\,.$$

Therefore, since $\omega(1) = \frac{1}{1+\gamma} < 1 = \omega(0)$, by induction the sequence $\omega(t)$ is strictly decreasing and converges to a limit $\bar{\omega} := \lim_{t\to\infty}\omega(t) \in (0,1)$. The limit is the fixed point of $f(x)$ in $[0,1]$, so:

$$\bar{\omega} = f(\bar{\omega})$$
$$\bar{\omega} = \frac{1}{1+\gamma+(d-1)(1-\bar{\omega})}$$

that gives the second order equation:

$$\bar{\omega}(d-1)(1-\bar{\omega}) - 1 + \bar{\omega} + \gamma\bar{\omega} = 0 \tag{6}$$
$$(d-1)\bar{\omega}^2 - d\bar{\omega} - \gamma\bar{\omega} + 1 = 0\,.$$

For $\gamma = 0$, the above equation has solutions $\frac{1}{d-1}$ and 1. For $\gamma > 0$ is is not difficult to see that the smaller solution becomes even smaller and tends to zero, while the larger solution becomes larger than 1, which is not acceptable. The statement contains the expression of the smaller, acceptable solution. It remains to prove that $(d-1)\bar{\omega} < 1$. As $\bar{\omega}$ belongs to $(0,1)$, so does $1-\bar{\omega}$. If we rewrite the equation (6) as:

$$\bar{\omega}(d-1)(1-\bar{\omega}) = 1 - \bar{\omega} - \gamma\bar{\omega} < 1 - \bar{\omega}$$

we can conclude that, since $\gamma > 0$, $(d-1)\bar{\omega} < 1$. □

We define, for every small $\epsilon > 0$, the following time:

$$t_{\omega,\epsilon} := \inf\{t : \omega(t) \leq \bar{\omega} + \epsilon\}\,.$$

By approximating how the sequence $\omega(t)$ approaches its limit $\bar{\omega}$, it is possible to estimate $t_{\omega,\epsilon}$.

Lemma 4. Given a small $\epsilon > 0$, it holds:

$$t_{\omega,\epsilon} \geq \left\lceil \frac{\log\epsilon - \log(1-\bar{\omega})}{\log f'(\bar{\omega})} \right\rceil$$
$$t_{\omega,\epsilon} \leq t^{\wedge}_{\omega,\epsilon}\,,$$

where:

$$t^{\wedge}_{\omega,\epsilon} := \left\lceil \frac{\log\epsilon}{\log\frac{\frac{1}{1+\gamma}-\bar{\omega}}{1-\bar{\omega}}} \right\rceil \leq \left\lceil \frac{-\log\epsilon}{\log(1+\gamma)} \right\rceil\,.$$

Proof. To find a lower bound, consider the first order Taylor expansion of $f(x)$, namely $g_1(x) = f'(\bar{\omega})(x-\bar{\omega})+\bar{\omega}$. Note that $g_1(x) < f(x) < x$ in $(\bar{\omega}, 1]$, because $f(x)$ is strictly convex. The sequence $\omega_1(t)$ defined by $\omega_1(0) = 1$ and $\omega_1(t+1) = g_1(\omega_1(t))$ is strictly decreasing and has limit $\bar{\omega}$. Moreover, for every $t \geq 1$, it satisfies:

$$\bar{\omega} \leq \omega_1(t) < \omega(t)\,,$$

so $t_{\omega_1,\epsilon} \leq t_{\omega,\epsilon}$. To compute $t_{\omega_1,\epsilon}$, we observe that:

$$\omega_1(t+1) - \bar{\omega} = f'(\bar{\omega})(\omega_1(t) - \bar{\omega})$$

so:

$$\omega_1(t) - \bar{\omega} = (f'(\bar{\omega}))^t(1-\bar{\omega})$$

and hence:

$$t_{\omega_1,\epsilon} = \inf\{t : (f'(\bar{\omega}))^t(1-\bar{\omega}) \leq \epsilon\}$$

which gives:

$$t_{\omega_1,\epsilon} = \left\lceil \frac{\log\epsilon - \log(1-\bar{\omega})}{\log f'(\bar{\omega})} \right\rceil\,.$$

To get the smaller of the upper bounds, consider the function:

$$g_2(x) = \frac{\frac{1}{1+\gamma} - \bar{\omega}}{1-\bar{\omega}}(x-\bar{\omega}) + \bar{\omega}$$

which interpolates the points $(\bar{\omega}, \bar{\omega})$ and $(1, \frac{1}{1+\gamma})$, i.e. the points $(\bar{\omega}, f(\bar{\omega}))$ and $(1, f(1))$. For every $x \in (\bar{\omega}, 1)$, we have that $f(x) < g_2(x) < x$ because $f(x)$ is convex and $\gamma > 0$. The sequence $\omega_2(t)$, defined by $\omega_2(t+1) = g_2(\omega_2(t))$ with initial condition $\omega_2(0) = 1$, is clearly strictly decreasing to the limit $\bar{\omega}$. Moreover, for every $t > 0$:

$$\bar{\omega} \leq \omega(t) < \omega_2(t)$$

thus $t_{\omega,\epsilon} \leq t_{\omega_2,\epsilon}$. By induction, we have that:

$$\omega_2(t) - \bar{\omega} = \left(\frac{\frac{1}{1+\gamma} - \bar{\omega}}{1-\bar{\omega}}\right)^t (1-\bar{\omega})\,,$$

therefore:

$$t_{\omega_2,\epsilon} = \inf\left\{t: \left(\frac{\frac{1}{1+\gamma}-\bar{\omega}}{1-\bar{\omega}}\right)^t (1-\bar{\omega}) \le \epsilon\right\}$$
$$\le \inf\left\{t: \left(\frac{\frac{1}{1+\gamma}-\bar{\omega}}{1-\bar{\omega}}\right)^t \le \epsilon\right\}$$
$$\le \inf\left\{t:\ t\log\left(\frac{\frac{1}{1+\gamma}-\bar{\omega}}{1-\bar{\omega}}\right) \le \log\epsilon\right\}$$
$$= \left\lceil \frac{\log\epsilon}{\log\frac{\frac{1}{1+\gamma}-\bar{\omega}}{1-\bar{\omega}}} \right\rceil =: t^{\wedge}_{\omega,\epsilon} \quad (7)$$

The quantity $t^{\wedge}_{\omega,\epsilon}$ can be further upper bounded, if we observe that:

$$\frac{\frac{1}{1+\gamma}-\bar{\omega}}{1-\bar{\omega}} \le \frac{1}{1+\gamma} < 1$$

so (7) is upper bounded by:

$$\left\lceil \frac{-\log\epsilon}{\log(1+\gamma)} \right\rceil,$$

which concludes the proof. □

The upper bound on $t^{\wedge}_{\omega,\epsilon}$ does not depend on $\bar{\omega}$; however, both $t^{\wedge}_{\omega,\epsilon}$ and its upper bound explode if γ approaches zero.

We now move on to study the behavior of the sequence $\eta(t)$. Thanks to the fact that $(d-1)\bar{\omega}$ is eventually smaller than one, the sequence $\eta(t)$ converges.

Lemma 5. Let $d \ge 2$ and $\gamma > 0$. The sequence $\eta(t)$ converges to the limit:

$$\bar{\eta} = \frac{1}{1-(d-1)\bar{\omega}}.$$

Proof. The product $(d-1)\omega(t)$ is monotonically decreasing and, given $\epsilon \in (0, \frac{1}{d-1}-\bar{\omega})$:

$$(d-1)\omega(t) < (d-1)\omega(t_{\omega,\epsilon}) < 1,$$

for every $t \ge t_{\omega,\epsilon}$. Therefore, the sequence $\eta(t)$ will eventually converge. To identify the limit value, we consider the recursions of $\omega(t)$ and $\eta(t)$ as a bi-dimensional dynamical system, of which we study the fixed points, i.e. pairs $(\bar{\omega}, \bar{\eta})$ such that:

$$\begin{cases} \bar{\omega} = f(\bar{\omega}) \\ \bar{\eta} = 1 + (d-1)\bar{\omega}\bar{\eta}. \end{cases}$$

The dynamic of $\omega(t)$ is independent and we already identified the unique acceptable solution. The corresponding fixed point of $\eta(t)$ is $\bar{\eta} = \frac{1}{1-(d-1)\bar{\omega}}$. □

It is more difficult to estimate a convergence time for $\eta(t)$, because this sequence is not monotonic. Actually, we observe that $\eta(t)$ is increasing for:

$$t \le t_{\omega,\frac{1}{d-1}-\bar{\omega}}$$

because $(d-1)\omega(t) \ge 1$. However, if $\eta(t)$ starts to be decreasing, then it continues to be strictly decreasing.

Lemma 6. If there exists t^* such that $\eta(t^*+1) \le \eta(t^*)$, then $\eta(t+1) < \eta(t)$, for every $t > t^*$.

Proof. Since $\omega(t)$ is strictly decreasing, we have:

$$\eta(t+2) = 1 + (d-1)\omega(t+1)\eta(t+1)$$
$$< 1 + (d-1)\omega(t)\eta(t) = \eta(t+1),$$

and similarly for the following times. □

In order to estimate:

$$t_{\eta,\epsilon} := \inf\{t : |\eta(t) - \bar{\eta}| \le \epsilon\},$$

we consider the sequence $\eta_1(t)$, defined by the recursion:

$$\eta_1(t+1) = 1 + (d-1)\bar{\omega}\eta_1(t),$$

with initial value $\eta_1(0) = 1$. We just substituted the limit $\bar{\omega}$ to $\omega(t)$ in the recursion of $\eta(t)$ and hence, clearly:

$$\eta_1(t) \le \eta(t),$$

because $\bar{\omega} \le \omega(t)$ for every t. If we forget about the transient, we can reasonably expect a slightly faster convergence from $\eta_1(t)$ than from $\eta(t)$.

Lemma 7. The sequence $\eta_1(t)$ is monotonically increasing and has limit $\bar{\eta}$. Given $\epsilon > 0$:

$$t_{\eta_1,\epsilon} = \left\lceil \frac{\log\epsilon - \log\bar{\eta}}{\log((d-1)\bar{\omega})} \right\rceil - 1$$

Proof. The sequence $\eta_1(t)$ can be rewritten as the geometric series:

$$\eta_1(t) = \sum_{i=0}^{t}((d-1)\bar{\omega})^i = \frac{1-((d-1)\bar{\omega})^{t+1}}{1-(d-1)\bar{\omega}},$$

and we observe that $\eta_1(t)$ is monotonically increasing and converges to the limit $\bar{\eta}$, because $(d-1)\bar{\omega} < 1$. Next, consider $|\eta_1(t) - \bar{\eta}| \le \epsilon$, equivalent to:

$$\bar{\eta}((d-1)\bar{\omega})^{t+1} \le \epsilon.$$

We can bound the convergence time as follows:

$$t_{\eta_1,\epsilon} = \inf\{t : ((d-1)\bar{\omega})^{t+1} \le \epsilon\bar{\eta}^{-1}\}$$
$$= \inf\left\{t:\ t \ge \frac{\log\epsilon - \log\bar{\eta}}{\log((d-1)\bar{\omega})} - 1\right\}$$
$$= \left\lceil \frac{\log\epsilon - \log\bar{\eta}}{\log((d-1)\bar{\omega})} \right\rceil - 1$$

where we used $(d-1)\bar{\omega} < 1$. □

We present a couple of numerical examples. In the first example, represented in Figure 2, we used $d = 5$ and $\gamma = 1$. The sequence $\omega(t)$ converges quickly and the bound $t^{\wedge}_{\omega,0.01} = 5$ captures the actual value $t_{\omega,0.01} = 5$. The value $(d-1)\bar{\omega} \approx 0.76$ is sufficiently smaller than one, and makes the sequence $\eta(t)$ converge quickly, with $t_{\eta,0.01} = 30$. The quantity $t_{\eta_1,0.01} = 22$ is an optimistic but fair estimate of the convergence time of η. In the second example, in Figure 3, we used $d = 5$ and $\gamma = 0.01$. The sequence $\omega(t)$ continues to converge quickly and $t_{\omega,0.01} = 9$. As expected, the bound $t^{\wedge}_{\omega,0.01} = 347$ is loose because $\frac{1}{1+0.01}$ is close to one. Also the value $(d-1)\bar{\omega}$ becomes close to one and this makes the sequence $\eta(t)$ converge slowly, with $t_{\eta,0.01} = 3436$. Again, the estimate $t_{\eta_1,0.01} = 3102$ is optimistic but fair.

The analysis of the fully homogeneous scenario confirms what we observed earlier on the MPA, and provides a quantitative insights about the gap between the convergence times of $\omega(t)$ and $\eta(t)$. The sequence $\omega(t)$ converges rather quickly, in a few steps, although the simple bound $t^{\wedge}_{\omega,0.01}$ becomes loose for small γs. A small γ makes $(d-1)\bar{\omega}$ close to one, which implies slow convergence and a large overshoot (and limit $\bar{\eta}$).

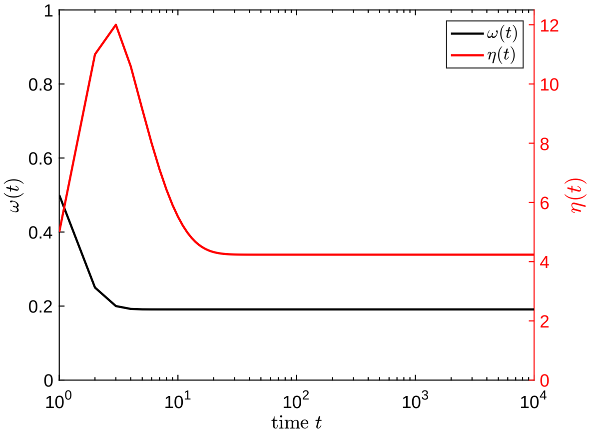

Fig. 2. The plot contains the sequence $\omega(t)$ (solid black, left y-axis) for $d = 5$ and $\gamma = 1$, whose convergence time is $t_{\omega,0.01} = 5$ upper bounded by $t^{\wedge}_{\omega,0.01} = 5$. The corresponding sequence $\eta(t)$ (solid red, right y-axis) converges with time $t_{\eta,0.01} = 30$ estimated by $t_{\eta_1,0.01} = 22$. The limits are $\bar{\omega} \approx 0.19$ and $\bar{\eta} \approx 4.24$.

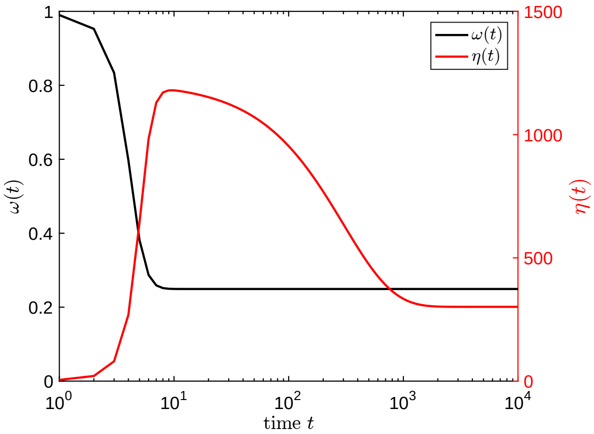

Fig. 3. The plot contains the sequence $\omega(t)$ (solid black, left y-axis) for $d = 5$ and $\gamma = 0.01$, whose convergence time is $t_{\omega,0.01} = 9$ upper bounded by $t^{\wedge}_{\omega,0.01} = 347$. The corresponding sequence $\eta(t)$ (solid red, right y-axis) converges with time $t_{\eta,0.01} = 3436$, estimated by $t_{\eta_1,0.01} = 3102$. The limits are $\bar{\omega} \approx 0.25$ and $\bar{\eta} \approx 301.33$.

4. NOT FULLY HOMOGENEOUS CASES

We move on to study the convergence time of the MPA in more general, non fully homogeneous, settings. In fact, we (mainly) consider non-uniform vectors $\boldsymbol{\gamma}$, while the graph $\mathcal{G}$ remains d-regular. We will try to estimate the convergence time using a heuristic mean-field approach, based on the recursions of the previous section.

To break the homogeneity, we consider a generic non-negative (and non-identically zero) vector $\boldsymbol{\gamma} \in \mathbb{R}^I_+$, while we assume the graph $\mathcal{G}$ is still d-regular, with $d \geq 2$. We consider the set $\Gamma = \{\gamma_i : i \in I\}$ containing the possible values of γ_i and introduce a vector $\mathbf{p} = [0,1]^\Gamma$ to describe the statistic of $\boldsymbol{\gamma}$, that is:

$$p_\gamma = \frac{|\{\gamma_i : \gamma_i = \gamma\}|}{|I|} .$$

We generalize the function $f(x)$, defined in (5), to the following:

$$f_{\boldsymbol{\gamma}}(x) := \sum_{\gamma \in \Gamma} p_\gamma \frac{1}{1 + \gamma + (d-1)(1-x)} , \tag{8}$$

and consequently the recursions become:

$$\omega(t+1) = f_{\boldsymbol{\gamma}}(\omega(t))$$
$$\eta(t+1) = 1 + (d-1)\omega(t)\eta(t)$$

with $\omega(0) = 1$ and $\eta(0) = 1$. The new sequence $\omega(t)$ captures the behaviour of an heuristic "average" message $W^{i\to j}(t)$, in a mean-field sense. In fact, if we assume the average of the messages $W^{i\to j}(t^*)$ is $\omega(t^*)$, by neglecting the interconnection topology and the correlations, we can expect the average of the messages $W^{i\to j}(t^*+1)$ to be $\omega(t^*+1)$. This is at least exact for the first time step.

Lemma 8. It holds: $\omega(1) = \frac{1}{|I|d} \sum_{(j,i):\{i,j\}\in E} W^{i\to j}(1)$.

Proof. $W^{i\to j}(0) = \omega(0) = 1$ for every (j,i). Hence:

$$\frac{1}{|I|d} \sum_{(j,i):\{i,j\}\in E} W^{i\to j}(1) = \frac{1}{|I|d} \sum_{(j,i):\{i,j\}\in E} \frac{1}{1+\gamma_i} = \frac{1}{|I|d} \sum_i d \frac{1}{1+\gamma_i} = f_{\boldsymbol{\gamma}}(1)$$

Recognizing $f_{\boldsymbol{\gamma}}(1) = \omega(1)$ concludes the proof. □

Consequently, also $\eta(1)$ and $\eta(2)$ represent the averages of the messages $H^{i\to j}(1)$ and $H^{i\to j}(2)$.

The function $f_{\boldsymbol{\gamma}}(x)$ has the same properties of $f(x)$, making the new sequence $\omega(t)$ strictly decreasing to its limit $\bar{\omega}$ which also satisfies $(d-1)\bar{\omega} < 1$.

Lemma 9. It holds: $(d-1)\bar{\omega} < 1$.

Proof. The function: $\frac{1}{1+(d-1)(1-x)}$ is convex in $[0,1]$ where it has two fixed points, namely $\frac{1}{d-1}$ and 1. The vector $\boldsymbol{\gamma}$ is non-identically zero, therefore:

$$f_{\boldsymbol{\gamma}}(x) < \frac{1}{1 + (d-1)(1-x)} ,$$

for every $x \in [0,1]$. Any fixed point of $f_{\boldsymbol{\gamma}}(x)$ is either strictly smaller than $\frac{1}{d-1}$ or larger than 1; a fact that means $\bar{\omega} \leq \frac{1}{d-1}$ and the result. □

To study the speed of convergence of the heuristic recursions, we keep the same definitions of convergence times provided in the previous section. We coherently adapt the bound $t^{\wedge}_{\omega,\epsilon}$, substituting to $\frac{1}{1+\gamma}$ the quantity $\omega(1) = f_{\boldsymbol{\gamma}}(1) = \sum_{\gamma\in\Gamma} p_\gamma \frac{1}{1+\gamma}$, to obtain:

$$t^{\wedge}_{\omega,\epsilon} := \left\lceil \frac{\log \epsilon}{\log \frac{\omega(1) - \bar{\omega}}{1 - \bar{\omega}}} \right\rceil$$

In order to dicuss the effectiveness of the heuristic recursions, we present a couple of numerical examples. We consider connected d-regular graphs $\mathcal{G} = (I, E)$ of size $n = |I|$ with non-uniform vectors $\boldsymbol{\gamma}$. We simulate the MPA and compute the averages of the messages:

$$w(t) := \frac{1}{nd} \sum W^{i\to j}(t) , \qquad h(t) := \frac{1}{nd} \sum H^{i\to j}(t),$$

where the sums are over all the messages, and compare them with the corresponding sequences $\omega(t)$ and $\eta(t)$.

The example of Figure 4 corresponds to a random regular graph with $n = 1000$ nodes and $d = 5$, with $\boldsymbol{\gamma}$ such that $\Gamma = \{0.1, 0.2, 0.4, 1\}$ and:

$$\mathbf{p} = [p_{0.1}, p_{0.2}, p_{0.4}, p_1] = [0.255, 0.258, 0.242, 0.245] .$$

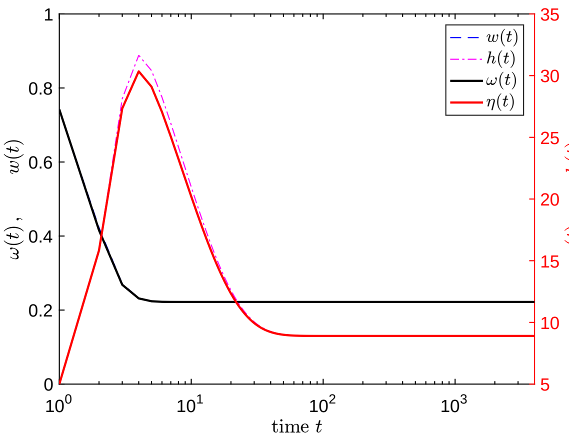

Fig. 4. Simulation on a random 5-regular graph with $n = 1000$ nodes (255 with $\gamma_i = 0.1$, 258 with $\gamma_i = 0.2$, 242 with $\gamma_i = 0.4$, 245 with $\gamma_i = 1$). The plots $w(t)$ (dashed blue, left y-axis) and $h(t)$ (dashed dotted magenta, right y-axis) represent the averages of the MPA messages, with final values 0.22 and 8.93 and convergence time $t_{w,0.01} = 7$ and $t_{h,0.01} = 71$ respectively. The sequence $\omega(t)$ (solid black, left y-axis) converges to the limit $\bar{\omega} \approx 0.22$ in $t_{\omega,0.01} = 6$ steps, upper bounded by $t^{\wedge}_{\omega,0.01} = 12$. The sequences $\eta(t)$ (solid red, right y-axis), converges to the limit $\bar{\eta} \approx 8.90$ with $t_{\eta,0.01} = 71$, estimated by $t_{\eta_1,0.01} = 57$.

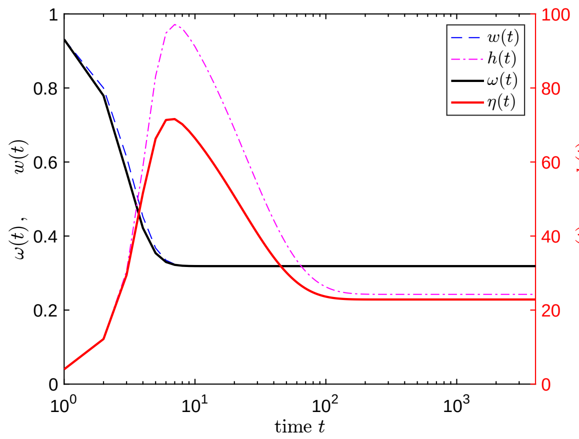

Fig. 5. Simulation on a 4-regular, bi-dimensional torus with $n = 13 \times 77 = 1001$ nodes (of which 605 have $\gamma_i = 0$, 316 have $\gamma_i = 0.1$ and 80 have $\gamma_i = 1$). The plots $w(t)$ (dashed blue, left y-axis) and $h(t)$ (dashed dotted magenta, right y-axis) represent the averages of the MPA messages, with final values 0.32 and 24.24 and convergence time $t_{w,0.01} = 9$ and $t_{h,0.01} = 250$ respectively. The sequence $\omega(t)$ (solid black, left y-axis) converges to the limit $\bar{\omega} \approx 0.32$ in $t_{\omega,0.01} = 9$ steps, upper bounded by $t^{\wedge}_{\omega,0.01} = 44$. The sequences $\eta(t)$ (solid red, right y-axis), converges to the limit $\bar{\eta} \approx 22.86$ with $t_{\eta,0.01} = 199$, estimated by $t_{\eta_1,0.01} = 172$.

The sequences $w(t)$ and $\omega(t)$ practically coincide and also $h(t)$ and $\eta(t)$ are very close to each other. Consequently, also convergence times are very well approximated. The example of Figure 5 corresponds to a bi-dimensional torus (with first neighbour connections, $d = 4$) of $n = 1001$ nodes with $\boldsymbol{\gamma}$ such that $\Gamma = \{0, 0.1, 1\}$ and:

$$\mathbf{p} = [p_0, p_{0.1}, p_1] = [0.604, 0.316, 0.080] .$$

The agreement between the sequences $w(t)$ and $h(t)$ and their mean-field approximation $\omega(t)$ and $\eta(t)$ is good, although it is possible to notice a small difference between the limit values of $h(t)$ and $\eta(t)$. The timing estimates are also good,despite the fact that quite a few nodes have null γ_i and the average γ_i is closer to zero.

These examples suggest that the mean-field dynamics captures quite well the average messages of the MPA for regular graphs $\mathcal{G}$ and non-uniform vector $\boldsymbol{\gamma}$.

5. CONCLUSION

Our work has been devoted to a recently proposed message-passing algorithm that computes the so-called harmonic influence of a node in a social network. In this paper, we have presented some preliminary facts and evidences towards the mean-field analysis of its convergence. Our key observation is that, on regular graphs with homogeneous external influence, the algorithm degenerates into two coupled scalar recursions that are amenable to a detailed analysis. Interestingly, simulations show that the insights obtained from this idealized case bear significance in more general cases that break the homogeneity. Further research is needed to characterize more precisely the set of networks where the mean-field dynamics is able to capture the convergence speed of the MPA.

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