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Shapley value for constant-sum games*

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Abstract. It is proved that Young's [4] axiomatization for the Shapley value by marginalism, efficiency, and symmetry is still valid for the Shapley value defined on the class of nonnegative constant-sum games with nonzero worth of grand coalition and on the entire class of constant-sum games as well.

Key words: cooperative TU game, value, axiomatic characterization, Shapley value.

Among single-valued solutions usually called values the most famous and the most appealing is the Shapley value [2]. Different axiomatizations for the Shapley value defined on the entire space of games with fixed set of players are known. Two main of them are the classical one given by Shapley [2] and that of Young [4]. The original Shapley's axiomatization exploits the additivity axiom that being a very beautiful mathematical statement does not express any fairness property. The axiomatization of Young that characterizes the Shapley value by marginalism, efficiency, and symmetry appears to be more attractive since all the axioms present different reasonable properties of fair division. However, not always we consider the entire space of games. Sometimes due to different reasons we restrict consideration to some subclass of games, e.g. to nonnegative or positive games, to simple games, to convex games, to constant-sum games, etc. And a reasonable question arises — is Young's axiomatization for the Shapley value on the subclass of games under scrutiny still valid? In

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general, the answer is negative. For instance, Young's axiomatization for the Shapley value considered on the subclass of all simple games (the Shapley-Shubik power index) is not correct. Indeed, in a simple game with three players the normalized Banzhaf power index presents a counterexample. In a simple game with three players the Shapley-Shubik and normalized Banzhaf indices agree with each other, except in the case where the Shapley-Shubik index assigns to one player the value 2/3 and to each of the other players 1/6; in this case the Banzhaf index assigns to these players the values 3/5 and 1/5 respectively. Yet in the case of simple games with three players the normalized Banzhaf index appears to satisfy the conditions of Young (marginalism, efficiency, and symmetry).[1] In this note we prove that Young's axiomatization is valid for the Shapley value defined on the class of nonnegative constant-sum games with nonzero worth of grand coalition and on the entire class of constant-sum games as well. One might argue that constant-sum games are not that appealing except for their nice mathematical properties. On the other hand, there are indeed relevant classes of nonnegative TU games satisfying the constant sum condition from outset, for example classes of simple majority games. In the literature, other solutions have also been related to the particular class of constant-sum games, for example in [3] it is shown that for this class of games the modified nucleolus coincides with the prenucleolus. The class of nonnegative constant-sum games with nonzero worth of grand coalition and in particular the Shapley value defined on this class appear in the study of semiproportional values in [1].

Now we recall some definitions and notation. A *cooperative game with transferable utility (TU game)* is a pair $\langle N, v\rangle$, where $N = \{1, \dots, n\}$ is a finite set of $n \geq 2$ players and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ (or $S \in 2^N$) of s players is called a *coalition*, and the associated real number $v(S)$ presents the *worth* of the coalition S. The set of all games with a fixed player set N we denote $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears, we write v instead of $\langle N, v\rangle$ when refer to a game. For any set of games $\mathcal{G} \subseteq \mathcal{G}_N$, a *value on* $\mathcal{G}$ is a mapping $\xi : \mathcal{G} \to \mathbb{R}^n$ that associates with each game $v \in \mathcal{G}$ a vector $\xi(v) \in \mathbb{R}^n$, where the real number $\xi_i(v)$ represents the *payoff* to player i in the game v.

A value ξ is *marginalist* if, for all $v \in \mathcal{G}$, for every $i \in N$, $\xi_i(v)$ depends only upon the ith marginal utility vector $\{v(S \cup i) - v(S)\}_{S \subseteq N\setminus i}$, i.e.,

$$\xi_i(v) = \phi_i(\{v(S \cup i) - v(S)\}_{S \subseteq N\setminus i}),$$

where $\phi_i : \mathbb{R}^{2^{n-1}} \to \mathbb{R}^1$.

A value ξ is *efficient* if, for all $v \in \mathcal{G}$,

$$\sum_{i \in N} \xi_i(v) = v(N).$$

A value ξ is *symmetric* if, for all $v \in \mathcal{G}$, for any permutation $\pi : N \to N$, and for all $i \in N$,

$$\xi_{\pi(i)}(v^\pi) = \xi_i(v),$$

where $v^\pi(S) = v(\pi(S))$ for all $S \subseteq N$, $S \neq \emptyset$.

[1]This counterexample is due to Moshe' Machover.

Throughout the remainder of the paper we restrict our consideration to nonnegative constant-sum games of the class

$$\mathcal{G}_N^{+c} = \{v \in \mathcal{G}_N | v(N) \neq 0, v(S) \geq 0, v(S) + v(N \backslash S) = v(N), \text{ for all } S \subseteq N\}.$$

We prove below that the Shapley value defined on $\mathcal{G}_N^{+c}$ can be characterized by three Young's axioms of marginalism, efficiency, and symmetry. Our proof strategy by induction is similar to that in Young. The proof of Young is not applicable directly to the considered case since an important moment in his proof is an expansion of a game $v \in \mathcal{G}_N$ via unanimity basis $\{u_T\}_{\substack{T\subseteq N\\T\neq\emptyset}}$ defined as

$$u_T(S) = \begin{cases} 1, & T \subseteq S, \\ 0, & T \not\subseteq S, \end{cases} \qquad \text{for all } S \subseteq N,$$

but unanimity games u_T, for all nonempty and non-singleton coalitions $T \subseteq N$, do not belong to $\mathcal{G}_N^{+c}$.

Theorem 1. *The only efficient, symmetric, and marginalist value defined on $\mathcal{G}_N^{+c}$ is the Shapley value.*

Proof: Every game $v \in \mathcal{G}_N^{+c}$, being a constant-sum game, appears to be a self-dual game, i.e., $v = v^*$, where for any $v \in \mathcal{G}_N$, a dual game v^* is defined as

$$v^*(S) = v(N) - v(N \backslash S), \quad \text{for all } S \subseteq N.$$

For any game $v \in \mathcal{G}_N$ presented via unanimity basis $\{u_T\}_{\substack{T\subseteq N\\T\neq\emptyset}}$,

$$v = \sum_{\substack{T\subseteq N\\T\neq\emptyset}} \lambda_T u_T,$$

the dual game v^* can be presented via dual unanimity basis $\{u_T^*\}_{\substack{T\subseteq N\\T\neq\emptyset}}$ with the same set of coefficients λ_T, $T \subseteq N$, $T \neq \emptyset$, i.e.,

$$v^* = \sum_{\substack{T\subseteq N\\T\neq\emptyset}} \lambda_T u_T^*,$$

since it is easy to check that, for any two games $v, v' \in \mathcal{G}_N$ and any real α,

$$(v + v')^* = v^* + v'^*,$$

$$(\alpha v)^* = \alpha v^*.$$

Therefore, every constant-sum game $v \in \mathcal{G}_N^{+c}$ being self-dual can be presented as a linear combination

$$v = \sum_{\substack{T\subseteq N\\T\neq\emptyset}} \lambda_T w_T \tag{1}$$

of games w_T, $T \subseteq N$, $T \neq \emptyset$, where for all $S \subseteq N$,

$$w_T(S) = \frac{u_T(S) + u_T^*(S)}{2} = \begin{cases} 1, & T \subseteq S, \\ 1/2, & T \cap S \neq \emptyset,\ T \not\subseteq S, \\ 0, & T \cap S = \emptyset\ . \end{cases}$$

For all $T \subseteq N$, $T \neq \emptyset$, $w_T \in \mathcal{G}_N^{+c}$. Similarly to unanimity game u_T, in any game w_T, $T \subseteq N$, $T \neq \emptyset$, every player $i \notin T$ is a null-player, i.e., all of his marginal contributions are equal to zero. The Shapley values of both games w_T and u_T coincide, i.e., for every $i \in N$,

$$Sh_i(w_T) = Sh_i(u_T) = \begin{cases} 1/t, & i \in T, \\ 0, & i \notin T. \end{cases}$$

However, we cannot apply the induction procedure directly to the expansion (1) since for a nonnegative constant-sum game $v \in \mathscr{G}_N^{+c}$, not all coefficients λ_T in (1) are necessarily nonnegative (we can state only that at least one of them is positive), and deletion of a term in (1) may lead out of the class $\mathscr{G}_N^{+c}$.

To overcome the problem we consider another than (1) expansion of a game $v \in \mathscr{G}_N^{+c}$ via games w_T, $T \subseteq N$, $T \neq \emptyset$. For an expression (1) and each $t = 1, \ldots, n$, define

$$\lambda_t = \max\{\max_{T:|T|=t} \lambda_T,\ 0\}, \quad \text{and} \quad \bar{\lambda}_T = \lambda_t - \lambda_T \geq 0.$$

Consider a symmetric game

$$u = \sum_{t=1}^{n} \lambda_t \sum_{\substack{T \subseteq N \\ T \neq \emptyset \\ |T|=t}} w_T.$$

One can easily see that $u \in \mathscr{G}_N^{+c}$. Since (1),

$$v = u - \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \bar{\lambda}_T w_T. \tag{2}$$

Observe that deletion of any term under the summation sign in (2) does not move out of $\mathscr{G}_N^{+c}$ since $\bar{\lambda}_T \geq 0$, $T \subseteq N$, $T \neq \emptyset$. Let now the index I of a game $v \in \mathscr{G}_N^{+c}$ be the minimum number of terms under the summation in an expression (2), i.e.,

$$v = u - \sum_{k=1}^{I} \bar{\lambda}_{T_k} w_{T_k},$$

where all $\lambda_{T_k} \neq 0$. We proceed the remaining part of the proof by induction on this index I.

Let ξ be an efficient, symmetric, and marginalist value on $\mathscr{G}_N^{+c}$.

If $I = 0$, then $v = u$, and for symmetric game u the result follows directly from efficiency and symmetry assumptions about both values ξ and the Shapley value.

Assume now that $\xi(v)$ is the Shapley value whenever the index of $v \in \mathscr{G}_N^{+c}$ is at most I, and consider some $v \in \mathscr{G}_N^{+c}$ with the index equal to $I+1$. Let $T = \cap_{k=1}^{I+1} T_k$. For all $i, j \in T$, symmetry implies that $\xi_i(v) = \xi_j(v)$; the similar statement is true for the Shapley value too. Hence, combined with the requirement of efficiency (both payoff vectors $\xi(v)$ and $Sh(v)$ sum up to $v(N)$) it is sufficient to prove that $\xi_i(v) = Sh_i(v)$ when $i \notin T$. Define a game

$$v^{(i)} = u - \sum_{k: i \in T_k} \bar{\lambda}_{T_k} w_{T_k}.$$

Obviously, the index of $v^{(i)}$ is at most I and, therefore, by induction hypothesis, $\xi(v^{(i)}) = Sh(v^{(i)})$. To complete the proof notice that both ith marginal utility vectors relevant to the games v and $v^{(i)}$ coincide and, so, by marginalism of both values ξ and the Shapley value, $\xi_i(v) = \xi_i(v^{(i)})$ and $Sh_i(v) = Sh_i(v^{(i)})$. $\square$

To conclude with it is reasonable to note that Young's axiomatization is valid as well for the Shapley value defined on the entire class of constant-sum games

$$\mathscr{G}_N^c = \{v \in \mathscr{G}_N \mid v(S) + v(N \backslash S) = v(N), \quad \text{for all } S \subseteq N\}.$$

Indeed, the last statement can be proved by the same way as it was done for the case of the subclass of nonnegative constant-sum games $\mathscr{G}_N^{+c}$ or one can exploit the same proof as in Young but with replacement of unanimity games u_T, $T \subseteq N$, $T \neq \emptyset$, via games w_T, $T \subseteq N$, $T \neq \emptyset$.

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