Directed Path Graphs

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Abstract

The concept of a line digraph is generalized to that of a directed path graph. The directed path graph $\overrightarrow{P_k}(D)$ of a digraph D is obtained by representing the directed paths on k vertices of D by vertices. Two vertices are joined by an arc whenever the corresponding directed paths in D form a directed path on $k+1$ vertices or form a directed cycle on k vertices in D. Several properties of $\overrightarrow{P_3}(D)$ are studied, in particular with respect to isomorphism and traversability.

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1 Introduction

We refer to [1] for any undefined terminology.

In [2] path graphs were introduced as a generalization of line graphs of (undirected) graphs. In the next section we shall introduce an analogous concept for directed graphs. But first we recall some basic definitions and notation concerning directed graphs.

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We define a directed graph or *digraph* D to be a pair $(V(D), A(D))$, where $V(D)$ is a finite non-empty set of elements called *vertices*, and $A(D)$ is a (finite) set of distinct ordered pairs of distinct elements of $V(D)$ called *arcs*. For convenience we shall denote the arc (v, w) (where $v, w \in V(D)$) by vw. If $a = vw$ is an arc of D, then we say that v and w are *adjacent*, and that a is an *out-arc* of v and an *in-arc* of w; we call w an *out-neighbour* of v and v an *in-neighbour* of w. The *in-degree* $d^-(v)$ of v is the number of in-arcs of v; the *out-degree* $d^+(v)$ of v is the number of out-arcs of v; v is a *source* or *sink* if $d^-(v) = 0$ or $d^+(v) = 0$, respectively. The *underlying graph* $U(D)$ of a digraph D is the graph (or multigraph) obtained from D by replacing each arc by an (undirected) edge joining the same pair of vertices. A digraph D is called *strongly connected* if, for each pair of vertices v and w, there is a directed path in D from v to w, and *connected* if there is a path from v to w in $U(D)$. A directed subgraph of D corresponding to a path of $U(D)$ is called a *semipath* of D. We denote by $\overrightarrow{P_k}$ a directed path on k vertices ($k \geq 1$), i.e. a semipath on k vertices with one source and one sink, in which all arcs are oriented from source to sink. A directed cycle $\overrightarrow{C_k}$ ($k \geq 2$) consists of a $\overrightarrow{P_k}$ with source v and sink w together with the arc wv. Two arcs $a, b \in A(D)$ are said to be *adjacent* if $\{a, b\} = \{vw, wz\}$ for some vertices $v, w, z \in V(D)$; to stress the head-to-tail adjacency, we say that *a hits b* if $a = vw$ and $b = wz$. We call two adjacent arcs $a, b \in A(D)$ a *$\overrightarrow{P_3}$-pair* or a *$\overrightarrow{C_2}$-pair* if they form a $\overrightarrow{P_3}$ or a $\overrightarrow{C_2}$ in D, respectively. If $\{a, b\} \subseteq A(D)$ is a $\overrightarrow{P_3}$-pair and a hits b, then we denote the $\overrightarrow{P_3}$ formed by a and b simply by ab.

2 Directed path graphs

Let k be a positive integer, and let D be a digraph containing at least one $\overrightarrow{P_k}$. Denote by $\overrightarrow{\Pi}_k(D)$ the set of all $\overrightarrow{P_k}$'s of D. Then the *$\overrightarrow{P_k}$-graph* of D, denoted by $\overrightarrow{P_k}(D)$, is the digraph with vertex set $\overrightarrow{\Pi}_k(D)$; pq is an arc of $\overrightarrow{P_k}(D)$ if and only if the $\overrightarrow{P_k}$'s corresponding to p and q in D together form a $\overrightarrow{P}_{k+1}$ or a $\overrightarrow{C_k}$. Note that $\overrightarrow{P_1}(D) = D$ and $\overrightarrow{P_2}(D) = \overrightarrow{L}(D)$, the line digraph of D, as it was introduced in [3].

For a nice survey of results on line graphs and line digraphs we refer to [5]. In the sequel we shall restrict ourselves to $\overrightarrow{P_3}$-graphs. In Section 3 we give some elementary results on $\overrightarrow{P_3}$-graphs, in Section 4 we discuss isomorphisms of $\overrightarrow{P_3}$-graphs, and in Section 5 we consider the traversability of $\overrightarrow{P_3}$-graphs. We close with some miscellaneous results and remarks in Section 6.

3 Elementary results

Let D be a digraph containing at least one $\overrightarrow{P_3}$ and let $G = \overrightarrow{P_3}(D)$. To express the number of vertices, the number of arcs, and the degrees of the vertices of G in terms of D, we first introduce some additional terminology.

For a vertex $v \in V(D)$, we set

$$\overline{A}_v = \{u \in V(D) \mid \{uv, vu\} \subseteq A(D)\},$$

and we define

$$\overline{A}(D) = \{uv \in A(D) \mid vu \in A(D)\}.$$

Now the number of $\overrightarrow{P_3}$'s in D with middle vertex v is equal to

$$(d^-(v) - |\overline{A}_v|)d^+(v) + |\overline{A}_v|(d^+(v) - 1) = d^-(v)d^+(v) - |\overline{A}_v|.$$

Hence

$$|V(G)| = \sum_{v \in V(D)} (d^-(v)d^+(v) - |\overline{A}_v|) = \sum_{v \in V(D)} d^-(v)d^+(v) - |\overline{A}(D)|.$$

The number of arcs of G can be counted by summing up, for each arc a of D, the number of $\overrightarrow{P_3}$'s of D "joined" together by having the arc a "in common", as follows: each arc $uv \in A(D)\setminus\overline{A}(D)$ joins $d^-(u)d^+(v)$ $\overrightarrow{P_3}$'s, while each arc $uv \in \overline{A}(D)$ joins $(d^-(u) - 1)(d^+(v) - 1)$ $\overrightarrow{P_3}$'s. Hence

$$|A(G)| = \sum_{uv \in A(D)} (d^-(u)d^+(v)) + |\overline{A}(D)| - \sum_{uv \in \overline{A}(D)} (d^-(u) + d^+(v)).$$

The in-degree and out-degree of a vertex in G corresponding to a $\overrightarrow{P_3}$ uvw in D are

$$\begin{aligned} &d^-(u) - |\{u\} \cap \overline{A}_v| \qquad \text{and} \\ &d^+(w) - |\{w\} \cap \overline{A}_v|, \end{aligned}$$

respectively.

4 Isomorphisms of $\overrightarrow{P_3}$-graphs

In this section we consider two questions:

(1) For which digraphs D is $\overrightarrow{P_3}(D) \cong D$?

(2) For which digraphs D_1 and D_2 does $\overrightarrow{P_3}(D_1) \cong \overrightarrow{P_3}(D_2)$ imply $D_1 \cong D_2$?

We refer to [5] for results related to similar questions concerning line (di)graphs, and to [2] for analogous results on P_3-graphs of (undirected) graphs.
In this section we shall see that the directed cycles $\overrightarrow{C_n}$ are "almost the only" digraphs for which $\overrightarrow{P_3}(D) \cong D$, and that $\overrightarrow{P_3}(D_1) \cong \overrightarrow{P_3}(D_2)$ "almost always" implies $D_1 \cong D_2$. Before we present the results we introduce some additional terminology.

Let D be a digraph. A direct tree T of D is an *out-tree* of D if $V(T) = V(D)$ and precisely one vertex of T has in-degree zero (the *root* of T), while all other vertices of T have in-degree one. An *in-tree* of D is defined analogously with respect to out-degrees. Note that any strongly-connected digraph contains an in-tree and an out-tree, and no sources or sinks.

Theorem 1

Let D be a connected digraph without sources or sinks. If D has an in-tree or an out-tree, then $\overrightarrow{P_3}(D) \cong D$ if and only if $D \cong \overrightarrow{C_n}$.

Proof If $D \cong \overrightarrow{C_n}$, then clearly $\overrightarrow{P_3}(D) \cong \overrightarrow{C_n} \cong D$.

For the converse, assume without loss of generality that D has an out-tree T with root v. Let t denote the number of vertices with out-degree zero in T. Denote $V(D) = \{v, v_1, \ldots, v_{n-1}\}$, where $v_1, \ldots, v_t$ are the vertices with out-degree zero in T. Note that $\overrightarrow{P_3}(D)$ does not contain $\overrightarrow{C_2}$. Since v has at least one in-arc in D, and each of $v_1, \ldots, v_t$ has at least one out-arc in D, we know that $\overrightarrow{P_3}(D)$ has at least

$$d_T^+(v) + t + d_T^+(v_{t+1}) + \ldots + d_T^+(v_{n-1})$$

vertices. From $\overrightarrow{P_3}(D) \cong D$ we obtain

$$n = |V(D)| = |V(\overrightarrow{P_3}(D))| \geq d_T^+(v) + t + d_T^+(v_{t+1}) + \ldots + d_T^+(v_{n-1}),$$

hence

$$d_T^+(v) + d_T^+(v_{t+1}) + \ldots + d_T^+(v_{n-1}) \leq n - t. \tag{1}$$

On the other hand, since T is an out-tree, we obtain

$$d_T^+(v) + (d_T^+(v_{t+1}) + 1) + \ldots + (d_T^+(v_{n-1}) + 1) + t = 2(n-1),$$

hence

$$d_T^+(v) + d_T^+(v_{t+1}) + \ldots + d_T^+(v_{n-1}) = n - 1. \tag{2}$$

Combining (1) and (2), we get that $t \leq 1$, implying that $t = 1$, and that $T = \overrightarrow{P_n}$. Similar arguments show that any in-tree of D is a $\overrightarrow{P_n}$. This is only possible if $D \cong \overrightarrow{C_n}$. ■

Let D be a digraph obtained from a $\overrightarrow{P_{m_1}}$ ($m_1 \geq 1$) and a $\overrightarrow{C_{m_2}}$ ($m_2 \geq 3$) by identifying the first or last vertex of $\overrightarrow{P_{m_1}}$ with one vertex of $\overrightarrow{C_{m_2}}$. Then one easily checks that $\overrightarrow{P_3}(D) \cong D$, that D contains either precisely one source or precisely one sink, and that D contains an in-tree or an out-tree.

Before we turn to Question 2 we introduce some additional terminology concerning isomorphisms.

Let D and D' be two digraphs. An *isomorphism* of D onto D' is a bijection $f : V(D) \to V(D')$ such that $uv \in A(D)$ if and only if $f(u)f(v) \in A(D')$. An *arc-isomorphism* of D onto D' is a bijection $f : A(D) \to A(D')$ such that $a \in A(D)$ hits $b \in A(D)$ if and only if $f(a) \in A(D')$ hits $f(b) \in A(D')$. Hence an arc-isomorphism of D onto D' is an isomorphism of $\overrightarrow{L}(D)$ onto $\overrightarrow{L}(D')$. A $\overrightarrow{P_3}$*-isomorphism* of D onto D' is an isomorphism of $\overrightarrow{P_3}(D)$ onto $\overrightarrow{P_3}(D')$.

Question 2 can be rephrased as follows.

(2′) Which $\overrightarrow{P_3}$-isomorphisms of D onto D' are induced by isomorphisms of D onto D'?

The related question for arc-isomorphisms was answered in [3].

Theorem 2
Let D and D' be two digraphs without sources or sinks. Then every arc-isomorphism of D onto D' is induced by an isomorphism of D onto D', hence $\overrightarrow{L}(D) \cong \overrightarrow{L}(D')$ if and only if $D \cong D'$.

We can prove a similar result on $\overrightarrow{P_3}$-isomorphisms if we make a "weak" additional assumption concerning the digraphs D and D'.

Theorem 3
Let D and D' be two connected digraphs without sources or sinks. If for each arc $a = uv \in A(D) \cup A(D')$ these exist arcs $b = xu$ and $c = vy$ in the same digraph with $x \neq v$ and $y \neq u$, then every $\overrightarrow{P_3}$-isomorphism of D onto D' is induced by an arc-isomorphism of D onto D'.

Proof Let f denote a $\overrightarrow{P_3}$-isomorphism of D onto D', where D and D' satisfy the conditions of the theorem. For any arc $x \in A(D)$, there exist two arcs $y, z \in A(D)$ such that yx and xz correspond to two $\overrightarrow{P_3}$'s of D. Since f is a $\overrightarrow{P_3}$-isomorphism, for some $\overrightarrow{P_3}$-pairs $\{a, b\}, \{c, d\} \subseteq A(D')$, $f(yx) = ab$ and $f(xz) = cd$. But this implies $b = c$, since adjacencies of $\overrightarrow{P_3}$'s are preserved by f. Considering, for the same $x \in A(D)$, two arbitrary arcs $y', z' \in A(D)$ such that $y'x$ and xz' are $\overrightarrow{P_3}$'s of D, by the same arguments we find $\overrightarrow{P_3}$-pairs $\{k, \ell\}, \{m, n\} \subseteq A(D')$ such that $f(y'x) = k\ell$ and $f(xz') = mn$. Combining $f(y'x) = k\ell$ and $f(xz) = bd$, we obtain $\ell = b$, and similarly combining $f(yx) = ab$ and $f(xz') = mn$, we obtain $m = b$.

Now define a function $f^* : A(D) \to A(D')$ by $f^*(x) = b$ if $\overrightarrow{P_3}$-pairs $\{y, x\}, \{x, z\} \subseteq A(D)$ and $\{a, b\}, \{c, d\} \subseteq A(D')$ exist with $f(yx) = ab$ and $f(xz) = cd$. (From the above discussion it follows that b is independent of the choice of the $\overrightarrow{P_3}$-pairs containing x.)
In the remainder of the proof we show that f^* is an arc-isomorphism of D onto D'.

Using that f is bijective, it is not difficult to check that f^* is bijective:

(a) f^* is surjective:
By the hypothesis of the theorem, for any $x' \in A(D')$ these exist $\overrightarrow{P_3}$-pairs $\{y', x'\}, \{x', z'\} \subseteq A(D')$. Since f is a $\overrightarrow{P_3}$-isomorphism, there exist $\overrightarrow{P_3}$-pairs $\{y, x''\}, \{x''', z\} \subseteq A(D)$ such that $f^{-1}(y'x') = yx''$ and $f^{-1}(x'z') = x'''z$. By similar arguments as before, this implies $x'' = x'''$, and $x' = f(x'')$, hence f^* is surjective.

(b) f^* is injective:
If $f^*(x) = b = f^*(y)$ for some $x, y \in A(D)$ and $b \in A(D')$, then for certain $\overrightarrow{P_3}$-pairs $\{x_1, x\}, \{y, y_1\} \subseteq A(D)$ and $\{b_1, b\}, \{b, b_2\} \subseteq A(D')$, we get $f^{-1}(b_1 b) = x_1 x$ and $f^{-1}(b b_2) = y y_2$, yielding $x = y$.

Denote by f_* the inverse of f^*. It is not difficult to see that f_* can be obtained from f^{-1} in the same way f^* was obtained from f, i.e. $(f^*)^{-1} = (f^{-1})^*$.

It remains to show that f^* preserves adjacencies and nonadjacencies of arcs in D and D'. It is sufficient to show that $f^*(x)$ and $f^*(y)$ are adjacent arcs in D' if x and y are adjacent arcs in D. This is easy if $\{x, y\}$ is a $\overrightarrow{P_3}$-pair: then, by the hypothesis of the theorem, there exist $\overrightarrow{P_3}$-pairs $\{z, x\}$ and $\{y, z'\}$ in D, and by similar arguments as before $\overrightarrow{P_3}$-pairs $\{a, b\}, \{b, c\}$, and $\{c, d\}$ in D' such that $f(zx) = ab$, $f(xy) = bc$, and $f(yz') = cd$, implying $f^*(x) = b$, $f^*(y) = c$, and hence $f(xy) = f^*(x)f^*(y)$, so that $\{f^*(x), f^*(y)\}$ is a $\overrightarrow{P_3}$-pair in D'. This implies that f^* preserves $\overrightarrow{P_3}$-pairs, and by the same arguments, that f_* preserves $\overrightarrow{P_3}$-pairs. Next assume $\{x, y\}$ is a $\overrightarrow{C_2}$-pair in D. If D is a digraph obtained from a cycle C_n by replacing each edge uv by two arcs uv and vu, then one easily checks that $D \cong D'$, unless D' consists of two disjoint $\overrightarrow{C_n}$'s, contradicting the connectivity of D'. In the other case, there exist arcs $x_1, x_2, \ldots, x_k$, $y_1, y_2, \ldots, y_k$ in D such that $\{x_1, y_1\}, \{y_k, x_k\}$, and $\{x_i, x_{i+1}\}, \{y_{i+1}, y_i\}$ $(i = 1, \ldots, k-1)$ are $\overrightarrow{P_3}$-pairs in D, while $\{x_i, y_i\}$ $(i = 2, \ldots, k-1)$ are $\overrightarrow{C_2}$-pairs in D, and $\{x, y\} = \{x_i, y_i\}$ for some $i \in \{2, \ldots, k-1\}$. We complete the proof by showing that, for each $i \in \{2, \ldots, k-1\}$, $\{f^*(x_i), f^*(y_i)\}$ is a $\overrightarrow{C_2}$-pair in D', in particular $\{f^*(x), f^*(y)\}$. Assume, to the contrary, that i is the smallest index in $\{2, \ldots, k-1\}$ such that $\{f^*(x_i), f^*(y_i)\}$ is not a $\overrightarrow{C_2}$-pair. Suppose first that $f^*(x_i)f^*(y_i)$ (or $f^*(y_i)f^*(x_i)$) is a $\overrightarrow{P_3}$ in D'. Then, since f_* preserves $\overrightarrow{P_3}$-pairs, x_iy_i is a $\overrightarrow{P_3}$ in D, a contradiction. Hence $f^*(x_i)$ and $f^*(y_i)$ are nonadjacent arcs in D'. Considering the $\overrightarrow{P_3}$'s $f^*(x_{i-1})f^*(x_i)$ and $f^*(y_i)f^*(y_{i-1})$ in D', it is clear that $\{f^*(x_{i-1}), f^*(y_{i-1})\}$ is not a $\overrightarrow{C_2}$-pair in D'. The choice of i implies that $i = 2$. Hence $\{f^*(x_2), f^*(y_2)\}$ is not a $\overrightarrow{C_2}$-pair (nor a $\overrightarrow{P_3}$-pair) in D'. Now, considering the $\overrightarrow{P_3}$'s $f^*(x_1)f^*(x_2), f^*(x_1)f^*(y_1)$, and $f^*(y_2)f^*(y_1)$ in D', we easily obtain a contradiction.

From the above discussion we conclude that f is induced by f^*. ■

From the above proof we already note that we cannot omit the connectivity condition in Theorem 3. The next two pairs of nonisomorphic digraphs with isomorphic $\overrightarrow{P_3}$-graphs show we cannot omit the condition on the sources and sinks, or the condition on the arcs, respectively.

The first digraph of the first pair consists of a $\overrightarrow{C_3}$, one additional vertex v, and arcs from v to two vertices of the $\overrightarrow{C_3}$; the second one of a $\overrightarrow{C_3}$, two additional vertices v_1, v_2, and arcs v_1u and v_2w to two vertices u and w of the $\overrightarrow{C_3}$.

The first digraph of the second pair is a $\overrightarrow{C_4}$; the second one is obtained from a $\overrightarrow{C_4}$ by replacing two vertex-disjoint arcs by $\overrightarrow{C_2}$'s.

Combining Theorems 2 and 3 it is clear we have the following consequences for digraphs D and D' satisfying the conditions in the hypothesis of Theorem 3.

Corollary 4
$\overrightarrow{P_3}(D) \cong \overrightarrow{P_3}(D')$ *if and only if* $D \cong D'$.

Corollary 5
$Aut(D) \cong Aut(\overrightarrow{L}(D)) \cong Aut(\overrightarrow{P_3}(D))$.

Remark

The isomorphism problem for (undirected) P_3-graphs seems to be more difficult, as it also is more difficult for line graphs than for line digraphs. Results on isomorphisms of P_3-graphs were obtained in [2], [4], and [6].

5 Traversability of $\overrightarrow{P_3}$-graphs

In this section we consider (*directed*) *Euler tours* and (*directed*) *Hamilton cycles* in $\overrightarrow{P_3}$-graphs.

First we present a useful relationship between $\overrightarrow{P_3}$-graphs and iterated line digraphs. Given a digraph D, we denote by $Asym(D)$ the graph obtained from D by deleting all $\overrightarrow{C_2}$'s, i.e. by deleting all $\overrightarrow{C_2}$-pairs $\{uv, vu\} \subseteq A(D)$.

Theorem 6

For any digraph D containing at least one $\overrightarrow{P_3}$, $\overrightarrow{P_3}(D) \cong \overrightarrow{L}(Asym(\overrightarrow{L}(D)))$.

Proof Let D be a digraph containing at least one $\overrightarrow{P_3}$. Then $\overrightarrow{P_3}(D)$ and $\overrightarrow{L}(Asym(\overrightarrow{L}(D)))$ exist, and $v \in V(\overrightarrow{L}(Asym(\overrightarrow{L}(D))))$ if and only if $v \in A(Asym(\overrightarrow{L}(D)))$. This is equivalent to saying that $v = xy$ for some $x, y \in V(Asym(\overrightarrow{L}(D)))$, or, equivalently, for some $x, y \in V(\overrightarrow{L}(D))$ such that $\{x, y\}$ is not a $\overrightarrow{C_2}$-pair of D. It is clear that this is equivalent to saying that v is a $\overrightarrow{P_3}$ of D, hence $v \in V(\overrightarrow{P_3}(D))$.

Moreover, u and v are adjacent vertices in $\overrightarrow{L}(Asym(\overrightarrow{L}(D)))$ if and only if uv corresponds to a $\overrightarrow{P_3}$ in $Asym(\overrightarrow{L}(D))$, and hence in $\overrightarrow{L}(D)$. It is again clear that this is equivalent to saying that u and v correspond to two adjacent $\overrightarrow{P_3}$'s in D, or, equivalently, that u and v are adjacent vertices in $\overrightarrow{P_3}(D)$. ∎

Corollary 7

For any digraph D containing at least one $\overrightarrow{P_3}$ and no $\overrightarrow{C_2}$, $\overrightarrow{P_3}(D) \cong \overrightarrow{L}(\overrightarrow{L}(D)) = \overrightarrow{L}^2(D)$.

Proof This follows immediately from Theorem 6 and the observation that D contain a $\overrightarrow{C_2}$ if and only if $\overrightarrow{L}(D)$ contains a $\overrightarrow{C_2}$. ∎

Corollary 8

For any digraph D containing at least one $\overrightarrow{P_3}$,
$\lim_{n\to\infty} |V(\overrightarrow{P_3}^n(D))| < \infty$ if and only if $\lim_{n\to\infty} |V(\overrightarrow{L}^n(D))| < \infty$.

Proof This follows from the fact that $\overrightarrow{P_3}^2(D) = \overrightarrow{L}(Asym(\overrightarrow{L}(Asym(\overrightarrow{L}(D))))) =$
$\overrightarrow{L}^2(Asym(\overrightarrow{L}(D)))$, hence $\overrightarrow{P_3}^n(D) = \overrightarrow{L}^n(Asym(\overrightarrow{L}(D)))$. ∎

Corollary 9

For any digraph D containing at least one $\overrightarrow{P_3}$, $\overrightarrow{P_3}(D)$ is strongly connected if and only if $Asym(\overrightarrow{L}(D))$ is strongly connected.

Proof This is an immediate consequence of Theorem 6 and [5, Theorem 7.4(i)]: $\overrightarrow{L}(D)$ is strongly connected if and only if D is strongly connected. ■

In particular, if $Asym(D)$ is strongly connected, then $Asym(\overrightarrow{L}(D))$ is strongly connected and so is $\overrightarrow{P_3}(D)$, but for the following digraph D, $Asym(D)$ is disconnected, while $Asym(\overrightarrow{L}(D))$ is strongly connected.

The digraph D consists of two vertex-disjoint $\overrightarrow{C_3}$'s and two additional arcs uv and vu between two vertices u and v of different $\overrightarrow{C_3}$'s.

For line digraphs of strongly connected digraphs, the following result ([5, Theorem 10.1]) characterizes the traversability.

Theorem 10
Let D be a strongly connected digraph. Then

(i) $\overrightarrow{L}(D)$ *is Eulerian if and only if* $d^-(v) = d^+(w)$ *for each arc* vw *in* D*;*

(ii) $\overrightarrow{L}(D)$ *is Hamiltonian if and only if* D *is Eulerian.*

Combining Theorems 6 and 10 we immediately obtain the following characterization of Eulerian and Hamiltonian $\overrightarrow{P_3}$-graphs of digraphs D in terms of properties of $Asym(\overrightarrow{L}(D))$.

Corollary 11
Let D be a digraph such that $Asym(\overrightarrow{L}(D))$ is strongly connected. Then

(i) $\overrightarrow{P_3}(D)$ *is Eulerian if and only if* $d^-(v) = d^+(w)$ *for each arc* vw *in* $Asym(\overrightarrow{L}(D))$*;*

(ii) $\overrightarrow{P_3}(D)$ *is Hamiltonian if and only if* $Asym(\overrightarrow{L}(D))$ *is Eulerian.*

The properties of $Asym(\overrightarrow{L}(D))$ in Corollary 11 can be translated into properties of D as follows.

For an arc $xy \in A(D)$, let

$$\begin{aligned} d^-(xy) &= d^-(x) - |\{yx\} \cap \overline{A}(D)| \qquad \text{and} \\ d^+(xy) &= d^+(y) - |\{yx\} \cap \overline{A}(D)|. \end{aligned}$$

Then $d^-(v) = d^+(w)$ for each arc vw in $Asym(\overrightarrow{L}(D))$ if and only if $d^-(ab) = d^+(bc)$ for each $\overrightarrow{P_3}$ abc in D.

We say that a Euler tour T of D is a $\overrightarrow{C_2}$*-tour* if the arcs of each $\overrightarrow{C_2}$ of D are successive arcs in T. Then $Asym(\overrightarrow{L}(D))$ is Eulerian if and only if $\overrightarrow{L}(D)$ has a $\overrightarrow{C_2}$-tour and $\overrightarrow{L}(D) \ncong \overrightarrow{C_2}$. Furthermore, $Asym(\overrightarrow{L}(D))$ is a vertex-disjoint union of Eulerian digraphs if $\overrightarrow{L}(D)$ is Eulerian (and $\ncong \overrightarrow{C_2}$). Hence we obtain the following result.

Theorem 12
Let D be a digraph such that $Asym(\overrightarrow{L}(D))$ is strongly connected. Then

(i) $\overrightarrow{P_3}(D)$ *is Eulerian if and only if* $d^-(ab) = d^+(bc)$ *for each* $\overrightarrow{P_3}$ abc *in* D*;*

(ii) $\overrightarrow{P_3}(D)$ *is Hamiltonian if and only if* $\overrightarrow{L}(D)$ *has a* $\overrightarrow{C_2}$*-tour;*

(iii) $\overrightarrow{P_3}(D)$ *contains a 2-factor if and only if* $\overrightarrow{L}(D)$ *is Eulerian, or, equivalently if* $d^-(v) = d^+(w)$ *for each arc* vw *in* D*;*

(iv) $\overrightarrow{P_3}(D)$ *is Hamiltonian if* $d^-(v) = d^+(w)$ *for each arc* vw *in* D*, and* D *contains no* $\overrightarrow{C_2}$*.*

6 Miscellaneous results

6.1 Cycle structure

By considering the possible adjacency structures of $\overrightarrow{P_3}$'s in a digraph D, one easily obtains the following result on (short) cycles in $\overrightarrow{P_3}(D)$. We omit the proof and remark that similar results can be deduced for longer cycles. Recall that $U(D)$ denotes the underlying (undirected) graph of D.

Theorem 13
Let D be a digraph. Then

(i) $\overrightarrow{P_3}(D)$ *contains no* $\overrightarrow{C_2}$*;*

(ii) Each C_3 *in* $U(\overrightarrow{P_3}(D))$ *is a* $\overrightarrow{C_3}$ *in* $\overrightarrow{P_3}(D)$*;*

(iii) Each C_4 *in* $U(\overrightarrow{P_3}(D))$ *is induced (has no chords) and is a* $\overrightarrow{C_4}$ *or is oriented with alternating arc directions in* $\overrightarrow{P_3}(D)$*;*

(iv) No C_k $(k \geq 5)$ *of* $U(\overrightarrow{P_3}(D))$ *is both induced and oriented with alternating arc directions in* $\overrightarrow{P_3}(D)$*.*

6.2 Splitting vertices

Let D be a digraph and $v \in V(D)$ a source with out-arcs $vu_1, \ldots, vu_k$. Suppose D' is obtained from D by replacing v by two (or more) new vertices v_1, v_2 and splitting the out-arcs $vu_1, \ldots, vu_k$ into two (or more) disjoint (non-empty) sets $v_1u_1, \ldots, v_1u_{k_1}$, $v_2u_{k_1+1}, \ldots, v_2u_k$. Then it is clear that $\overrightarrow{P_3}(D') \cong \overrightarrow{P_3}(D)$. A similar splitting preserving the $\overrightarrow{P_3}$-structure can be applied to sinks. Of course the reverse operation of combining sources or sinks is also preserving the $\overrightarrow{P_3}$-structure, as long as sources or sinks do not have common out-neighbours or in-neighbours, respectively.

Splitting an arbitrary vertex v of D into two new vertices v_1, v_2 and dividing the in-arcs and out-arcs at v among v_1 and v_2, we obtain a digraph D' with the property that $\overrightarrow{P_3}(D')$ is an induced subgraph of $\overrightarrow{P_3}(D)$. We leave the details to the reader.

These splitting operations can be useful in studying structural properties of $\overrightarrow{P_3}$-graphs. We give an example.

Suppose D contains a $\overrightarrow{C_2}$ with arcs uv and vu and such that $d^-(v) = d^+(v) = 2$. Let D' be obtained from D by splitting v into v_1 and v_2 such that $d^-(v_1) = d^+(v_1) = d^-(v_2) = d^+(v_2) = 1$. Then $\overrightarrow{L}(D') = \overrightarrow{L}(D)$ minus the $\overrightarrow{C_2}$ between the vertices of $\overrightarrow{L}(D)$ corresponding to uv and vu. If all $\overrightarrow{C_2}$'s in D are "nicely" distributed, then, repeating the above procedure, could yield a graph D^* such that $\overrightarrow{L}(D^*) = Asym(\overrightarrow{L}(D))$, hence such that $\overrightarrow{P_3}(D^*) = \overrightarrow{P_3}(D)$. This shows, for instance, that the digraph consisting of two vertex-disjoint $\overrightarrow{C_n}$'s has the same $\overrightarrow{P_3}$-graph as the digraph obtained from a C_n by replacing each edge uv by two arcs uv and vu.

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