SURVEY PAPER

Optimality Conditions in Smooth Nonlinear Programming^{1,2}

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Abstract. This survey is concerned with necessary and sufficient optimality conditions for smooth nonlinear programming problems with inequality and equality constraints. These conditions deal with strict local minimizers of order one and two and with isolated minimizers. In most results, no constraint qualification is required. The optimality conditions are formulated in such a way that the gaps between the necessary and sufficient conditions are small and even vanish completely under mild constraint qualifications.

Key Words. Nonlinear programming, necessary and sufficient optimality conditions, strong and strict local minimizers, isolated minimizers.

1. Introduction

With this paper, we intend to give an up-to-date treatment of the classical problem of characterizing local minimizers of finite optimization problems with equality and inequality constraints. From the wealth of previous contributions on this subject, we will only give the restrictive list (Refs. 1–10). We present first- and second-order, primal and dual, necessary and sufficient optimality conditions without a constraint qualification. In this regard, we have been greatly influenced by the article of Ben-Tal (Ref. 10).

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We have tried to make the relationship between primal and equivalent dual conditions as clear as possible by a consequent use of appropriate theorems of the alternative. Emphasis has been placed on the fact that first-order sufficient conditions are related to strict local minimizers of order one, whereas second-order conditions are related to strict local minimizers of order two. Besides that, we offer a sufficient condition for an isolated minimizer. We emphasize that we have used a classical approach and that the main results are essentially known.

The results of this paper have been derived under weak regularity assumptions, leading to statements with small gaps between necessary and corresponding sufficient optimality conditions. We show that such gaps do not occur in the case where a suitable Mangasarian–Fromovitz constraint qualification holds. We have not dealt with perturbations and parametric aspects. In this context, appropriate regularity assumptions are much stronger, such as the linear independence constraint qualification (LICQ) at the minimizer and strict positivity for all multipliers corresponding to an active inequality. Here, the reader is referred to Refs. 11–16. Investigations into generic properties of these optimization problems can be found in Refs. 17–20.

2. Basic Notations

Consider the finite optimization problem

$$\begin{aligned} \text{(P)} \quad & \min \quad g_0(x), \\ \text{s.t.} \quad & g_j(x) \leq 0, \quad j \in \mathcal{J}, \\ & h_i(x) = 0, \quad i \in \mathcal{I}, \end{aligned}$$

where $\mathcal{I}$ and $\mathcal{J}$ are finite index sets, $0 \notin \mathcal{J}$, $x \in \mathbb{R}^n$. In the following, we suppose that the functions g_0 , g_j , $j \in \mathcal{J}$, and h_i , $i \in \mathcal{I}$, are sufficiently smooth. In short, we suppose that these functions are from $C^1(\mathbb{R}^n, \mathbb{R})$ for first-order statements and from $C^2(\mathbb{R}^n, \mathbb{R})$ for statements concerning second-order conditions. With this agreement in mind, we can omit the smoothness assumptions for the problem functions in all stated results. The cases where $\mathcal{J} = \emptyset$ and/or $\mathcal{I} = \emptyset$ are allowed. The feasible set $\mathcal{M}$ is the set

$$\mathcal{M} = \{x \in \mathbb{R}^n \mid g_j(x) \leq 0, j \in \mathcal{J}; h_i(x) = 0, i \in \mathcal{I}\}.$$

At all times, we will assume that $\mathcal{M} \neq \emptyset$. The point $\bar{x} \in \mathcal{M}$ is a local minimizer for (P) if there is an open neighborhood $\mathcal{U}$ of $\bar{x}$ such that

$$g_0(x) \geq g_0(\bar{x}), \quad \text{for all } x \in \mathcal{M} \cap \mathcal{U}.$$

If this inequality is strictly satisfied for $x \neq \bar{x}$, then $\bar{x}$ is said to be a strict local minimizer. If, with $p \in \mathbb{N}$, there is a constant $\kappa > 0$ such that

$$g_0(x) \geq g_0(\bar{x}) + \kappa \|x - \bar{x}\|^p, \quad \text{for all } x \in \mathcal{M} \cap \mathcal{U}, \quad (1)$$

then $\bar{x}$ is called a strict local minimizer of order p . In the case $p=1$, the point $\bar{x}$ is more commonly called a strong local minimizer. If there is a neighborhood $\mathcal{U}$ such that $\bar{x}$ is the only local minimizer in $\mathcal{U}$, then $\bar{x}$ is called an isolated local minimizer. If (1) holds for p (with some $\kappa > 0$, $\mathcal{U}$) and there is no $\kappa > 0$, $\mathcal{U}$ such that (1) is valid for $p-1$, then $\bar{x}$ is called a strict local minimizer of exact order p .

For $x \in \mathcal{M}$, we define the activity maps

$$\begin{aligned} \mathcal{J}_*(x) &= \{j \in \mathcal{J} \mid g_j(x) = 0\}, \\ \mathcal{J}_*^0(x) &= \mathcal{J}_*(x) \cup \{0\}. \end{aligned}$$

Definition 2.1. Let $x \in \mathcal{M}$. We define the following cones:

$$\begin{aligned} C_x &= \{\tilde{x} \in \mathbb{R}^n \mid \nabla g_j(x) \tilde{x} < 0, j \in \mathcal{J}_*(x); \nabla h_i(x) \tilde{x} = 0, i \in \mathcal{I}\}, \\ C_x^0 &= \{\tilde{x} \in \mathbb{R}^n \mid \nabla g_j(x) \tilde{x} < 0, j \in \mathcal{J}_*^0(x); \nabla h_i(x) \tilde{x} = 0, i \in \mathcal{I}\}, \\ \bar{C}_x &= \{\tilde{x} \in \mathbb{R}^n \mid \nabla g_j(x) \tilde{x} \leq 0, j \in \mathcal{J}_*(x); \nabla h_i(x) \tilde{x} = 0, i \in \mathcal{I}\}, \\ \bar{C}_x^0 &= \{\tilde{x} \in \mathbb{R}^n \mid \nabla g_j(x) \tilde{x} \leq 0, j \in \mathcal{J}_*^0(x); \nabla h_i(x) \tilde{x} = 0, i \in \mathcal{I}\}. \end{aligned}$$

Remark 2.1. We emphasize that in general $\bar{C}_x$ [resp. $\bar{C}_x^0$] is not equal to the closure $\text{cl } C_x$ [resp. $\text{cl } C_x^0$] of C_x [resp. C_x^0]. However, it can be easily seen, e.g., that $\text{cl } C_x \subset \bar{C}_x$ and that $\text{cl } C_x \neq \bar{C}_x \Rightarrow C_x = \emptyset$. Consider for example an optimization problem with the feasible set $\mathcal{M} \subset \mathbb{R}^2$ defined by the inequalities

$$g_1(x) = -x_2 + x_1^2 \leq 0, \quad g_2(x) = x_2 - 2x_1^2 \leq 0.$$

Then for $x=0$, we get

$$C_x = \emptyset \quad \text{and} \quad \bar{C}_x = \{\tilde{x} \mid \tilde{x}_2 = 0\}.$$

It will turn out that the cone C_x is the set of all $\tilde{x}$ such that $x + \epsilon \tilde{x}$ points into the feasible set for $0 < \epsilon$ sufficiently small. Analogously, C_x^0 is the set of all $\tilde{x}$ in C_x such that

$$g_0(x + \epsilon \tilde{x}) < g_0(x), \quad \text{for } 0 < \epsilon \text{ sufficiently small};$$

i.e., $\tilde{x}$ is a feasible direction of descent.

Let $x \in \mathcal{M}$, $\tilde{x} \in \bar{C}_x$ [resp. $\tilde{x} \in \bar{C}_x^0$]. Suppose that $\mathcal{J}_*(x) \neq \emptyset$ [resp. $\mathcal{J}_*^0(x) \neq \emptyset$]. If $\tilde{x}$ is on the relative boundary of $\bar{C}_x$ [resp. of $\bar{C}_x^0$], then

$\nabla g_j(x)\tilde{x}=0$ for some indices $j \in \mathcal{J}_*(x)$ [resp. $j \in \mathcal{J}_*^0(x)$]. This motivates the introduction of the index set $\mathcal{J}_{**}(x, \tilde{x})$ [resp. $\mathcal{J}_{**}^0(x, \tilde{x})$],

$$\mathcal{J}_{**}(x, \tilde{x}) = \{j \in \mathcal{J}_*(x) \mid \nabla g_j(x)\tilde{x} = 0\}$$

$$[\mathcal{J}_{**}^0(x, \tilde{x}) = \{j \in \mathcal{J}_*^0(x) \mid \nabla g_j(x)\tilde{x} = 0\}].$$

The cones in Definition 2.1 could be called first-order cones. For the analysis of second-order effects, we need the following polyhedra in $\mathbb{R}^n$, which could be called second-order polyhedra.

Definition 2.2. Let $x \in \mathcal{M}$. For every $\tilde{x} \in \bar{C}_x$, we define

$$C_{x,\tilde{x}} = \{\tilde{x} \mid \tilde{x}' \nabla^2 g_j(x)\tilde{x} + \nabla g_j(x)\tilde{x} < 0, j \in \mathcal{J}_{**}(x, \tilde{x}),$$

$$\tilde{x}' \nabla^2 h_i(x)\tilde{x} + \nabla h_i(x)\tilde{x} = 0, i \in \mathcal{I}\},$$

$$C_{x,\tilde{x}}^0 = \{\tilde{x} \mid \tilde{x}' \nabla^2 g_j(x)\tilde{x} + \nabla g_j(x)\tilde{x} < 0, j \in \mathcal{J}_{**}^0(x, \tilde{x}),$$

$$\tilde{x}' \nabla^2 h_i(x)\tilde{x} + \nabla h_i(x)\tilde{x} = 0, i \in \mathcal{I}\},$$

$$\bar{C}_{x,\tilde{x}} = \{\tilde{x} \mid \tilde{x}' \nabla^2 g_j(x)\tilde{x} + \nabla g_j(x)\tilde{x} \leq 0, j \in \mathcal{J}_{**}(x, \tilde{x}),$$

$$\tilde{x}' \nabla^2 h_i(x)\tilde{x} + \nabla h_i(x)\tilde{x} = 0, i \in \mathcal{I}\},$$

$$\bar{C}_{x,\tilde{x}}^0 = \{\tilde{x} \mid \tilde{x}' \nabla^2 g_j(x)\tilde{x} + \nabla g_j(x)\tilde{x} \leq 0, j \in \mathcal{J}_{**}^0(x, \tilde{x}),$$

$$\tilde{x}' \nabla^2 h_i(x)\tilde{x} + \nabla h_i(x)\tilde{x} = 0, i \in \mathcal{I}\}.$$

Remark 2.2. It is easily verified that

$$\mathcal{J}_{**}(x, 0) = \mathcal{J}_*(x), \quad \mathcal{J}_{**}^0(x, 0) = \mathcal{J}_*^0(x),$$

and that

$$C_{x,0} = C_x, \quad C_{x,0}^0 = C_x^0,$$

$$\bar{C}_{x,0} = \bar{C}_x, \quad \bar{C}_{x,0}^0 = \bar{C}_x^0.$$

The geometrical meaning of the previous definitions is contained in the following lemma, which is proved in Section 5.

Lemma 2.1. Let $\bar{x} \in \mathcal{M}$. Suppose that the gradients $\nabla h_i(\bar{x})$, $i \in \mathcal{I}$, are linearly independent.

- (i) Let $\tilde{x} \in C_{\bar{x}}$. Then, there exist a positive real number S and a curve $\gamma \in C^1[0, S]$ such that

- (a) $\gamma(s) \in \mathcal{M}$, for $0 \leq s < S$,

- (b) $\gamma(0) = \bar{x}$,
 (c) $\gamma'(0) = \tilde{x}$.
- (ii) Let $\bar{x} \in \bar{C}_{\bar{x}}$, and $\tilde{x} \in C_{\bar{x}, \bar{x}}$. Then, there exist a positive real number S and a curve $\gamma \in C^2[0, S]$ such that
- (a) $\gamma(s) \in \mathcal{M}$, for $0 \leq s < S$,
 (b) $\gamma(0) = \bar{x}$,
 (c) $\gamma'(0) = \tilde{x}$,
 (d) $\gamma''(0) = \tilde{\tilde{x}}$.

Remark 2.3. In general, the requirement that the set $\{\nabla h_i(\bar{x}), i \in \mathcal{J}\}$ is a linearly independent set cannot be circumvented by replacing an equality constraint $h_i(x) = 0$ by two inequality constraints $h_i(x) \leq 0$ and $-h_i(x) \leq 0$. In that case, in part (i) a vector $\tilde{x} \in C_{\bar{x}}$ must satisfy $\pm \nabla h_i(\bar{x})\tilde{x} < 0$, which is impossible. In part (ii), for $\bar{x} \in \bar{C}_{\bar{x}}$, a vector $\tilde{x} \in C_{\bar{x}, \bar{x}}$ has to satisfy at least the two inequalities

$$\nabla h_i(\bar{x})\tilde{x} + \tilde{x}'\nabla^2 h_i(\bar{x})\tilde{x} < 0,$$

$$-\nabla h_i(\bar{x})\tilde{x} - \tilde{x}'\nabla^2 h_i(\bar{x})\tilde{x} < 0,$$

a contradiction. Hence, $C_{\bar{x}, \bar{x}} = \emptyset$. In both cases, the statement of Lemma 2.1 has become void.

We give an example illustrating the lemma and Definitions 2.1 and 2.2.

Example 2.1. In $\mathbb{R}^2$, consider an optimization problem with feasible set $\mathcal{M}$ given by the inequalities (see Fig. 1a)

$$g_1(x) = -x_1 + x_2^2 \leq 0, \quad g_2(x) = -x_2 + x_1^2 \leq 0.$$

Fig. 1. Illustration of Lemma 2.1.

Let $\bar{x}=0$. Then,

$$C_{\bar{x}} = \{\tilde{x} \mid -\tilde{x}_1 < 0, -\tilde{x}_2 < 0\}, \quad (2)$$

$$\bar{C}_{\bar{x}} = \{\tilde{x} \mid -\tilde{x}_1 \leq 0, -\tilde{x}_2 \leq 0\}. \quad (3)$$

Consider a curve

$$\gamma(s) = \bar{x} + \tilde{x}s + (1/2)\tilde{x}s^2 + o(s^2),$$

with $\tilde{x}, \tilde{x} \in \mathbb{R}^2$. If $\tilde{x} \in C_{\bar{x}}$, then, for all $\tilde{x} \in \mathbb{R}^2$, the points $\gamma(s)$ lie in $\mathcal{M}$ for $0 < s$ sufficiently small (see Fig. 1b), since then

$$g_j(\gamma(s)) = \nabla g_j(\bar{x})\tilde{x}s + o(s) < 0, \quad \text{for } j = 1, 2.$$

For these $\tilde{x}$,

$$\mathcal{J}_{**}(\bar{x}, \tilde{x}) = \emptyset \quad \text{and} \quad C_{\bar{x}, \tilde{x}} = \bar{C}_{\bar{x}, \tilde{x}} = \mathbb{R}^2.$$

If $\tilde{x} \in \bar{C}_{\bar{x}} \setminus C_{\bar{x}}$ then $\gamma(s)$ is tangent to $\mathcal{M}$ at 0 (see Fig. 1c). In that case, we have $\nabla g_j(\bar{x})\tilde{x} = 0$ for $j = 1$ or $j = 2$. We consider the case $j = 1$. Then $\tilde{x} = (0, \tilde{x}_2)$ with $\tilde{x}_2 \geq 0$, and if $\tilde{x}_2 > 0$ we have $\mathcal{J}_{**}(\bar{x}, \tilde{x}) = \{1\}$. We need the second-order term in the Taylor series for $g_1(\gamma(s))$ to decide whether $\gamma(s) \in \mathcal{M}$ for s sufficiently small. Then, as one can readily verify,

$$g_1(\gamma(s)) = (1/2)(\tilde{x}'\nabla^2 g_1(\bar{x})\tilde{x} + \nabla g_1(\bar{x})\tilde{x})s^2 + o(s^2).$$

So, $\gamma(s) \in \mathcal{M}$ for s sufficiently small if the factor in front of s^2 is negative, which is the case if $\tilde{x} \in C_{\bar{x}, \tilde{x}}$; see the curve γ_1 in Fig. 1c. If the factor in front of s^2 is positive, then $\gamma(s) \notin \mathcal{M}$; see the curve γ_2 in Fig. 1c. Explicitly, for $\tilde{x} = (0, \tilde{x}_2)$ with $\tilde{x}_2 > 0$,

$$C_{\bar{x}, \tilde{x}} = \{\tilde{x} \mid 2\tilde{x}_2^2 - \tilde{x}_1 < 0\}.$$

Finally, if $\tilde{x} = (0, 0)$, we have

$$\mathcal{J}_{**}(\bar{x}, \tilde{x}) = \mathcal{J}_*(\bar{x}) = \{1, 2\} \quad \text{and} \quad C_{\bar{x}, \tilde{x}} = C_{\bar{x}}.$$

3. Optimality Conditions: Survey of the Results

In this section, we present the results on optimality conditions. The proofs are to be found in Section 5. In Section 3.1, necessary and sufficient optimality conditions are given which do not require a constraint qualification with respect to the inequality constraints. They are stated in both a primal form and an equivalent dual form. In the primal case, the optimality conditions can be formulated in terms of the cones and polyhedra introduced in Definitions 2.1 and 2.2. In order to formulate the dual form of the optimality conditions, we define some cones which can be considered as dual companions to these former cones and polyhedra. Section 3.2 deals with

optimality conditions for the case where, at candidate local minimizers, a so-called Mangasarian–Fromovitz constraint qualification is satisfied. Then, the necessary and sufficient conditions coincide. Finally, Section 3.3 is devoted to some remarks on the case where the (stronger) linear independence constraint qualification holds.

3.1. Optimality Conditions without Constraint Qualifications. We start with the primal necessary conditions; see also Ref. 3, Theorem 3.1, and Ref. 10, Theorem 3.1.

Theorem 3.1. Primal Necessary Optimality Conditions. Let $\bar{x} \in \mathcal{M}$ be a local minimizer for problem (P). Assume that the gradients $\nabla h_i(\bar{x})$, $i \in \mathcal{I}$, are linearly independent. Then, the following results hold:

- (i) First-Order Conditions. $C_{\bar{x}}^0 = \emptyset$, i.e., there is no $\tilde{x} \in \mathbb{R}^n$ solving the system

$$\nabla g_j(\bar{x})\tilde{x} < 0, \quad j \in \mathcal{J}_*^0(\bar{x}),$$

$$\nabla h_i(\bar{x})\tilde{x} = 0, \quad i \in \mathcal{I}.$$

- (ii) Second-Order Conditions. For any $\tilde{x} \in \bar{C}_{\bar{x}}^0$, we have $C_{\tilde{x}, \bar{x}}^0 = \emptyset$; i.e., for any $\tilde{x} \in \bar{C}_{\bar{x}}^0$, there is no $\tilde{\tilde{x}} \in \mathbb{R}^n$ solving the system

$$\tilde{x}' \nabla^2 g_j(\bar{x})\tilde{x} + \nabla g_j(\bar{x})\tilde{\tilde{x}} < 0, \quad j \in \mathcal{J}_{**}^0(\bar{x}, \tilde{x}),$$

$$\tilde{x}' \nabla^2 h_i(\bar{x})\tilde{x} + \nabla h_i(\bar{x})\tilde{\tilde{x}} = 0, \quad i \in \mathcal{I}.$$

Remark 3.1. The requirement in Theorem 3.1 that, at the point $\bar{x}$, the gradients of the equality constraints are linearly independent can be considered as a constraint qualification. However, this is a natural requirement, as is illustrated by the following example:

$$\min \quad g_0(x) = x_1,$$

$$\text{s.t.} \quad h_1(x) = x_2^2 - x_1^3 = 0.$$

The point $\bar{x} = (0, 0)$ is a (global) minimizer. In this case,

$$C_{\bar{x}} = \mathbb{R}^2 \quad \text{and} \quad C_{\bar{x}}^0 = \{\tilde{x} \mid \tilde{x}_1 < 0\} \neq \emptyset.$$

Thus, without the condition that the gradients $\nabla h_i(\bar{x})$ are linearly independent, Theorem 3.1 fails to be true.

In the primal sufficient conditions, only the strict inequalities are replaced by weak ones; see also Ref. 21, Theorem 3.1.4 for (i), even for semi-infinite problems, and Ref. 10, Theorem 4.2 for (ii).

Theorem 3.2. Primal Sufficient Optimality Conditions. Let $\bar{x} \in \mathcal{M}$.

- (i) First-Order Conditions. Suppose that $\bar{C}_{\bar{x}}^0 = \{0\}$, i.e., there is no nonzero solution $\tilde{x} \in \mathbb{R}^n$ to the system

$$\nabla g_j(\bar{x})\tilde{x} \leq 0, \quad j \in \mathcal{J}_*^0(\bar{x}),$$

$$\nabla h_i(\bar{x})\tilde{x} = 0, \quad i \in \mathcal{I}.$$

Then, $\bar{x}$ is a strict local minimizer of order one for problem (P); see also Remark 3.2.

- (ii) Second-Order Conditions. Suppose that $\bar{C}_{\bar{x}}^0 \neq \{0\}$ and for any nonzero $\tilde{x} \in \bar{C}_{\bar{x}}^0$ we have $\bar{C}_{\bar{x}, \tilde{x}}^0 = \emptyset$; i.e., no $\tilde{\tilde{x}} \in \mathbb{R}^n$ solves the system

$$\tilde{x}' \nabla^2 g_j(\bar{x})\tilde{x} + \nabla g_j(\bar{x})\tilde{\tilde{x}} \leq 0, \quad j \in \mathcal{J}_{**}^0(\bar{x}, \tilde{x}),$$

$$\tilde{x}' \nabla^2 h_i(\bar{x})\tilde{x} + \nabla h_i(\bar{x})\tilde{\tilde{x}} = 0, \quad i \in \mathcal{I}.$$

Then, $\bar{x}$ is a strict local minimizer of order two for problem (P); see also Remark 3.2.

Remark 3.2. Later on in the proofs, the following stronger version of the statements of Theorem 3.2 and of Theorem 3.4 below will be shown:

Assume that the first-order [resp. second-order] conditions of Theorem 3.2 or Theorem 3.4 hold. Then, with the constant c_1 [resp. c_2] defined in (21) [resp. in (22)], for any κ less than c_1 [resp. c_2], there is a neighborhood $\mathcal{U}$ of $\bar{x}$ such that, for all $x \in \mathcal{U} \cap \mathcal{M}$,

$$g_0(x) \geq g_0(\bar{x}) + \kappa \|x - \bar{x}\|, \text{ in the first-order case}$$

$$[\text{resp. } g_0(x) \geq g_0(\bar{x}) + \kappa \|x - \bar{x}\|^2, \text{ in the second-order case}].$$

We proceed with the formulation of the equivalent dual conditions. To this aim, it is convenient to introduce dual companions to the cones C_x^0 , $\bar{C}_x^0$ and the polyhedra $C_{x,\bar{x}}^0$, $\bar{C}_{x,\bar{x}}^0$.

Definition 3.1. Let $x \in \mathcal{M}$. Then, the following dual cones can be defined:

$$\mathcal{D}_x^0 = \left\{ d = \sum_{j \in \mathcal{J}_*^0(x)} \mu_j \nabla g_j(x) \right. \\ \left. + \sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(x) \mid \mu_j \geq 0, j \in \mathcal{J}_*^0(x), \text{ not all } 0; \lambda_i \in \mathbb{R}, i \in \mathcal{I} \right\},$$

$$\bar{\mathcal{D}}_x^0 = \left\{ d = \sum_{j \in \mathcal{J}_*^0(x)} \mu_j \nabla g_j(x) + \sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(x) \mid \mu_j \geq 0, j \in \mathcal{J}_*^0(x); \lambda_i \in \mathbb{R}, i \in \mathcal{I} \right\}.$$

Remark 3.3. Note that, by using Lemma 4.1, it follows that

$$\mathcal{D}_x^0 = \{d \in \mathbb{R}^n \mid d'y < 0, \text{ for all } y \in C_x^0\}, \text{ provided } C_x^0 \neq \emptyset,$$

$$\bar{\mathcal{D}}_x^0 = \{d \in \mathbb{R}^n \mid d'y \leq 0, \text{ for all } y \in \bar{C}_x^0\}.$$

This justifies the term dual cone for $\mathcal{D}_x^0$, $\bar{\mathcal{D}}_x^0$, and similarly for the cones $\mathcal{D}_{x,\tilde{x}}^0$, $\bar{\mathcal{D}}_{x,\tilde{x}}^0$ below.

Definition 3.2. Let $x \in \mathcal{M}$. For any $\tilde{x} \in \bar{C}_x^0$ we define the following cones in $\mathbb{R}^{n+1}$:

$$\mathcal{D}_{x,\tilde{x}}^0 = \left\{ \alpha \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sum_{j \in \mathcal{J}_*^0(x)} \mu_j \begin{bmatrix} \nabla g_j(x) \\ \tilde{x}' \nabla^2 g_j(x) \tilde{x} \end{bmatrix} + \sum_{i \in \mathcal{I}} \lambda_i \begin{bmatrix} \nabla h_i(x) \\ \tilde{x}' \nabla^2 h_i(x) \tilde{x} \end{bmatrix} \mid \alpha, \mu_j \geq 0, j \in \mathcal{J}_*^0(x), \text{ not all zero}; \lambda_i \in \mathbb{R}, i \in \mathcal{I} \right\},$$

$$\bar{\mathcal{D}}_{x,\tilde{x}}^0 = \left\{ \alpha \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sum_{j \in \mathcal{J}_*^0(x)} \mu_j \begin{bmatrix} \nabla g_j(x) \\ \tilde{x}' \nabla^2 g_j(x) \tilde{x} \end{bmatrix} + \sum_{i \in \mathcal{I}} \lambda_i \begin{bmatrix} \nabla h_i(x) \\ \tilde{x}' \nabla^2 h_i(x) \tilde{x} \end{bmatrix} \mid \alpha > 0, \mu_j \geq 0, j \in \mathcal{J}_*^0(x); \lambda_i \in \mathbb{R}, i \in \mathcal{I} \right\}.$$

It will be convenient to define the set $\bar{\mathcal{D}}_x^0$ of all multipliers which can be used to represent 0 as an element of $\bar{\mathcal{D}}_x^0$,

$$\bar{\mathcal{D}}_x^0 = \left\{ \left\{ \mu_j \geq 0, j \in \mathcal{J}_*^0(x), \lambda_i \in \mathbb{R}, i \in \mathcal{I} \right\} \mid \sum_{j \in \mathcal{J}_*^0(x)} \mu_j \nabla g_j(x) + \sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(x) = 0 \right\}.$$

For the sake of brevity, the elements of $\bar{\mathcal{D}}_x^0$ will often be denoted by (μ, λ) .

As might be expected, the primal cone $\bar{C}_x^0$ and the cone $\bar{\mathcal{D}}_x^0$ are related, as is pointed out in the following remark.

Remark 3.4. Let $x \in \mathcal{M}$ and $(\mu, \lambda) \in \bar{\mathcal{Q}}_x^0$. Then,

$$0 = \sum_{j \in \mathcal{J}_*^0(x)} \mu_j \nabla g_j(x) + \sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(x). \quad (4)$$

Multiplying (4) by an element $\tilde{x} \in \bar{C}_x^0$, we obtain

$$\mu_j \nabla g_j(x) \tilde{x} = 0, \quad j \in \mathcal{J}_*^0(x). \quad (5)$$

So, if $\nabla g_j(x) \tilde{x} < 0$ for some index j and for some $\tilde{x} \in \bar{C}_x^0$, then the corresponding multiplier vanishes. Relation (5) holds for all $\tilde{x} \in \bar{C}_x^0$ and all $(\mu, \lambda) \in \bar{\mathcal{Q}}_x^0$, and is a sort of complementarity condition. So, roughly speaking, the bigger $\bar{C}_x^0$, the smaller $\bar{\mathcal{Q}}_x^0$ and vice versa.

Now, we formulate the dual optimality conditions; see also, e.g., Ref. 10, Theorem 3.2.

Theorem 3.3. Dual Necessary Optimality Conditions. Let $\bar{x} \in \mathcal{M}$ be a local minimizer for problem (P). Assume that the gradients $\nabla h_i(\bar{x})$, $i \in \mathcal{I}$, are linearly independent. Then:

- (i) First-Order Conditions (Fritz-John Conditions). $0 \in \mathcal{D}_{\bar{x}}^0$; i.e., there exist multipliers $\bar{\mu}_j \geq 0$, $j \in \mathcal{J}_*^0(\bar{x})$, not all zero, $\bar{\lambda}_i \in \mathbb{R}$, $i \in \mathcal{I}$, such that

$$\sum_{j \in \mathcal{J}_*^0(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla h_i(\bar{x}) = 0. \quad (6)$$

- (ii) Second-Order Conditions. For any $\tilde{x} \in \bar{C}_{\bar{x}}^0$, we have $0 \in \mathcal{D}_{\tilde{x}, \bar{x}}^0$; i.e., for any $\tilde{x} \in \bar{C}_{\bar{x}}^0$, there are multipliers $\bar{\mu}_j \geq 0$, $j \in \mathcal{J}_*^0(\bar{x})$, not all zero, $\bar{\lambda}_i \in \mathbb{R}$, $i \in \mathcal{I}$, such that (6) is satisfied, and moreover,

$$\tilde{x}' \left[\sum_{j \in \mathcal{J}_*^0(\bar{x})} \bar{\mu}_j \nabla^2 g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla^2 h_i(\bar{x}) \right] \tilde{x} \geq 0. \quad (7)$$

An example of an optimization problem where, at a minimizer $\bar{x}$, different multipliers are really necessary to get (7) for different $\tilde{x} \in \bar{C}_{\bar{x}}^0$, is to be found in Ref. 10, Example 2.1.

Remark 3.5. In view of Remark 3.4, the optimality conditions in Theorem 3.3 can be slightly strengthened in the sense that the summations over $j \in \mathcal{J}_*^0(\bar{x})$ can be replaced by summations over $j \in \mathcal{J}_{**}^0(\bar{x})$, where

$$\mathcal{J}_{**}^0(\bar{x}) = \bigcap_{\tilde{x} \in \bar{C}_{\bar{x}}^0} \mathcal{J}_{**}^0(\bar{x}, \tilde{x}).$$

As in the primal case, there is only a small gap between the necessary and the following sufficient conditions; see Ref. 21, Theorem 3.1.16 for a result as in Theorem 3.4(i) in semi-infinite optimization, and, e.g., Refs. 5, 10 in connection with Theorem 3.4(ii).

Theorem 3.4. Dual Sufficient Optimality Conditions. Let $\bar{x} \in \mathcal{M}$.

(i) First-Order Conditions. Suppose that

$$0 \in \text{int } \bar{\mathcal{D}}_{\bar{x}}^0.$$

Then, $\bar{x}$ is a strict local minimizer of order one; see Remark 3.2.

(ii) Second-Order Conditions. Suppose that $\bar{C}_{\bar{x}}^0 \neq \{0\}$ and that, for any nonzero $\tilde{x} \in \bar{C}_{\bar{x}}^0$, we have $0 \in \bar{\mathcal{D}}_{\bar{x}, \tilde{x}}^0$; i.e., for any nonzero $\tilde{x} \in \bar{C}_{\bar{x}}^0$, there are multipliers $\bar{\mu}_j \geq 0, j \in \mathcal{J}_*^0(\bar{x}), \bar{\lambda}_i \in \mathbb{R}, i \in \mathcal{I}$, such that (6) is satisfied, and moreover,

$$\tilde{x}' \left[\sum_{j \in \mathcal{J}_*^0(\bar{x})} \bar{\mu}_j \nabla^2 g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla^2 h_i(\bar{x}) \right] \tilde{x} > 0. \quad (8)$$

Then, $\bar{x}$ is a strict local minimizer of order two for (P); see Remark 3.2.

Remark 3.6. In the optimality conditions given in Section 3.1, the objective function g_0 and the constraint functions $g_j, j \in \mathcal{J}$, are treated in the same way. Because of this symmetric treatment of all functions $g_j, j \in \mathcal{J}_*^0(\bar{x})$, we preferred to use as first-order sufficient condition in Theorem 3.4 the assumption $0 \in \text{int } \bar{\mathcal{D}}_{\bar{x}}^0$, instead of the commonly used equivalent condition (see, e.g., Ref. 21)

$$-\nabla g_0(\bar{x}) \in \text{int} \left\{ \sum_{j \in \mathcal{J}_*^0(\bar{x})} \mu_j \nabla g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(\bar{x}) \mid \mu_j \geq 0, j \in \mathcal{J}_*^0(\bar{x}), \lambda_i \in \mathbb{R}, i \in \mathcal{I} \right\}.$$

Remark 3.7. Now, we can conveniently summarize the optimality conditions in Table 1.

Table 1. Summary of optimality conditions.

Conditions	Order	Primal problem	Dual problem
Necessary	1st	$C_{\bar{x}}^0 = \emptyset$	$0 \in \bar{\mathcal{D}}_{\bar{x}}^0$
	2nd	$\forall \tilde{x} \in \bar{C}_{\bar{x}}^0: C_{\bar{x}, \tilde{x}}^0 = \emptyset$	$\forall \tilde{x} \in \bar{C}_{\bar{x}}^0: 0 \in \bar{\mathcal{D}}_{\bar{x}, \tilde{x}}^0$
Sufficient	1st	$\bar{C}_{\bar{x}}^0 = \{0\}$	$0 \in \text{int } \bar{\mathcal{D}}_{\bar{x}}^0$
	2nd	$\forall \tilde{x} \in \bar{C}_{\bar{x}}^0, \tilde{x} \neq 0: \bar{C}_{\bar{x}, \tilde{x}}^0 = \emptyset$	$\forall \tilde{x} \in \bar{C}_{\bar{x}}^0, \tilde{x} \neq 0: 0 \in \bar{\mathcal{D}}_{\bar{x}, \tilde{x}}^0$

3.2. Optimality Conditions Under Mangasarian–Fromovitz Constraint Qualifications.

Definition 3.3. We say that, for $x \in \mathcal{M}$, the Mangasarian–Fromovitz constraint qualification (MFCQ) is fulfilled if:

- (i) $\nabla h_i(x)$, $i \in \mathcal{I}$, are linearly independent;
- (ii) $C_x \neq \emptyset$, i.e., there exists a vector $\xi \in \mathbb{R}^n$ such that $\nabla h_i(x)\xi = 0$, $i \in \mathcal{I}$, and $\nabla g_j(x)\xi < 0$, $j \in \mathcal{J}_*(x)$.

We define a related second-order constraint qualification which is weaker than the MFCQ.

Definition 3.4. Let $x \in \mathcal{M}$ with $\bar{C}_x \neq \{0\}$. We say that for x the second-order Mangasarian–Fromovitz constraint qualification (SOMFCQ) is fulfilled if:

- (i) $\nabla h_i(x)$, $i \in \mathcal{I}$, are linearly independent;
- (ii) for any nonzero $\tilde{x} \in \bar{C}_x$, we have $C_{x,\tilde{x}} \neq \emptyset$; i.e., there exists a vector $\xi \in \mathbb{R}^n$ such that

$$\begin{aligned} \tilde{x}' \nabla^2 g_j(x) \tilde{x} + \nabla g_j(x) \xi &< 0, & j \in \mathcal{J}_{**}(x, \tilde{x}), \\ \tilde{x}' \nabla^2 h_i(x) \tilde{x} + \nabla h_i(x) \xi &= 0, & i \in \mathcal{I}. \end{aligned}$$

Remark 3.8. The fact that MFCQ implies SOMFCQ can be seen as follows. Since $\nabla h_i(x)$, $i \in \mathcal{I}$, are linearly independent, for given $\tilde{x} \in \bar{C}_x$ there exists $d \in \mathbb{R}^n$, such that

$$\tilde{x}' \nabla^2 h_i(x) \tilde{x} + \nabla h_i(x) d = 0, \quad i \in \mathcal{I}.$$

Then, with a vector $\xi \in \mathbb{R}^n$ satisfying condition (ii) of the definition of the MFCQ, putting $\xi(t) = d + t\xi$, we find that

$$\begin{aligned} \tilde{x}' \nabla^2 g_j(x) \tilde{x} + \nabla g_j(x) \xi(t) &< 0, & j \in \mathcal{J}_*(x, \tilde{x}), \\ \tilde{x}' \nabla^2 h_i(x) \tilde{x} + \nabla h_i(x) \xi(t) &= 0, & i \in \mathcal{I}, \end{aligned}$$

for sufficiently large $t > 0$; i.e., $\xi(t)$ is a vector satisfying the relations of condition (ii) of the definition of the SOMFCQ.

Remark 3.9. The validity at a point $x \in \mathcal{M}$ of a constraint qualification can be considered as extra information concerning the structure of the feasible set at x . Note that, by imposing the MFCQ or the SOMFCQ, the

objective function g_0 and the constraint functions g_j , $j \in \mathcal{J}$, are no longer treated in the same way.

Without a constraint qualification, it is possible that the feasible set consists of only a single point. Take, e.g., the feasible set $\mathcal{M} \subset \mathbb{R}^2$ given by

$$x_1^2 + x_2^2 \leq 0.$$

A situation where SOMFCQ is satisfied, but not MFCQ, occurs in the following example.

Example 3.1. Consider the feasible set

$$\mathcal{M} = \{(x_1, x_2) \in \mathbb{R}^2 \mid -x_2 + x_1^2 \leq 0, x_2 - 2x_1^2 \leq 0\};$$

see also Remark 2.1. At $x=0$, we have

$$\bar{C}_x = \{\tilde{x} \in \mathbb{R}^2 \mid \tilde{x}_2 = 0\}, \quad C_x = \emptyset,$$

$$C_{x,\tilde{x}} = \{\tilde{\tilde{x}} \in \mathbb{R}^2 \mid 2\tilde{x}_1^2 < \tilde{\tilde{x}}_2 < 4\tilde{x}_1^2\} \neq \emptyset, \quad \text{for all } 0 \neq \tilde{x} \in \bar{C}_x.$$

Thus, at $x=0$, the SOMFCQ is satisfied, but not the MFCQ.

In the following remark, we show that, by assuming MFCQ or even SOMFCQ in the Fritz-John conditions (6), we always can choose $\bar{\mu}_0 = 1$.

Remark 3.10. Let $\bar{x} \in \mathcal{M}$.

(a) Let the MFCQ hold at $\bar{x}$. If $(\bar{\mu}, \bar{\lambda}) \in \bar{\mathcal{D}}_{\bar{x}}^0$ with $\bar{\mu} \neq 0$, as occurring in the Fritz-John conditions of Theorem 3.3, then we must have $\bar{\mu}_0 > 0$; i.e., we can assume that the so-called Kuhn-Tucker condition holds,

$$\nabla g_0(\bar{x}) + \sum_{j \in \mathcal{J}_*(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla h_i(\bar{x}) = 0. \quad (9)$$

To prove this, let us assume that $\bar{\mu}_0 = 0$. If $\xi \in \mathbb{R}^n$ satisfies the requirements of Definition 3.3, then

$$\begin{aligned} 0 &= \left[\sum_{j \in \mathcal{J}_*(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla h_i(\bar{x}) \right] \xi \\ &= \sum_{j \in \mathcal{J}_*(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) \xi < 0, \end{aligned}$$

a contradiction. The inequality on the right-hand side follows, since the $\bar{\mu}_j$ are nonnegative and not all zero.

(b) Suppose that at $\bar{x}$, with $\bar{C}_{\bar{x}} \neq \{0\}$, the SOMFCQ is satisfied. Then, if the second-order conditions of Theorem 3.3 hold at $\bar{x}$, in (6) and (7) we

can assume that $\bar{\mu}_0 = 1$, i.e., (6) can be replaced by (9). To prove this, assume that $\bar{\mu}_0 = 0$. Then, take any $\bar{x} \in \bar{C}_{\bar{x}}$, $\bar{x} \neq 0$, note that $\bar{C}_{\bar{x}} \neq \{0\}$. In view of Remark 3.4, the index set $\mathcal{J}_{**}^0(\bar{x})$ in (6) and (7) can be replaced by $\mathcal{J}_{**}^0(\bar{x}, \bar{x})$. By multiplying (6) by a vector ξ of Definition 3.4, and adding to (7), we obtain

$$0 \leq \sum_{j \in \mathcal{J}_{**}^0(\bar{x}, \bar{x})} \bar{\mu}_j (\bar{x}' \nabla^2 g_j(\bar{x}) \bar{x} + \nabla g_j(\bar{x}) \xi) \\ + \sum_{i \in \mathcal{I}} \bar{\lambda}_i (\bar{x}' \nabla^2 h_i(\bar{x}) \bar{x} + \nabla h_i(\bar{x}) \xi) < 0,$$

a contradiction.

In view of this remark, the validity of MFCQ motivates the introduction of the following set of Kuhn–Tucker multipliers:

$$\bar{\mathcal{Q}}_x = \{(\mu, \lambda) \in \bar{\mathcal{Q}}_x^0 \mid \mu_0 = 1\} \\ = \left\{ \left\{ \mu_j \geq 0, j \in \mathcal{J}_*(x), \lambda_i \in \mathbb{R}, i \in \mathcal{I} \right\} \right\} \\ \nabla g_0(x) + \sum_{j \in \mathcal{J}_*(x)} \mu_j \nabla g_j(x) + \sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(x) = 0 \Big\}.$$

The boundedness of the set $\bar{\mathcal{Q}}_x$ can be completely characterized by the following result due to Gauvin (Ref. 22). This result is a direct consequence of the theorem of the alternative in Lemma 4.4, due to Mangasarian (Ref. 23).

Theorem 3.5. Let $x \in \mathcal{M}$, and let $\bar{\mathcal{Q}}_x \neq \emptyset$. Then:

$\bar{\mathcal{Q}}_x$ is bounded (more precisely, a compact convex polyhedron)

$\Leftrightarrow$ MFCQ holds at x .

Under the Mangasarian–Fromovitz constraint qualifications, we can completely close the gap between the necessary and sufficient conditions in the theorems of the previous subsection.

Theorem 3.6. Primal and Dual Necessary and Sufficient Conditions with Constraint Qualification. Let $\bar{x} \in \mathcal{M}$. Then, we have:

- (i) **First-Order Condition.** Suppose that the MFCQ holds at $\bar{x}$. Then, $\bar{x}$ is a (isolated, cf. Theorem 3.7) strict local minimizer of order one for problem (P) if and only if the first-order sufficient conditions of Theorem 3.2 or Theorem 3.4 hold.

- (ii) **Second-Order Condition.** Suppose that $\bar{C}_{\bar{x}}^0 \neq \{0\}$ and the SOMFCQ is satisfied at $\bar{x}$. Then, $\bar{x}$ is a strict local minimizer of exact order two for problem (P) if and only if the second-order sufficient conditions of Theorem 3.2 or of Theorem 3.4 hold.

See Ref. 24, Theorem 3.3 for a result as in (ii) under a SMFCQ condition, which is stronger than MFCQ; and see Ref. 25, Remark 3.14 for a corresponding statement for semi-infinite programming. For characterizations of strict local minima of order p in a more general setting of nonsmooth programs, see Refs. 8, 26.

The second-order sufficient optimality conditions of Theorem 3.4 are not strong enough to guarantee that the strict local minimizer $\bar{x}$ is also an isolated local minimizer.

Example 3.2. Consider the following optimization problem in $\mathbb{R}$ (this is a slight modification of a counterexample given in Ref. 13):

$$\begin{aligned} \min \quad & g_0(x) = x^2, \\ \text{s.t.} \quad & g_1(x) = \begin{cases} x^6(1 + \sin(1/x)), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \leq 0. \end{aligned}$$

For this problem, we have

$$\mathcal{M} = \{x \mid x = 2/[(4k-1)\pi], k \in \mathbb{Z}\} \cup \{0\},$$

and for $\bar{x} = 0$,

$$\bar{C}_{\bar{x}}^0 = \mathbb{R}, \quad C_{\bar{x}} = \emptyset, \quad C_{\bar{x}, \bar{x}} = \emptyset, \quad \text{for all } \bar{x} \in \bar{C}_{\bar{x}}^0.$$

Hence, the SOMFCQ is not fulfilled at $\bar{x}$. However, at $\bar{x} = 0$ the second-order optimality condition of Theorem 3.4 is valid with those multipliers $(\bar{\mu}_0, \bar{\mu}_1)$ for which $\bar{\mu}_0 > 0$. Hence, $\bar{x}$ is a strict local minimizer of order two. Since all points in $\mathcal{M} \setminus \{0\}$ are local minimizers, $\bar{x}$ is not an isolated minimizer.

However, if (8) holds for any $\bar{x} \in \bar{C}_{\bar{x}}^0$ and any multiplier vector $(\bar{\mu}, \bar{\lambda}) \in \bar{\mathcal{Q}}_{\bar{x}}^0$ with $\bar{\mu} \neq 0$, and if MFCQ or SOMFCQ is satisfied, then $\bar{x}$ can be shown to be an isolated minimizer. For a result as in the following theorem in a more general context, we refer to Robinson (Ref. 13, Theorem 2.4).

Theorem 3.7. Let $\bar{x} \in \mathcal{M}$. Suppose that either (i) or (ii) holds, where

- (i) $\bar{C}_{\bar{x}}^0 = \{0\}$ and the MFCQ is fulfilled at $\bar{x}$;
 (ii) $\bar{C}_{\bar{x}}^0 \neq \{0\}$, $\bar{\mathcal{Q}}_{\bar{x}}^0 \neq \{0\}$, the SOMFCQ is satisfied at $\bar{x}$ and, for any nonzero $\bar{x} \in \bar{C}_{\bar{x}}^0$, and any multiplier $(\bar{\mu}, \bar{\lambda}) \in \bar{\mathcal{Q}}_{\bar{x}}^0$, $(\bar{\mu}, \bar{\lambda}) \neq 0$, we

have

$$\tilde{x}' \left[\sum_{j \in \mathcal{J}_*^0(\bar{x})} \bar{\mu}_j \nabla^2 g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla^2 h_i(\bar{x}) \right] \tilde{x} > 0.$$

Then, there exists a neighborhood $\mathcal{U}$ of $\bar{x}$ such that

$$0 \notin \mathcal{D}_x^0, \text{ i.e., the Fritz-John conditions are not fulfilled,} \\ \text{for all } x \in \mathcal{U} \cap \mathcal{M} \setminus \{\bar{x}\}. \quad (10)$$

In particular, $\bar{x}$ is an isolated strict local minimizer of order one in case (i) and of order two in case (ii).

3.3. Linear Independence Constraint Qualification: Geometrical Illustration. In this section, we will devote some attention to the case where the linear independence constraint qualification holds at a point $\bar{x} \in \mathcal{M}$. The motivation is that this case can serve as a geometrical interpretation of the optimality conditions of the preceding sections. The approach follows the line set in the books by Jongen, Jonker, and Twilt (Refs. 17, 18).

Definition 3.5. Let $x \in \mathcal{M}$. We say that the linear independence constraint qualification (LICQ) holds at x if the gradients $\{\nabla h_i(x), i \in \mathcal{I}, \nabla g_j(x), j \in \mathcal{J}_*(x)\}$ are linearly independent.

Remark 3.11. If $\bar{x} \in \mathcal{M}$ and LICQ holds at $\bar{x}$, then it is easily seen that also MFCQ is fulfilled. Hence, if x is a local minimizer of problem (P), the multiplier $\bar{\mu}_0$ for the objective function g_0 as occurring in the necessary optimality conditions (Theorem 3.3) can be chosen equal to one; see Remark 3.10.

In this section, we will not restrict ourselves to local minimizers, but instead we will consider all points $\bar{x}$ which fulfill a generalization of the Kuhn-Tucker condition.

Definition 3.6. The point $\bar{x} \in \mathcal{M}$ is called a critical point if LICQ holds at $\bar{x}$, and if there are multipliers $\bar{\mu}_j, j \in \mathcal{J}_*(\bar{x})$, and $\bar{\lambda}_i, i \in \mathcal{I}$, such that

$$\nabla g_0(\bar{x}) + \sum_{j \in \mathcal{J}_*(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla h_i(\bar{x}) = 0. \quad (11)$$

Remark 3.12. Note that, in Definition 3.6, the multipliers are uniquely determined (hence $\bar{\mathcal{D}}_{\bar{x}}$ is a singleton), and that the multipliers for the inequality constraints are not required to be nonnegative.

First, we show that, locally around $\bar{x}$, the feasible set has a nice structure if LICQ holds at $\bar{x}$. Let $\bar{x} \in \mathcal{M}$ (not necessarily a critical point), and suppose that LICQ holds at $\bar{x}$. Without loss of generality, we assume that $\mathcal{J} = \{1, \dots, q\}$ and $\mathcal{J}_*(\bar{x}) = \{q+1, \dots, p\}$, $p \leq n$. Then, there exist vectors $\xi_{p+1}, \dots, \xi_n \in \mathbb{R}^n$ such that⁵

$$\{\nabla h_i(\bar{x}), i=1, \dots, q, \nabla g_j(\bar{x}), j=q+1, \dots, p, \xi'_k, k=p+1, \dots, n\},$$

is a basis of $\mathbb{R}^n$. Consider the mapping $\Phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$, given by

$$\Phi: x \mapsto \begin{bmatrix} h_i(x), & i=1, \dots, q \\ -g_j(x) & j=q+1, \dots, p \\ \xi'_k(x-\bar{x}), & k=p+1, \dots, n \end{bmatrix}. \quad (12)$$

Clearly, $\Phi(\bar{x})=0$ and $\nabla_x \Phi(\bar{x})$ is regular. Hence, there is a neighborhood $\mathcal{U}$ of $\bar{x}$ such that Φ is a local diffeomorphism on $\mathcal{U}$. Further, there is a neighborhood $\mathcal{V}$ of the origin such that

$$\Phi: \mathcal{U} \cap \mathcal{M} \mapsto \mathcal{V} \cap (\{0\}^q \times \mathbb{H}^{n-q} \times \mathbb{R}^{n-p}),$$

where $\mathbb{H} = \{x \in \mathbb{R} \mid x \geq 0\}$.

Now, we will consider the case where $\bar{x}$ is a critical point for problem (P). First, we clarify the role of the Lagrange parameters. We put

$$\hat{x} = \Phi(x) \quad \text{and} \quad \hat{g}_0 = g_0 \circ \Phi^{-1}.$$

In these new coordinates, the point $\Phi(\bar{x})=0$ is a critical point for $\hat{g}_0$, and we have

$$\frac{\partial \hat{g}_0}{\partial \hat{x}_i}(0) = -\bar{\lambda}_i, \quad i=1, \dots, q,$$

$$\frac{\partial \hat{g}_0}{\partial \hat{x}_j}(0) = \bar{\mu}_j, \quad j=q+1, \dots, p,$$

$$\frac{\partial \hat{g}_0}{\partial \hat{x}_k}(0) = 0, \quad k=p+1, \dots, n.$$

Note that

$$x \in \mathcal{M} \cap \mathcal{U} \Rightarrow \hat{x} \in \hat{\mathcal{M}} := \{\hat{x} \mid \hat{x}_i = 0, i=1, \dots, q; \hat{x}_j \geq 0, j=q+1, \dots, p\}.$$

⁵Note that here the subindices define the indexed elements ξ_k in $\mathbb{R}^n$. In the sequel, subindices are used sometimes to denote components of vectors and sometimes for indexing elements in $\mathbb{R}^n$. However, the actual meaning will be clear from the context.

Now, we assume that we have $\bar{\mu}_j \neq 0$, for all $j = q+1, \dots, p$, and that the matrix

$$\frac{\partial^2 \hat{g}_0}{\partial \hat{x}_k \partial \hat{x}_l}(0), \quad k, l = p+1, \dots, n,$$

is regular. Then, a Taylor expansion of $\hat{g}_{0|\hat{\mathcal{H}}}$ at 0, where in any local coordinate $\hat{x}_l$ we consider only the first nonvanishing term, can be written as

$$\hat{g}_0(\hat{x}) = \sum_{j=q+1}^p \bar{\mu}_j \hat{x}_j + (1/2) \sum_{k,l=p+1}^n \hat{x}_k \frac{\partial^2 \hat{g}_0}{\partial \hat{x}_k \partial \hat{x}_l}(0) \hat{x}_l + \rho(\hat{x}), \quad (13a)$$

$$\hat{x}_i = 0, \quad i = 1, \dots, q, \quad (13b)$$

$$\hat{x}_j \geq 0, \quad j = q+1, \dots, p, \quad (13c)$$

where $\rho(\hat{x})$ contains all other terms of the Taylor expansion.

The number of negative [positive] $\bar{\mu}_j$ is called the linear index LI [linear coindex LCI] of $\bar{x}$. The number of negative [positive] eigenvalues of $(\partial^2 \hat{g}_0 / (\partial \hat{x}_k \partial \hat{x}_l))(0)$ is called the quadratic index QI [quadratic coindex QCI]. The numbers LI, LCI, QI, QCI are called the Morse indices of $\bar{x}$.

Now, the generalized Morse lemma (cf. Ref. 17, Theorem 3.2.1) states that, under the conditions mentioned above, there exists a local diffeomorphism $\hat{\Phi}: \hat{x} \mapsto \hat{\hat{x}}$ such that the term $\rho(\hat{x})$ in (13) is transformed to zero. In these local coordinates, we have

$$\begin{aligned} \hat{g}_0(\hat{\hat{x}}) &= \hat{g}_0 \circ \hat{\Phi}^{-1}(\hat{\hat{x}}) \\ &= - \sum_{j=1}^{\text{LI}} \hat{\hat{x}}_{j+q} + \sum_{j=1}^{\text{LCI}} \hat{\hat{x}}_{q+\text{LI}+j} - \sum_{k=1}^{\text{QI}} \hat{\hat{x}}_{k+p}^2 + \sum_{k=1}^{\text{QCI}} \hat{\hat{x}}_{k+p+\text{QI}}^2, \end{aligned}$$

with

$$\hat{\hat{x}}_i = 0, \quad i = 1, \dots, q,$$

$$\hat{\hat{x}}_j \geq 0, \quad j = q+1, \dots, p,$$

and $\hat{\hat{x}}$ in a suitable neighborhood of 0. Hence, in these local coordinates, the optimization problem (P) is locally equivalent with

$$\begin{aligned} (P_{\text{local}}) \quad & \min \hat{g}_0(\hat{\hat{x}}), \\ \text{s.t.} \quad & \hat{\hat{x}}_i = 0, \quad i = 1, \dots, q, \\ & \hat{\hat{x}}_j \geq 0, \quad j = q+1, \dots, p. \end{aligned}$$

Now, $\hat{\hat{x}} = 0$ is a local minimizer [resp. maximizer] for P_{local} , hence $\bar{x}$ is a local minimizer [resp. maximizer] for (P), iff $\text{LI} + \text{QI} = 0$ [resp. $\text{LCI} + \text{QCI} = 0$]. The point $\hat{\hat{x}} = 0$ is a strict local minimizer of order one iff $\text{LI} + \text{QI} + \text{QCI} =$

0, which implies $p=n$, and a strict local minimizer of exact order two iff $LI+QI=0$ and $QCI \neq 0$. If $\bar{x}$ is neither a minimizer nor a maximizer, $\bar{x}$ is called a saddle point. Further, it can be easily seen that $\hat{x}=0$ is an isolated critical point for P_{local} , hence $\bar{x}$ is an isolated critical point for (P).

4. Theorems of the Alternative

In this section, we present a review of the various theorems of the alternative which will be used in the proofs of the optimality conditions to switch between primal optimality conditions and equivalent dual ones. We start with the well-known Farkas' lemma in a homogeneous version and an inhomogeneous version. The homogeneous version is applied in the proof of first-order conditions, and the inhomogeneous one is useful for the second-order conditions.

Lemma 4.1. Let $a_{k_1}, b_{k_2}, c_{k_3} \in \mathbb{R}^n, k_1 \in \mathcal{K}_1, k_2 \in \mathcal{K}_2, k_3 \in \mathcal{K}_3$, with $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ finite index sets. Suppose that $\mathcal{K}_1 \neq \emptyset$. Then, one and only one of the alternatives (i) and (ii) holds:

- (i) There exists a vector $\xi \in \mathbb{R}^n$ satisfying

$$\xi' a_{k_1} < 0, \quad k_1 \in \mathcal{K}_1,$$

$$\xi' b_{k_2} \leq 0, \quad k_2 \in \mathcal{K}_2,$$

$$\xi' c_{k_3} = 0, \quad k_3 \in \mathcal{K}_3.$$

- (ii) There exist numbers $\mu_{k_1} \geq 0, k_1 \in \mathcal{K}_1$, not all zero, $\mu_{k_2} \geq 0, k_2 \in \mathcal{K}_2$, and $\lambda_{k_3} \in \mathbb{R}, k_3 \in \mathcal{K}_3$, such that

$$\sum_{k_1 \in \mathcal{K}_1} \mu_{k_1} a_{k_1} + \sum_{k_2 \in \mathcal{K}_2} \mu_{k_2} b_{k_2} + \sum_{k_3 \in \mathcal{K}_3} \lambda_{k_3} c_{k_3} = 0.$$

Proof. See, e.g., Ref. 27, p. 30. □

The following lemma gives a slight generalization of a nonhomogeneous Farkas' lemma; see, e.g., Ref. 27, p. 32.

Lemma 4.2. Let $a_{k_1}, b_{k_2}, c_{k_3} \in \mathbb{R}^n, k_1 \in \mathcal{K}_1, k_2 \in \mathcal{K}_2, k_3 \in \mathcal{K}_3$, with $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ finite index sets. Let furthermore be given $\alpha_{k_1}, \beta_{k_2}, \gamma_{k_3} \in \mathbb{R}, k_1 \in \mathcal{K}_1, k_2 \in \mathcal{K}_2, k_3 \in \mathcal{K}_3$. Then, one and only one of the alternatives (i) and (ii)

holds:

- (i) There exists a vector $\xi \in \mathbb{R}^n$ satisfying

$$\xi' a_{k_1} < a_{k_1}, \quad k_1 \in \mathcal{K}_1,$$

$$\xi' b_{k_2} \leq \beta_{k_2}, \quad k_2 \in \mathcal{K}_2,$$

$$\xi' c_{k_3} = \gamma_{k_3}, \quad k_3 \in \mathcal{K}_3.$$

- (ii) There exist numbers $\mu_{k_1} \geq 0$, $k_1 \in \mathcal{K}_1$, $\mu_0 \geq 0$, not all zero, $\mu_{k_2} \geq 0$, $k_2 \in \mathcal{K}_2$, and $\lambda_{k_3} \in \mathbb{R}$, $k_3 \in \mathcal{K}_3$, such that

$$\sum_{k_1 \in \mathcal{K}_1} \mu_{k_1} a_{k_1} + \sum_{k_2 \in \mathcal{K}_2} \mu_{k_2} b_{k_2} + \sum_{k_3 \in \mathcal{K}_3} \lambda_{k_3} c_{k_3} = 0,$$

$$\sum_{k_1 \in \mathcal{K}_1} \mu_{k_1} a_{k_1} + \sum_{k_2 \in \mathcal{K}_2} \mu_{k_2} \beta_{k_2} + \sum_{k_3 \in \mathcal{K}_3} \lambda_{k_3} \gamma_{k_3} = -\mu_0 \leq 0.$$

Proof. By introducing a new variable ξ_{n+1} , condition (i) can be written equivalently as:

- (i') There exists a solution (ξ, ξ_{n+1}) of

$$-\xi_{n+1} < 0,$$

$$\xi' a_{k_1} - \xi_{n+1} a_{k_1} < 0, \quad k_1 \in \mathcal{K}_1,$$

$$\xi' b_{k_2} - \xi_{n+1} \beta_{k_2} \leq 0, \quad k_2 \in \mathcal{K}_2,$$

$$\xi' c_{k_3} - \xi_{n+1} \gamma_{k_3} = 0, \quad k_3 \in \mathcal{K}_3.$$

By Lemma 4.1, the corresponding alternative reads (note that now the index set $\mathcal{K}_1$ corresponding to the strict inequalities is nonvoid):

- (ii') There exist numbers $\mu_{k_1} \geq 0$, $k_1 \in \mathcal{K}_1$, $\mu_0 \geq 0$, not all zero, $\mu_{k_2} \geq 0$, $k_2 \in \mathcal{K}_2$, and $\lambda_{k_3} \in \mathbb{R}$, $k_3 \in \mathcal{K}_3$, such that

$$\begin{aligned} \mu_0 \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \sum_{k_1 \in \mathcal{K}_1} \mu_{k_1} \begin{bmatrix} a_{k_1} \\ -a_{k_1} \end{bmatrix} \\ + \sum_{k_2 \in \mathcal{K}_2} \mu_{k_2} \begin{bmatrix} b_{k_2} \\ -\beta_{k_2} \end{bmatrix} + \sum_{k_3 \in \mathcal{K}_3} \lambda_{k_3} \begin{bmatrix} c_{k_3} \\ -\gamma_{k_3} \end{bmatrix} = 0, \end{aligned}$$

which is equivalent with (ii). $\square$

To deal with strong optimality, we need another version of such a theorem of the alternative. This so-called stable theorem of the alternative is a special case of Theorem 1(i) $\Leftrightarrow$ (i') in Ref. 23. For completeness, we will give a short proof based on Lemma 4.2.

Lemma 4.3. Let $b_{k_2}, c_{k_3} \in \mathbb{R}^n$, $k_2 \in \mathcal{K}_2$, $k_3 \in \mathcal{K}_3$, with finite index sets $\mathcal{K}_2$, $\mathcal{K}_3$. Then, the following conditions (i) and (ii) are equivalent:

(i) There exists no vector $\xi \in \mathbb{R}^n$, $\xi \neq 0$, satisfying

$$\begin{aligned}\xi' b_{k_2} &\leq 0, & k_2 \in \mathcal{K}_2, \\ \xi' c_{k_3} &= 0, & k_3 \in \mathcal{K}_3.\end{aligned}$$

(ii) For the cone

$$\mathcal{D} = \left\{ \sum_{k_2 \in \mathcal{K}_2} \mu_{k_2} b_{k_2} + \sum_{k_3 \in \mathcal{K}_3} \lambda_{k_3} c_{k_3} \mid \mu_{k_2} \geq 0, k_2 \in \mathcal{K}_2, \lambda_{k_3} \in \mathbb{R}, k_3 \in \mathcal{K}_3 \right\},$$

we have

$$0 \in \text{int } \mathcal{D}.$$

Proof. Condition (i) is equivalent to: For any fixed $d \in \mathbb{R}^n$, there is no $\xi \in \mathbb{R}^n$, satisfying

$$-\xi' d < 0, \quad \xi' b_{k_2} \leq 0, \quad k_2 \in \mathcal{K}_2, \quad \xi' c_{k_3} = 0, \quad k_3 \in \mathcal{K}_3.$$

By applying Lemma 4.1, this is equivalent to: For any fixed $d \in \mathbb{R}^n$, there exist numbers $\mu_0 > 0$, $\mu_{k_2} \geq 0$, $k_2 \in \mathcal{K}_2$, $\lambda_{k_3} \in \mathbb{R}$, $k_3 \in \mathcal{K}_3$ such that

$$-\mu_0 d + \sum_{k_2 \in \mathcal{K}_2} \mu_{k_2} b_{k_2} + \sum_{k_3 \in \mathcal{K}_3} \lambda_{k_3} c_{k_3} = 0.$$

By dividing this relation by μ_0 , we get a representation $d \in \mathcal{D}$. Since d is arbitrary, this is equivalent to $\mathcal{D} = \mathbb{R}^n$, or since $\mathcal{D}$ is a cone, with the condition $0 \in \text{int } \mathcal{D}$. $\square$

The following version of a theorem of the alternative is appropriate to characterize the boundedness of the set of Kuhn–Tucker multipliers; see Theorem 3.5. The statement is a special case of Theorem 3 in Ref. 23.

Lemma 4.4. Let $a_0, a_{k_1}, c_{k_3} \in \mathbb{R}^n$, $k_1 \in \mathcal{K}_1$, $k_3 \in \mathcal{K}_3$, with finite index sets $\mathcal{K}_1$, $\mathcal{K}_3$ satisfying $|\mathcal{K}_1| + |\mathcal{K}_3| \geq 1$. Suppose that $\mathcal{Q} \neq \emptyset$, where $\mathcal{Q}$ is defined by

$$\mathcal{Q} = \left\{ (\mu, \lambda), \mu \in \mathbb{R}^{|\mathcal{K}_1|}, \mu \geq 0, \lambda \in \mathbb{R}^{|\mathcal{K}_3|} \mid a_0 + \sum_{k_1 \in \mathcal{K}_1} \mu_{k_1} a_{k_1} + \sum_{k_3 \in \mathcal{K}_3} \lambda_{k_3} c_{k_3} = 0 \right\}.$$

Then, the following conditions (i) and (ii) are equivalent:

- (i) The vectors $c_{k_3}, k_3 \in \mathcal{K}_3$ are linearly independent and there exists a vector $\xi \in \mathbb{R}^n$ satisfying

$$\xi' a_{k_1} < 0, \quad k_1 \in \mathcal{K}_1,$$

$$\xi' c_{k_3} = 0, \quad k_3 \in \mathcal{K}_3.$$

- (ii) The set $\mathcal{Q}$ is bounded.

5. Proofs

In this section, we present the proofs of the results stated in the Sections 2 and 3. The most convenient way to do this is by using the following strategy. First, we prove Lemma 2.1. Then, the primal necessary optimality conditions of Theorem 3.1 directly follow. By means of Lemma 4.2, the inhomogeneous theorem of the alternative, the dual necessary conditions are easily obtained. Then, the dual sufficient conditions are established by contraposition. Next, again using Lemmas 4.1 and 4.2, the primal sufficient conditions are easily proved.

The rest of the section is devoted to the proofs of Theorems 3.6 and 3.7, concerning situations where MFCQ or SOMFCQ holds at a (candidate) local minimizer $\bar{x}$.

Proof of Lemma 2.1(i). We put $\mathcal{I} = \{1, \dots, m\}$, and we can choose $\xi_{m+1}, \dots, \xi_n \in \mathbb{R}^n$ such that $\{\nabla h_i(\bar{x}), (i \in \mathcal{I}), \xi_{m+1}, \dots, \xi_n\}$ is a basis for $\mathbb{R}^n$. Let $\Phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined as

$$\Phi: x \mapsto \begin{bmatrix} h_1(x) \\ \vdots \\ h_m(x) \\ \xi'_{m+1}(x - \bar{x}) \\ \vdots \\ \xi'_n(x - \bar{x}) \end{bmatrix}.$$

It is clear that $\nabla\Phi(\bar{x})$ has full rank. So, by the inverse function theorem, Φ is locally a diffeomorphism around $\bar{x}$. Now choose, with the given $\tilde{x} \in C_{\bar{x}}$,

$$\hat{\gamma}(s) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \xi'_{m+1}\tilde{x} \\ \vdots \\ \xi'_n\tilde{x} \end{bmatrix} s, \quad s \in \mathbb{R},$$

and let

$$\gamma(s) = \Phi^{-1}(\hat{\gamma}(s)). \quad (14)$$

By definition,

$$\gamma(0) = \bar{x},$$

and locally around $s=0$ the curve γ is continuously differentiable. Note that the k th component of $\hat{\gamma}(s)$ can be written as

$$\hat{\gamma}_k(s) = \nabla\Phi_k(\bar{x})\tilde{x}s. \quad (15)$$

Because the first m components of $\hat{\gamma}$ vanish for all s ,

$$h_i(\Phi^{-1}(\hat{\gamma}(s))) = 0, \text{ for } i = 1, \dots, m, \text{ and for all } s, 0 \leq s < S,$$

with S sufficiently small. By the chain rule, we obtain

$$\hat{\gamma}'(s) = \nabla\Phi(\gamma(s))\gamma'(s),$$

hence

$$\nabla\Phi(\bar{x})\tilde{x} = \nabla\Phi(\bar{x})\gamma'(0),$$

and because $\nabla\Phi(\bar{x})$ is regular, it follows that

$$\gamma'(0) = \tilde{x}, \text{ i.e., } \gamma(s) = \bar{x} + s\tilde{x} + o(s).$$

Then, by a Taylor expansion, using $\tilde{x} \in C_{\bar{x}}$, we find that

$$g_j(\gamma(s)) = g_j(\bar{x}) + \nabla g_j(\bar{x})\tilde{x}s + o(s) \leq 0,$$

for $j \in \mathcal{J}$; hence, $\gamma(s) \in \mathcal{M}$, for $0 \leq s$ sufficiently small. $\square$

Proof of Lemma 2.1(ii). With the vectors $\xi_{m+1}, \dots, \xi_n$ and the function Φ in the proof of part (i), we define, with the given $\bar{x} \in \bar{C}_{\bar{x}}$ and $\tilde{x} \in C_{\bar{x}, \tilde{x}}$,

$$\hat{\gamma}(s) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \xi'_{m+1} \tilde{x} \\ \vdots \\ \xi'_n \tilde{x} \end{bmatrix} s + (1/2) \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \xi'_{m+1} \tilde{x} \\ \vdots \\ \xi'_n \tilde{x} \end{bmatrix} s^2, \quad s \in \mathbb{R},$$

and again

$$\gamma(s) = \Phi^{-1}(\hat{\gamma}(s)).$$

As before,

$$\gamma(0) = \bar{x},$$

and locally around $s=0$ the curve γ is twice continuously differentiable, where now

$$\hat{\gamma}_k(s) = \nabla \Phi_k(\bar{x}) \tilde{x} s + (1/2) (\tilde{x}' \nabla^2 \Phi_k(\bar{x}) \tilde{x} + \nabla \Phi_k(\bar{x}) \tilde{x}) s^2. \quad (16)$$

As in the proof of part (i), the first m components of $\hat{\gamma}$ vanish for all s . Hence,

$$h_i(\Phi^{-1}(\hat{\gamma}(s))) = 0, \text{ for } i = 1, \dots, m, \text{ and for all } s, 0 \leq s < S,$$

with S sufficiently small. Moreover, as before we find that

$$\gamma'(0) = \tilde{x}.$$

Further, twice differentiating the k th component $\hat{\gamma}_k$ of $\hat{\gamma}$ as given by (14) yields

$$\hat{\gamma}_k''(s) = (\gamma'(s))' \nabla(\nabla' \Phi_k(\gamma(s))) \gamma'(s) + \nabla \Phi_k(\gamma(s)) \gamma''(s), \quad (17)$$

and then

$$\hat{\gamma}_k''(0) = \tilde{x}' \nabla^2 \Phi_k(\bar{x}) \tilde{x} + \nabla \Phi_k(\bar{x}) \gamma''(0). \quad (18)$$

On the other hand, by (16) we get

$$\hat{\gamma}_k''(0) = \tilde{x}' \nabla^2 \Phi_k(\bar{x}) \tilde{x} + \nabla \Phi_k(\bar{x}) \tilde{x}, \quad k = 1, \dots, n,$$

hence

$$\gamma''(0) = \tilde{x}.$$

Finally, by a Taylor expansion, we find that

$$\begin{aligned} g_j(\gamma(s)) &= g_j(\bar{x}) + \nabla g_j(\bar{x})\tilde{x}s, \\ &+ (1/2)(\tilde{x}'\nabla^2 g_j(\bar{x})\tilde{x} + \nabla g_j(\bar{x})\tilde{x})s^2 + o(s^2) \leq 0, \end{aligned}$$

for $j \in \mathcal{J}$; hence, $\gamma(s) \in \mathcal{M}$, for $0 \leq s$ sufficiently small. $\square$

Proof of Theorem 3.1. To prove part (i), let us assume that there is a vector $\tilde{x} \in C_{\bar{x}}^0$. Then $\tilde{x} \in C_{\bar{x}}$, and by Lemma 2.1(i) there exists a curve $\gamma(s)$ with the properties (a) to (c) of that lemma. Since along γ we have

$$g_0(\gamma(s)) = g_0(\bar{x}) + \nabla g_0(\bar{x})\tilde{x}s + o(s) < g_0(\bar{x}),$$

for $s > 0$ small enough, $\bar{x}$ cannot be a local minimizer.

We prove part (ii). Assume that there is a vector $\tilde{x} \in \bar{C}_{\bar{x}}^0$ and a vector $\tilde{\tilde{x}} \in C_{\bar{x}, \tilde{x}}^0$. Then, $\tilde{x} \in \bar{C}_{\bar{x}}$ and $\tilde{\tilde{x}} \in C_{\bar{x}, \tilde{x}}$, and by Lemma 2.1(ii) there exists a curve $\gamma(s)$ with the properties (a) to (d) of that lemma. But along this curve, $g_0(\gamma(s))$ decreases,

$$\begin{aligned} g_0(\gamma(s)) &= g_0(\bar{x}) + \nabla g_0(\bar{x})\tilde{x}s + (1/2)[\tilde{x}'\nabla^2 g_0(\bar{x})\tilde{x} + \nabla g_0(\bar{x})\tilde{\tilde{x}}]s^2 + o(s^2) \\ &< g_0(\bar{x}), \end{aligned}$$

for $s > 0$ small enough, contradicting the assumption that $\bar{x}$ is a local minimizer. $\square$

Proof of Theorem 3.3. Part (i) follows by Theorem 3.1(i), since by Lemma 4.1 the conditions of Theorem 3.1(i) and Theorem 3.3(i) are equivalent.

Now, we prove part (ii). For a local minimizer $\bar{x}$, Theorem 3.1(ii) states that, for any $\tilde{x} \in \bar{C}_{\bar{x}}^0$, there is no solution $\tilde{\tilde{x}}$ of the system

$$\begin{aligned} \tilde{x}'\nabla^2 g_j(\bar{x})\tilde{x} + \nabla g_j(\bar{x})\tilde{\tilde{x}} &< 0, \quad j \in \mathcal{J}_{**}^0(\bar{x}, \tilde{x}), \\ \tilde{x}'\nabla^2 h_i(\bar{x})\tilde{x} + \nabla h_i(\bar{x})\tilde{\tilde{x}} &= 0, \quad i \in \mathcal{I}. \end{aligned}$$

Application of the inhomogeneous Farkas lemma yields, for any $\tilde{x} \in \bar{C}_{\bar{x}}^0$, the existence of multipliers $\bar{\alpha}, \bar{\mu}_j \geq 0, j \in \mathcal{J}_{**}^0(\bar{x}, \tilde{x})$, not all zero, and $\bar{\lambda}_i \in \mathbb{R}, i \in \mathcal{I}$, such that

$$\sum_{j \in \mathcal{J}_{**}^0(\bar{x}, \tilde{x})} \bar{\mu}_j \nabla g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla h_i(\bar{x}) = 0, \quad (19)$$

$$\tilde{x}' \left[\sum_{j \in \mathcal{J}_{**}^0(\bar{x}, \tilde{x})} \bar{\mu}_j \nabla^2 g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla^2 h_i(\bar{x}) \right] \tilde{x} = \bar{\alpha} \geq 0. \quad (20)$$

If in (20) $\bar{\alpha} = 0$, then $\bar{\mu}_j, j \in \mathcal{J}_{**}^0(\bar{x}, \tilde{x})$, are not all zero. But if $\bar{\alpha} > 0$, then obviously not all $\bar{\mu}_j, j \in \mathcal{J}_{**}^0(\bar{x}, \tilde{x}), \bar{\lambda}_i, i \in \mathcal{I}$, can be zero. By the linear

independence of the gradients $\nabla h_i(\bar{x})$, $i \in \mathcal{I}$, the relation (19) implies $\bar{\mu} \neq 0$ in this case too. Since $\mathcal{J}_{**}^0(\bar{x}, \bar{x}) \subset \mathcal{J}_*^0(\bar{x})$, the result follows. Note that the multipliers in (19), (20) need not be unique. $\square$

Proof of Theorem 3.2(i). We prove the stronger statement of Remark 3.2. Define

$$c_1 := \min_{\bar{x} \in \bar{C}_{\bar{x}}, \|\bar{x}\|=1} \nabla g_0(\bar{x})\bar{x}. \quad (21)$$

By assumption, we have $c_1 > 0$. Now, suppose that the statement is not valid. Then, there exists a constant $\kappa < c_1$ and a sequence of points $x_k \in \mathcal{M}$ with

$$g_0(x_k) \leq g_0(\bar{x}) + \kappa \|x_k - \bar{x}\|, \quad x_k \neq \bar{x}, \quad \lim_{k \rightarrow \infty} x_k = \bar{x}.$$

We can assume

$$x_k = \bar{x} + t_k \tilde{x}_k, \quad \text{with } t_k > 0, \|\tilde{x}_k\| = 1.$$

Then, by passing to a subsequence, we infer that

$$\lim_{k \rightarrow \infty} \tilde{x}_k = \tilde{x}, \quad \lim_{k \rightarrow \infty} t_k = 0,$$

with

$$g_0(x_k) - g_0(\bar{x}) = t_k \nabla g_0(\bar{x})\tilde{x}_k + o(t_k) \leq \kappa \|x_k - \bar{x}\| = \kappa t_k.$$

Hence,

$$t_k \nabla g_0(\bar{x})\tilde{x}_k \leq \kappa t_k + o(t_k),$$

and dividing by t_k and letting $k \rightarrow \infty$, we obtain

$$\nabla g_0(\bar{x})\tilde{x} \leq \kappa.$$

Similarly, we get

$$\begin{aligned} \nabla g_j(\bar{x})\tilde{x} &\leq 0, & j \in \mathcal{J}_*(\bar{x}), \\ \nabla h_i(\bar{x})\tilde{x} &= 0, & i \in \mathcal{I}, \end{aligned}$$

i.e.,

$$\tilde{x} \in \bar{C}_{\bar{x}}, \quad \|\tilde{x}\| = 1.$$

This gives a contradiction to the definition of c_1 . $\square$

Proof of Theorem 3.4. Part (i) follows by Theorem 3.2(i), because by Lemma 4.3 the primal condition $\bar{C}_{\bar{x}}^0 = \{0\}$ and the dual condition $0 \in \text{int } \bar{\mathcal{D}}_{\bar{x}}^0$ of Theorem 3.4 are equivalent.

Now, we turn to part (ii). We prove the stronger statement of Remark 3.2 and put

$$c_2 := \min_{\tilde{x} \in \bar{\mathcal{C}}_{\tilde{x}}^0, \|\tilde{x}\| = 1} \max_{(\mu, \lambda) \in \bar{\mathcal{D}}_{\tilde{x}}^0, \|(\mu, \lambda)\| = 1} (1/2) \tilde{x}' \nabla_x^2 L_{\tilde{x}}(\tilde{x}, \mu, \lambda) \tilde{x}, \quad (22)$$

where we have introduced the so-called (local) Lagrange function at $\tilde{x}$,

$$L_{\tilde{x}}(x, \mu, \lambda) = \sum_{j \in \mathcal{J}_*^0(\tilde{x})} \mu_j g_j(x) + \sum_{i \in \mathcal{I}} \lambda_i h_i(x).$$

The expression (22) is well defined, and by assumption $c_2 > 0$. Now, choose $c < c_2$ arbitrarily, and suppose that the statement of Remark 3.2 is not valid. Then, there is a sequence of points $x_k \in \mathcal{M}$ such that $\lim_{k \rightarrow \infty} x_k = \bar{x}$ and

$$g_0(x_k) - g_0(\bar{x}) \leq c \|x_k - \bar{x}\|^2. \quad (23)$$

We can write

$$x_k = \bar{x} + t_k \tilde{x}_k, \quad \text{with } \|\tilde{x}_k\| = 1.$$

Then,

$$\lim_{k \rightarrow \infty} t_k = 0,$$

and by choosing a convergent subsequence, we can assume that

$$\lim_{k \rightarrow \infty} \tilde{x}_k = \tilde{x}, \quad \|\tilde{x}\| = 1.$$

We will show that $\tilde{x} \in \bar{\mathcal{C}}_{\tilde{x}}^0$. To do so, we make use of the following Taylor expansions (recall that x_k is feasible):

$$\begin{aligned} ct_k^2 &\geq g_0(x_k) - g_0(\bar{x}) \\ &= t_k \nabla g_0(\bar{x}) \tilde{x}_k + (1/2) t_k^2 \tilde{x}_k' \nabla^2 g_0(\bar{x}) \tilde{x}_k + o(t_k^2), \end{aligned} \quad (24)$$

$$\begin{aligned} 0 &\geq g_j(x_k) - g_j(\bar{x}) \\ &= t_k \nabla g_j(\bar{x}) \tilde{x}_k + (1/2) t_k^2 \tilde{x}_k' \nabla^2 g_j(\bar{x}) \tilde{x}_k + o(t_k^2), \end{aligned} \quad (25)$$

for all $j \in \mathcal{J}_*(\bar{x})$, and

$$\begin{aligned} 0 &= h_i(x_k) - h_i(\bar{x}) \\ &= t_k \nabla h_i(\bar{x}) \tilde{x}_k + (1/2) t_k^2 \tilde{x}_k' \nabla^2 h_i(\bar{x}) \tilde{x}_k + o(t_k^2), \end{aligned} \quad (26)$$

for all $i \in \mathcal{I}$. Dividing these relations by t_k and taking the limit for $k \rightarrow \infty$ yields $\tilde{x} \in \bar{\mathcal{C}}_{\tilde{x}}^0$. We now choose a multiplier $(\bar{\mu}, \bar{\lambda}) \in \bar{\mathcal{D}}_{\tilde{x}}^0$ such that it solves

$$\max_{(\mu, \lambda) \in \bar{\mathcal{D}}_{\tilde{x}}^0, \|(\mu, \lambda)\| = 1} (1/2) \tilde{x}' \nabla_x^2 L_{\tilde{x}}(\tilde{x}, \mu, \lambda) \tilde{x},$$

and put

$$c(\tilde{x}) = (1/2)\tilde{x}'\nabla_{\tilde{x}}^2 L_{\tilde{x}}(\tilde{x}, \bar{\mu}, \bar{\lambda})\tilde{x}.$$

By definition, $c(\tilde{x}) \geq c_2$. By multiplying the relations (24), (25), (26) with the corresponding components of $(\bar{\mu}, \bar{\lambda})$ and adding, we find that, by using $\bar{\mu}_0 \leq \|(\bar{\mu}, \bar{\lambda})\| = 1$,

$$\begin{aligned} L_{\tilde{x}}(x_k, \bar{\mu}, \bar{\lambda}) - L_{\tilde{x}}(\tilde{x}, \bar{\mu}, \bar{\lambda}) &= t_k \nabla_{\tilde{x}} L_{\tilde{x}}(\tilde{x}, \bar{\mu}, \bar{\lambda}) \tilde{x}_k \\ &\quad + (1/2)t_k^2 \tilde{x}_k' \nabla_{\tilde{x}}^2 L_{\tilde{x}}(\tilde{x}, \bar{\mu}, \bar{\lambda}) \tilde{x}_k + o(t_k^2) \\ &\leq \mu_0 c t_k^2 \leq c t_k^2. \end{aligned}$$

Using

$$\nabla_{\tilde{x}} L_{\tilde{x}}(\tilde{x}, \bar{\mu}, \bar{\lambda}) = 0,$$

division by t_k^2 yields, for $k \rightarrow \infty$,

$$c(\tilde{x}) = (1/2)\tilde{x}'\nabla_{\tilde{x}}^2 L_{\tilde{x}}(\tilde{x}, \bar{\mu}, \bar{\lambda})\tilde{x} \leq c < c_2,$$

in contradiction to $c(\tilde{x}) \geq c_2$. □

Proof of Theorem 3.2(ii). By Lemma 4.2, the condition of Theorem 3.2(ii) implies: For any nonzero $\tilde{x} \in \bar{C}_{\tilde{x}}^0$, there exist multipliers $\bar{\mu}_j \geq 0$, $j \in \mathcal{J}_*^0(\tilde{x})$, $\lambda_i \in \mathbb{R}$, $i \in \mathcal{I}$, such that (6) holds and (8) holds, even with $\mathcal{J}_*^0(\tilde{x})$ replaced by the possibly smaller set $\mathcal{J}_{**}^0(\tilde{x}, \tilde{x})$. So, the result follows by Theorem 3.4(ii). □

Proof of Theorem 3.6. First, we prove part (i). We only have to prove the direction $\Rightarrow$, because the other direction follows from Theorem 3.2 or 3.4. Let us assume that there is a vector $\tilde{x} \in \bar{C}_{\tilde{x}}^0$, $\tilde{x} \neq 0$. By MFCQ, there exists a $\xi \in \mathbb{R}^n$ such that

$$\nabla g_j(\tilde{x})\xi \leq c < 0, \quad j \in \mathcal{J}_*(\tilde{x}),$$

$$\nabla h_i(\tilde{x})\xi = 0, \quad i \in \mathcal{I},$$

Hence, for any fixed $\epsilon > 0$, we have

$$\nabla g_j(\tilde{x})(\tilde{x} + \epsilon\xi) \leq \epsilon c, \quad j \in \mathcal{J}_*(\tilde{x}), \tag{27a}$$

$$\nabla h_i(\tilde{x})(\tilde{x} + \epsilon\xi) = 0, \quad i \in \mathcal{I}, \tag{27b}$$

i.e.,

$$(\tilde{x} + \epsilon\xi) \in C_{\tilde{x}}.$$

Here, we can assume that

$$\|\tilde{x} + \epsilon \xi\| \geq 1.$$

By applying Lemma 2.1(c), we deduce that there exists a curve γ_ϵ and a constant $S_\epsilon > 0$ such that

$$\begin{aligned}\gamma_\epsilon(s) &\in \mathcal{M}, & \text{for } 0 \leq s < S_\epsilon, \\ \gamma_\epsilon(0) &= \tilde{x}, & \gamma'_\epsilon(0) = (\tilde{x} + \epsilon \xi).\end{aligned}$$

Moreover, using $\gamma_\epsilon(s) = \tilde{x} + s(\tilde{x} + \epsilon \xi) + o(s)$ and $\nabla g_0(\tilde{x})\tilde{x} \leq 0$, we find that

$$\begin{aligned}g_0(\gamma_\epsilon(s)) - g_0(\tilde{x}) &= s \nabla g_0(\tilde{x})(\tilde{x} + \epsilon \xi) + o(s) \\ &\leq s \epsilon |\nabla g_0(\tilde{x})\xi| + o(s) \\ &\leq \epsilon |\nabla g_0(\tilde{x})\xi| \|s(\tilde{x} + \epsilon \xi)\| + o(s) \\ &= \epsilon |\nabla g_0(\tilde{x})\xi| \|\gamma_\epsilon(s) - \tilde{x}\| + o(s).\end{aligned}$$

This is true for any $\epsilon > 0$. Consequently, for $\epsilon \rightarrow 0$, we have a contradiction to the Definition (1) of a strict local minimizer $\tilde{x}$ of order $p = 1$.

(ii) We prove the direction. By assumption, there exist $\kappa > 0$ and a neighborhood $\mathcal{U}$ of $\tilde{x}$ such that, for $\hat{g}(x) := g_0(x) - \kappa \|x - \tilde{x}\|^2$, we have

$$\hat{g}(x) - \hat{g}(\tilde{x}) = g_0(x) - g_0(\tilde{x}) - \kappa \|x - \tilde{x}\|^2 \geq 0, \quad \text{for all } x \in \mathcal{M} \cap \mathcal{U}.$$

Hence, $\tilde{x}$ is a local minimizer of problem $(\hat{P})$, to minimize $\hat{g}$ on the feasible set of (P). Note that, since $\nabla \hat{g}(\tilde{x}) = \nabla g_0(\tilde{x})$, the first-order conditions for $(\hat{P})$ and (P) coincide. By Theorem 3.3(ii), for any $\tilde{x} \in \bar{C}_{\tilde{x}}^0$, (6) is satisfied with some multipliers $\bar{\mu}_j \geq 0$, $j \in \mathcal{J}_*(\tilde{x})$, $\bar{\lambda}_i \in \mathbb{R}$, $i \in \mathcal{I}$, and

$$\begin{aligned}\tilde{x}' \left[\bar{\mu}_0 \nabla^2 \hat{g}(\tilde{x}) + \sum_{j \in \mathcal{J}_*(\tilde{x})} \bar{\mu}_j \nabla^2 g_j(\tilde{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla^2 h_i(\tilde{x}) \right] \tilde{x} \\ = \tilde{x}' \left[\sum_{j \in \mathcal{J}_*(\tilde{x})} \bar{\mu}_j \nabla^2 g_j(\tilde{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla^2 h_i(\tilde{x}) \right] \tilde{x} - 2\bar{\mu}_0 \kappa \tilde{x}' \tilde{x} \geq 0,\end{aligned} \quad (28)$$

where we have used

$$\nabla^2 \hat{g}(\tilde{x}) = \nabla^2 g_0(\tilde{x}) - 2\kappa I.$$

Since SOMFCQ is satisfied at $\tilde{x}$, by Remark 3.10 we can assume $\bar{\mu}_0 = 1$. Then, (28) gives the second-order condition (8).

In the other direction, in view of Theorem 3.2, it only remains to show that, under the assumptions of the theorem, $\tilde{x}$ cannot be a local minimizer of order one. To do so, take an element $\tilde{x} \in \bar{C}_{\tilde{x}}^0$, with $\tilde{x} \neq 0$, and a vector

$\xi \in C_{\bar{x}, \bar{x}}$ (such a ξ exists by SOMFCQ). Then, by Lemma 2.1(ii), there is a curve

$$\gamma(s) = \bar{x} + s\tilde{x} + (1/2)s^2\xi + o(s^2)$$

and $S_0 > 0$, such that $\gamma(s) \in \mathcal{M}$ for $0 \leq s < S_0$. Using $\nabla g_0(\bar{x})\tilde{x} \leq 0$, as above, we obtain

$$\begin{aligned} g_0(\gamma(s)) - g_0(\bar{x}) &\leq (1/2)s^2(\tilde{x}'\nabla^2 g_0(\bar{x})\tilde{x} + \nabla g_0(\bar{x})\xi) + o(s^2) \\ &\leq \mathcal{O}(\|\gamma(s) - \bar{x}\|^2); \end{aligned}$$

i.e., $\bar{x}$ cannot be a local minimizer of order one. $\square$

Proof of Theorem 3.7. Suppose that the statement (10) does not hold. Then, there exists a sequence $x_k \in \mathcal{M}$, $k \in \mathbb{N}$, such that $\lim_{k \rightarrow \infty} x_k = \bar{x}$, and corresponding multiplier vectors $(\mu^k, \lambda^k) \in \bar{\mathcal{Q}}_{x_k}^0$, $(\mu^k, \lambda^k) \neq 0$, such that

$$\nabla_x L_k := \mu_0^k \nabla g_0(x_k) + \sum_{j \in \mathcal{J}_*(x_k)} \mu_j^k \nabla g_j(x_k) + \sum_{i \in \mathcal{I}} \lambda_i^k \nabla h_i(x_k) = 0. \quad (29)$$

We can write

$$x_k = \bar{x} + t_k \tilde{x}_k, \quad \text{with } \|\tilde{x}_k\| = 1 \text{ and } t_k > 0.$$

Then,

$$\lim_{k \rightarrow \infty} t_k = 0,$$

and by choosing a convergent subsequence, we can assume that

$$\lim_{k \rightarrow \infty} \tilde{x}_k = \tilde{x}, \quad \|\tilde{x}\| = 1.$$

By continuity, for the activity maps we get

$$\mathcal{J}_*(x_k) \subset \mathcal{J}_*(\bar{x}), \quad \text{for } k \text{ sufficiently large.} \quad (30)$$

Thus, for k sufficiently large, (29) can be replaced by

$$\begin{aligned} \nabla_x L_k &= \mu_0^k \nabla g_0(x_k) + \sum_{j \in \mathcal{J}_*(\bar{x})} \mu_j^k \nabla g_j(x_k) + \sum_{i \in \mathcal{I}} \lambda_i^k \nabla h_i(x_k) = 0, \\ &\text{with } \mu_j^k = 0 \text{ for } k \in \mathcal{J}_*(\bar{x}) \setminus \mathcal{J}_*(x_k). \end{aligned} \quad (31)$$

Thus, the vectors (μ^k, λ^k) can be seen as possible elements of $\bar{\mathcal{Q}}_{\bar{x}}^0$. Now, in (31), we can assume some boundedness for the vectors (μ^k, λ^k) such as $\|(\mu^k, \lambda^k)\| = 1$, $k \in \mathbb{N}$, and by passing to a convergent subsequence,

$$\lim_{k \rightarrow \infty} (\mu^k, \lambda^k) = (\bar{\mu}, \bar{\lambda}).$$

By taking in (31) the limit for $k \rightarrow \infty$, we get

$$\nabla_x L := \bar{\mu}_0 \nabla g_0(\bar{x}) + \sum_{j \in \mathcal{J}_*(\bar{x})} \bar{\mu}_j \nabla g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla h_i(\bar{x}) = 0, \quad (32)$$

i.e., $(\bar{\mu}, \bar{\lambda}) \in \bar{\mathcal{Q}}_{\bar{x}}^0$. By MFCQ [or SOMFCQ in case (ii)], it follows that $\bar{\mu}_0 > 0$; see Remark 3.10. Using

$$\lim_{k \rightarrow \infty} \mu_j^k = \bar{\mu}_j,$$

we see that, for any index j such that $\bar{\mu}_j > 0$, the relation

$$g_j(x_k) - g_j(\bar{x}) = 0$$

must hold for k sufficiently large. Consequently we get, for $j \in \mathcal{J}_*(\bar{x})$,

$$g_j(x_k) - g_j(\bar{x}) = t_k \nabla g_j(\bar{x}) \tilde{x} + o(t_k) \begin{cases} = 0, & \text{for } \bar{\mu}_j > 0, \\ \leq 0, & \text{for } \bar{\mu}_j = 0, \end{cases}$$

and

$$h_i(x_k) - h_i(\bar{x}) = t_k \nabla h_i(\bar{x}) \tilde{x} + o(t_k) = 0, \quad i \in \mathcal{I},$$

giving

$$\nabla g_j(\bar{x}) \tilde{x} \leq 0, \quad \bar{\mu}_j \nabla g_j(\bar{x}) \tilde{x} = 0, \quad j \in \mathcal{J}_*(\bar{x}), \quad (33a)$$

$$\nabla h_i(\bar{x}) \tilde{x} = 0, \quad i \in \mathcal{I}. \quad (33b)$$

Moreover, multiplying (32) by $\tilde{x}$ and using $\bar{\mu}_0 > 0$ yields

$$\nabla g_0(\bar{x}) \tilde{x} = 0 \quad \text{and} \quad \tilde{x} \in \bar{C}_{\bar{x}}^0.$$

This gives a contradiction in case (i).

In case (ii), we note that the relation

$$\mu_j^k \nabla g_j(\bar{x}) \tilde{x} = 0, \quad j \in \mathcal{J}_*(\bar{x}), \quad (34)$$

is valid for sufficiently large k . In fact, the condition $\nabla g_j(\bar{x}) \tilde{x} < 0$ implies, using $t_k > 0$,

$$g_j(x_k) - g_j(\bar{x}) = t_k \nabla g_j(\bar{x}) \tilde{x} + o(t_k) < 0, \quad \text{for } k \rightarrow \infty,$$

and then $\mu_j^k = 0$. Now, by using (31), (32), a Taylor expansion yields

$$\begin{aligned} 0 = \nabla_x L_k - \nabla_x L &= \sum_{j \in \mathcal{J}_*(\bar{x})} (\mu_j^k - \bar{\mu}_j) \nabla g_j(\bar{x}) + \sum_{i \in \mathcal{I}} (\lambda_i^k - \bar{\lambda}_i) \nabla h_i(\bar{x}) \\ &\quad + t_k \left(\sum_{j \in \mathcal{J}_*(\bar{x})} \mu_j^k \nabla^2 g_j(\bar{x}) \tilde{x} + \sum_{i \in \mathcal{I}} \lambda_i^k \nabla^2 h_i(\bar{x}) \tilde{x} \right) + o(t_k). \end{aligned}$$

After multiplication by $\tilde{x}'$, using (33), (34) and $\nabla g_0(\bar{x})\bar{x}=0$, we obtain

$$t_k \tilde{x}' \left(\sum_{j \in \mathcal{J}_*^0(\bar{x})} \bar{\mu}_j \nabla^2 g_j(\bar{x}) + \sum_{i \in \mathcal{I}} \bar{\lambda}_i \nabla^2 h_i(\bar{x}) \right) \tilde{x} = o(t_k), \quad \text{for } k \rightarrow \infty.$$

Division by t_k and letting $k \rightarrow \infty$ gives a contradiction in case (ii). Since the gradients $\nabla h_i(\bar{x})$, $i \in \mathcal{I}$, are linearly independent, by continuity of $\nabla h_i(x)$ there exists a neighborhood U of $\bar{x}$ such that, for all $x \in U$, the gradients $\nabla h_i(x)$, $i \in \mathcal{I}$ are linearly independent. Thus by Theorem 3.3, the relation (10) implies that U cannot contain any local minimizer $x \neq \bar{x}$. This completes the proof. $\square$

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