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Joint Synthesis of Robust Dynamic State Feedback and Dynamic Disturbance Feed-Forward for Uncertain Systems

Hakan Köroğlu

Abstract— **Joint synthesis of dynamic state feedback is considered together with dynamic disturbance feedforward. First a novel parameter transformation is introduced to derive a new LMI condition for synthesis in the case of LTI systems. This condition forms the basis of a new synthesis method, which can easily be specialized to combination of static or dynamic sate feedback with static or dynamic disturbance feedforward. Moreover, it can also be used to synthesize LTI or scheduled controllers for systems that depend on uncertain time-varying parameters, some of which are not measurable online.**

I. Introduction

Control systems in various domains are usually designed with an ingenious combination of feedback (for stability and robustness) and feedforward (for improved tracking and disturbance rejection). Linear matrix inequality (LMI) optimization forms a convenient setting for a unified synthesis of feedback and feedforward controllers for single as well as multiple ($\mathscr{H}_2$, $\mathscr{H}_\infty$ etc.) performance objectives [2], [21], [15]. LMI based synthesis can also be used to design scheduled (or linear parameter-varying - LPV) controllers for systems that depend on online-measurable parameters (see [16] and the references therein). Synthesis becomes more challenging in the case of uncertain and unmeasurable parameters, nonlinearities and dynamic uncertainties.

Luckily though, robust controller synthesis (in the presence of unmeasurable parameters) can be rendered convex in a number of specific cases and typically by admitting to some unknown degree of conservatism. In particular, one can perform robust syntheses based on LMI optimization in the cases of static state feedback [19], dynamic disturbance feedforward [7], [23], [5], [3], [12], output filtering/estimation [13], [6], [1], [25], [22] and dynamic output feedback with uncertainty solely in the disturbance filter [4], [11]. From a technical point of view, it is the specific structure that makes robust synthesis possible for a particular problem. A generic structure is identified in [20] for which robust synthesis is possible within the setting of dynamic IQCs (see also [26]).

In this paper, we are concerned with the synthesis of dynamic state feedback jointly with dynamic disturbance feedforward especially for uncertain systems. A joint dynamic feedback/feedforward synthesis is straightforward for known linear time-invariant (LTI) systems based on the LMI approach of [21], which would typically deliver controllers with an order that is equal to the order of the generalized plant. It is, however, not possible to apply this method to uncertain systems with unmeasurable parameters. On the other hand, it is possible to synthesize static state feedback jointly with static disturbance feedforward for uncertain systems as presented in [17]. To the best of the author's knowledge, there exists no LMI method in the literature that one can use for a concurrent synthesis of dynamic state feedback with dynamic disturbance feedforward for uncertain systems. One can though first perform a robust static state feedback and then complement it with dynamic disturbance feedforward. Such a sequential approach might lead to suboptimal design for uncertain systems as pointed out by [27] and would be quite inconvenient especially when the goal is to synthesize a parameter-dependent controller. A joint synthesis of static feedback with dynamic feedforward was considered in the previous work [9].

We develop in this paper a novel LMI approach that makes it possible to synthesize dynamic state feedback concurrently with dynamic disturbance feedforward. The method can also be specialized easily to all types of combinations (static/dynamic) of state-feedback and disturbance-feedforward. In the next section, we first consider the joint synthesis problem for LTI systems and introduce a new variable transformation as the key to the proposed approach. We then extend the method to uncertain systems in Section III through an application of the full-block S-procedure [18]. The paper is concluded after a simple example that illustrates in which interesting ways the new approach can be used.

II. Joint Synthesis for LTI Systems

In this section, we formulate the problem for an LTI system and derive a new LMI condition for joint synthesis.

A. Problem Statement

Let us consider an LTI plant as

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t)+Bu(t)+Hw(t),\ x(0)=0, \\ z(t) & = & Cx(t)+Du(t)+Ew(t), \end{array} \tag{1}$$

where $x(t)\in\mathbb{R}^{n_x}$ denotes the state, while $u(t)\in\mathbb{R}^{n_u}$, $w(t)\in\mathbb{R}^{n_w}$ and $z(t)\in\mathbb{R}^{n_z}$ represent the control input, the disturbance and the performance signals respectively. The system realization $(A,[B\,H],C,[D\,E])\in\mathbb{R}^{n_x\times n_x}\times\mathbb{R}^{n_x\times(n_u+n_w)}\times\mathbb{R}^{n_z\times n_x}\times\mathbb{R}^{n_z\times(n_u+n_w)}$ is fixed and known. We assume that the states are available for feedback without any measurement noise. We also consider for feedforward a (reference or disturbance) signal $r(t)\in\mathbb{R}^{n_r}$ that relates to w as

$$r(t)=\Gamma w(t), \tag{2}$$

where $\Gamma\in\mathbb{R}^{n_r\times n_w}$ is a fixed and known matrix.

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In this section, we consider the joint synthesis of dynamic state feedback and dynamic feedforward with an LTI controller. The control input is hence generated as

$$\begin{aligned} \dot{\xi}(t) &= A_f\xi(t)+B_f r(t)+B_s x(t),\ \ \xi(0)=0, \\ u(t) &= \underbrace{C_f\xi(t)+D_f r(t)}_{u_1}+\underbrace{D_s x(t)}_{u_2}, \end{aligned} \tag{3}$$

where $(A_f,[\,B_f\ \ B_s\,],C_f,[\,D_f\ \ D_s\,]) \in \mathbb{R}^{n_\xi\times n_\xi}\times\mathbb{R}^{n_\xi\times(n_r+n_x)}\times\mathbb{R}^{n_u\times n_\xi}\times\mathbb{R}^{n_u\times(n_r+n_x)}$ represents a controller realization.

By combining equations (1)-(3), we obtain the closed-loop dynamics (with suppressed time dependence) as follows:

$$\begin{aligned} \dot{\varkappa} &= \underbrace{\begin{bmatrix} A+BD_s & BC_f \\ B_s & A_f \end{bmatrix}}_{\mathscr{A}}\underbrace{\begin{bmatrix} x \\ \xi \end{bmatrix}}_{\varkappa}+\underbrace{\begin{bmatrix} H+BD_f\Gamma \\ B_f\Gamma \end{bmatrix}}_{\mathscr{B}}w, \\ z &= \underbrace{\begin{bmatrix} C+DD_s & DC_f \end{bmatrix}}_{\mathscr{C}}\varkappa+\underbrace{(E+DD_f\Gamma)}_{\mathscr{D}}w. \end{aligned} \tag{4}$$

The LTI joint synthesis problem then reads as follows:

Problem 1: Given the LTI plant of (1)-(2), find an LTI controller as in (3) such that the closed loop system in (4) is stable and satisfies the following $\mathscr{L}_2$-gain condition:

$$\|z\|_2 \triangleq \sqrt{\int_0^\infty z^T(t)z(t)dt} < \gamma\|w\|_2,\ \forall w(\cdot): 0<\|w\|_2<\infty. \tag{5}$$

B. A New LMI Condition for Joint Synthesis

In this section, we derive an LMI solution to the joint synthesis problem in the case of a full-order controller (i.e. $n_\xi = n_x$). We start by recalling the well-known matrix inequality condition for stability and $\mathscr{L}_2$-gain performance (see e.g. [21]), expressed in terms of a positive-definite matrix $\mathscr{X}\in\mathbb{S}_+^{n_x+n_\xi}\triangleq\{\mathscr{X}=\mathscr{X}^T\in\mathbb{R}^{(n_x+n_\xi)\times(n_x+n_\xi)}:\mathscr{X}\succ 0\}$ as

$$\mathsf{He}\begin{bmatrix} \mathscr{X}\mathscr{A} & \mathscr{X}\mathscr{B} & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \mathscr{C} & \mathscr{D} & -\frac{\gamma}{2}I \end{bmatrix} \prec 0, \tag{6}$$

where $\mathsf{He}(\mathscr{M})\triangleq\mathscr{M}+\mathscr{M}^T$.

We now consider a specific parametrization for $\mathscr{X}$ and then introduce a congruence transformation to arrive at an LMI condition for joint synthesis. The specific choice of $\mathscr{X}$ is expressed compatibly with the closed-loop of (4) with $n_\xi = n_x$ and in terms of $Y,X\in\mathbb{S}_+^{n_x}$ as follows:

$$\mathscr{X}=\begin{bmatrix} Y^{-1} & -Y^{-1} \\ -Y^{-1} & Y^{-1}+X^{-1} \end{bmatrix}=\begin{bmatrix} Y+X & X \\ X & X \end{bmatrix}^{-1}. \tag{7}$$

We stress that this parametrization is without loss of generality, since we can always bring $\mathscr{X}$ and its inverse into the specific forms in (7) by a suitable state transformation. A specific congruence transformation translates (6) to

$$\mathsf{He}\begin{bmatrix} \mathscr{Y}^T\mathscr{X}\mathscr{A}\mathscr{Y} & \mathscr{Y}^T\mathscr{X}\mathscr{B} & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \mathscr{C}\mathscr{Y} & \mathscr{D} & -\frac{\gamma}{2}I \end{bmatrix} \prec 0, \tag{8}$$

where $\mathscr{Y}$ is an invertible matrix. The suitable choice of $\mathscr{Y}$ for our derivations is identified as

$$\mathscr{Y}=\begin{bmatrix} Y & X \\ 0 & X \end{bmatrix}. \tag{9}$$

With this choice, we obtain

$$\mathscr{Y}^T\mathscr{X}=\begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}. \tag{10}$$

By now introducing new matrix variables as

$$K\triangleq(A_f+B_s)X,\ L\triangleq B_sY,\ M\triangleq(C_f+D_s)X,\ N\triangleq D_sY, \tag{11}$$

we arrive at the following solution of Problem 1:

Theorem 1: There is a solution to Problem 1 if and only if there exist $X,Y\in\mathbb{S}_+^{n_x}$; $K,L\in\mathbb{R}^{n_x\times n_x}$; $M,N\in\mathbb{R}^{n_u\times n_x}$; $B_f\in\mathbb{R}^{n_x\times n_r}$ and $D_f\in\mathbb{R}^{n_u\times n_r}$ that satisfy

$$\mathsf{He}\begin{bmatrix} AY+BN-L & AX+BM-K & H+(BD_f-B_f)\Gamma & 0 \\ L & K & B_f\Gamma & 0 \\ 0 & 0 & -\frac{\gamma}{2}I & 0 \\ CY+DN & CX+DM & E+DD_f\Gamma & -\frac{\gamma}{2}I \end{bmatrix} \prec 0. \tag{12}$$

The controller can then be constructed from a solution as

$$\begin{bmatrix} A_f & B_f & B_s \\ C_f & D_f & D_s \end{bmatrix}=\begin{bmatrix} KX^{-1}-B_s & B_f & LY^{-1} \\ MX^{-1}-D_s & D_f & NY^{-1} \end{bmatrix}. \tag{13}$$

Remark 1: Theorem 1 is specialized to different cases of joint synthesis as in Table I. The controller can be ensured to be strictly proper simply by choosing $D_f=0$ and $N=0$.

	no SFB	static SFB	dynamic SFB
no DFF	none	only N,Y	$B_f=0$ $D_f=0$
static DFF	only D_f	only D_f,N,Y	$D_f=0$
dynamic DFF	$L=0$ $N=0$	$L=0$	all

TABLE I

SPECIAL CASES OF JOINT SYNTHESIS (SFB: STATE FEEDBACK, DFF: DISTURBANCE FEEDFORWARD)

The controller obtained via the approach of [21] would depend explicitly on the system data and thus any uncertain parameter as well. On the other hand, Theorem 1 is based on a computation of the controller as in (13) without any explicit dependence on system matrices. Thanks to this property, it becomes possible to extend the synthesis of Theorem 1 to uncertain systems with unmeasurable parameters.

III. JOINT SYNTHESIS FOR UNCERTAIN SYSTEMS

In this section we consider the joint synthesis problem for a system that depends on uncertain parameters. A solution is then presented for the synthesis of a robust controller with no dependence on unmeasurable parameters.

A. Problem Statement for an Uncertain Plant

Let us recall the plant model in (1) and assume that the system matrices are uncertain. This uncertainty is expressed in the form of a linear fractional transformation (LFT)

$$\begin{bmatrix} A(\Delta(\delta)) & B(\Delta(\delta)) & H(\Delta(\delta)) \\ C(\Delta(\delta)) & D(\Delta(\delta)) & E(\Delta(\delta)) \end{bmatrix}=\begin{bmatrix} A_0 & B_0 & H_0 \\ C_0 & D_0 & E_{00} \end{bmatrix} + \begin{bmatrix} H_1 \\ E_{01} \end{bmatrix}\Delta(\delta)\,(I-E_{11}\Delta(\delta))^{-1}\begin{bmatrix} C_1 & D_1 & E_{10} \end{bmatrix}, \tag{14}$$

where $\Delta(\delta)$ represents a matrix that has affine dependence on an uncertain (and possibly time-varying) parameter vector

δ. As is well known, such a system can also be represented as the following LFT interconnection:

$$\left[\begin{array}{c}\dot{x}(t)\\ \hline z(t)\\ q(t)\end{array}\right]\left[\begin{array}{c|ccc}A_0 & B_0 & H_0 & H_1\\ \hline C_0 & D_0 & E_{00} & E_{01}\\ C_1 & D_1 & E_{10} & E_{11}\end{array}\right]\left[\begin{array}{c}x(t)\\ \hline u(t)\\ w(t)\\ p(t)\end{array}\right]$$
$$\text{and } p(t)=\Delta(\delta(t))q(t). \quad (15)$$

The uncertainty is described by

$$\delta(t)\in\mathscr{U}_\delta,\ \forall t\geq 0, \quad (16)$$

where $\mathscr{U}_\delta$ represents a compact uncertainty set. In order to provide conditions for a concrete case, we restrict our interest to polytopic uncertainty sets described as the convex hull of a set of known vertices $\{\delta^1,..,\delta^\eta\}$ as

$$\mathscr{U}_\delta=\mathsf{conv}\left(\{\delta^1,..,\delta^\eta\}\right) \triangleq \left\{\sum_{i=1}^{\eta}\alpha_i\delta^i:\sum_{i=1}^{\eta}\alpha_i=1,\alpha_i\in[0,1],\forall i\right\}. \quad (17)$$

We also assume that the uncertain parameter vector is composed of measurable and unmeasurable components as

$$\delta=\begin{bmatrix}\mu\\ \psi\end{bmatrix};\ \begin{array}{l}\mu: \text{ vector of measurable parameters}\\ \psi: \text{ vector of unmeasurable parameters}\end{array} \quad (18)$$

In this paper, we do not consider any specific upper bounds on the entries of $\varphi\triangleq\dot{\psi}(t)\in\mathbb{R}^{n_\psi}$, while the elements of $\nu(t)\triangleq\dot{\mu}(t)\in\mathbb{R}^{n_\mu}$ are assumed to satisfy $|\nu_i(t)|\leq\nu_i^{\max}$. We hence characterize the uncertainty for $\dot{\delta}$ as

$$\dot{\delta}(t)\in\mathscr{U}_{\dot{\delta}}=\left\{\begin{bmatrix}\nu\\ \varphi\end{bmatrix}\in\mathbb{R}^{n_\mu+n_\psi}:\nu\in\mathscr{U}_\nu\right\},\ \forall t\geq 0, \quad (19)$$

where the uncertainty set for ν is identified as a hyper-rectangular region with $\kappa=2^{n_\mu}$ corners:

$$\mathscr{U}_\nu=\{\nu\in\mathbb{R}^{n_\mu}:\nu_i=\pm\nu_i^{\max}\}=\mathsf{conv}\left(\{\nu^1,..,\nu^\kappa\}\right). \quad (20)$$

In this section, we consider the synthesis of an LPV controller (with suppressed time dependence) as

$$\begin{array}{rcl}\dot{\xi}&=&A_f(\mu)\xi+B_f(\mu)r+B_s(\mu)x,\ \ \xi(0)=0,\\ u&=&C_f(\mu)\xi+D_f(\mu)r+D_s(\mu)x.\end{array} \quad (21)$$

The problem formulation then reads as follows:

Problem 2: Given an uncertain plant described by (15)-(20), find a scheduled controller of the form (21) such that the closed loop system described by (4) and (14) is stable and satisfies the $\mathscr{L}_2$-gain condition of (5) for all $\delta(\cdot)$ trajectories that are admissible according to (16) and (19).

B. LMI Conditions for Joint Synthesis for Uncertain Systems

To derive a solution for Problem 2, we first recall the $\mathscr{L}_2$-gain performance condition for parameter-dependent systems (see e.g. [28]). With $\mathscr{X}$ and the controller realization allowed to depend on the measurable parameters, the only modification needed in (6) would be the addition of a differential term to the (1,1) block. The same lines of derivation then lead to (12) with μ-dependent matrix variables and with differential terms added to (1,1)-(2,2) blocks.

We now restrict our interest to the case in which the problem variables are chosen to have affine μ-dependence:

$$\begin{bmatrix}Y(\mu) & X(\mu) & 0\\ N(\mu) & M(\mu) & D_f(\mu)\\ L(\mu) & K(\mu) & B_f(\mu)\end{bmatrix}=\begin{bmatrix}Y_0 & X_0 & 0\\ N_0 & M_0 & D_{f0}\\ L_0 & K_0 & B_{f0}\end{bmatrix} + \underbrace{\begin{bmatrix}\mu^T\otimes I_{n_x} & 0 & 0\\ 0 & \mu^T\otimes I_{n_u} & 0\\ 0 & 0 & \mu^T\otimes I_{n_x}\end{bmatrix}}_{\Lambda(\mu)}\begin{bmatrix}Y_1 & X_1 & 0\\ N_1 & M_1 & D_{f1}\\ L_1 & K_1 & B_{f1}\end{bmatrix}. \quad (22)$$

Here we have organized the variables and introduced $\Lambda(\mu)$ in a way that is convenient for our future derivations. The notation is better understood by an expression of X as

$$X(\mu)=X_0+\underbrace{\begin{bmatrix}\mu_1 I_{n_x} & \cdots & \mu_{n_\mu}I_{n_x}\end{bmatrix}}_{\mu^T\otimes I_{n_x}}\underbrace{\begin{bmatrix}X_{11}\\ \vdots\\ X_{1n_\mu}\end{bmatrix}}_{X_1}. \quad (23)$$

The subscript notation $(\cdot)_{1i}$ will thus be assumed for all problem variables in (22). Since X and Y need to be symmetric, we also require $X_0,Y_0,X_{1i},Y_{1i}\in\mathbb{S}^{n_x}\triangleq\{X=X^T\in\mathbb{R}^{n_x\times n_x}\}$.

Let us next derive for this choice of problem variables the modification needed in the $\mathscr{L}_2$-gain condition for LPV systems. We first recall again that a derivative term would emerge in the (1,1) block of the matrix in the left hand side of (6) if $\mathscr{X}$ has time-dependence. With $\mathscr{X}$ chosen to have μ dependence, this would indeed be the case when μ is a time-varying parameter. In particular, when X and Y have affine μ dependence, we would have in the (1,1) block of the matrix in the left hand side of (8) a term as

$$\mathscr{Y}^T\frac{d\mathscr{X}}{dt}\mathscr{Y}=-\mathscr{Y}^T\mathscr{X}\frac{d\mathscr{X}^{-1}}{dt}\mathscr{X}\mathscr{Y}=\begin{bmatrix}\frac{dY}{dt} & 0\\ 0 & \frac{dX}{dt}\end{bmatrix} =\sum_{i=1}^{n_\mu}\begin{bmatrix}\frac{\partial Y(\mu)}{\partial\mu_i}\nu_i & 0\\ 0 & \frac{\partial X(\mu)}{\partial\mu_i}\nu_i\end{bmatrix}=\sum_{i=1}^{n_\mu}\begin{bmatrix}\nu_i Y_{1i} & 0\\ 0 & \nu_i X_{1i}\end{bmatrix}. \quad (24)$$

The matrix in the left hand side of (12) should hence be modified by adding $0.5\sum_{i=1}^{n_\mu}\nu_i Y_{1i}$ to the (1,1) block and $0.5\sum_{i=1}^{n_\mu}\nu_i X_{1i}$ to the (2,2) block. In order develop a synthesis method, we will hence have to derive finitely many (and typically sufficient) conditions for the LMI of (12), which has rational/affine dependence on $\delta\in\mathscr{U}_\delta/\nu\in\mathscr{U}_\nu$ respectively.

As a first step towards the derivation of finitely many LMI conditions for synthesis, we first deal with ν dependence. Recall that the dependence on ν emerges due to (24) and is affine. Moreover, the associated uncertainty set $\mathscr{U}_\nu$ is a hyper-rectangular region as in (20). Hence condition (12) will be satisfied for all $\nu\in\mathscr{U}_\nu$ if and only if it is satisfied for all $\nu=\nu^k$, $k=1,\ldots,\kappa$. This brings us to 2^{n_μ} LMI conditions which all have rational dependence on $\delta\in\mathscr{U}_\delta$.

We now consider applying the full-block S-procedure by [18], which is a convenient way to resolve rational parameter dependence. We use a version of this procedure that considers parameter-dependent LMIs of the form

$$\mathsf{He}\underbrace{\left(\mathscr{M}_{00}^k+\mathscr{M}_{01}\Theta(I-\mathscr{M}_{11}\Theta)^{-1}\mathscr{M}_{10}\right)}_{\mathscr{M}^k(\Theta)} \prec 0, \forall \Theta \in \mathscr{U}_\Theta, \quad (25)$$

where $\mathscr{U}_\Theta$ represents a compact set, which is formed by all admissible values of the uncertain parameter matrix Θ. Condition (12) can indeed be expressed in the generic form of (25) with a suitably defined Θ and with the superscript ^{k} referring to the LMI obtained with $\nu = \nu^k$. The structure of Θ and the associated uncertainty set are identified as

$$\mathscr{U}_\Theta \triangleq \left\{ \Theta(\delta) \triangleq \begin{bmatrix} \Lambda(\mu) & 0 \\ 0 & \Delta(\delta) \end{bmatrix} : \delta \in \mathscr{U}_\delta \right\}. \quad (26)$$

When the uncertainty set for δ is assumed to be as in (17), the uncertainty set for Θ can be expressed as

$$\mathscr{U}_\Theta = \mathsf{conv}\left(\{\Theta^1, \ldots, \Theta^\eta\}\right), \text{ where } \Theta^i \triangleq \Theta(\delta^i). \quad (27)$$

As a major ingredient of the full-block S-procedure, we also introduce the scaling matrices $P = P^T$ via the condition

$$\begin{bmatrix} \Theta^T \\ I \end{bmatrix}^T \underbrace{\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}}_{P} \begin{bmatrix} \Theta^T \\ I \end{bmatrix} \succcurlyeq 0, \forall \Theta \in \mathscr{U}_\Theta. \quad (28)$$

The set of scaling matrices is thus defined by

$$\mathscr{P}_{\mathscr{U}_\Theta} \triangleq \left\{ P = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} : Q \in \mathbb{S}^{n_q+n_\mu(2n_x+n_u)}, \right.$$
$$\left. R \in \mathbb{S}^{n_p+2n_x+n_u}, S \in \mathbb{R}^{(n_q+n_\mu(2n_x+n_u))\times(n_p+2n_x+n_u)} : (28) \right\}. \quad (29)$$

We can now cite the version of the full-block S-procedure that is convenient for our derivations from [8] as follows:

Lemma 1: [8] The LFT of $\mathscr{M}^k(\Theta)$ is well-posed, i.e. $(I - \mathscr{M}_{11}\Theta)$ is nonsingular for all $\Theta \in \mathscr{U}_\Theta$ and (25) holds if and only if there exists a $P_k \in \mathscr{P}_{\mathscr{U}_\Theta}$ that satisfies

$$\mathsf{He}\begin{bmatrix} \mathscr{M}_{00}^k & 0 \\ \mathscr{M}_{10} & 0 \end{bmatrix} + \begin{bmatrix} 0 & \mathscr{M}_{01} \\ I & \mathscr{M}_{11} \end{bmatrix} \underbrace{\begin{bmatrix} Q_k & S_k \\ S_k^T & R_k \end{bmatrix}}_{P_k} \begin{bmatrix} 0 & \mathscr{M}_{01} \\ I & \mathscr{M}_{11} \end{bmatrix}^T \prec 0. \quad (30)$$

The full-block S-procedure resolves the issue of rational parameter dependence by introducing extra matrix variables. These matrix variables should however satisfy the LMI of (28), which in fact has quadratic parameter dependence. Finitely many and typically sufficient conditions hence need to be derived for (28) by employing a suitable relaxation scheme for the considered uncertainty set. For the polytopic set of (27), a simple approach would be to ensure concavity by requiring $Q \preccurlyeq 0$ and then to impose (28) only at the vertices $\Theta = \Theta^i$. A potentially less conservative approach would be based on Pólya's method. Referring the reader to [10] for further details, we provide here the sufficient conditions based on zeroth order Pólya relaxation as

$$\Omega_{ii}(P) \succcurlyeq 0, \; i = 1,..,\eta, \quad (31)$$
$$\mathsf{He}(\Omega_{ij}(P)) \succcurlyeq 0, \; i = 1,..,\eta;\; j = i+1,..,\eta, \quad (32)$$

where Ω_{ij}s are defined by

$$\Omega_{ij}(P) \triangleq \begin{bmatrix} (\Theta^i)^T \\ I \end{bmatrix}^T P \begin{bmatrix} (\Theta^j)^T \\ I \end{bmatrix}. \quad (33)$$

An inner approximation of the set of scaling matrices is thus characterized by $\eta(\eta+1)/2$ LMIs as follows:

$$\mathscr{P}_0 = \{P : (31) \text{ and } (32)\} \subseteq \mathscr{P}_{\mathscr{U}_\Theta}. \quad (34)$$

Higher order Pólya relaxations might reduce conservatism at the cost of increased computational complexity.

In order to arrive at a tractable synthesis procedure after the application of the full-block S-procedure, we also need to be able to express (30) as an LMI. To achieve this without introducing additional conservatism, we need to find a representation of (12) as in (25) where $\mathscr{M}_{01}$ and $\mathscr{M}_{11}$ have no dependence on matrix variables. We are indeed able to obtain such a representation and state the second main result of this paper as follows:

Theorem 2: There is a solution to Problem 2 if there exist $X_0, Y_0, X_{1i}, Y_{1i} \in \mathbb{S}^{n_x}$; $i = 1, \ldots, n_\mu$ with

$$X_0 + \sum_{i=1}^{n_\mu} \mu_i^j X_{1i} \succ 0, \quad j = 1, \ldots, \kappa, \quad (35)$$
$$Y_0 + \sum_{i=1}^{n_\mu} \mu_i^j Y_{1i} \succ 0, \quad j = 1, \ldots, \kappa, \quad (36)$$

and $K_0, K_{1i} \in \mathbb{R}^{n_x \times n_x}$; $B_{f0}, B_{f1i} \in \mathbb{R}^{n_x \times n_r}$; $M_0, N_0, M_{1i}, N_{1i} \in \mathbb{R}^{n_u \times n_x}$; $D_{f0}, D_{f1i} \in \mathbb{R}^{n_u \times n_r}$; $i = 1, \ldots, n_\mu$ and $P_k \in \mathscr{P}_0$; $k = 1, \ldots, \kappa$ that satisfy (30) for all $k = 1, \ldots, \kappa$, with ingredients of $\mathscr{M}^k$ as in (37). The controller can be constructed as in (13) with the parameter-dependent terms obtained from (22).

Proof: Omitted for reasons of space. ■

Remark 2: Robust LTI controllers would be obtained with constant matrix variables. The LMI condition in this case can be expressed by removing the fifth, sixth and seventh row/column blocks in (37) and by setting $\sum_{i=1}^{n_\mu} \nu_i^k Y_{1i} = \sum_{i=1}^{n_\mu} \nu_i^k X_{1i} = 0$. Since we then have $\Theta = \Delta$, the set of scaling matrices should be defined compatibly. If, moreover, the system has affine parameter dependence ($E_{11} = 0$), the use of the full-block S-procedure causes no additional conservatism. Indeed the set of scaling matrices could then be defined as in (34) based solely on (31) and would be identical to $\mathscr{P}_{\mathscr{U}_\Theta}$. Nevertheless, we can also consider a solution without applying the full-block S-procedure. We would then directly ensure (12) simply by imposing it at the extreme points $\delta = \delta^j$. In this case, now new matrix variable is introduced, but several LMI conditions of a large size are imposed.

IV. Illustrative Example

In this section, we illustrate the proposed synthesis method in a cooperative adaptive cruise control (CACC) system. The dynamics of this system are expressed via

$$\dot{e}(t) = v(t) - ha(t), \, \dot{v}(t) = a_0(t) - a(t), \, \dot{a}(t) = \varpi\left(u(t) - a(t)\right). \quad (38)$$

In these equations, e and $v = v_0 - v_1$ represent the spacing error and the velocity of the leader relative to the follower. On the other hand, $a_0 = \dot{v}_0$ and $a = \dot{v}_1$ represent the acceleration signals of the leading and the following vehicles respectively.

$$\begin{bmatrix} \mathscr{M}_{00}^k & \mathscr{M}_{01} \\ \mathscr{M}_{10} & \mathscr{M}_{11} \end{bmatrix} = \left[\begin{array}{cccc|cccc} A_0Y_0+B_0N_0-L_0+\frac{1}{2}\sum_{i=1}^{n_\mu}\nu_i^k Y_{1i} & A_0X_0+B_0M_0-K_0 & H_0+(B_0D_{f0}-B_{f0})\Gamma & 0 & A_0 & B_0 & -I & H_1 \\ L_0 & K_0+\frac{1}{2}\sum_{i=1}^{n_\mu}\nu_i^k X_{1i} & B_{f0}\Gamma & 0 & 0 & 0 & I & 0 \\ 0 & 0 & -\frac{\gamma}{2}I & 0 & 0 & 0 & 0 & 0 \\ C_0Y_0+D_0N_0 & C_0X_0+D_0M_0 & E_{00}+D_0D_{f0}\Gamma & -\frac{\gamma}{2}I & C_0 & D_0 & 0 & E_{01} \\ \hline Y_1 & X_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ N_1 & M_1 & D_{f1}\Gamma & 0 & 0 & 0 & 0 & 0 \\ L_1 & K_1 & B_{f1}\Gamma & 0 & 0 & 0 & 0 & 0 \\ C_1Y_0+D_1N_0 & C_1X_0+D_1M_0 & E_{10}+D_1D_{f0}\Gamma & 0 & C_1 & D_1 & 0 & E_{11} \end{array}\right]. \quad (37)$$

The headway time h is typically fixed to a constant value. On the other hand, we consider an uncertain bandwidth as

$$\varpi(t) = \varpi_\circ(1+\psi(t)),\ \psi(t) \in [\psi_{\min}, \psi_{\max}] \subset (-1, \infty), \quad (39)$$

where $\varpi_\circ$ is fixed and known, while $\psi(t) \in [\psi_{\min}, \psi_{\max}]$ is uncertain. The acceleration signal hence evolves as

$$\dot{a} = -\varpi_\circ a + \varpi_\circ u - \varpi_\circ \cdot \underbrace{\psi \cdot \overbrace{a}^{q_1}}_{p_1}. \quad (40)$$

The goal of a CACC system is to maintain small e. Though this is to be achieved with a reasonable acceleration a, we focus on a single objective with $z = e$ to make a fair comparison of two alternative syntheses. Thanks to the simple vehicle model, acceleration signals turn out to be quite acceptable as well. We assume that e, v and a are available, while the leader acceleration a_0 is communicated with some minor distortion (due to a small delay or noise).

We now build a state-space model to be used for controller synthesis. To this end, we first express the leader acceleration as a weighted sum of two signal components:

$$a_0 = (1-\rho)\cdot \underbrace{a_r}_{r} + \rho d \text{ (Model-1)}, \quad (41)$$

$$= (1-\rho)\cdot \overbrace{\underbrace{\mathsf{sign}(a_r)}_{\mu}\cdot \underbrace{|a_r|}_{q_2=r}}^{p_2} + \rho d \text{ (Model-2)}. \quad (42)$$

In these expressions, a_r represents the signal that is received by the following vehicle, while d captures communication inaccuracies. The desired level of robustness against communication problems can hence be adjusted by the scalar $\rho \in [0,1]$. In Model-1 based on (41), ψ would be the single uncertain and unmeasurable parameter. For the purpose of our illustration, we introduce Model-2 based on (42), where $\mu(t) \in [-1,1]$ can be treated as an additional uncertain and yet online-measurable parameter. For reasons of space, we provide the state-space description of only Model-2 as

$$\left[\begin{array}{c} \dot{e} \\ \dot{v} \\ \dot{a} \\ \hline z \\ \hdashline q_1 \\ q_2 \end{array}\right] = \left[\begin{array}{ccc|c:c:c:cc} 0 & 1 & -h & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & \rho & 0 & 1-\rho \\ 0 & 0 & -\varpi & \varpi & 0 & 0 & -\varpi & 0 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} e \\ v \\ a \\ \hline u \\ \hdashline r \\ d \\ \hdashline p_1 \\ p_2 \end{array}\right]$$

$$\text{and } \underbrace{\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}}_{p} = \underbrace{\begin{bmatrix} \psi & 0 \\ 0 & \mu \end{bmatrix}}_{\Delta} \underbrace{\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}}_{q}, \quad \begin{array}{c} \psi \in [\psi_{\min}, \psi_{\max}] \\ \mu \in \{-1, 0, 1\} \end{array}. \quad (43)$$

Scheduling on μ would in fact mean using a switching controller. Since the energies of the disturbance inputs would be identical for Model-1 and Model-2, it would also be meaningful to compare the performance levels achieved with: (i) a robust LTI controller designed for Model-1; and (ii) a scheduled (switching) controller designed for Model-2.

In various design and simulation exercises, we have observed that the scheduled synthesis can indeed lead to improved performance levels and hence smaller spacing errors. Perhaps more interestingly, we have also noticed that scheduled synthesis can also be used to obtain robust LTI controllers. Though this might be coincidental for our rather simple example, it nicely exemplifies in what ways the joint synthesis method developed in this paper can be used. We hence present the results for a joint synthesis exercise in which two robust LTI controllers are compared: (i) Controller-1 synthesized as a robust LTI controller for Model-1; and (ii) Controller-2 derived from a scheduled synthesis for Model-2 in a way that will be explained shortly. In both syntheses, L is set to zero to have only static state feedback, i.e. $B_s = 0$. We also set $N_1 = 0$ in the scheduled synthesis for Model-2 to avoid μ dependence in the static state feedback gain. Since μ would assume only three values, we could compute and compare the corresponding feedforward filters (i.e. the transfer function from r to u_1 in (3)). Perhaps not so surprisingly, the filters obtained for $\mu = -1$ and $\mu = +1$ turned out to be the same except for a sign change, while the filter for $\mu = 0$ happened to be practically zero. Motivated by this observation, we considered the filter obtained with $\mu = 1$ and the associated (μ-independent) static state feedback as a candidate robust LTI controller (i.e. Controller-2) that can be compared with the robust LTI controller synthesized for Model-1 (Controller-1). An analysis based on (6) indeed confirmed that Controller-2 ensures a better level of performance than Controller-1, as presented in Table II. This is also verified by a set of simulation results presented in Figure 1. In this scenario, the leading vehicle first performs a harsh braking and then accelerates persistently to get back to the initial velocity as seen in the middle plot. As is visible from the top plot, Controller-2 maintains a significantly smaller spacing error. This is achieved even in the face of a sharp change in the bandwidth parameter of the vehicle around a speed of 15 m/s, as can be observed from the bottom plot. The acceleration profile of the following vehicle is visibly identical to the acceleration profile of the leader. With a more realistic vehicle model, the following vehicle will have a delayed acceleration profile and hence increased spacing errors.

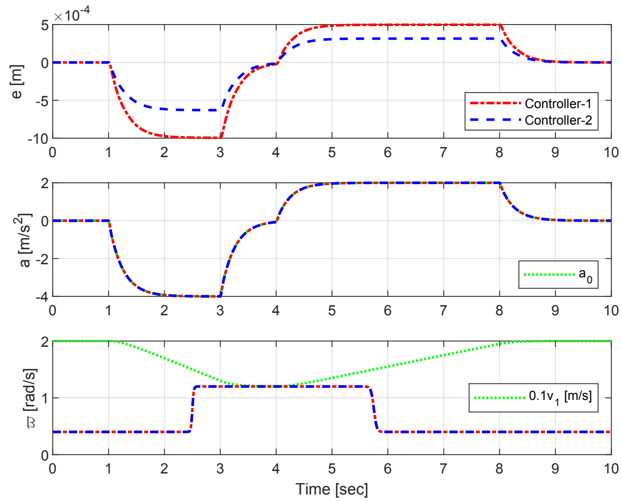

Fig. 1. Simulation results with Controller-1 (red-dash-dotted) and Controller-2 (blue-dashed) ($\varpi_\circ = 0.8$ rad/s, $h = 0$, $\rho = 0.1$).

DFF Filter - 1	$\frac{0.015635(s-2.9\cdot 10^6)(s^2+68.32s+3792)}{(s+1229)(s^2+83.34s+2853)}$
SFB Gain - 1	$\begin{bmatrix} 4.34\cdot 10^6 & 1.0\cdot 10^5 & -1.0\cdot 10^3 \end{bmatrix}$
Performance γ - 1	$2.3\cdot 10^{-4}$
DFF Filter - 2	$\frac{-0.0027414(s+2.0\cdot 10^8)(s^2+45.28s+8786)}{(s+3100)(s^2+96.16s+3846)}$
SFB Gain - 2	$\begin{bmatrix} 4.66\cdot 10^7 & 8.0\cdot 10^5 & -6.9\cdot 10^3 \end{bmatrix}$
Performance γ - 2	$1.4\cdot 10^{-4}$

TABLE II

CONTROLLER-1 AND CONTROLLER-2 (OBTAINED IN MATLAB® VIA YALMIP [14] AS THE PARSER AND SEDUMI [24] AS THE SOLVER)

V. CONCLUDING REMARKS

A novel LMI-based approach is developed for the synthesis of dynamic state feedback concurrently with dynamic disturbance feedforward. In addition to facilitating various specialized syntheses, the method extends easily to systems that depend on uncertain time-varying parameters. Potential conservatism needs to be reduced especially in the case when unmeasurable parameters have bounded derivatives. Conditions based on dilated LMIs might be useful in this respect as well as for improved multi-objective synthesis. It possibly requires more investigation to extend the method in a way to consider more general uncertainties via dynamic integral quadratic constraints.

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