On sufficient spectral radius conditions for hamiltonicity

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A R T I C L E   I N F O

Article history:
Received 8 March 2019
Received in revised form 19 December 2019
Accepted 28 January 2020
Available online 11 February 2020

Keywords:
Hamiltonian graph
Sufficient condition
Spectral radius
Minimum degree

A B S T R A C T

During the last decade several research groups have published results on sufficient conditions for the hamiltonicity of graphs in terms of their spectral radius and their signless Laplacian spectral radius. Here we extend some of these results. All of our results involve the characterization of the exceptional graphs, i.e., all the nonhamiltonian graphs that satisfy the condition. The proofs of our main results are based on the Bondy–Chvátal closure, a degree sequence condition due to Chvátal, and an operation on the edges that is known as Kelmans’ transformation.

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1. Introduction

We use the textbook of Bondy and Murty [3] for any terminology and notation not defined here, and we consider undirected simple graphs only.

Let G be a graph with vertex set V(G) and edge set E(G). We use e(G) = |E(G)| to denote the number of edges of G, N_G(v) to denote the set of vertices adjacent to a vertex v ∈ V(G), and d_G(v) = |N_G(v)| to denote the degree of v. Moreover, we define N_G[v] = N_G(v) ∪ {v}, and we use δ(G) to denote the minimum degree of (the vertices of) G. We use G + H and G ∨ H to denote the disjoint union and the join of two vertex-disjoint graphs G and H, respectively. The union of k vertex-disjoint copies of the same graph G is denoted by kG. For two distinct graphs G and H, we often use G = H to indicate that G and H are isomorphic. Similarly, we use G ∈ K to denote that G is isomorphic to a graph in the set of graphs K. If S is a nonempty subset of V(G), then G[S] denotes the subgraph of G induced by S, i.e., the graph with vertex set S containing all edges of G that join two vertices of S. For two vertex-disjoint nonempty subsets S and T of V(G), G[S, T] denotes the bipartite graph on vertex set S ∪ T with all edges of G that join a vertex of S and a vertex of T.

Let G be a graph with vertex set {v_1, v_2, . . . , v_n}. Then the adjacency matrix A(G) of G is the symmetric n × n matrix with entries A(i, j) = 1 if and only if v_i v_j ∈ E(G) and zeros elsewhere. The diagonal degree matrix D(G) of G is the n × n matrix with entries D(i, i) = d(v_i) and zeros elsewhere. The matrix Q(G) = D(G) + A(G) is known as the signless Laplacian matrix of G. The largest eigenvalue of A(G), denoted by ρ(A(G)) or ρ(G), is called the spectral radius of G. The largest eigenvalue of Q(G), denoted by q(Q(G)) or q(G), is called the signless Laplacian spectral radius of G.
The graph $G$ is said to be hamiltonian if it admits a Hamilton cycle, i.e., a cycle containing all the vertices of $G$. It is well-known that the problem of deciding whether a given graph is hamiltonian or not is an NP-complete problem. Many scholars have focused on finding sufficient conditions for graphs to be hamiltonian. Here we focus on sufficient conditions for the hamiltonicity of $G$ in terms of lower bounds on the spectral radius and the signless Laplacian spectral radius of $G$.

To put our results in the right context, in the next two subsections we start with a brief summary of a number of sufficient conditions for hamiltonicity in terms of the spectral radius, and signless Laplacian spectral radius, respectively, that were obtained in the last decade. Our aim is to extend these results.

1.1. Sufficient conditions in terms of the spectral radius

In 2010, Fiedler and Nikiforov [8] presented the following sufficient condition for hamiltonicity.

**Theorem 1.1** ([8]). Let $G$ be a graph of order $n$. If $\rho(G) > n - 2$, then $G$ is hamiltonian unless $G = K_1 \lor (K_{n-2} + K_1)$.

Note that the graph $G = K_1 \lor (K_{n-2} + K_1)$ is clearly not hamiltonian: it has a vertex with degree 1. This also implies that deleting the vertex $v$ corresponding to the first $K_1$, the graph $G - v$ has two components, so $G$ is not 1-tough. In general, a graph $G$ is called 1-tough if and only if the number of components of $G - S$ is at most $|S|$ for every cut set $S$ of $G$. Being 1-tough is an obvious necessary condition for being hamiltonian (See [3] for details).

The work of [8] spurred the interest of several research groups. In 2015, Ning and Ge [18] obtained the following result for graphs on $n$ vertices with $\rho(G)$.

**Theorem 1.2** ([18]). Let $G$ be a graph of order $n \geq 14$ with $\delta(G) \geq 2$. If $\rho(G) \geq \rho(K_2 \lor (K_{n-4} + 2K_1))$, then $G$ is hamiltonian unless $G = K_2 \lor (K_{n-4} + 2K_1)$.

Note that $K_2 \lor (K_{n-4} + 2K_1)$ is another example of a graph that is not 1-tough. Excluding a family of four classes of graphs that are not 1-tough, Benediktovich [1] obtained the following extension of the result of Fiedler and Nikiforov.

**Theorem 1.3** ([1]). Let $G$ be a graph of order $n \geq 9$ with $\delta(G) \geq 2$. If $\rho(G) \geq n - 3$, then $G$ is hamiltonian unless $G \in \{K_4 \lor 5K_1, K_3 \lor (K_{1,4} + K_1), K_1 \lor (K_{n-3} + 2K_2), K_2 \lor (K_{n-4} + 2K_1)\}$.

By imposing the minimum degree condition $\delta(G) \geq k$, for general integers $k \geq 1$, Li and Ning [14] established the following result.

**Theorem 1.4** ([14]). Let $k$ be an integer, and let $G$ be a graph of order $n$. If $\delta(G) \geq k + 1$ and $\rho(G) \geq \rho(K_k \lor (K_{n-2k} + kK_1))$, where $n \geq \max\{6k + 5, (k^2 + 6k + 4)/2\}$, then $G$ is hamiltonian unless $G = K_k \lor (K_{n-2k} + kK_1)$.

Recently, Nikiforov [17] extended and strengthened Theorem 1.4 in the following sense.

**Theorem 1.5** ([17]). Let $G$ be a graph of order $n$ with $\delta(G) \geq k$. If $k \geq 2$, $n \geq k^2 + k + 4$ and $\rho(G) \geq n - k - 1$, then $G$ is hamiltonian unless $G = K_k \lor (K_{n-k-1} + K_k)$ or $G = K_k \lor (K_{n-2k} + kK_1)$.

More recently, Ge and Ning [9] showed that the statement in the above theorem due to Nikiforov also holds for $k \geq 1$ and $n \geq \max\{1/2k^2 + k + \frac{3}{2}, 6k + 5\}$.

1.1.1. Our results on the spectral radius

Motivated by the above results, in this paper we continue the study of sufficient conditions for hamiltonicity in terms of the spectral radius. Our main result in this section involves a slightly different type of lower bound on the spectral radius, and it involves the well-known Bondy–Chvátal closure [2]. We postpone all the proofs of our contributions to Section 3.

**Theorem 1.6.** Let $G$ be a graph of order $n \geq 6k^2 + 4k + 2$ with $\delta(G) \geq k \geq 1$. If

$$\rho(G) > \frac{k - 1}{2} + \sqrt{n^2 - (3k + 1)n + \frac{(k+1)^2 - 4}{4}},$$

then $G$ is hamiltonian unless $\text{cl}_n(G) = K_1 \lor (K_{n-k-1} + K_k)$ or $\text{cl}_n(G) = K_k \lor (K_{n-2k} + kK_1)$.

In the above statement, $\text{cl}_n(G)$ denotes the Bondy–Chvátal closure of $G$, obtained from $G$ by recursively joining pairs of non-adjacent vertices by an edge whose degree sum is at least $n$ (in the current graph) until no such pair remains. The main results in [2] show that $\text{cl}_n(G)$ is well-defined and that $\text{cl}_n(G)$ is hamiltonian if and only if $G$ is hamiltonian.

It is easy to check that $n - k - 1 > \frac{k-1}{2} + \sqrt{n^2 - (3k + 1)n + \frac{(k+1)^2 - 4}{2}}$. Hence Theorem 1.6 extends Theorem 1.5, in the sense that the lower bound on $\rho(G)$ is generally better.

We also obtained the following result for graphs on $n \geq 5$ vertices with $\delta(G) \geq 1$. 
Theorem 1.7. Let \( G \) be a graph of order \( n \geq 5 \) with \( \delta(G) \geq 1 \). If
\[
\rho(G) > \sqrt{n^2 - 4n},
\]
then \( G \) is hamiltonian unless \( G \in \{ G_1, G_1', G_2^3, G_2^4, G_3^{17}, G_3^{18}, G_3^{21}, G_3^{22}, G_4^{24}, G_2^{26} \} \).

Here, \( G_1 = K_1 \lor (K_{n-2} + K_1) \), and \( G_1' \) and \( G_2^3 \) are obtained from \( G_1 \) by deleting one edge (as depicted in Fig. 1). The other exceptional graphs in the above theorem belong to a larger set of graphs on 5–7 vertices that are defined in Section 3 and depicted in Figs. 3–5. Note that our result is an improvement of Ge and Ning’s result in [9], i.e., the aforementioned statement of Theorem 1.5 in the case \( k = 1 \).

We now turn to sufficient conditions in terms of the signless Laplacian spectral radius, and we start again with a brief summary of related results.

1.2. Sufficient conditions in terms of the signless Laplacian spectral radius

For the signless Laplacian radius, several sufficient conditions for hamiltonian properties have appeared in [15,20], and [21]. We start with the following results due to Yu and Fan [20], and Liu et al. [15], respectively.

Theorem 1.8 ([20]). Let \( G \) be a graph of order \( n \geq 3 \). If \( q(G) > 2n - 4 \), then \( G \) is hamiltonian unless \( G = K_2 \lor 3K_1 \) or \( G = K_1 \lor (K_{n-2} + K_1) \).

Theorem 1.9 ([15]). Let \( G \) be a graph of order \( n \geq 4 \) with \( \delta(G) \geq 2 \). If \( q(G) \geq 2n - 5 + \frac{3}{n-1} \), then \( G \) is hamiltonian unless \( G = K_2 \lor 4K_1 \) or \( G = K_3 \lor 3K_1 \).

By imposing the minimum degree condition \( \delta(G) \geq k \), for general integers \( k \geq 1 \), Li and Ning [14] established the following counterpart of Theorem 1.4.

Theorem 1.10 ([14]). Let \( k \) be an integer, and let \( G \) be a graph of order \( n \). If \( \delta(G) \geq k \geq 1 \) and \( q(G) \geq q(K_k \lor (K_{n-k} + kK_1)) \), where \( n \geq \max(6k + 5, (3k^2 + 5k + 4)/2) \), then \( G \) is hamiltonian unless \( G = K_k \lor (K_{n-k} + kK_1) \).

The most recent result in this area that we are aware of is the following result due to Li et al. [13].

Theorem 1.11 ([13]). Assume \( k > 1 \) and \( n \geq k^4 + k^3 + 4k^2 + k + 6 \). Let \( G \) be a connected graph with \( n \) vertices and \( \delta(G) \geq k \). If \( q(G) \geq 2(n - k - 1) \), then \( G \) is hamiltonian unless \( G \in \mathcal{M}_1(n, k) \) or \( G \in \mathcal{L}_1(n, k) \).

This result involves the two exceptional classes \( \mathcal{M}_1(n, k) \) and \( \mathcal{L}_1(n, k) \) that we will not define here. We refer the interested reader to [13] for the definitions.

1.2.1. Our results on the signless Laplacian spectral radius

Our main result in this section is the following counterpart of Theorem 1.6.

Theorem 1.12. Let \( G \) be a graph of order \( n \geq 6k^2 + 4k + 3 \) with \( \delta(G) \geq k \geq 1 \). If
\[
q(G) > 2n - 2k - 2 - \frac{2k + 1}{n-1},
\]
then \( G \) is hamiltonian unless \( c_{ln}(G) = K_1 \lor (K_{n-k-1} + K_k) \) or \( c_{ln}(G) = K_k \lor (K_{n-2k} + kK_1) \).

It is obvious that the lower bound condition on \( q(G) \) of Theorem 1.12 is weaker than that of Theorem 1.11. Hence Theorem 1.12 strengthens Theorem 1.11 in that sense. We also obtained the following result.

Theorem 1.13. Let \( G \) be a graph of order \( n \geq 6 \) with \( \delta(G) \geq 1 \). If
\[
q(G) > 2n - 4 - \frac{3}{n-1},
\]
then \( G \) is hamiltonian unless \( G \in \{ G_1, G_1', G_2^3, G_3^{17}, G_3^{21}, G_3^{24}, G_2^{26} \} \).

Here, as in Theorem 1.7, \( G_1 = K_1 \lor (K_{n-2} + K_1) \), and \( G_1' \) and \( G_2^3 \) are obtained from \( G_1 \) by deleting one edge (as depicted in Fig. 1). The other exceptional graphs in the above theorem belong to a larger set of graphs on 5–7 vertices that are defined in Section 3 and depicted in Figs. 2–5. Noting that \( K_{n-2} \) is a proper subgraph of \( G_1 \), using Lemma 2.4 of Section 2, we obtain that \( q(G_1) > q(K_{n-2}) = 2n - 4 \). Therefore our result improves Theorem 1.10 in the case \( k = 1 \).

The rest of the paper is organized as follows. In Section 2, we will give some useful techniques and state some known results that we need in our proofs as lemmas. In Section 3, we present the proofs of our results.
2. Preliminaries

We start this section by introducing a technique that was first introduced in [12], and is known as Kelmans’ transformation. Let \( G \) be a graph and let \( u, v \in V(G) \). We construct a new graph \( G^* \) by replacing all edges \( uv \) by \( xu \) for every \( x \in N(v) \setminus N(u) \). So, Kelmans’ transformation is realized as a sequence of edge rotations (around \( x \) from \( v \) to \( u \)). The relevance of Kelmans’ transformation in the context of radii is expressed in the following two lemmas.

**Lemma 2.1** ([6]). Let \( G \) be a graph, and let \( G^* \) be a graph obtained from \( G \) by some Kelmans’ transformation. Then \( \rho(G) \leq \rho(G^*) \).

**Lemma 2.2** ([14]). Let \( G \) be a graph, and let \( G^* \) be a graph obtained from \( G \) by a Kelmans’ transformation. Then \( q(G) \leq q(G^*) \).

Our next auxiliary result is based on equitable partitions. Suppose \( M \) is a symmetric real \( n \times n \) matrix whose rows and columns are indexed by \( X = \{1, \ldots, n\} \). Let \( \pi = \{X_1, \ldots, X_m\} \) be a partition of \( X \). Let \( M \) be partitioned according to \( \{X_1, \ldots, X_m\} \), i.e.,

\[
M = \begin{pmatrix}
M_{11} & \cdots & M_{1m} \\
\vdots & & \vdots \\
M_{m1} & \cdots & M_{mm}
\end{pmatrix},
\]

where \( M_{ij} \) denotes the block of \( M \) formed by the rows in \( X_i \) and the columns in \( X_j \). Let \( b_{ij} = \frac{1^T M_{ij}}{|X_i|} \), i.e., the average row sum of \( M_{ij} \), where \( 1 \) is the column vector (of the correct dimension) with all entries equal to 1. Then the matrix \( M/\pi = (b_{ij})_{m \times m} \) is called the quotient matrix of \( M \). If the row sum of each block \( M_{ij} \) is a constant, then the partition is called equitable.

If \( M \) denotes the adjacency matrix \( A(G) \) (or signless Laplacian matrix \( Q(G) \)) of a graph \( G \), and \( M \) admits an equitable partition \( \{X_1, \ldots, X_m\} \), then the corresponding partition of \( V(G) \) is called an equitable partition of \( G \). This partition has the property that for every vertex \( v \in X_i \), the number of neighbors of \( v \) in \( X_j \) is the same, for all \( i, j \in \{1, 2, \ldots, m\} \). Any partition with the latter property is clearly also an equitable partition of \( G \).

The next result shows in which sense the use of equitable partitions can facilitate determining the radii of a graph \( G \).

**Lemma 2.3** ([10]). Let \( G \) be a graph, and let \( \pi \) be an equitable partition of \( G \). Then \( \rho(G) = \rho(A(G)) = \rho(A(G)/\pi) \) and \( q(G) = q(Q(G)) = q(Q(G)/\pi) \).

Another useful result involves the radii of subgraphs.

**Lemma 2.4** ([4,10]). Let \( G \) be a connected graph. If \( H \) is a subgraph of \( G \), then \( \rho(H) \leq \rho(G) \) and \( q(H) \leq q(G) \), with strict inequalities in case \( H \) is a proper subgraph of \( G \).

We will also frequently use the following two known inequalities for \( \rho(G) \) and \( q(G) \), respectively.

**Lemma 2.5** ([16]). Let \( G \) be a graph on \( n \) vertices and \( m \) edges with minimum degree \( \delta \). Then \( \rho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}} \).

**Lemma 2.6** ([7,20]). Let \( G \) be a connected graph on \( n \) vertices and \( m \) edges. Then \( q(G) \leq \frac{2m}{n-1} + n - 2 \).

In conjunction with **Lemma 2.5**, we also use the following property.

**Lemma 2.7** ([11,16]). For nonnegative integers \( p \) and \( q \) with \( 2q \leq p(p - 1) \) and \( 0 \leq x \leq p - 1 \), the function \( f(x) = \frac{x^2}{2} + \sqrt{2q - px + \frac{(1+q)^2}{4}} \) is decreasing with respect to \( x \).

We complete this section with two known results on hamiltonicity that date back to the 1970s. The first one is a well-known degree sequence result due to Chvátal [5]. Here, the degrees of the \( n \) vertices of a graph \( G \) are ordered in a non-decreasing way as a degree sequence \( (d_1, d_2, \ldots, d_n) \). We say that a sequence of non-decreasing integers is graphical if there exists a graph with that degree sequence.

**Lemma 2.8** ([5]). Let \( G \) be a graph on \( n \geq 3 \) vertices with degree sequence \( (d_1, d_2, \ldots, d_n) \). If there is no integer \( k < n/2 \) such that \( d_k \leq k \) and \( d_{n-k} \leq n - k - 1 \), then \( G \) is hamiltonian.

The final result is a result due to Bondy and Chvátal [2] that we have mentioned earlier. It deals with the closure \( cl_n(G) \) obtained from a graph \( G \) by recursively adding edges between non-adjacent pairs of vertices with degree sum at least \( n \), until no such pair remains in the resulting graph.

**Lemma 2.9** ([2]). A graph \( G \) is hamiltonian if and only if \( cl_n(G) \) is hamiltonian.
3. The proofs of our results

The basis of the proofs of Theorems 1.6 and 1.12 is the following structural result.

**Theorem 3.1.** Let $G$ be a graph of order $n \geq 6k^2 + 4k + 2$ with $\delta(G) \geq k \geq 1$. If $$e(G) \geq \frac{n^2 - (2k + 1)n}{2},$$ then $G$ is hamiltonian unless $cl_n(G) = K_1 \lor (K_k + K_{n-k-1})$ or $cl_n(G) = K_k \lor (K_{n-2k} + kK_1)$.

**Proof.** Let $H = cl_n(G)$, and let $H$ have degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$ and $d_i = d(v_i)$ for all $i \in \{1, 2, \ldots, n\}$. The main consequence of this assumption is that $d_i + d_j < n$ for every two non-adjacent vertices $v_i$ and $v_j$. Suppose $G$ is not hamiltonian. By Lemma 2.9, $H$ is not hamiltonian. Then, by Lemma 2.8, there is an integer $s < n/2$ such that $d_i \leq s$ and $d_{n-s} \leq n - s - 1$. Then

$$2e(H) = \sum_{i=1}^{n} d_i \leq s \cdot s + (n - 2s)(n - s - 1) + s(n - 1) = n^2 - (2s + 1)n + 3s^2 + s.$$

Thus $n^2 - (2k + 1)n \leq 2e(G) \leq 2e(H) \leq n^2 - (2s + 1)n + 3s^2 + s$, i.e., $(2k - 2s)n + 3s^2 + s \geq 0$. Let $f(x) = (2k - 2s)n + 3s^2 + s$. So $f(s) \geq 0$. The rest of the proof is based on the following three claims that are each followed by short proofs.

**Claim 1.** $s = k$ and $d_1 = d_2 = \cdots = d_k = k$.

**Proof of Claim 1.** Note that $k \leq \delta(G) \leq \delta(H) \leq d_k \leq s$. Now suppose that $s \geq k + 1$. Then, since $f(x)$ is convex in $x$,

$$f(s) \leq \max(f(k + 1), f\left(\frac{n-1}{2}\right))$$

$$= \max(3k^2 + 7k + 4 - 2n, -\frac{n^2}{4} + 2kn + \frac{1}{4})< 0,$$

a contradiction. Thus $s = k$ and $\delta(H) = k$, hence $d_1 = d_2 = \cdots = d_k = k$.

**Claim 2.** $H\{v_{k+1}, \ldots, v_n\} = K_{n-k}$.

**Proof of Claim 2.** Firstly, we show $d_{k+1} \geq n - 3k^2 - 2k - 1$. In fact, if $d_{k+1} < n - 3k^2 - 2k - 1$, then

$$2e(H) = \sum_{i=1}^{k} d_i + d_{k+1} + \sum_{i=k+2}^{n-k} d_i + \sum_{i=n-k+1}^{n} d_i < k \cdot k + n - 3k^2 - 2k - 1 + (n - 2k - 1)(n - k - 1) + k(n - 1) = n^2 - (2k + 1)n,$$

a contradiction. Thus $d_i \geq n - 3k^2 - 2k - 1$ for $k + 1 \leq i \leq n$.

Suppose that $H\{v_{k+1}, \ldots, v_n\} \neq K_{n-k}$. Then there exist two non-adjacent distinct vertices $v_i, v_j$, where $k + 1 \leq i, j \leq n$. Then we have

$$d_i + d_j \geq 2(n - 3k^2 - 2k - 1) \geq n,$$

a contradiction. This completes the proof of Claim 2.

Let $X_1 = \{v_1, \ldots, v_k\}$. Since $d_1 = \cdots = d_k = k$ and $|X_1| = k$, every vertex in $X_1$ must have a neighbor and has at most $k$ neighbors in $\{v_{k+1}, \ldots, v_n\}$. Let $X_2 \subseteq \{v_{k+1}, \ldots, v_n\}$ be all the neighbors of vertices in $X_1$. Let $N_{X_1}(x) = X_2 \cap N_H(x)$ for $x \in V(H)$. Set $|X_2| = \ell$. So we have $1 \leq \ell \leq k$.

**Claim 3.** $H[X_1, X_2] = K_{\ell, \ell}$.

**Proof of Claim 3.** Suppose that $H[X_1, X_2] \neq K_{\ell, \ell}$. Then there exist two distinct vertices $u, v \in X_1$, and a vertex $w \in X_2$ such that $w \in N_{X_1}(u) \setminus N_{X_1}(v)$. 
Since \( d_H(w) \geq n - k \), we have
\[ d_H(v) + d_H(w) \geq k + n - k = n, \]
a contradiction. Hence \( H[X_1, X_2] = K_{k, \ell}. \)

Combining the above three claims, if \( \ell = 1 \), then one easily concludes that \( H = K_1 \lor (K_k + K_{n-k-1}) \), and if \( \ell = k \), that \( H = K_k \lor (K_{n-2k} + kK_1) \). For the case that \( 2 \leq \ell \leq k - 1 \), we use the following lemma that is implicit in the proof of Lemma 2 in [14].

**Lemma 3.2.** Let \( 2 \leq \ell \leq k - 1 \), and let \( F \) be a \((k - \ell)\)-regular graph of order \( k \). Then \( K_\ell \lor (F \lor K_{n-k-1}) \) is hamiltonian.

This completes the proof of Theorem 3.1. □

Now it is a relatively easy exercise to deliver the proof of Theorems 1.6 and 1.12.

**Proof of Theorem 1.6.** By Lemmas 2.5 and 2.7, and the fact that \( \delta(G) \geq k \), we have \( \rho(G) \leq \frac{k-1}{2} + \frac{\sqrt{2e(G) - kn + \frac{(k+1)^2}{4}}}{2} \).

Since \( \rho(G) > \frac{k-1}{2} + \sqrt{n^2 - (3k+1)n + \frac{(k+1)^2}{4}} \), we have \( e(G) \geq \frac{n^2 - (2k+1)n}{2} \). Supposing that \( G \) is not hamiltonian, using Theorem 3.1, we obtain that \( cl(G) = K_1 \lor (K_k + K_{n-k-1}) \) or \( cl(G) = K_k \lor (K_{n-2k} + kK_1) \). □

**Proof of Theorem 1.12.** Suppose \( G \) is not hamiltonian. Using Lemma 2.6, we have
\[
2n - 2k - 2 = 2k + 1 - q(G) \leq \frac{2e(G)}{n-1} + n - 2,
\]
thus \( e(G) \geq \frac{n^2 - (2k+1)n}{2} \). Using Theorem 3.1, we conclude that \( cl(G) = K_1 \lor (K_k + K_{n-k-1}) \) or \( cl(G) = K_k \lor (K_{n-2k} + kK_1) \). □

Before we continue with the proofs of our other results, we first introduce a number of graphs that appear in the formulation of these results and in our next auxiliary result. We will mainly refer to the figures that follow, and not give rigorous definitions of these graphs. As we mentioned before \( G_1 = K_1 \lor (K_{n-2} \lor K_1) \), and \( G_1^1 \) and \( G_1^2 \) are obtained from \( G_1 \) by deleting one edge, as depicted in Fig. 1, in which the ellipses indicate a \( K_{n-1} \) of sufficiently large order. The labels of the vertices will be used to indicate an equitable partition in the later proofs.

The graphs \( G_1^2 - G_2^2 \) are obtained in a similar way from \( G_1 = K_1 \lor (K_{n-2} + K_1) \) by deleting two edges, as indicated in Fig. 2, in which the ellipses again indicate a \( K_{n-1} \) of sufficiently large order, and the labels will be used later.

Next to the infinite classes of graphs that have been sketched in Figs. 1 and 2, we need to define a lot of small graphs on 5–7 vertices. The graphs \( G_1^1 - G_2^1 \) on 7 vertices have been drawn in Fig. 3, in which the numerical labels indicate the degrees of the vertices. Similarly, the graphs \( G_2^1 - G_2^5 \) on 6 vertices have been drawn in Fig. 4, in which the numerical labels again indicate the degrees of the vertices. Finally, the graphs \( G_3^1 - G_3^5 \) on 5 vertices have been drawn in Fig. 5, in which the numerical labels again indicate the degrees of the vertices.

We note here that some of the depicted graphs in Figs. 3–5 are hamiltonian, while others are not, and that some have a complete closure, while others have not. These properties are relatively easy to check because the graphs have a small number of vertices.

One of the key ingredients in the proofs of Theorems 1.7 and 1.13 is the following structural result.

**Theorem 3.3.** Let \( G \) be a graph of order \( n \geq 5 \) with \( \delta(G) \geq 1 \). If
\[
e(G) \geq \left( \frac{n-1}{2} \right) - 1,
\]
then \( G \) is hamiltonian unless \( G \in \{G_1, G_1^1, G_1^2, G_2^1, G_2^2, G_2^3, G_2^4, G_2^5, G_2^6, G_2^7, G_2^8, G_2^9, G_2^{10}, G_3, G_3^1, G_3^2, G_3^3, G_3^4, G_3^5, G_3^6, G_3^7, G_3^8, G_3^9, G_3^{10}, G_4^1, G_4^2, G_4^3, G_4^4, G_5^1, G_5^2, G_5^3, G_5^4, G_5^5, G_6, G_6^1, G_6^2, G_6^3, G_6^4, G_6^5, G_6^6, G_6^7, G_6^8, G_6^9, G_6^{10}, G_7 \}.\)
Proof. Suppose that $G$ is not hamiltonian and that $G$ has degree sequence $(d_1, d_2, \ldots, d_n)$, where $d_1 \leq d_2 \leq \cdots \leq d_n$ and $d_i = d(u_i)$ for $i \in \{1, 2, \ldots, n\}$. By Lemma 2.8, there exists an integer $k < n/2$ such that $d_k \leq k$ and $d_{n-k} \leq n-k-1$. Thus

$$n^2 - 3n \leq 2e(G) = \sum_{i=1}^{n} d_i \leq k \cdot k + (n - 2k)(n - k - 1) + k(n - 1)$$

$$= n^2 - (2k + 1)n + 3k^2 + k.$$

Since $n \geq 2k + 1$, we have $k^2 - 3k - 2 \leq 0$, i.e., $1 \leq k \leq 3$. We distinguish three main cases according to the value of $k$.

Case 1. $k = 1$. Then $d_1 = 1$, $d_{n-1} \leq n - 2$, so $\left(\frac{n-1}{2}\right) - 1 \leq e(G) \leq \left(\frac{n-1}{2}\right) + 1$. If $e(G) = \left(\frac{n-1}{2}\right) + 1$, then $d_1 = 1$, $d_2 = \cdots = d_{n-1} = n - 2$, $d_n = n - 1$, which implies that $G = K_1 \vee (K_{n-2} + K_1) = G_1$. If $e(G) = \left(\frac{n-1}{2}\right)$, then $G$ is a graph
obtained from $G_1$ by deleting one edge. Since $\delta(G) \geq 1$, $G \in \{G_1, G_2\}$ (See Fig. 1). If $e(G) = \binom{n-1}{2} - 1$, then $G$ is a graph obtained from $G_1$ by deleting two edges. Since $\delta(G) \geq 1$, $G \in \{G_1, G_2, G_3, G_4\}$ (See Fig. 2).

**Case 2.** $k = 2$. Then $d_2 \leq 2$, $d_{n-2} \leq n - 3$, and so $n^2 - 3n \leq 2e(G) \leq n^2 - 5n + 14$. Since $n \geq 2k + 1 = 5$, we have $5 \leq n \leq 7$. We distinguish three subcases.

**Subcase 2.1.** $n = 7$. Then $d_5 \leq 4$ and $e(G) = 14$. If $d_1 = 1$, then $d_6 \leq 5$, and there is no graphical sequence satisfying these conditions. Hence $d_1 = d_2 = 2$. Then the degree sequence is $(2, 2, 4, 4, 6, 6)$, which implies $G = K_2 \lor (2K_1 + K_3) = G_3$ (See Fig. 3).

**Subcase 2.2.** $n = 6$. Then $d_4 \leq 3$ and $9 \leq e(G) \leq 10$. If $d_1 = 1$, then $d_5 \leq 4$. In this case, we can get only one graphical sequence $(1, 2, 3, 4, 5, 5)$, which implies $G = G_3^6$ (See Fig. 4 for this graph and the other graphs in this subcase).

If $d_1 = d_2 = 2$, then we have $d_5 + d_6 = 2m - \sum_{i=1}^{4} d_i \geq 18 - 10 = 8$. Also note that $\sum d_i$ is even and $18 \leq \sum d_i \leq 20$. If $d_5 = d_6 = 5$, then the only graphical sequences are $(2, 2, 2, 5, 5)$ and $(2, 2, 3, 3, 5, 5)$. The first sequence implies that $G = K_2 \lor 4K_1 = G_3^{10}$ and the second sequence implies that $G = K_2 \lor (K_2 + 2K_1) = G_3^{11}$.

If $d_5 = 4$ and $d_6 = 5$, then the only graphical sequence is $(2, 2, 2, 3, 4, 5)$, which implies that $G = G_3^{10}$.

If $d_5 = d_6 = 4$, then the only graphical sequence is $(2, 2, 3, 3, 4, 4)$, which implies that $G \in \{G_3^{11}, G_3^{12}, G_3^{13}, G_3^{14}\}$. Note that each of $\{G_3^{12}, G_3^{13}, G_3^{14}\}$ is Hamiltonian, hence $G = G_3^{11}$.
If \( d_5 = 3 \) and \( d_6 = 5 \), then the only graphical sequence is \((2, 2, 3, 3, 5)\), which implies that \( G \in \{G_3^{15}, G_3^{16}\} \). Note that \( G_3^{15} \) is hamiltonian. Hence \( G = G_3^{16} \).

**Subcase 2.3.** \( n = 5 \). Then \( d_3 \leq 2 \) and \( 5 \leq e(G) \leq 7 \). If \( d_1 = d_2 = d_3 = 1 \), then \( d_4 \leq 3 \), and there is no graphical sequence.

If \( d_1 = d_2 = d_3 = 2 \), then \( d_4 \leq 3 \). In this case there are two graphical sequences \((1, 1, 2, 2, 4)\) and \((1, 1, 2, 3, 3)\), which implies \( G = G_3^{17} \) and \( G = G_3^{18} \), respectively (See Fig. 5 for these two graphs and the other graphs in this subcase).

If \( d_1 = 1 \) and \( d_2 = d_3 = 2 \), then there are two graphical sequences \((1, 2, 2, 2)\) and \((1, 2, 2, 3)\). The first sequence implies that \( G \in \{G_3^{19}, G_3^{20}\} \). The second sequence implies that \( G = G_3^{21} \).

If \( d_1 = d_2 = d_3 = 2 \), then \( 4 \leq d_4 + d_5 \leq 8 \). If \( d_4 = d_5 = 2 \), then the graphical sequence is \((2, 2, 2, 2)\), which implies that \( G = G_3^{22} \) is a contradiction.

If \( d_4 = 2 \) and \( d_5 = 4 \), then the graphical sequence is \((2, 2, 2, 2, 4)\), which implies that \( G = K_1 \lor 2K_3 = G_3^{23} \).

If \( d_4 = d_5 = 3 \), then the graphical sequence is \((2, 2, 2, 3, 3)\), which implies that \( G \in \{G_3^{24}, G_3^{25}\} \). Note that \( G_3^{25} \) is hamiltonian. Hence \( G = G_3^{24} \).

If \( d_4 = d_5 = 4 \), then the graphical sequence is \((2, 2, 2, 4, 4)\), which implies that \( G = K_2 \lor 3K_3 = G_3^{26} \). This completes Case 2.

**Case 3.** \( k = 3 \). Then \( d_3 \leq 3 \), \( d_{n-3} \leq n - 4 \) and \( n^2 - 3n \leq 2e(G) \leq n^2 - 7n + 30 \). Since \( n \geq 3k + 1 = 7 \), we obtain that \( n = 7 \). Hence \( d_4 \leq 3 \) and \( 14 \leq e(G) \leq 15 \). If \( d_1 = 1 \), then \( d_5 \leq 5 \) and there is no graphical sequence. If \( d_1 = d_2 = d_3 = d_4 = 2 \), then \( d_5 \leq 5 \), and there is also no graphical sequence.

If \( d_1 = d_2 = d_3 = 2 \) and \( d_4 = 3 \), then \( d_5 \leq 5 \) and there is again no graphical sequence. If \( d_1 = d_2 = 2 \) and \( d_3 = d_4 = 4 \), then \( d_5 \leq 5 \) and there is no graphical sequence either.

If \( d_1 = 2 \) and \( d_2 = d_3 = d_4 = 3 \), then \( d_5 \leq 5 \) and the only graphical sequence is \((2, 3, 3, 3, 5, 6, 6)\), which implies \( G = K_2 \lor (K_1 \lor K_1) = G_3^{27} \).

If \( d_1 = d_2 = d_3 = d_4 = 3 \), then \( d_5 \leq 6 \). Note that \( 28 \leq \sum_{i=1}^{7} d_i \leq 30 \) and \( \sum_{i=1}^{7} d_i \) is an even number. Hence the only graphical sequences are \((3, 3, 3, 3, 4, 6, 6), (3, 3, 3, 3, 5, 5, 6)\) and \((3, 3, 3, 3, 6, 6, 6)\). From the first sequence, we obtain that \( G = G_3^{28} \), which is hamiltonian, a contradiction.

For the second sequence, there are two possibilities: if \( v_5 \) is not adjacent to \( v_6 \), then \( G = K_1 \lor 2K_4 = G_3^{29} \). If \( v_5 \) is adjacent to \( v_6 \), then \( G = G_3^{30} \). Noting that \( G_3^{30} \) is hamiltonian, the second possibility yields a contradiction. From the third sequence, we obtain that \( G = K_3 \lor 4K_1 = G_3^{31} \).

This completes the proof of Theorem 3.3. □

Next, we prove two lemmas in order to deduce lower bounds on the radii of the graphs of Fig. 1 and upper bounds on the radii of the graphs of Fig. 2, respectively.

**Lemma 3.4.** For \( n \geq 3 \) and \( 1 \leq i \leq 2 \),

(i) \( \rho(G_i) \geq \sqrt{n^2 - 4n} \)

(ii) \( q(G_i) \geq 2n - 4 - \frac{3}{n-1} \).

**Proof.** Firstly, we claim that \( \rho(G_i^2) \geq \rho(G_i^1) \) and \( q(G_i^2) \geq q(G_i^1) \). Indeed, let \( u, v, w \) be the three vertices corresponding to the labels in Fig. 1. It is clear that \( G_i^1 = G_i^2 - uv + uw \) (with the obvious meaning that we do not define formally). Using Lemmas 2.1 and 2.2, it is clear that the claim holds. Hence it is sufficient to show only that \( \rho(G_i^1) > \sqrt{n^2 - 4n} \) and \( q(G_i^1) > 2n - 4 - \frac{3}{n-1} \).

Let us consider the following partition \( \pi \) of \( V(G_i^1) \) in \( X_1 = \{y\}, X_2 = \{u\}, X_3 = \{w\} \), and \( X_4 = V(G_i^1) \setminus \{y, u, w\} \). It is easy to check that this partition is equitable, and that the corresponding adjacency matrix and signless Laplacian matrix of its quotient \( G_i^1/\pi \) are as follows:

\[
A(G_i^1/\pi) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & n - 3 \\
0 & 0 & 0 & n - 3 \\
0 & 1 & 1 & n - 4
\end{pmatrix},
\]

\[
Q(G_i^1/\pi) = \begin{pmatrix}
1 & 1 & 0 & 0 \\
1 & n - 2 & 0 & n - 3 \\
0 & 0 & n - 3 & n - 3 \\
0 & 1 & 1 & 2n - 6
\end{pmatrix}.
\]

Let \( f_1(x) = \det(xI_A - A(G_i^1/\pi)) \) and \( g_1(x) = \det(xI_A - Q(G_i^1/\pi)) \) be the characteristic polynomials of \( A(G_i^1/\pi) \) and \( Q(G_i^1/\pi) \), respectively. Then, using the Laplace expansion for the calculation of the determinants, after some standard arithmetic we obtain:

\[
f_1(x) = x^4 - (n - 4)x^3 - (2n - 5)x^2 + (n - 4)x + n - 3;
\]

\[
g_1(x) = x^4 - (4n - 10)x^3 + (5n^2 - 25n + 30)x^2 - (2n^3 - 13n^2 + 21n)x + 2n^3 - 20n^2 + 66n - 72.
\]
This implies $f_2(n - 2 - \frac{2}{n}) = -n - \frac{16}{n} - \frac{4}{n^2} + \frac{32}{n^3} + \frac{16}{n^4} + 7$, and that $f_2(n - 2 - \frac{2}{n}) < 0$ for $n \geq 3$. We also get that $g_1(2n - 4 - \frac{3}{n^2}) = -\frac{11n^2 + 55n^4 - 267n^5 + 3152n^6 + 100n - 5}{16n - 1}$, so that $g_1(2n - 4 - \frac{3}{n^3}) < 0$ for $n \geq 3$. Using Lemma 2.3, we conclude that $\rho(G^1_1) > n - 2 - \frac{2}{n} > \sqrt{n^2 - 4n}$ and $q(G^1_1) > 2n - 4 - \frac{3}{n^2}$, confirming (i) and (ii), respectively. This completes the proof. □

**Lemma 3.5.** For $n \geq 6$ and $1 \leq i \leq 5$,

(i) $\rho(G^2_i) < \sqrt{n^2 - 4n}$ and

(ii) $q(G^2_i) < 2n - 4 - \frac{3}{n-1}$.

**Proof.** Firstly, we claim that $\rho(G^2_i) \leq \rho(G^2_2)$ and $q(G^2_i) \leq q(G^2_2)$ for $1 \leq i \leq 4$. Indeed, let $u$, $v$, $w$ be the three vertices of $G^2_i$ corresponding to the labels in Fig. 2 for $1 \leq i \leq 4$. For $1 \leq i \leq 4$, we have $N_{G^2_i}(v) \cap N_{G^2_i}[u] = \{w\}$ and $N_{G^2_i}(u) \cap N_{G^2_i}[v] \neq \emptyset$.

It is easy to check that

- $G^2_1 = G^1_1 - vw + uw$,
- $G^2_2 = G^2_i - vw + uw$,
- $G^2_3 = G^1_1 - vw + uw$,
- $G^2_4 = G^2_i - vw + uw$.

Using Lemmas 2.1 and 2.2, it is clear that the claim holds. Hence it is sufficient to show only that $\rho(G^2_2) < \sqrt{n^2 - 4n}$ and $q(G^2_2) < 2n - 4 - \frac{3}{n-1}$.

Let us consider the following partition $\pi$ of $V(G^2_i)$ in $X_1 = \{x_1\}$, $X_2 = \{x_2\}$, $X_3 = \{x_3, x_4\}$, $X_4 = \{x_5\}$, and $X_5 = V(G^2_i) \setminus \{x_1, x_2, x_3, x_4, x_5\}$. It is easy to check that this partition is equitable, and that the corresponding adjacency matrix and signless Laplacian matrix of its quotient $G^2_i/\pi$ are as follows:

$$A(G^2_i/\pi) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 1 & n - 5 \\ 0 & 1 & 1 & 0 & n - 5 \\ 0 & 1 & 0 & 0 & n - 5 \\ 0 & 1 & 2 & 1 & n - 6 \end{pmatrix}.$$  

$$Q(G^2_i/\pi) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & n - 1 & 2 & 1 & n - 5 \\ 0 & 1 & n - 2 & 0 & n - 5 \\ 0 & 1 & 0 & 0 & n - 5 \\ 0 & 1 & 2 & 1 & 2n - 8 \end{pmatrix}.$$  

Let $f_2(x) = \det(xI - A(G^2_i/\pi))$ and $g_2(x) = \det(xI - Q(G^2_i/\pi))$ be the characteristic polynomials of $A(G^2_i/\pi)$ and $Q(G^2_i/\pi)$, respectively. Then

$$f_2(x) = x^5 - (n - 5)x^4 - (3n - 10)x^3 - 2x^2 + (3n - 13)x - n + 5$$

and

$$g_2(x) = x^5 - (5n - 14)x^4 + (9n^2 - 51n + 71)x^3 - (7n^3 - 58n^2 + 150n - 110)x^2$$

$$+ (2n^4 - 19n^3 + 51n^2 - 108n)x - 2n^4 + 28n^3 - 146n^2 + 336n - 288.$$  

To complete the proof of (i), first of all we obtain the following derivatives of $f_2(x)$ with respect to $x$, by standard analysis:

$$f^{(1)}_2(x) = 5x^4 - 4(n - 5)x^3 - 3(3n - 10)x^2 - 4x + 3n - 13,$$

$$f^{(2)}_2(x) = 20x^3 - 12(n - 5)x^2 - 6(3n - 10)x - 4,$$

$$f^{(3)}_2(x) = 60x^2 - 24(n - 5)x - 6(3n - 10),$$

$$f^{(4)}_2(x) = 120x - 24(n - 5),$$

$$f^{(5)}_2(x) = 120.$$  

By substitution and simple but tedious calculations, for $n \geq 6$, we obtain

$$f_2(n - 2 - \frac{7}{2n}) = \frac{1}{2}n^3 - \frac{1}{2}n^2 + 22n - \frac{141}{4} - \frac{343}{4n} + \frac{2989}{8n^2} + \frac{343}{2n^3} - \frac{12005}{16n^4} - \frac{16807}{32n^5}$$

$$> \frac{1}{2}n^3 - \frac{1}{2}n^2 + 22n - 41.$$
\[ f_2'(n - 2 - \frac{7}{2n}) = n^4 - 5n^3 - 10n^2 + 60n - \frac{1589}{4n} - \frac{637}{2n^2} + \frac{1715}{2n^3} + \frac{12005}{16n^4} + \frac{175}{2} > n^4 - 5n^3 - 10n^2 + 60n + 16 = (n^2 - 4)(n - 6)(n + 1) + 40n - 8 > 0, \]
\[ f_2''(n - 2 - \frac{7}{2n}) = 8n^3 - 30n^2 - 78n + \frac{378}{n} - \frac{735}{n^2} - \frac{1715}{2n^3} + 271 > 8n^3 - 30n^2 - 78n + 169 = 2n(2n - 12)(2n + 4) + 2n^2 + 18n + 169 > 0, \]
\[ f_2'''(n - 2 - \frac{7}{2n}) = 36n^2 - 90n + \frac{420}{n} + \frac{735}{n^2} - 276 > 36n^2 - 90n - 276 = 6(2n + 2)(3n - 18) + 6(15n - 10) > 0, \]
\[ f_2''''(n - 2 - \frac{7}{2n}) = 96n - \frac{420}{n} - 120 > 0, \]
\[ f_2'''''(n - 2 - \frac{7}{2n}) = 120. \]

Hence, by the Fourier–Budan Theorem (See, e.g., [19]), there is no root of \( f_2(x) \) in the interval \([n - 2 - \frac{7}{2n}, +\infty)\). Thus, using Lemma 2.3, we obtain that \( \rho(G_2^2) < n - 2 - \frac{7}{2n} < \sqrt{n^2 - 4n} \).

To prove (ii), for the derivatives of \( g_2(x) \) with respect to \( x \), we obtain:

\[ g_2(x) = 5x^4 - 4(5n - 14)x^3 + 3(9n^2 - 51n + 71)x^2 - 2(7n^3 - 58n^2 + 150n - 110)x \]
\[ + (2n^4 - 19n^3 + 51n^2 - 108), \]
\[ g_2'(x) = 20x^3 - 12(5n - 14)x^2 + 6(9n^2 - 51n + 71)x - 2(7n^3 - 58n^2 + 150n - 110), \]
\[ g_2''(x) = 60x^2 - 24(5n - 14)x + 6(9n^2 - 51n + 71), \]
\[ g_2'''(x) = 120x - 24(5n - 14), \]
\[ g_2''''(x) = 120. \]

By substitution, for \( n \geq 6 \), we obtain

\[ g_2(2n - 4 - \frac{3}{n - 1}) = \frac{1}{(n - 1)^3}(8n^8 - 111n^7 + 572n^6 - 1327n^5 + 1116n^4 + 587n^3 \]
\[ - 1174n^2 - 42n + 128) \]
\[ > \frac{1}{n^2}(8n^8 - 111n^7 + 572n^6 - 1327n^5 + 1116n^4 + 587n^3 \]
\[ - 1174n^2 - 42n + 128) \]
\[ = 8n^3 - 111n^2 + 572n - 1327 + \frac{1116}{n} + \frac{587}{n^2} - \frac{1174}{n^3} - \frac{42}{n^4} + \frac{128}{n^5} > 0, \]
\[ g_2'(2n - 4 - \frac{3}{n - 1}) = \frac{1}{(n - 1)^4}(2n^8 - 15n^7 + 29n^6 - 48n^5 + 425n^4 - 1252n^3 \]
\[ + 889n^2 + 541n - 166) \]
\[ > \frac{1}{n^3}(2n^8 - 15n^7 + 29n^6 - 48n^5 + 425n^4 - 1252n^3 \]
\[ + 889n^2 + 541n - 166) \]
\[ = 2n^4 - 15n^3 + 29n^2 - 48n + 425 - \frac{1252}{n} + \frac{889}{n^2} + \frac{541}{n^3} - \frac{166}{n^4} > 2n^4 - 15n^3 + 29n^2 - 48n + 243 \]
\[ = n^2(n - 2)(2n - 11) + (7n - 6)(n - 6) + 207 > 0, \]
\[ g_2''(2n - 4 - \frac{3}{n - 1}) = \frac{2(7n^6 - 41n^5 + 24n^4 + 156n^3 - 10n^2 - 435n + 29)}{(n - 1)^3} \]
we obtain that

\[
q = \frac{1}{\sqrt{\rho(G)}}.
\]

The following table with the values of \( \rho \) remaining 18 exceptional graphs on 5–7 vertices in the statement of Theorem 3.3, we used a computer to produce the

**Proof of Theorem 1.7.**

Theorem 1.13, respectively.

Combining all the above ingredients, we complete this paper by presenting our short proofs of Theorem 1.7 and Theorem 1.13, respectively.

\[
\sqrt{n^2 - 4n} < \rho(G) \leq \sqrt{2e(G)} - n + 1,
\]

hence \( e(G) \geq \binom{n-1}{2} - 1 \). Using Theorem 3.3, this implies \( G \in \{G_1, G_1^1, G_2^1, G_2^2, G_2^3, G_2^4, G_3^1, G_3^2, G_3^3, G_3^4, G_3^5, G_3^6, G_3^7, G_3^8, G_3^9, G_3^{10}, G_3^{11}, G_3^{12}, G_3^{13}, G_3^{14}, G_3^{15}, G_3^{16}, G_3^{17}, G_3^{18}, G_3^{19}, G_3^{20}, G_3^{21}, G_3^{22}, G_3^{23}, G_3^{24}, G_3^{25}\} \).

Since \( G_1 \) is a proper subgraph of \( G_i (i = 1, 2) \), by Lemma 2.4, we have \( \rho(G_1) > \rho(G_i) (i = 1, 2) \). Then using Lemmas 3.4(i), 3.5(i), and the numerical results in Table 1, we conclude that \( G \in \{G_1, G_1^1, G_2^1, G_2^2, G_3^1, G_3^2, G_3^3, G_3^4, G_3^5, G_3^6, G_3^7, G_3^8, G_3^9, G_3^{10}, G_3^{11}, G_3^{12}, G_3^{13}, G_3^{14}, G_3^{15}, G_3^{16}, G_3^{17}, G_3^{18}, G_3^{19}, G_3^{20}, G_3^{21}, G_3^{22}, G_3^{23}, G_3^{24}, G_3^{25}\} \).

**Proof of Theorem 1.13.** Suppose that \( G \) is not hamiltonian. Using Lemma 2.6, we obtain

\[
2n - 4 - \frac{3}{n - 1} < q(G) \leq \frac{2e(G)}{n - 1} + n - 2.
\]
hence $e(G) \geq \binom{n-1}{2} - 1$. Using Theorem 3.3, this implies $G \in \{G_1, G_1', G_2, G_2', G_3, G_3', G_4, G_5, G_5', G_6, G_6', G_7, G_7', G_8, G_8', G_9, G_9', G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}, G_{16}, G_{17}, G_{18}, G_{19}, G_{20}, G_{21}, G_{22}, G_{23}, G_{24}, G_{25}\}$.

Since $G_1'$ is a proper subgraph of $G_1 (i = 1, 2)$, by Lemma 2.4, we have $q(G_1) > q(G_1') (i = 1, 2)$. By Lemma 3.4(ii), it is possible that $G \in \{G_1, G_1', G_2, G_2', G_3, G_3', G_4, G_5, G_5', G_6, G_6', G_7, G_7', G_8, G_8', G_9, G_9', G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}, G_{16}, G_{17}, G_{18}, G_{19}, G_{20}, G_{21}, G_{22}, G_{23}, G_{24}, G_{25}\}$.

Using Table 1, we conclude that $G \in \{G_2, G_2', G_3, G_3', G_5, G_5', G_7, G_7', G_9, G_9', G_{10}, G_{11}, G_{12}, G_{13}, G_{14}, G_{15}, G_{16}, G_{17}, G_{18}, G_{19}, G_{20}, G_{21}, G_{22}, G_{23}, G_{24}, G_{25}\}$. \hfill \Box

CRediT authorship contribution statement

All authors contributed to the concepts, discussions, proofs and write up. Qiannan Zhou has put in most efforts in the proofs as a PhD student.

Acknowledgments

We thank the two anonymous referees for pointing out some mistakes and for their suggestions on an earlier version that improved the presentation.

References