Combined Risk Measures:
Representation Results and Applications

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Chapter 1

Introduction

In certain financial markets, it is possible to price and hedge a contingent claim by a trading strategy which perfectly replicates the payoff of the claim. In that case, almost all the risk is reduced by trading. Many problems in the financial industry, however, are characterized by the fact that an exposure to risk can not be offset completely by an appropriate trading strategy. Moreover, the regulators for the financial industry often require financial institutions to deposit a collateral to cover some or all of their risk exposure. This set-up can be modeled as an optimization problem where pricing and hedging involves a trade-off between trader and regulator. If the objective functions of the trader and the regulator in this optimization problem are chosen to be convex risk measures then combined risk measures have to be analyzed in order to solve the problem. This motivates our research into characterization of linear combinations and convolutions of convex risk measures.

1.1 Background

In this section we give a discussion of recent developments in the field of risk measure theory and pricing in complete and incomplete markets to provide some background for the thesis. For a extensive introduction to the theory of coherent and convex risk measures we refer to Föllmer and Schied [32]. An overview of various approaches to pricing and hedging in incomplete markets is given in Cont and Tankov [20].

1.1.1 Risk Measures

Risk measures play an important role in the description of decision making under uncertainty. Generally speaking, in finance a risk measure attempts to assign a single

numerical value to future random outcomes. We review the recent developments in the theory of measuring risk.

In practice, *Value at Risk* at level λ ($V@R_\lambda$) is the most widely used risk measure in financial institutions. Value at Risk allows for a very simple interpretation and can be easily implemented in practice. In financial terms, $V@R_\lambda$ is the smallest amount of capital which, if added to a position, keeps the probability of a negative outcome below the level λ . Mathematically, $V@R_\lambda$ is the upper λ -quantile of the distribution of the position with negative sign. Value at Risk, however, fails to satisfy some natural consistency requirements. It has two serious deficiencies. First, it is ineffective in recognizing the dangers of concentrated risk or 'tail risk'. Secondly, it fails to measure diversification effects properly. Value at Risk has been seriously criticized in the academic literature as a risk measurement and management tool since the middle of the 1990s (Acerbi and Tasche [2], Artzner et al. [7]) and by governmental authorities (Turner Review [73] and Committee on Banking Supervision [19]).

It is an advantage if a risk measure of a financial position can be interpreted in monetary terms, i.e. as a minimal amount of money, which if added makes a position acceptable. This property, which is called translation invariance (Artzner et al. [6], [7]) introduces an axiomatic approach to risk measures. The set of economically desirable properties consists of monotonicity, translation invariance, positive homogeneity and subadditivity. A risk measure having these four properties is called a *coherent risk measure*. In Artzner et al. [6], [7] representation results are deduced on a finite probability space. Later, Delbaen [23] extended the theory to arbitrary probability spaces. Föllmer and Schied [31] and Frittelli and Rosazza Gianin [33] relaxed the axioms of coherent risk measures and replaced positive homogeneity and subadditivity by the weaker condition convexity. The corresponding risk measures are called *convex risk measure*. The risk measure *Average Value at Risk at level λ* ($AV@R_\lambda$) is a better alternative to $V@R_\lambda$, since it satisfies all the properties of a coherent risk measure and it has the potential to replace $V@R_\lambda$ as a standard risk measure in the near future. $AV@R_\lambda$ is defined as the average of the Value at Risks with level γ , for all γ smaller than λ . Sometimes Average Value at Risk is also called Conditional Value at Risk or Expected Shortfall. The concept of a convex risk measure led to a rich theory and became a basis for various generalizations. For example Filipović and Svindland [28], Svindland [71] and Kaina and Rüschendorf [44] discussed convex risk measures on L^p -spaces. Acerbi [1] introduced the concept of *spectral risk measures*. A spectral risk measure is defined as a weighted average of Value at Risks, giving larger weights to Value at Risks with smaller levels. Thus larger losses, which are deeper in the tail of the distribution, are multiplied by a larger weight. We consider a special class of spectral risk measures in the optimization problem given in Chapter 5. Clearly, $V@R_\lambda$, $AV@R_\lambda$ and spectral risk measure only involve the distribution of a position under a given probability measure. The class of risk measures which only depend on the distribution is called *law-invariant risk measures*. As shown by Kusuoka [49] in the coherent case and by Kunze [48], Dana [21] and Frittelli and Rosazza Gianin [34] in the general convex case, any law-invariant convex risk measure

can be constructed by using Average Value at Risk as building blocks. In many situations, the risk of combined positions will be strictly lower than the sum of the individual risks. If, on the other hand, there are two positions which are comonotone, (i.e they have perfect positive dependence between the components) then the risk should just add up. These risk measures are called *comonotonic risk measures*. All comonotone and law-invariant risk measures are precisely the class of risk measures which can be represented as a spectral risk measure. This result was proven by Kusuoka [49] for L^∞ and by Shapiro [70] for L^p .

There are further useful extension of convex risk measures. For instance, Cheridito and Li [16], [17] considered convex risk measures on Orlicz spaces. El Karoui and Ravanelli [27] study cash sub-additive risk measures, which satisfy a weaker condition than translation invariance. Cerreia-Vioglio et al. [15] and Frittelli and Rosazza Gianin [35] consider quasi-convex risk measures, a generalization of convexity, and derived its dual representation.

In the thesis we will characterize linear combinations and convolutions of convex risk measures on L^p -spaces with $1 < p < +\infty$. So far, only the *inf-convolution of risk measures* has been studied. Delbaen [22] considered the inf-convolution of coherent risk measures in the framework of L^∞ . Barrieu and El Karoui [8], [9] extended these results to convex risk measures. Toussaint and Sircar [72] analyzed the inf-concolution on L^2 and Arai [4] derived the inf-convolution of convex risk measures on Orlicz spaces.

1.1.2 Pricing in complete and incomplete markets

The theory on the pricing and hedging of contingent claims forms an integral part of modern finance. The foundation was laid by Black and Scholes [11] and Merton [55] in the early 1970s. They developed a model, which is now known as the *Black-Scholes Model* and derived an analytical formula for the price of an European option. The analysis is based on two keys assumptions, the principles of *no-arbitrage* and *complete markets*.

An arbitrage opportunity is an investment strategy with zero initial investment that yields with strictly positive probability a strictly positive profit without any downside risk. It is thus essentially a riskless money making machine on the financial market. An example of an arbitrage opportunity is if two traders quote different prices for the same financial product. Then buying the product from the trader which quotes the lower price and selling it to the other trader would produce a sure positive profit. All investors who see this would take advantage of this riskless profit and start trading this strategy until the price has moved back to its equilibrium value. We will therefore assume the absence of these possibilities in the thesis.

The basic idea of valuing an option is to construct a hedging portfolio, which in general consists of a bank account and a position in underlying assets, which are continuously rebalanced in such a way that at any time the option is worth exactly as much as the hedging portfolio. If such a strategy exists, then the market is *complete*. One may show that in complete markets, the price of a contingent claim is given by its expectation with respect

to a unique equivalent martingale measure. In such markets, options are redundant, since it is always possible to find a replicating strategy. The Black-Scholes model is an example of this, but its assumptions do not hold in the real world, a fact which is acknowledged by both practitioners and academics.

In more realistic models it is not always possible to find a replicating portfolio. The market is then *incomplete*, meaning there are risks in the market which can not be hedged away. These risks can emerge if jumps of the underlying process are incorporated in the model or if there are more random sources than there are traded assets. In an incomplete market there is a set of different equivalent martingale measures. Thus, pricing a claim bears an intrinsic risk that cannot be hedged away completely. Therefore we are faced with an optimization problem to choose a suitable hedging strategy which minimizes the residual risk as much as possible.

If one cannot replicate a contingent claim, a conservative approach is to look for a replicating portfolio which is in any case larger than the payoff of the contingent claim, a so-called *superhedging strategy*, see Gushchin and Mordecki [36] and Kramkov [47]. Unfortunately, a superhedging strategy may lead to prices which are too high for practical usage. As shown by Eberlein and Jacod [25] in the example of a call option the superhedging strategy is to buy and hold the stock, which is excessively expensive. Since hedging in incomplete markets does not offset all risks, one rather has to reduce the risk under a certain objective function. Different functions lead to different prices and different hedging strategies. We will briefly present the most common approaches.

Quadratic hedging means we find an optimal hedging strategy by minimizing the difference between the terminal value of the hedging portfolio and the payoff of the claim with respect to the L^2 -norm. An overview of the quadratic hedging approach can be found in Schweizer [69]. The drawback of this method is that it is symmetric, meaning losses and gains are contributing to the error in the same way.

An optimal hedging strategy can also be defined by maximizing the *expected utility*. A utility function is a concave and increasing function representing the weights of different outcomes, where the concavity represents the risk aversion of the trader. Utility functions are well known in mathematical economics and date back to the work of von Neumann and Morgenstern [75]. Later Markowitz [52] and Samuelson [68] used utility functions to find an optimal strategy for the consumption portfolio optimization problem. It can be used for pricing claims by *utility indifference pricing*, first proposed by Hodges and Neuberger [42]. The selling price of a claim is given by the amount of money which makes the trader indifferent between (1) selling the claim and receiving the money and then optimizing her utility and (2) maximizing her utility without the claim and the extra money. The pitfall of this method is that it requires the trader to know her utility function, which is quite difficult in practice.

Minina [56] and Minina and Vellekoop [57] studied a model where the cost of risk is incorporated. In this *capital reserve model* the trader maximizes her profit, but a limitation is imposed on the trader by a risk function that depends on the market state and the portfolio. According to the value of this function, the trader is required to set aside some

money as a reserve. The higher the risk is, the more money the trader has to set aside. The prices can then be determined by indifference pricing.

A useful alternative is *risk indifference pricing*, see Xu [76] and Øksendal and Sulem [60], where the criterion of maximizing utility is replaced by minimizing the risk exposure measured by a convex risk measure. The main advantage of this method is the axiomatic set up of convex risk measures which enables one to solve an optimization problem without explicitly choosing a specific risk measure.

In Chapter 4 we combine the approaches of the capital reserve model and risk indifference pricing to price and hedge contingent claims in an incomplete market as a trade-off between trader and regulator.

If the initial capital is given, utility maximization can be replaced by risk minimization. This leads to the problem of partial hedging. The problem has been studied using different risk measures. Föllmer and Leukert [29] used a *quantile hedging* approach to determine a hedging strategy which minimizes the probability of the losses. In this setting very large losses could occur, although they occur with small probabilities. Therefore, Föllmer and Leukert [30] generalized their approach by studying the expected shortfall of the losses. Nakano [58] uses a coherent risk measure to quantify the losses due to shortfall. Rudloff [65], [66], [67] further improves the result of Nakano and generalizes the results by introducing a convex risk measure as an objective function.

In Chapter 5 we derive an optimal hedging strategy for a claim such that risk of the difference of the hedging portfolio and the claim is minimized.

1.2 Outline

The rest of the thesis consists of two parts each having two chapters. In the first part we provide theoretical results to the field of risk measures. Then two applications are given in the second part, which are quite independent from each other.

In Chapter 2 we review the concept of risk measures on L^p -spaces with $1 < p < +\infty$. Various aspects of convex risk measures have appeared before in the literature. Risk measures have been defined in different ways and studied on different spaces. The main focus of the thesis is on combined risk measure. Keeping this in mind, we provide a structural basis by stating and proving the different characterization results and adjusting them to our definitions and notations.

The aim of Chapter 3 is to characterize linear combinations and convolutions of convex risk measures. The inf-convolution of convex risk measures was introduced in Delbaen [22] and Barrieu and El Karoui [8], [9] in the L^∞ framework and extended by many other authors. We study other combinations and convolutions of convex risk measures. Because of our heavy reliance on convex analysis, in particular on the duality correspondence, we dedicate Section 3.1 to this field. In this section we perform operations such as adding, subtraction, inf-convolution and deconvolution for given functions and show that these operations arrange themselves in dual pairs. Furthermore, we investigate the epi-multiplication of a function and a scalar. As we will see, multiplication of scalars and

epi-multiplication, addition and inf-convolution and subtraction and deconvolution will be these pairs. This enables us to use the elegant dual theory for combinations of convex risk measures. These results are well known in convex analysis for finite dimensional spaces. See for example Rockafellar [62] for the epi-multiplication, inf-convolution and sum and Hiriart-Urruty [40] for the deconvolution and the difference. The results can be carried out in more general settings, for example Van Tiel [74] treats the inf-convolution on normed linear spaces. We adjust these dual operation to our setting and prove them on reflexive Banach spaces. In Section 3.2 we derive basic properties of combinations of a convex risk measure and a convex set, and between two convex risk measures. We start with the epi-multiplication, review the results on the inf-convolution, and additionally derive the dual representation results of the sum, the deconvolution and the difference. Some examples, including the combination and convolution of Average Value at Risk, entropic risk measure and spectral risk measure are given in Section 3.3.

In Chapter 4 an application of the theory and results deduced in Chapter 3 is given. We study the pricing and hedging problem for contingent claims in an incomplete market as a trade-off between a trader and a regulator. In our model the regulator allows the trader to take some risk, but insists that the residual risk, which is not hedged away, has to be covered. To achieve this, the regulator introduces an extra bank account which serves as a capital reserve to cover for eventual losses of the trader and is dependent on the risk of the trader's portfolio. The risk attitudes of the trader and the regulator are reflected by different risk measures. This differs from the existing results of Minina and Vellekoop [57], where the price was determined by the portfolio's Greeks. We employ two pricing methods: risk measure pricing in Section 4.2 and risk indifference pricing in Section 4.3.

In Chapter 5 the problem of partial hedging of a contingent claim is considered. Under the assumption of a complete market, it is always possible to replicate the claim. In this case, the claim can be priced using the unique equivalent martingale measure. The question is of a different nature when the initial capital is less than the expectation under the equivalent martingale measure. The aim of this chapter is to find a suitable hedging strategy such that the risk of the difference of the hedging portfolio and the claim is minimized under a simple spectral risk measure, which is a special class of spectral risk measures where the spectrum is given as a step function. Minimizing the risk of the difference of the hedging portfolio and the claim is a more natural alternative to minimizing the risk of losses due to shortfall, which is often considered in the literature, see Föllmer and Leukert [30], Nakano [58] and Rudloff [65], [66], [67]. In Section 5.2 we solve the problem for the case when the risk measure is given by Average Value at Risk. The results are illustrated by solving the problem for a call and a put option in the Black-Scholes model. In Section 5.3 we extend the results to simple spectral risk measures and derive a solution for the call option in the Black-Scholes model.

In Chapter 6 the main conclusions are drawn and we present recommendations of possible directions for future research.

The main contributions are the following:

- *Representation results of combined risk measures.*

We characterize linear combinations and convolutions of convex risk measures in terms of their penalty functions using the duality correspondence and investigate the basic properties. These results are the main contribution of Chapter 3. We consider four cases. In Theorem 3.2.1 we prove that the epi-multiplication of a risk measure is again a convex risk measure. The sum of two risk measures is considered in Theorem 3.2.4. We adopt the notion of deconvolution and introduce it as an operation in risk analysis. We consider two different types of deconvolutions. First, the deconvolution of two risk measures and second the deconvolution of a risk measure and a set. In Theorem 3.2.5 and Theorem 3.2.6 we derive the dual representation of these deconvolutions. Furthermore, we characterize the risk measure defined by the difference of two convex risk measures in Theorem 3.2.8.

- *New results for pricing and hedging in incomplete markets.*

We introduce an extra bank account which serves as a capital reserve in Chapter 4. This leads to the capital reserve model. We employ two pricing methods, risk measure pricing in Section 4.2 and risk indifference pricing in Section 4.3, to price a financial claim with a fixed maturity in this new model. We assume that the regulator and the trader have different risk measures reflecting their different attitude towards risk. The resulting pricing operator in both pricing methods is given by a weighted sum of the regulator's and trader's risk measures, see Theorem 4.2.5 and Theorem 4.3.5.

- *New approach for partial hedging problems.*

We rewrite Average Value at Risk in terms of expected shortfall using the Fenchel-Legendre transform in Chapter 5. This approach allows us to find a hedging strategy that minimizes the risk of the difference between the hedging portfolio and a claim, where the risk is given by a simple spectral risk measure. The problem can be solved stepwise. First, this dynamic optimization problem can be reduced to an n -dimensional optimization problem by exploiting the Neyman-Pearson lemma. This n -dimensional problem is then analyzed. In case the risk measure is given by Average Value at Risk, we provide an explicit solution in Theorem 5.2.10. One of the key findings is that the optimal solution might partly exceed the value of the claim, see Proposition 5.2.11. We illustrate our results by solving the problem for vanilla options in the Black-Scholes model.

Part I

Representation of Convex Risk Measures

Convex Risk Measures on L^p

In this chapter we introduce the concept of risk measures. We present the dual representation of a convex risk measure and review the relation between such measures and their acceptance sets. Further, we give examples of risk measures, as we introduce Average Value at Risk ($AV@R$), entropic and spectral risk measures.

We provide a definition of a convex risk measure on L^p with $1 < p < +\infty$ and to ensure that the dual representation of a convex risk measure ρ exists, we assume that ρ is lower semi-continuous and proper. Therefore we include lower semi-continuity and finiteness at 0, that is $\rho(0) < +\infty$, as properties of a convex risk measure. This differs from other publications on this topic. To ensure properness Frittelli and Rosazza Gianin [33] consider finite valued risk measures, Filipović and Svindland [28] consider $\rho(0) < +\infty$ and Rudloff [65] consider $\rho(0) = 0$. Kaina and Rüschendorf [44] do not assume properness in their definition of a convex risk measure. In none of the aforementioned publications lower semi-continuity is assumed to be a property of a convex risk measure.

Although various definitions of convex risk measures on L^p have appeared before, our definition of a convex risk measure seems to be new. Therefore we will state and prove the different characterization results and adjust them to our definitions and notations.

The chapter is structured as follows. In Section 2.1 we review some basic results from convex analysis on reflexive Banach spaces. These results will be used to characterize convex risk measure in the following sections and to represent linear combinations and convolutions of convex risk measures in Chapter 3. A broad introduction to convex analysis on Banach Spaces can be found in the books of Boţ et al. [13], Ekeland and Témam [26], Luenberger [51] or Van Tiel [74]. For a more general overview, we refer to the book of Dunford and Schwartz [24]. In Section 2.2, we characterize convex and coherent risk measures on L^p -spaces with $1 < p < +\infty$ and discuss several important properties of these risk measures. Using the tools of convex analysis we link the proper-

ties of risk measures to the corresponding properties in the dual space and derive the dual representation of convex risk measures. Furthermore, we give some examples of convex and coherent risk measures. In Section 2.3 we state some results on the continuity and differentiability of convex risk measures. These results are needed to characterize the difference of two convex risk measures in Chapter 3. In Section 2.4 we discuss the relation between convex risk measures and their acceptance sets. These results are well known and mostly similar to the case of L^∞ which can be found in Föllmer and Schied [32], Section 4.1. They have been generalized in many publications. Especially noteworthy is Hamel [38], a work on which we base the section, although we do not treat this topic in the same generality. The last section, Section 2.5, focuses on a special class of coherent risk measures, spectral risk measures, which only involve the distribution of a position, so they are law-invariant. Since spectral risk measures can be characterized by their spectrum it is easy to add and subtract these risk measures. We introduce a subclass of spectral risk measures called simple spectral risk measures. For this class of risk measures the spectrum is given by a step function. For the optimization problem stated in Chapter 5 the objective function is given by a such simple spectral risk measure.

2.1 Preliminaries of Convex Analysis

Let V be a reflexive Banach space with topological dual V^* . We designate by V and V^* two dual vector spaces with bilinear pairing denoted by $\langle \cdot, \cdot \rangle$. Consider mappings of V into $\mathbb{R} \cup \{+\infty\}$, meaning the value $+\infty$ is allowed to the function with the convention $(+\infty) - (+\infty) = +\infty$. Additionally, we define by $\div$ the lower extension of subtraction, that is, $(+\infty) \div (+\infty) = -\infty$. This notation is needed to define the deconvolution given in Chapter 3. We continue with some general definitions of convex analysis.

A function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *convex* if for every $X, Y \in V$ we have

$$f(\gamma X + (1 - \gamma)Y) \leq \gamma f(X) + (1 - \gamma)f(Y) \quad \text{for all } \gamma \in [0, 1].$$

For every function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$, we call the section

$$\text{dom}(f) := \{X; f(X) < +\infty\}$$

the *effective domain* of f . A function f is called *proper* if $\text{dom}(f) \neq \emptyset$. By $\text{int}(\text{dom}(f))$ we denote the interior of the domain of f . A function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be *lower semi-continuous* on V if it satisfies the following condition

$$\liminf_{X \rightarrow X_0} f(X) \geq f(X_0) \quad \text{for all } X_0 \in V.$$

Given a function f , there exists a greatest lower semi-continuous function (not necessarily finite) majorized by f . This function is called *lower semi-continuous hull*. The *closure* $\text{cl}(f)$ of f is defined to be the lower semi-continuous hull of f if f nowhere has the value $-\infty$, and in the other case it is defined to be constant and equal to $-\infty$. f is said to be

closed if $\text{cl}(f) = f$. The *convex hull* $\text{co}(f)$ of a function f is the largest convex minorant of f .

The *epigraph* of a function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is the set

$$\text{epi}(f) := \{(X, a) \in V \times \mathbb{R}; f(X) \leq a\}. \quad (2.1)$$

If we replace " $\leq$ " by " $<$ " in (2.1), then the set is called *strict epigraph* and is denoted by $\text{epi}_s(f)$.

An epigraph is the set of points of $V \times \mathbb{R}$ which lie above the graph of f . The epigraph is a useful concept in the study of convex function due to the one-to-one correspondence of f being lower semi-continuous and $\text{epi}(f)$ being closed. Additionally, f is convex if and only if $\text{epi}(f)$. This is shown in the following two propositions. Further information about epigraphs can be found in Van Tiel [74] and Boř et al. [13].

Proposition 2.1.1. (Boř et al. [13], Theorem 2.2.9) Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a function. The following statements are equivalent:

- (1) f is lower semi-continuous.
- (2) $\text{epi}(f)$ is closed.
- (3) The level set $S_a := \{X \in V; f(X) \leq a\}$ is closed for all $a \in \mathbb{R}$. □

Proposition 2.1.2. (Van Tiel [74], Theorem 5.10) A function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is convex if and only if its epigraph is convex. □

The epigraph can be seen as the vertical closure of the strict epigraph. In fact the closure of the epigraph and the strict epigraph are equal as we will show in the following proposition. The proof can be found in Hess [39].

Proposition 2.1.3. (Hess [39], Section 4) For any function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ we have the following equality of sets

$$\text{cl}(\text{epi}(f)) = \text{cl}(\text{epi}_s(f)). \quad \square$$

The basic tool for the dual representation of a convex risk measures is the Fenchel-Moreau theorem which for the sake of completeness we restate here. First, we give the definition of the conjugate of a function f .

Definition 2.1.4. The *conjugate* f^* and the *biconjugate* f^{**} of a function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ are given by

$$\begin{aligned} f^* : V^* &\rightarrow \mathbb{R} \cup \{+\infty\}, \quad f^*(Z) := \sup_{X \in V} \{\langle X, Z \rangle - f(X)\} \quad \text{for all } Z \in V^*. \\ f^{**} : V &\rightarrow \mathbb{R} \cup \{+\infty\}, \quad f^{**}(X) := \sup_{Z \in V^*} \{\langle X, Z \rangle - f^*(Z)\} \quad \text{for all } X \in V. \quad \diamond \end{aligned}$$

The conjugate functions f^* and f^{**} are lower semi-continuous and convex. Additionally, f^* and f^{**} are proper whenever f is proper. It follows from the definition that $f^{**} \leq f$. The reverse is known as the Fenchel-Moreau theorem, which we formulate as follows:

Theorem 2.1.5. (Van Tiel [74], Theorem 6.18) Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper convex function. If f is lower semi-continuous, then

$$f = f^{**}, \text{ i.e. } f(X) = \sup_{Z \in V^*} \{\langle X, Z \rangle - f^*(Z)\} \quad \text{for all } X \in V. \quad \square$$

If f is neither convex nor lower semi-continuous, then we still have equality between the biconjugate and the convex closure of f .

Theorem 2.1.6. (Van Tiel [74], Theorem 6.15) Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function. Then

$$f^{**} = \text{cl}(\text{co}(f)). \quad \square$$

Next, we shall introduce several properties of sets and the indicator function of sets. The conjugate of a coherent risk measures is in fact an indicator function on a specific set of probability measures as we will see in Section 2.2 and therefore these results are of interest.

Definition 2.1.7. Let $\mathcal{C} \subset V$. The *indicator function* $\delta_{\mathcal{C}} : V \rightarrow \mathbb{R} \cup \{+\infty\}$ of $\mathcal{C}$ is defined by

$$\delta_{\mathcal{C}}(X) = \begin{cases} 0, & \text{if } X \in \mathcal{C}, \\ +\infty, & \text{if } X \notin \mathcal{C}. \end{cases}$$

The *support function* of $\mathcal{C}$ is the conjugate $\delta_{\mathcal{C}}^*$ of the indicator function $\delta_{\mathcal{C}}$ of the set $\mathcal{C}$

$$\begin{aligned} \delta_{\mathcal{C}}^*(Z) &= \sup_{X \in V} \{\langle X, Z \rangle - \delta_{\mathcal{C}}(X)\} \\ &= \sup_{X \in \mathcal{C}} \langle X, Z \rangle. \end{aligned} \quad \diamond$$

We have the following relation between a given set and the indicator function of the same set.

Proposition 2.1.8. (Van Tiel [74], Example 5.15) Let $\mathcal{C} \subset V$. Then

- (1) $\mathcal{C}$ is convex if and only if $\delta_{\mathcal{C}}$ is convex.
- (2) $\mathcal{C}$ is closed if and only if $\delta_{\mathcal{C}}$ is lower semi-continuous. $\square$

We continue with the definitions of subgradients, Fréchet (sub-)differentials as well as Gâteaux differentials on Banach spaces. For further reading on this topic we refer to Boţ et al. [13] and Borwein and Zhu [12]. These results are needed to characterize the subdifferentials of convex risk measures; we will state these results in Section 2.3.

Definition 2.1.9. Let f be a function $V \rightarrow \mathbb{R} \cup \{+\infty\}$, and let X be a point of V where f is finite. Let $Z \in V^*$. Then Z is said to be a *subgradient* of f at X if

$$f(Y) \geq f(X) + \langle Y - X, Z \rangle,$$

whenever $Y \in V$. The set of all subgradients of f at X is called the *subdifferential* of f at X . It is denoted by $\partial f(X)$. The function f is said to be *subdifferentiable* at X if $\partial f(X) \neq \emptyset$. If $X \notin \text{dom}(f)$ we take $\partial f(X) = \emptyset$. $\diamond$

The interpretation of a subgradient is that Z defines a continuous and affine function $h(Y) := f(X) + \langle Y - X, Z \rangle$ which is less or equal than f and equal to f at the point $X \in V$. As a direct consequence we have the following characterization

Proposition 2.1.10. (Boţ et al. [13], Theorem 2.3.12) Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be given an $X \in V$. Then

$$f^*(Z) + f(X) = \langle X, Z \rangle \Leftrightarrow Z \in \partial f(X) \quad \text{and} \quad f(X) < +\infty. \quad \square$$

The next result displays the connection between the subdifferential of a given function f and the subdifferential of the conjugate f^* .

Proposition 2.1.11. (Boţ et al. [13], Theorem 2.3.17) Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be given an $X \in V$. Then

- (1) If $Z \in \partial f(X)$, then $X \in \partial f^*(Z)$.
- (2) If f is proper, convex and lower semi-continuous, then $Z \in \partial f(X)$ if and only if $X \in \partial f^*(Z)$. $\square$

An assertion on the existence of a subgradient is given in the following statement.

Proposition 2.1.12. (Boţ et al. [13], Theorem 2.3.18) Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, convex and continuous at some point $X \in V$. Then $\partial f(X) \neq \emptyset$, i.e. f is subdifferentiable at X . $\square$

Definition 2.1.13. A function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ is *Fréchet differentiable* at X and $f'(X) \in V^*$ is the *Fréchet derivative* of f at X if

$$\lim_{\|Y\| \rightarrow 0} \frac{|f(X+Y) - f(X) - \langle f'(X), Y \rangle|}{\|Y\|} = 0.$$

We say f is C^1 at X if $f' : V \rightarrow V^*$ is norm continuous at X . We say a Banach space is *Fréchet smooth* provided that it has an equivalent norm that is differentiable, indeed C^1 , for all $X \neq 0$. $\diamond$

As an example we have the L^p -spaces ($1 < p < +\infty$) being Fréchet smooth in their original norm, see Borwein and Zhu [12], Chapter 3.

We continue with notion of Fréchet-subdifferentiability. This is a subset of the subdifferentials defined in Definition 2.1.9. In Chapter 3 we will use the Fréchet-subdifferentials to characterize the difference of risk measures.

Definition 2.1.14. Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous function. We say f is *Fréchet-subdifferentiable* and Z is a *Fréchet-subderivative* of f at X if $X \in \text{dom}(f)$ and

$$\liminf_{\|Y\| \rightarrow 0} \frac{f(X+Y) - f(X) - \langle Z, Y \rangle}{\|Y\|} \geq 0.$$

We denote the set of all Fréchet-subderivatives of f at X by $\partial_F f(X)$ and call this object the *Fréchet subdifferential* of f at X . For convenience we define $\partial_F f(u) = \emptyset$ if $u \notin \text{dom}(f)$. $\diamond$

We notice that for a lower semi-continuous and convex function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ and $u \in V$, we have

$$\partial f(u) = \partial_F f(u).$$

Definition 2.1.15. We say f is *Gâteaux-differentiable* at X if there exists a $Z \in V^*$ such that for all $Y \in V$

$$\lim_{\varepsilon \rightarrow 0} \frac{f(X + \varepsilon Y) - f(X)}{\varepsilon} = \langle Y, Z \rangle. \quad (2.2)$$

Z is uniquely determined by (2.2). It is called the *Gâteaux-differential* of f at X . We shall denote it by $\nabla f(X)$. $\diamond$

Fréchet-differentiability implies Gâteaux-differentiability, but the converse is not true for the general case.

For convex functions, Gâteaux-differentiability and uniqueness of the subgradient are closely related, as stated below.

Proposition 2.1.16. (Boj et al. [13], Theorem 2.3.19) Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, convex and continuous at $X \in \text{dom}(f)$ and its subdifferential $\partial f(X)$ be a singleton. Then f is Gâteaux-differentiable at X and $\partial f(X) = \{\nabla f(X)\}$. $\square$

In Section 2.3 we will exploit this assertion to characterize the subdifferentials of convex risk measures.

2.2 The Convex Risk Measure and its Dual Representation

In this section, we study the concept of convex risk measures on L^p . The definition of convex and coherent risk measures is given by an axiomatic formalization to characterize

a measure of risk. Coherent risk measures were first introduced in the seminal paper of Artzner et. al [7] on finite probability spaces. The set of economical desirable properties which characterize a measure of risk consists of monotonicity, translation invariance, positive homogeneity and subadditivity. A risk measure having these four properties is called coherent risk measure. Later, Delbaen [23] extended the theory to general probability spaces. Föllmer and Schied [31] and Frittelli and Rosazza Gianin [33] relaxed the axioms of coherent risk measures and replaced positive homogeneity and subadditivity by convexity. The corresponding risk measure is called convex risk measure. This is the class of risk measures we will study on L^p -spaces with $1 < p < +\infty$. We introduce several important properties of a convex risk measure, which are essential to derive its dual representation. Furthermore, we link the properties of a convex risk measure to the corresponding properties in the dual space. At the end of this section, we give some examples such as Average Value at Risk and entropic risk measure. We shall start with some notations.

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space. We denote by $L^p(\mathcal{F})$ the space of all (equivalent classes of) $\mathcal{F}$ -measurable random variables whose absolute value raised to the p -th power has a finite expectation and $\|\cdot\|_p$ the respective (strong) norm. We write $L^p := L^p(\mathcal{F})$ for $1 \leq p < \infty$. Let us introduce the space $L^\infty(\mathcal{F})$, defined as the set of all $\mathcal{F}$ -measurable and bounded random variables with norm $\|\cdot\|_\infty := \inf\{c \geq 0; \mu[|\cdot| > c] = 0\}$. It is well known that the topological dual space of L^p is given by L^q with $q = \frac{p}{p-1}$ for $1 \leq p < +\infty$. We shall write $\mathbb{E}[XZ] = \langle X, Z \rangle$ for the bilinear pairing on $L^p \times L^q$. Let $\mathcal{M}_a$ denote the class of all absolutely continuous probability measures with respect to μ on $(\Omega, \mathcal{F})$. We identify the positive part of the dual space of L^p with

$$\mathcal{M}_a^q := \left\{ \mathbb{P} \in \mathcal{M}_a; \frac{d\mathbb{P}}{d\mu} \in L^q \right\},$$

where $q = \frac{p}{p-1}$ is the conjugate index.

As mentioned, convex and coherent risk measures have to satisfy some properties. We impose extra to these conditions and include lower semi-continuity as a property of a convex risk measure and assume that the risk measure is finite at 0. This property ensures that the convolution of two risk measures is finite at 0 as well. For our purpose it is too restrictive to assume normality, i.e. $\rho(0) = 0$, since it is difficult to ensure that the convolution of risk measures is normalized. We therefore define a convex risk measure in the following way.

Definition 2.2.1. A *convex risk measure* is a function $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying the following properties:

- (M) Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- (T) Translation invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
- (C) Convexity: $\rho(\gamma X + (1 - \gamma)Y) \leq \gamma \rho(X) + (1 - \gamma)\rho(Y)$ for $0 \leq \gamma \leq 1$.

(L) Lower semi-continuity: $\liminf_{Y \rightarrow X} \rho(Y) \geq \rho(X)$.

(F) Finiteness at 0: $\rho(0) < +\infty$.

A convex risk measure is called a *coherent risk measure* if it fulfills:

(P) Positive homogeneity: $\rho(\gamma X) = \gamma \rho(X)$ for all $\gamma \geq 0$. $\diamond$

Under a risk measure we understand a function ρ which assigns to an uncertain outcome X a real value $\rho(X)$. The random variable X can be seen as the (risk-free) discounted payoff of a financial position at some future date. The number $\rho(X)$ can be understood as a capital requirement for X ; if $\rho(X) \leq 0$ then the risk is acceptable, otherwise it is not acceptable. Monotonicity (M) means that the capital requirement is reduced if the payoff profile is increased. Translation invariance (T) means if a constant amount of money m is added to the position X and invested in a risk-free manner, the capital requirement for X is reduced by m . In particular, translation invariance implies $\rho(X + \rho(X)) = 0$ if $\rho(X) < +\infty$. This means, if $\rho(X)$ is added to the position X , then we obtain a risk neutral position, so the risk becomes acceptable. Convexity (C) means that the diversification of a position should not increase the risk. Finiteness at 0 (F) and lower semi-continuity (L) are technical conditions which ensure that we can use the methods of convex analysis discussed in the previous section. In this thesis, we focus on representation results of combined risk measures. Therefore we added finiteness at 0 and lower semi-continuity to the definition of a convex risk measure to ensure the existence of the dual representation. If positive homogeneity (P) holds, then the capital requirements scale linearly when the position is multiplied with a positive scalar. Then (F) is equivalent to

(N) Normality: $\rho(0) = 0$,

and (C) is equivalent to

(S) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

The interpretation of subadditivity (S) is that the capital requirement of the aggregate position is bounded by the sum of the capital requirements of the individual risk.

Remark 2.2.2. We are using the standard convention that X describes the payoff of a financial position after discounting. This implies the simple representation of the translation invariance property. This approach is equivalent to measure the risk of an undiscounted position while taking the return of the risk free investment into account. Let the return of one unit invested into the risk free account at time T be e^{rT} where $r \in \mathbb{R}$ is the constant interest rate. Define $\psi(e^{rT}X) := \rho(X)$ for all $X \in L^p$ where X describes the discounted payoff. Then the translation invariance property is replaced by $\psi(e^{rT}X + e^{rT}m) = \psi(e^{rT}X) - m$ so we have $\rho(X + m) = \psi(e^{rT}X + e^{rT}m) = \psi(e^{rT}X) - m = \rho(X) - m$. This convention also applies to Chapter 4 and Chapter 5. $\diamond$

From Definition 2.2.1 a convex risk measure is proper, convex and lower semi-continuous. As a consequence we can employ the Fenchel-Moreau theorem given by Theorem 2.1.5 and characterize the convex risk measure ρ by its dual representation. Additionally, a convex risk measure has the properties of monotonicity (M), translation invariance (T) and finiteness at 0 (F). We would like to characterize these properties in terms of the conjugate function. Similar for positive homogeneity (P) in case of a coherent risk measure. The results are known, see for example Föllmer and Schied [32], Remark 4.17, for the space of all bounded random variables, or Rudloff [65], Theorem 1.5. for more general L^p -spaces. Nevertheless, this characterization plays an important role in understanding the dual representation of a convex risk measure. Therefore we will state and prove the characterization and adjust them to our definitions and notations of convex risk measures.

Theorem 2.2.3. Let $f : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be convex and lower semi-continuous with $f(0) < +\infty$. By Theorem 2.1.5, f has the dual representation

$$f(X) = \sup_{Z \in L^q} \{\mathbb{E}[XZ] - f^*(Z)\},$$

where we write $\mathbb{E}[XZ]$ as the bilinear pairing on $L^p \times L^q$. The following conditions are equivalent:

- (1) (i) Monotonicity: $f(X) \geq f(Y)$ for all $X \leq Y$.
(ii) $\text{dom}(f^*) \subset \{Z \in L^q; Z \leq 0\}$.
- (2) (i) Translation invariance: $f(X + m) = f(X) - m$ for all $m \in \mathbb{R}$.
(ii) $\text{dom}(f^*) \subset \{Z \in L^q; \mathbb{E}[Z] = -1\}$.
- (3) (i) Finiteness at 0: $f(0) < +\infty$.
(ii) $\inf_{Z \in L^q} f^*(Z) > -\infty$.
- (4) (i) Positive homogeneity: If $\gamma \geq 0$, then $f(\gamma X) = \gamma f(X)$.
(ii) $f^*(Z) = \delta_{\mathcal{C}}(Z)$ with $\mathcal{C} = \text{dom}(f^*)$.

PROOF. (1) Let f be monotone, then we have for given $\gamma \geq 0$ and $X \geq 0$ that $\gamma X \geq 0$ and by monotonicity for all $Z \in L^p$

$$f(0) \geq f(\gamma X) \geq \mathbb{E}[\gamma XZ] - f^*(Z).$$

It follows that for all $\gamma \geq 0$ and $X \geq 0$

$$f(0) + f^*(Z) \geq \gamma \mathbb{E}[XZ]. \quad (2.3)$$

We see that for $\gamma \rightarrow \pm\infty$ equation (2.3) can only be true if $\text{dom}(f^*) \subset \{Z \in L^q; Z \leq 0\}$ since $f(0) < +\infty$ and $X \geq 0$.

To prove the converse statement, let $X \leq Y$. We have $\mathbb{E}[XZ] \geq \mathbb{E}[YZ]$ for all $Z \leq 0$. If $\text{dom}(f^*) \subset \{Z \in L^q; Z \leq 0\}$ it follows that

$$f(X) = \sup_{Z \in \text{dom}(f^*)} \{\mathbb{E}[XZ] - f^*(Z)\} \geq \sup_{Z \in \text{dom}(f^*)} \{\mathbb{E}[YZ] - f^*(Z)\} = f(Y).$$

- (2) Let f be translation invariant. We have for all $X \in L^p$, $m \in \mathbb{R}$ and $Z \in L^q$

$$\begin{aligned} f(X) - m &= f(X + m) \\ &\geq \mathbb{E}[(X + m)Z] - f^*(Z) \\ &= \mathbb{E}[XZ] + m\mathbb{E}[Z] - f^*(Z). \end{aligned}$$

It follows that

$$f(X) + f^*(Z) - \mathbb{E}[XZ] \geq m\mathbb{E}[Z] + m. \quad (2.4)$$

By choosing an $X \in \text{dom}(f)$, for example $X = 0$, we see that for $m \rightarrow \pm\infty$ inequality (2.4) can only be true if $\mathbb{E}[Z] = -1$ for all $Z \in \text{dom}(f^*)$.

Conversely, assume that $\mathbb{E}[Z] = -1$ for all $Z \in \text{dom}(f^*)$. Then

$$\begin{aligned} f(X + m) &= \sup_{Z \in L^q} \{\mathbb{E}[(X + m)Z] - f^*(Z)\} \\ &= \sup_{Z \in L^q} \{\mathbb{E}[XZ] + m\mathbb{E}[Z] - f^*(Z)\} \\ &= \sup_{Z \in L^q} \{\mathbb{E}[XZ] - m - f^*(Z)\} \\ &= f(X) - m. \end{aligned}$$

- (3) We have by the Fenchel-Moreau theorem

$$f(0) = \sup_{Z \in L^q} \{\mathbb{E}[0Z] - f^*(Z)\} = - \inf_{Z \in L^q} f^*(Z).$$

The equivalence of (i) and (ii) follows.

- (4) First, we assume that f is positive homogeneous. By positive homogeneity of f we have $g_\gamma(X) := \gamma f(\gamma^{-1}X) = f(X)$ for all $\gamma > 0$. It follows from the definition of the conjugate that $g_\gamma^* = f^*$. A small calculation shows for all $\gamma > 0$ and $Z \in L^q$

$$\begin{aligned} g_\gamma^*(Z) &= \sup_{X \in L^p} \{\mathbb{E}[XZ] - g_\gamma(X)\} \\ &= \sup_{X \in L^p} \{\mathbb{E}[XZ] - \gamma f(\gamma^{-1}X)\} \\ &= \sup_{X \in L^p} \{\gamma \mathbb{E}[\gamma^{-1}XZ] - \gamma f(\gamma^{-1}X)\} \\ &= \gamma \sup_{X \in L^p} \{\mathbb{E}[XZ] - f(X)\} \\ &= \gamma f^*(Z). \end{aligned} \quad (2.5)$$

Equation (2.5) yields $f^*(Z) = \gamma f^*(Z)$ for all $Z \in L^q$ and $\gamma > 0$. On the domain of f^* this equation can just be true if f^* is an indicator function on the set $\mathcal{C} = \text{dom}(f^*)$.

On the other hand, if f^* is the indicator function of $\mathcal{C}$ then f is the support function of $\mathcal{C}$, as we have seen in the previous section. Thus

$$f(\gamma X) = \sup_{\text{dom}(f^*)} \mathbb{E}[\gamma X Z] = \gamma \sup_{\text{dom}(f^*)} \mathbb{E}[X Z] = \gamma f(X).$$

□

We have seen in the Theorem 2.2.3 that the elements $Z \in \text{dom}(f^*)$ are negative and integrate to -1. Therefore we can rewrite these elements as negative Radon-Nikodym derivatives as the following theorem shows.

Theorem 2.2.4. Assume conditions (1) and (2) of Theorem 2.2.3 hold, i.e. f is monotone and translation invariant. Then the domain of f^* is a subset of all negative Radon-Nikodym derivatives

$$\text{dom}(f^*) \subset \left\{ -\frac{d\mathbb{P}}{d\mu} \in L^q \right\}.$$

In this case f has a dual representation

$$f(X) = \sup_{Z \in L^q} \left\{ \mathbb{E}[X Z] - f^*(Z) \right\} = \sup_{\mathbb{P} \in \mathcal{M}_a^q} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - f^*\left(-\frac{d\mathbb{P}}{d\mu}\right) \right\}.$$

PROOF. For every $Z \in \text{dom}(f^*)$ we have by monotonicity $-Z \geq 0$ and by translation invariance $\mathbb{E}[-Z] = 1$. Thus $-Z$ is a Radon-Nikodym derivative and we can define a probability measure $\mathbb{P}$, which is absolutely continuous with respect to μ , such that $d\mathbb{P}/d\mu = -Z$. □

As a consequence of the previous theorems, we can characterize a convex or coherent risk measure ρ by its dual representation. Following the notation of Föllmer and Schied [32], we have the following representation results.

Theorem 2.2.5. A function $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex risk measure if and only if ρ admits the following representation

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_{\rho}(\mathbb{P}) \right\} \quad \text{for all } X \in L^p, \quad (2.6)$$

and we have $\inf_{\mathbb{P} \in \mathcal{P}} \alpha_{\rho}(\mathbb{P}) > -\infty$. Here $\mathcal{P} := \{\mathbb{P} \in \mathcal{M}_a^q : \rho^*(-d\mathbb{P}/d\mu) < +\infty\}$ and the penalty function α_{ρ} is defined by

$$\alpha_{\rho}(\mathbb{P}) := \rho^*\left(-\frac{d\mathbb{P}}{d\mu}\right).$$

PROOF. If ρ is a convex risk measure, then by Theorem 2.2.3 and by Theorem 2.2.4 ρ admits the dual representation (2.6). And by Theorem 2.2.3 (3) we have that finiteness at 0 and the condition $\inf_{\mathbb{P} \in \mathcal{M}_a^q} \alpha_\rho(\mathbb{P}) > -\infty$ are equivalent.

To show the converse, the dual representation (2.6) yields that ρ is proper, convex and lower semi-continuous. The function ρ is monotone and translation invariant since the domain of ρ^* is a subset of all negative Radon-Nikodym derivatives. By Theorem 2.2.4 (3), the condition $\inf_{\mathbb{P} \in \mathcal{M}_a^q} \alpha_\rho(\mathbb{P}) > -\infty$ is equivalent to ρ being finite at 0. $\square$

Similarly, we can describe the dual representation of a coherent risk measure.

Theorem 2.2.6. A function $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ is a coherent risk measure if and only if ρ admits the following representation

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[-X] \quad \text{for all } X \in L^p, \quad (2.7)$$

for some non-empty set $\mathcal{P} \subset \mathcal{M}_a^q$.

PROOF. The proof is as in Theorem 2.2.5 except positive homogeneity. Therefore we will just focus on this property. If ρ is positive homogeneous, then by Theorem 2.2.3 (4) the conjugate ρ^* is an indicator function on some set $\mathcal{P}$ and representation (2.7) follows. The converse implication is a consequence of definition of the support function and Theorem 2.2.3 (4). $\square$

In Section 3.2.4 we consider the deconvolution of two risk measures and in Section 4.2 we analyze the good deal valuation. To characterize these objects we need the following definition.

Definition 2.2.7. Let $f_1, f_2 : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$. We say that f_1 is *dominating* f_2 and write $f_1 \succcurlyeq f_2$, if

$$f_1(X) \geq f_2(X) \quad \text{for all } X \in L^p. \quad \diamond$$

The property that f_1 is dominating f_2 can be characterized by the conjugate functions f_1^* and f_2^* .

Proposition 2.2.8. Let $\rho_1, \rho_2 : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be two convex risk measures with penalty functions α_{ρ_1} and α_{ρ_2} , respectively. Then the following statements are equivalent

- (1) $\rho_1 \succcurlyeq \rho_2$.
- (2) $\alpha_{\rho_1} \preccurlyeq \alpha_{\rho_2}$.

PROOF. The proof follows directly from the definition of the conjugate. $\square$

We conclude the section with some examples of convex and coherent risk measures. We give Filipović and Svindland [28] as a reference. If the function space is restricted to L^∞ then the examples can be found in Föllmer and Schied [31].

Example 2.2.9. (Filipović and Svindland [28], Example 4.2) Consider the penalty function $\alpha : \mathcal{M}_a^\infty \rightarrow (0, +\infty]$ defined by

$$\alpha_{\text{Entr}_\beta}(\mathbb{P}) := \frac{1}{\beta} H(\mathbb{P}|\mu),$$

where $\beta > 0$ is a given constant and

$$H(\mathbb{P}|\mu) = \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right]$$

is the relative entropy of $\mathbb{P}$ with respect to μ . The corresponding *entropic risk measure*, denoted as Entr_β on L^p , is given by

$$\text{Entr}_\beta(X) = \sup_{\mathbb{P} \in \mathcal{M}_a^\infty} \left\{ \mathbb{E}_\mathbb{P}[-X] - \frac{1}{\beta} H(\mathbb{P}|\mu) \right\} = \frac{1}{\beta} \log \mathbb{E}[e^{-\beta X}]. \quad \diamond$$

Example 2.2.10. (Filipović and Svindland [28], Example 4.1) Define

$$\mathcal{P}_\lambda := \left\{ \mathbb{P} \in \mathcal{M}_a^1 : \frac{d\mathbb{P}}{d\mu} \leq \frac{1}{\lambda} \right\}$$

for some $\lambda \in (0, 1]$. The corresponding coherent risk measure

$$AV@R_\lambda(X) := \sup_{\mathbb{P} \in \mathcal{P}_\lambda} \mathbb{E}_\mathbb{P}[-X]$$

is defined on L^p and called *Average Value at Risk at level λ* . According to the Banach-Alaoglu theorem $\mathcal{P}_\lambda$ is compact, see Rudin [64], Section 3.15. Average Value at Risk can be written in terms of the quantile function of X

$$AV@R_\lambda(X) = -\frac{1}{\lambda} \int_0^\lambda q_X^+(s) ds,$$

where q_X^+ denotes the upper quantile function of X , defined as $q_X^+(s) := \inf\{x \in \mathbb{R}; \mu[X \leq x] > s\} = \sup\{x \in \mathbb{R}; \mu[X < x] \leq s\}$. If $\lambda = 1$, then one obtains $AV@R_1(X) = \mathbb{E}[-X]$. Average Value at Risk is a continuous and finite risk measure on L^p . For $X \in L^\infty$, we have

$$AV@R_0(X) := \lim_{\lambda \downarrow 0} AV@R_\lambda(X) = \text{ess sup}(-X)$$

which is the worst-case risk measure on L^∞ . Obviously, if we choose $p \in (1, \infty)$, then $\{X \in L^p : \text{ess sup}(-X) = +\infty\} \neq \emptyset$. $\diamond$

The concept of the Fenchel-Legendre transform of a convex function can be used to characterize Average Value at Risk

Proposition 2.2.11. (Föllmer and Schied [32] Lemma 4.46) For $\lambda \in (0, 1)$ and any λ -quantile q of X ,

$$AV@R_\lambda(X) = \frac{1}{\lambda} \mathbb{E}[(q - X)^+] - q = \inf_{y \in \mathbb{R}} \left\{ \frac{1}{\lambda} \mathbb{E}[(y - X)^+] - y \right\}. \quad \square$$

2.3 Continuity and (Sub-)differentiability of Risk Measures

We state some continuity conditions of convex risk measures. The results are needed in the following chapters to characterize the difference of two convex risk measures.

First, we review the result that convex and lower semi-continuous functions are continuous on the interior of their domain in L^p .

Theorem 2.3.1. (Biagini and Frittelli [10], Theorem 1) Let $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex risk measure. Then

- (1) ρ is continuous on $\text{int}(\text{dom}(\rho))$.
- (2) If ρ is a finite risk measure, then it is continuous. $\square$

For a given $X \in L^p$ we want to characterize the subdifferentials of a convex risk measure ρ at X . Under the condition that $\rho(X)$ is finite this is possible.

Proposition 2.3.2. Let $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex risk measure with penalty function α_ρ and assume that $\rho(X) < +\infty$ and $\partial\rho(X) \neq \emptyset$. We have

$$\partial\rho(X) = \left\{ -\frac{d\tilde{\mathbb{P}}}{d\mu} : \tilde{\mathbb{P}} \in \arg \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[-X] - \alpha_\rho(\mathbb{P}) \right\} \quad \text{for all } X \in L^p. \quad (2.8)$$

PROOF. Let $\tilde{Z} \in \partial\rho(X)$ with $\rho(X) < +\infty$. We have by Proposition 2.1.10 that $\rho(X) = \mathbb{E}[X\tilde{Z}] - \rho^*(\tilde{Z})$. Monotonicity and translation invariance of the risk measure ρ yield $\text{dom}(\rho^*) \subset \{\tilde{Z} \in L^q : \tilde{Z} \leq 0\}$ and $\mathbb{E}_{\mathbb{P}}[\tilde{Z}] = -1$. Therefore we can define $\tilde{\mathbb{P}}$ with $d\tilde{\mathbb{P}}/d\mu = -\tilde{Z}$. And in addition

$$\rho(X) = \mathbb{E}_{\tilde{\mathbb{P}}}[-X] - \alpha_\rho(\tilde{\mathbb{P}}). \quad (2.9)$$

The risk measure ρ admits a dual representation

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_\rho(\mathbb{P}) \right\}.$$

Due to (2.9) the supremum is attained at $\tilde{\mathbb{P}}$ and we have

$$-\tilde{Z} = \frac{d\tilde{\mathbb{P}}}{d\mu} \in \arg \max_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}}[-X] - \alpha_\rho(\mathbb{P}).$$

Conversely, suppose $d\tilde{\mathbb{P}}/d\mu$ is such that $\rho(X) = \mathbb{E}_{\tilde{\mathbb{P}}}[-X] - \alpha(\tilde{\mathbb{P}})$. By the dual representation of risk measures we have for any $Y \in L^p$

$$\rho(Y) \geq \mathbb{E}_{\tilde{\mathbb{P}}}[-Y] - \alpha(\tilde{\mathbb{P}}),$$

and therefore $\rho(Y) \geq \rho(X) + \mathbb{E}_{\tilde{\mathbb{P}}}[-(Y - X)]$. This means that $-d\tilde{\mathbb{P}}/d\mu \in \partial\rho(X)$. $\square$

As a consequence we obtain the following corollary.

Corollary 2.3.3. Let $\rho : L^p \rightarrow \mathbb{R}$ be a finite convex risk measure. We have the following two statements.

- (1) We have $\partial\rho(X) \neq \emptyset$, i.e. ρ is subdifferentiable at X and (2.8) holds.
- (2) If $\partial\rho(X)$ is a singleton, then ρ is Gâteaux-differentiable at X and $\partial\rho(X) = \{\nabla\rho(X)\}$.

PROOF. (1) Theorem 2.3.1 yields that any finite risk measure is continuous. Since a risk measure is proper and convex, we obtain by Proposition 2.1.12 the existence of a subgradient.

- (2) The second statement follows from Proposition 2.1.16. $\square$

2.4 Acceptance Sets

Let ρ be a convex risk measure on L^p . The acceptance set of ρ is given by $\mathcal{A}_\rho = \{X \in L^p; \rho(X) \leq 0\}$. In this section, we give relations between convex risk measures on L^p and their acceptance sets $\mathcal{A}_\rho$. Furthermore, we give conditions on a set $\mathcal{A}$ such that the risk measure $\rho_{\mathcal{A}}$ induced by $\mathcal{A}$ is a convex risk measure. The situation is mostly similar to the one of bounded random variables which can be found in Föllmer and Schied [32], Section 4.1. These results have been generalized in many publications. Especially noteworthy is Hamel [38] on which this section is based, although we do not treat this topic in the same generality. A broad summary can also be found in Rudloff [65], Section 1.1.4. We first give two definitions of Hamel [38].

Definition 2.4.1.

- (1) Let $\mathcal{B} \subset L^p$ be a non-empty set. A set $\mathcal{C} \subset L^p$ is called $\mathcal{B}$ -upward if $\mathcal{C} + \mathcal{B} \subset \mathcal{C}$.
- (2) A set $\mathcal{C} \subset L^p$ is called *translative* if $\mathcal{C} + m \subset \mathcal{C}$, for all $m \geq 0$. $\diamond$

We notice that the definition of L^p_+ -upward is equivalent to solidness, which is a condition used in many other publications.

We give the concept of acceptance sets for some function f on L^p instead of focusing on convex risk measure on L^p .

Definition 2.4.2. Let $f : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$. We define

$$\mathcal{A}_f := \{X \in L^p; f(X) \leq 0\}.$$

The set $\mathcal{A}_f \subset L^p$ is called the *acceptance set* of f . ◇

Of course, the term acceptance set is more connected to a risk measure ρ than an arbitrary function f , since $\mathcal{A}_\rho = \{X \in L^p; \rho(X) \leq 0\}$ is the class of positions which are acceptable in the sense that they do not require additional capital.

We have the following relation between a function f and its acceptance set $\mathcal{A}_f$.

Proposition 2.4.3. (Hamel [38], Proposition 3, 6 and 8) Let $f : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$. Then each property of Definition 2.2.3 yields a property of the acceptance set $\mathcal{A}_f$:

- (1) If f is monotone, then $\mathcal{A}_f$ is L^p_+ -upward.
- (2) If f is convex, then $\mathcal{A}_f$ is convex.
- (3) If f is translation invariant and lower semi-continuous, then $\mathcal{A}_f$ is closed.
- (4) If f is finite at 0, then there exists an $m \in \mathbb{R}$ such that $m \in \mathcal{A}_f$.
- (5) If f is positively homogeneous, then $\mathcal{A}_f$ is a convex cone.
- (6) If ρ is a convex risk measure then $\mathcal{A}_\rho$ is L^p_+ -upward, convex and lower semi-continuous.
- (7) If ρ is a coherent risk measure then $\mathcal{A}_\rho$ is L^p_+ -upward, convex conic and lower semi-continuous. □

Conversely, we can take a given non-empty set $\mathcal{A} \subset L^p$ of acceptable positions as the primary object. For a position $X \in L^p$ we can then define the capital requirement as the minimal amount m such that $m + X$ becomes acceptable. This means we can define the function $\rho_{\mathcal{A}} : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ by $\rho_{\mathcal{A}} := \inf\{m \in \mathbb{R}; X + m \in \mathcal{A}\}$. Again, we generalize the statement for any f on L^p .

Definition 2.4.4. Let $\mathcal{A} \subset L^p$, $X \in L^p$ and define

$$f_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R}; X + m \in \mathcal{A}\}. \quad (2.10)$$

The function $f_{\mathcal{A}}$ is called *f induced by the set $\mathcal{A}$* . ◇

We notice that for any set $\mathcal{A}$, the function $f_{\mathcal{A}}$ is translation invariant. Similarly to Proposition 2.4.3 each property of the set $\mathcal{A}$ yields a property of the function $f_{\mathcal{A}}$.

Proposition 2.4.5. (Hamel [38], Proposition 3, 6 and 8) Let $\mathcal{A} \subset L^p$. We have the following connections between the set $\mathcal{A}$ and the function $f_{\mathcal{A}}$.

- (1) If $\mathcal{A}$ is L^p_+ -upward, then $f_{\mathcal{A}}$ is monotone.

- (2) If $\mathcal{A}$ is convex, then $f_{\mathcal{A}}$ is convex.
- (3) If $\mathcal{A}$ is translative and closed, then $f_{\mathcal{A}}$ is lower semi-continuous.
- (4) If $\inf\{m \in \mathbb{R}; m \in \mathcal{A}\} \in \mathbb{R}$, then $f_{\mathcal{A}}$ is finite at 0.
- (5) If $\mathcal{A}$ is a cone, then $f_{\mathcal{A}}$ is positively homogeneous.
- (6) If $\mathcal{A}$ satisfies the condition (1)-(4), then $f_{\mathcal{A}}$ is a convex risk measure.
- (7) If $\mathcal{A}$ satisfies the condition (1)-(5), then $f_{\mathcal{A}}$ is a coherent risk measure. $\square$

We now discuss the relationship between the functions f and $f_{\mathcal{A}_f}$ and the sets $\mathcal{A}$ and $\mathcal{A}_{f_{\mathcal{A}}}$. In the following proposition we state sufficient conditions such that these objects are equal.

Proposition 2.4.6. (Hamel [38], Proposition 3)

- (1) Let $f : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be translation invariant and lower semi-continuous. Then $f = f_{\mathcal{A}_f}$.
- (2) Let $\mathcal{A}$ be translative and closed. Then $\mathcal{A} = \mathcal{A}_{f_{\mathcal{A}}}$. $\square$

In Chapter 3, special interest is paid to the relationship between the acceptance set $\mathcal{A}_f$ and the conjugate of f as well as the set $\mathcal{A}$ and the conjugate of the function $f_{\mathcal{A}}$ induced by the set $\mathcal{A}$. This proof is based on Rudloff [65], Theorem 1.5 (c) and Hamel [38], Theorem 4.

Proposition 2.4.7.

- (1) Let $f : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, convex, lower semi-continuous and translation invariant. Then the conjugate f^* can be represented by

$$f^*(Z) = \sup_{X \in \mathcal{A}_f} \mathbb{E}[XZ] = \delta_{\mathcal{A}_f}^*(Z) \quad \text{for all } Z \in \text{dom}(f^*).$$

Thus, the conjugate of f is equal to the support function of the acceptance set $\mathcal{A}_f$.

- (2) Let $\mathcal{A}$ be a non-empty set satisfying condition (3) of Proposition 2.4.5. Then the conjugate $f_{\mathcal{A}}^*$ can be represented by

$$f_{\mathcal{A}}^*(Z) = \sup_{X \in \mathcal{A}} \mathbb{E}[XZ] = \delta_{\mathcal{A}}^*(Z) \quad \text{for all } Z \in \text{dom}(f^*).$$

PROOF. (1) We observe that for all $X \in \mathcal{A}_f$ we have $f(X) \leq 0$ and it follows that for all $Z \in \text{dom}(f^*)$

$$f^*(Z) \geq \sup_{X \in \mathcal{A}_f} \{\mathbb{E}[XZ] - f(X)\} \geq \sup_{X \in \mathcal{A}_f} \mathbb{E}[XZ]. \quad (2.11)$$

To show the converse, we have for any $X \in \text{dom}(f)$ that $X + f(X) \in \mathcal{A}_f$, since by translation invariance $f(X + f(X)) = f(X) - f(X) = 0$. Therefore we have for all $Z \in \text{dom}(f^*)$

$$\sup_{\tilde{X} \in \mathcal{A}_f} \mathbb{E}[\tilde{X}Z] \geq \mathbb{E}[(X + f(X))Z] = \mathbb{E}[XZ] + f(X)\mathbb{E}[Z] = \mathbb{E}[XZ] - f(X). \quad (2.12)$$

The last equality sign follows from the translation invariance property of f , see Theorem 2.2.3 (2). The inequality is trivially satisfied for any $X \notin \text{dom}(f)$. Thus we can take the supremum over all $X \in L^p$ in (2.12), which gives us

$$\sup_{\tilde{X} \in \mathcal{A}_f} \mathbb{E}[\tilde{X}Z] \geq \sup_{X \in L^p} \{\mathbb{E}[XZ] - f(X)\} = f^*(Z). \quad (2.13)$$

We combine (2.11) and (2.13) and the statement follows.

- (2) Since $\mathcal{A}$ is non-empty it follows from the definition of f induced by the set $\mathcal{A}$ (2.10) that $f_{\mathcal{A}}$ is proper and translation invariant. It follows by (1) that the conjugate $f_{\mathcal{A}}^*$ is given by

$$f^*(Z) = \sup_{X \in \mathcal{A}_{f_{\mathcal{A}}}} \mathbb{E}[XZ].$$

By Proposition 2.4.6 (2), we have $\mathcal{A} = \mathcal{A}_{f_{\mathcal{A}}}$, which completes the proof of the proposition. $\square$

Any convex risk measure satisfies the requirements of Proposition 2.4.7. Therefore we have the following corollary.

Corollary 2.4.8.

- (1) Let $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex risk measure with acceptance set $\mathcal{A}_\rho$. Then the penalty function α_ρ can be represented by

$$\alpha_\rho(\mathbb{P}) := \rho^*(-d\mathbb{P}/d\mu) = \sup_{X \in \mathcal{A}_\rho} \mathbb{E}_{\mathbb{P}}[-X] = \delta_{\mathcal{A}_\rho}^*(-d\mathbb{P}/d\mu) \quad \text{for all } \mathbb{P} \in \mathcal{P}.$$

Thus, the conjugate of the risk measure ρ is equal to the support function of the acceptance set $\mathcal{A}_\rho$.

- (2) Let $\mathcal{A}$ be a set satisfying the condition (3) of Proposition 2.4.5. Then the penalty function $\alpha_{\rho_{\mathcal{A}}}$ can be represented by

$$\alpha_{\rho_{\mathcal{A}}}(\mathbb{P}) := \rho_{\mathcal{A}}^*(-d\mathbb{P}/d\mu) = \sup_{X \in \mathcal{A}} \mathbb{E}_{\mathbb{P}}[-X] = \delta_{\mathcal{A}}^*(-d\mathbb{P}/d\mu) \quad \text{for all } \mathbb{P} \in \mathcal{P}. \quad \square$$

2.5 Spectral Risk Measures

In this section, we characterize the dual representation of a spectral risk measure. A spectral risk measure is a risk measure given as a weighted average of the quantile function, which assigns higher weights on the lower part of the distribution. Spectral risk measures have been introduced by Acerbi [1]. Kusuoka [49] proved on L^∞ that any law-invariant and comonotonic coherent risk measure can be represented as a spectral risk measure. This result was further extended to L^p -spaces by Shapiro [70]. We will use spectral risk measures as examples in Chapter 3 and Chapter 4 since it is very simple to check whether the weighted sum or difference of two spectral risk measures is again a spectral risk measure.

Additionally, we introduce a special class of spectral risk measures, called *simple spectral risk measure*, where the spectrum is given as a step function. These measures can be characterized as weighted sum of Average Value at Risk. In our optimization problem stated in Chapter 5, the objective function is a simple spectral risk measure.

We start with the definition of law-invariant and comonotone risk measures on L^p . Throughout this section we will assume that the underlying probability space $(\Omega, \mathcal{F}, \mu)$ is atomless.

Definition 2.5.1. A convex risk measure $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ is called law-invariant if $\rho(X) = \rho(Y)$ whenever $X, Y \in L^p$ have the same distribution under μ . $\diamond$

Definition 2.5.2. Two measurable functions X and Y on $(\Omega, \mathcal{F})$ are called *comonotone* if

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \quad \text{for all } (\omega, \omega') \in \Omega \times \Omega.$$

A convex risk measure $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ is called comonotone if

$$\rho(X + Y) = \rho(X) + \rho(Y),$$

whenever $X, Y \in L^p$ are comonotone. $\diamond$

We formulate the main result of Shapiro [70], which states that any law-invariant and comonotonic coherent risk measure can be represented in a form of integrals of Average Value at Risk measures.

Theorem 2.5.3. (Shapiro [70], Theorem 3.1) Let $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be a law-invariant coherent risk measure. The following statements are equivalent:

- (1) There exists a unique probability measure ν on the interval $(0, 1]$ such that

$$\rho_\nu(X) = \int_0^1 AV@R_\lambda(X) \nu(d\lambda). \quad (2.14)$$

(2) The risk measure ρ is comonotonic. $\square$

We note that the measure ν in the above theorem is defined on the interval $(0, 1]$; we define $\nu(\{0\}) := 0$. In order to prove the relation between spectral risk measures and law invariant risk measures we need the following lemma.

Lemma 2.5.4. (Föllmer and Schied [32], Lemma 4.63) Let Ψ be a concave function and Ψ'_+ its right-continuous right-hand derivative. The identity

$$\Psi'_+(t) = \int_t^1 s^{-1} \nu(ds) \quad \text{for all } t \in [0, 1],$$

defines a one-to-one correspondence between probability measures ν on $(0, 1]$ and increasing concave functions $\Psi : [0, 1] \rightarrow [0, 1]$ with $\Psi(0) = 0$ and $\Psi(1) = 1$. Moreover, we have $\lim_{t \downarrow 0} \Psi(t) = \nu(\{0\}) = 0$.

PROOF. The proof is identical to Föllmer and Schied [32], Lemma 4.63, except that we excluded the case of ν having positive mass at 0. $\square$

Next, we give the definition of a spectral risk measure.

Definition 2.5.5. (Acerbi [1], Definition 2.4, Theorem 2.5) Let $\Upsilon : [0, 1] \rightarrow \mathbb{R}$ be non-negative, decreasing and $\int_0^1 \Upsilon(s) ds = 1$. Then

$$\rho_\Upsilon(X) := - \int_0^1 q_X^+(t) \Upsilon(t) dt \quad \text{for all } X \in L^p$$

is called a *spectral risk measure* with risk spectrum Υ . Here q_X^+ is the upper quantile function of X , see Example 2.2.10 for the definition. $\diamond$

A spectral risk measure is a coherent risk measure. In fact, every spectral risk measure is of the form (2.14) as the following proposition shows.

Proposition 2.5.6. The risk measure ρ_ν is a law-invariant and comonotonic with unique measure ν if and only if ρ_Υ is a spectral risk measure with spectrum $\Upsilon(t) := \Psi'_+(t)$ for all $t \in [0, 1]$, where

$$\Psi'_+(t) = \int_t^1 s^{-1} \nu(ds) \quad \text{for all } t \in [0, 1].$$

PROOF. The one-to-one correspondence between Ψ and ν follows from Lemma 2.5.4. By Example 2.2.10, the risk measure Average Value at Risk at level $\lambda \in (0, 1]$ is defined as $AV@R_\lambda(X) = -\frac{1}{\lambda} \int_0^\lambda q_X^+(s)ds$, for any $X \in L^p$. Therefore we have

$$\begin{aligned} \rho_\nu(X) &= \int_0^1 AV@R_\lambda(X) \nu(d\lambda) = \int_0^1 -\frac{1}{\lambda} \int_0^\lambda q_X^+(t)dt \nu(d\lambda) \\ &= - \int_0^1 \int_0^1 \frac{1}{\lambda} \mathbf{1}_{[0, \lambda]}(t) q_X^+(t) dt \nu(d\lambda) = - \int_0^1 \int_t^1 \frac{1}{\lambda} \nu(d\lambda) q_X^+(t) dt \\ &= - \int_0^1 q_X^+(t) \Psi'_+(t) dt. \end{aligned}$$

Since $\Upsilon(t) := \Psi'_+(t)$ the proposition follows. $\square$

We have seen in Theorem 2.5.3 that any spectral risk measure can be represented by (2.14). Of special interest for us is the case when ν is a simple measure.

Definition 2.5.7. We define ν_n to be a simple measure if we can write $\nu_n := \sum_{i=1}^n \alpha_i \mathbb{I}_{s_i}$, where $\mathbb{I}_s$ denotes the probability measure of mass one at s , and α_i are positive numbers such that $\sum_{i=1}^n \alpha_i = 1$. A coherent risk measure $\rho_{\nu_n} : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ is called *simple spectral risk measure* if there exists a simple measure ν_n on the interval $(0, 1]$ such that ρ_{ν_n} has the following representation

$$\rho_{\nu_n}(X) = \int_0^1 AV@R_\lambda(X) \nu_n(d\lambda). \quad (2.15)$$

$\diamond$

Lemma 2.5.8. Let ν_n be a simple measure. Then the risk measure ρ_{ν_n} has the following equivalent representation

- (1) The coherent risk measure ρ_{ν_n} is given by

$$\rho_{\nu_n}(X) = - \int_0^1 q_X^+(t) \Upsilon_n(t) dt,$$

with spectrum $\Upsilon_n(t) = \sum_{i=1}^n \frac{\alpha_i}{s_i} \mathbf{1}_{[0, s_i]}(t)$.

- (2) The coherent risk measure ρ_{ν_n} can be represented as a weighted sum of Average Value at Risk

$$\rho_{\nu_n}(X) = \sum_{i=1}^n \alpha_i AV@R_{s_i}(X).$$

PROOF. (1) From Proposition 2.5.6 it follows

$$\Upsilon_n(t) = \int_0^t s^{-1} \nu_n(ds) = \sum_{i=1}^n \frac{\alpha_i}{s_i} \mathbf{1}_{[0, s_i]}(t).$$

(2) Follows from Theorem 2.5.3 (2). □

In Chapter 5 we will focus on a dynamic optimization problem of finding a self-financing strategy that minimizes the risk. The objective function will be given as a simple spectral risk measure.

In this chapter we introduced notations and the setting of this thesis. We gave a formal definition of convex and coherent risk measures. We showed that these risk measures can always be characterized by their penalty functions and their acceptance sets. The analysis carried out is of importance since we are studying linear combinations and convolutions of two given risk measures in Chapter 3. Our aim is to derive the penalty functions and acceptance sets of these new risk measure from the information given by the old risk measures.

Further, we stated examples of convex and coherent risk measures, such as Average Value at Risk, entropic and spectral risk measures. These examples will be used in the following chapters.

Linear Combinations and Convolutions of Convex Risk Measures

Let us start this chapter with a motivating example to show why we are interested in using linear combinations and convolutions of two convex risk measures. Assume that we have two agents A and B with respective risk measures ρ_A and ρ_B , reflecting different risk attitudes (for instance, a supervisor that is more risk averse than a trader). The two agents are exposed to a common position X . Instead of using their respective risk measures to measure the position X , the agents may agree to apply a weighted average of their risk measures $\phi(X) = (1 - \lambda)\rho_A(X) + \lambda\rho_B(X)$, where $\lambda \in (0, 1)$ is the relative weight of agent B 's perspective. The first question is, whether or not ϕ is a risk measure and if yes, what is the penalty function of ϕ in terms of the penalty functions of ρ_A and ρ_B . Thus we are dealing with the sum of two risk measures.

We can also see this problem from a different perspective, as an optimal design problem. Assume now that ρ_B is given. Which risk measure ρ_A and which constant λ should the agent A use to generate a given measure ϕ . This problem leads to the difference of two convex risk measures and we want to ensure that ρ_A is indeed a convex risk measure and further characterize this function.

We can create a similar example for the inf-convolution. Again, we have two agents A and B with risk measures ρ_A and ρ_B and a common position X . The agents are dividing the position such that the overall risk ϕ is minimized. We can write this as $\phi(X) = \inf_Y \{\rho_A(X - Y) + \rho_B(Y)\}$ and ϕ is nothing else but the inf-convolution of ρ_A and ρ_B .

The inf-convolution is a well known operator in convex analysis. Delbaen [22] was the first to introduce this operator to the field of risk measures theory. He studied the inf-convolution of coherent risk measures in L^∞ and derived the penalty function and the acceptance set of the inf-convolution of two risk measures. Barrieu and El Karoui [8], [9] extended these results to convex risk measures and studied the inf-convolution of a risk measure and a set as well. Toussaint and Sircar [72] analyzed the inf-convolution on L^2 , and recently Arai [4] derived the inf-convolution of convex risk measures on Orlicz spaces. In this chapter, we study risk measures on L^p -spaces $1 < p < +\infty$ taking values in $\mathbb{R} \cup \{+\infty\}$.

In their paper, Hiriart-Urruty and Mazure [41] introduced the deconvolution operation. An English introduction to the topic can be found in Hiriart-Urruty [40]. The deconvolution is the inverse operation to the well known inf-convolution. The deconvolution has the nice geometrical interpretation that the epigraph of the deconvolution of two functions is equal to the star-difference of the epigraphs of the two respective functions. We have a comparable result for the inf-convolution which has an interpretation in terms of addition of epigraphs. The deconvolution appears as the conjugate of the difference of two convex functions, similar to the inf-convolution which is the conjugate of the addition of two convex functions. Additionally, we have a duality correspondence between multiplication of scalars and epi-multiplication.

The aim of this chapter is to characterize linear combinations and convolutions of convex risk measures in terms of their penalty functions using the duality correspondence of the just mentioned operations and investigate basic properties. To our knowledge these representation results regarding the sum, deconvolution and difference are new. We also study the properties of the deconvolution between a convex risk measure and a set. We further derive epi-multiplication of a risk measure, which has been introduced by Barrieu and El Karoui [9], but was not treated in detail. Additionally, we state the known results of the inf-convolution of convex risk measures.

We start with the simplest example, the epi-multiplication of a risk measure $\phi := \lambda * \rho$, with ρ being a convex risk measure. Even though the structure of this risk measure is very simple, it is useful, since the entropic risk measure can be characterized by an epi-multiplication. In Theorem 3.2.1 we prove that ϕ itself is a convex risk measure with a penalty function which is the multiplication of the scalar λ and the penalty function of ρ .

As mentioned, the inf-convolution of convex risk measures was introduced by Delbaen [22] and Barrieu and El Karoui [8], [9]. The penalty function of the new measure is given by a sum of the penalty functions of the two risk measures. We review the main results in Theorem 3.2.2, analyzing the inf-convolution between two convex risk measures and in Theorem 3.2.3 the inf-convolution between a convex risk measure and a set.

We consider the sum of two risk measures $\phi(X) = (1 - \lambda)\rho_1(X) + \lambda\rho_2(X)$ and derive the dual representation of ϕ in Theorem 3.2.4. The penalty function is given by the inf-convolution of epi-multiplication of penalty function α_{ρ_1} with the scalar $(1 - \lambda)$ and the epi-multiplication of α_{ρ_2} and λ .

We then adopt the notion of the deconvolution and introduce it as an operation in risk analysis. We consider the following two types of deconvolution. The first type of deconvolution is defined for two risk measures ρ_1 and ρ_2 as $(\rho_1 \boxminus \rho_2)(X) := \sup_{Y \in L^p} \{\rho_1(X + Y) \dot{-} \rho_2(Y)\}$, where the symbol $\dot{-}$ is defined as the lower extension of subtraction, meaning $(-\infty) \dot{-} (-\infty) = -\infty$. In Theorem 3.2.5 we derive the dual representation of the deconvolution of two convex risk measures. The penalty function of deconvolution is given by the difference of the penalty functions of the two risk measures ρ_1 and ρ_2 . The second type of deconvolution is defined for a set $\mathcal{B}$ and a risk measure ρ , either by $(\rho \boxminus \mathcal{B})(X) := \sup_{Y \in \mathcal{B}} \{\rho(X + Y)\}$ or $(\mathcal{B} \boxminus \rho)(X) := \sup_{Y \in L^p} \{\delta_{\mathcal{B}}(X + Y) \dot{-} \rho(Y)\}$. We shall see in Theorem 3.2.6 and Remark 3.2.7 that $\rho \boxminus \mathcal{B}$ is a risk measure, while $\mathcal{B} \boxminus \rho$ can never be a risk measure.

In Theorem 3.2.8, we characterize the risk measure defined as the difference of two convex risk measures, $\phi(X) = (1 + \lambda)\rho_1(X) - \lambda\rho_2(X)$. We derive its dual representation. The penalty function is given by the deconvolution of epi-multiplication of penalty function α_{ρ_1} and scalar $(1 + \lambda)$ and the epi-multiplication of α_{ρ_2} and λ .

We structure this chapter in the following way. In the next section we introduce the methods of convex analysis which are necessary to derive representation results for convex risk measures on reflexive Banach spaces. To be more specific, we introduce the inf-convolution and deconvolution and link these operations to the Minkowski sum and star-difference of sets. Furthermore, we derive properties of these functions. We characterize the conjugates of linear combinations and convolutions of functions. These results are crucial for the dual representation of linear combinations and convolutions of convex risk measures. In Section 3.2 we present our main theorems, we state sufficient conditions for the dual representation of the sum, deconvolution and difference of two risk measures as well as the epi-multiplication convex risk measures. We characterize the penalty function and the acceptance set of the newly derived risk measures. To complete the section, we review the results of the inf-convolution. Some examples, including the combination and convolution of Average Value at Risk, entropic risk measure and spectral risk measure are analyzed in Section 3.3.

3.1 Convex Analysis of Combined Functions

In this Section we review some statements from the field of convex analysis. We introduce the inf-convolution and deconvolution and derive conjugates of linear combinations and convolutions of functions. The main results in Section 3.2 are defined on L^p -spaces, but since the results stated in this sequel are valid on more general function spaces we will use reflexive Banach space V instead. The terminology and notation are the same as in Section 2.1.

3.1.1 Inf-convolution and Deconvolution

The deconvolution was introduced by Hiriart-Urruty and Mazure [41]. We show the relationship between deconvolution of functions and the star-difference of sets. We start with the well known definition of the inf-convolution.

Definition 3.1.1.

- (1) For functions $f_1, f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$, the *inf-convolution* is the function $f_1 \boxplus f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned} (f_1 \boxplus f_2)(X) &:= \inf_{\substack{X_1 + X_2 = X \\ X_1, X_2 \in V}} \{f_1(X_1) + f_2(X_2)\} = \inf_{Y \in V} \{f_1(X - Y) + f_2(Y)\} \\ &= \inf_{Y \in \text{dom}(f_2)} \{f_1(X - Y) + f_2(Y)\} \quad \text{for all } X \in V. \end{aligned}$$

- (2) For a function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ and a non-empty set $\mathcal{B} \subset V$, the inf-convolution of the function f and the set $\mathcal{B}$ is the function $f \boxplus \mathcal{B} : V \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$(f \boxplus \mathcal{B})(X) := \inf_{Y \in \mathcal{B}} \{f(X - Y)\}. \quad \diamond$$

Essentially, $f \boxplus \mathcal{B}$ is the inf-convolution of the function f with the indicator function of the set $\mathcal{B}$.

In similar manner to the inf-convolution we can define the deconvolution. First, we recall the lower extension of subtraction, represented with the symbol $\div$, defined as $(+\infty) \div (+\infty) = -\infty$.

Definition 3.1.2.

- (1) For functions $f_1, f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$, the *deconvolution* is the function $f_1 \boxminus f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$\begin{aligned} (f_1 \boxminus f_2)(X) &:= \sup_{\substack{X_1 - X_2 = X \\ X_1, X_2 \in V}} \{f_1(X_1) \div f_2(X_2)\} = \sup_{Y \in V} \{f_1(X + Y) \div f_2(Y)\} \\ &= \sup_{Y \in \text{dom}(f_2)} \{f_1(X + Y) - f_2(Y)\} \quad \text{for all } X \in V. \end{aligned}$$

- (2) For a function $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ and a non-empty set $\mathcal{B} \subset V$, the deconvolution of the function f and the set $\mathcal{B}$ is the function $f \boxminus \mathcal{B} : V \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$(f \boxminus \mathcal{B})(X) := \sup_{Y \in \mathcal{B}} \{f(X + Y)\}.$$

Additionally, the deconvolution of the set $\mathcal{B}$ and the function f is the function $\mathcal{B} \boxminus f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$(\mathcal{B} \boxminus f)(X) := \sup_{Y \in V} \{\delta_{\mathcal{B}}(X + Y) \div f(Y)\}. \quad \diamond$$

The deconvolution operation is strongly linked to the star-difference of sets. The star-difference was first defined by Hadwiger [37], although he called the operator Minkowski difference. The author prefers the name star-difference, since $\mathcal{B} - \mathcal{C} := \mathcal{B} + (-\mathcal{C})$ is often considered to be the Minkowski difference.

Definition 3.1.3. The *star-difference* of two subsets $\mathcal{B}, \mathcal{C} \subset V$ is defined by

$$\mathcal{B} \mathbin{\dot{-}} \mathcal{C} := \{X \in V; X + \mathcal{C} \subset \mathcal{B}\}. \quad \diamond$$

We will state some useful properties of the star-difference. Let $\mathcal{B}$ be convex, then $\mathcal{B} \mathbin{\dot{-}} \mathcal{C}$ is convex as well. Moreover, $\mathcal{B} \mathbin{\dot{-}} \mathcal{C} = \mathcal{B} \mathbin{\dot{-}} \text{cl}(\mathcal{C})$ whenever $\mathcal{B}$ is closed.

Closedness of the sets $\mathcal{B}$ and $\mathcal{C}$ and compactness of the set $\mathcal{C}$ ensure that the Minkowski sum, i.e. $\mathcal{B} + \mathcal{C} := \{X + Y; X \in \mathcal{B}, Y \in \mathcal{C}\}$, and the star-difference are inverse operations, for our purpose this is sufficient. Nonetheless, the condition can be weakened, see Martinez-Legaz and Penot [53] for more details. We have the following regularization of sets.

Proposition 3.1.4. Let $\mathcal{B}, \mathcal{C} \subset V$. If $\mathcal{B}$ and $\mathcal{C}$ are closed and $\mathcal{C}$ is compact then

$$(\mathcal{B} + \mathcal{C}) \mathbin{\dot{-}} \mathcal{C} = \mathcal{B}.$$

If in addition $\mathcal{B} \mathbin{\dot{-}} \mathcal{C} \neq \emptyset$ then

$$(\mathcal{B} \mathbin{\dot{-}} \mathcal{C}) + \mathcal{C} = \mathcal{B}.$$

PROOF. The first part follows from the definition of the Minkowski sum and the star-difference. The second part is direct consequence of Proposition 2.5 of Martinez-Legaz and Penot [53] and is therefore omitted. $\square$

As we have seen in Chapter 2, there is a one-to-one relation between properties of the (strict) epigraph of a function f and properties of f itself. Next, we will show that the strict epigraph of the inf-convolution of two functions is equal to the Minkowski sum of strict epigraphs of these two functions. Similarly, the deconvolution of two function is equal to the star-difference of the epigraphs.

Proposition 3.1.5. Let $f_1, f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be two functions. Then $\text{epi}_s(f_1) + \text{epi}_s(f_2) = \text{epi}_s(f_1 \boxplus f_2)$ and $\text{epi}(f_1) \mathbin{\dot{-}} \text{epi}(f_2) = \text{epi}(f_1 \boxminus f_2)$.

PROOF. We want to prove the first relation. Let $(X, a) \in \text{epi}_s(f_1 \boxplus f_2)$. By definition of the inf-convolution this means that $\inf_{Y \in V} \{f_1(X - Y) + f_2(Y)\} < a$ which is equivalent to the existence of $\varepsilon > 0$ such that $f_1(X - Y) + f_2(Y) = a - \varepsilon$ for some $Y \in V$. Define $a_1 := f_1(X - Y) + \varepsilon/2$ and $a_2 := f_2(Y) + \varepsilon/2$, then $(X - Y, a_1) \in \text{epi}_s(f_1)$ and $(Y, a_2) \in \text{epi}_s(f_2)$ with $(X - Y, a_1) + (Y, a_2) = (X, a)$ and the reverse inclusion of the first claim follows.

On the other hand, let $(X, a) \in \text{epi}_s(f_1) + \text{epi}_s(f_2)$. Choose $X, Y \in V$ and respective $a_1, a_2 \in \mathbb{R}$ such that $(X - Y, a_1) \in \text{epi}_s(f_1)$ and $(Y, a_2) \in \text{epi}_s(f_2)$, it follows that

$f_1 \boxplus f_2(X) \leq f_1(X - Y) + f_2(Y) < a$ and therefore $(X, a) \in \text{epi}_s(f_1 \boxplus f_2)$.

To prove the second statement, assume that $(X, a_1) \in \text{epi}(f_1 \boxplus f_2)$. We have for all $Y \in \text{dom}(f_2)$ that $f_1(X + Y) - f_2(Y) \leq a_1$ which implies that $(X, a_1) + (Y, f_2(Y)) \in \text{epi}(f_1)$. Hence also $(X, a_1) + (Y, a_2) \in \text{epi}(f_1)$ for all $(Y, a_2) \in \text{epi}(f_2)$. By the definition of the star-difference it follows that $(X, a_1) \in \text{epi}(f_1) \circledast \text{epi}(f_2)$.

To prove the converse, let $(X, a) \in \text{epi}(f_1) \circledast \text{epi}(f_2)$. This statement implies that for all $Y \in \text{dom}(f_2)$ we have $(X, a) + (Y, f_2(Y)) \in \text{epi}(f_1)$. We can rewrite this equation to $f_1(X + Y) - f_2(Y) \leq a$ for all $Y \in \text{dom}(f_2)$ and therefore $(X, a) \in \text{epi}(f_1 \boxplus f_2)$. $\square$

Similarly to the results of Proposition 3.1.4 we would like to show that the inf-convolution and the deconvolution are in fact inverse operations. The equality $\text{epi}_s(f_1) + \text{epi}_s(f_2) = \text{epi}_s(f_1 \boxplus f_2)$ only holds true since we considered strict epigraphs. To get $\text{epi}(f_1) + \text{epi}(f_2) = \text{epi}(f_1 \boxplus f_2)$ we need additional assumptions on f_1 and f_2 . We notice that due to Proposition 2.1.3 we have

$$\begin{aligned} \text{cl}(\text{epi}(f_1)) + \text{cl}(\text{epi}(f_2)) &= \text{cl}(\text{epi}_s(f_1)) + \text{cl}(\text{epi}_s(f_2)) = \text{cl}(\text{epi}_s(f_1) + \text{epi}_s(f_2)) \\ &= \text{cl}(\text{epi}_s(f_1 \boxplus f_2)) = \text{cl}(\text{epi}(f_1 \boxplus f_2)). \end{aligned} \quad (3.1)$$

If we assume that f_1 and f_2 are lower semi-continuous, we would like to drop the closure operator on the right-hand side of (3.1). It is sufficient to show that $f_1 \boxplus f_2$ is lower semi-continuous. We use a similar argument as in Arai [4].

Lemma 3.1.6.

- (1) Let $f_1, f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous functions such that $\text{dom}(f_2)$ is compact. Then $f_1 \boxplus f_2$ is lower semi-continuous.
- (2) Let $f_1 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be lower semi-continuous functions. Then $f_1 \boxminus f_2$ is lower semi-continuous.

PROOF. (1) Let $(X_n)_{n \geq 1} \subset V$ be a sequence converging to some $X \in V$. Due to the generalized Weierstrass theorem, we can find for any $n \geq 1$ a minimizing sequence $(Y_k^n)_{k \geq 1} \subset \text{dom}(f_2)$ of $\inf_{\tilde{Y} \in V} \{f_1(X_n - \tilde{Y}) - f_2(\tilde{Y})\}$. Since $\text{dom}(f_2)$ is compact for any $n \geq 1$ there exists a $Y^n \in \text{dom}(f_2)$ such that the sequence $(Y_k^n)_{k \geq 1}$ converges to Y^n . The lower semi-continuity of f_1 and f_2 implies for any $n \geq 1$

$$\begin{aligned} f_1(X_n - Y^n) + f_2(Y^n) &\leq \liminf_{k \rightarrow \infty} f_1(X_n - Y_k^n) + \liminf_{k \rightarrow \infty} f_2(Y_k^n) \\ &\leq \liminf_{k \rightarrow \infty} (f_1(X_n - Y_k^n) + f_2(Y_k^n)) \\ &= \inf_{\tilde{Y} \in V} \{f_1(X_n - \tilde{Y}) + f_2(\tilde{Y})\}. \end{aligned}$$

Consider the sequence $(Y^n)_{n \geq 1} \subset \text{dom}(f_2)$. Since $\text{dom}(f_2)$ is compact there exists a $Y \in \text{dom}(f_2)$ such that $Y^n \rightarrow Y$ as $n \rightarrow \infty$. Since $X_n \rightarrow X$ also $X_n - Y^n$ converges to $X - Y$. Therefore we have

$$\begin{aligned} f_1(X - Y) + f_2(Y) &\leq \liminf_{n \rightarrow \infty} f_1(X_n - Y^n) + \liminf_{n \rightarrow \infty} f_2(Y^n) \\ &\leq \liminf_{n \rightarrow \infty} (f_1(X_n - Y^n) + f_2(Y^n)). \end{aligned}$$

In conclusion we get

$$\begin{aligned} (f_1 \boxplus f_2)(X) &= \inf_{\tilde{Y} \in V} \{f_1(X - \tilde{Y}) + f_2(\tilde{Y})\} \\ &\leq f_1(X - Y) + f_2(Y) \\ &\leq \liminf_{n \rightarrow \infty} (f_1(X_n - Y^n) + f_2(Y^n)) \\ &= \liminf_{n \rightarrow \infty} \inf_{\tilde{Y} \in V} (f_1(X_n - \tilde{Y}) + f_2(\tilde{Y})) \\ &= \liminf_{n \rightarrow \infty} (f_1 \boxplus f_2)(X_n). \end{aligned}$$

- (2) The lower semi-continuity of f_1 implies that for a sequence $(X_n)_{n \geq 1} \subset V$ converging to some $X \in V$ we have

$$f_1(X - Y) \div f_2(Y) \leq \liminf_{n \rightarrow \infty} (f_1(X_n - Y) \div f_2(Y)) \quad \text{for all } Y \in V.$$

Thus

$$\begin{aligned} (f_1 \boxminus f_2)(X) &= \sup_{Y \in V} \{f_1(X - Y) \div f_2(Y)\} \\ &\leq \sup_{Y \in V} \{\liminf_{n \rightarrow \infty} \{f_1(X_n - Y) \div f_2(Y)\}\} \\ &= \sup_{Y \in V} \left\{ \lim_{m \rightarrow \infty} \left\{ \inf_{n \geq m} \{f_1(X_n - Y) \div f_2(Y)\} \right\} \right\}. \end{aligned} \tag{3.2}$$

Since the supremum of limits is smaller than the limit of suprema and the supremum of infima is smaller than the infimum of suprema we have that the right-hand side of equation (3.2) is smaller than

$$\liminf_{n \rightarrow \infty} \left\{ \sup_{Y \in V} \{f_1(X_n - Y) \div f_2(Y)\} \right\} = \liminf_{n \rightarrow \infty} (f_1 \boxminus f_2)(X_n).$$

It follows that $f_1 \boxminus f_2$ is lower semi-continuous. □

We extend the regularization result of Proposition 3.1.4 to lower semi-continuous functions by using epigraphs.

Proposition 3.1.7. Let $f_1, f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be two lower semi-continuous functions.

- (1) If $\text{dom}(f_2)$ is compact then

$$(f_1 \boxplus f_2) \boxminus f_2 = f_1. \quad (3.3)$$

- (2) If $\text{dom}(f_2)$ is compact and $f_1 \boxminus f_2$ is a proper function, then we have

$$(f_1 \boxminus f_2) \boxplus f_2 = f_1. \quad (3.4)$$

PROOF. (1) We prove the statement by using epigraphs. It follows from Proposition 3.1.5 that $\text{epi}((f_1 \boxplus f_2) \boxminus f_2) = \text{epi}(f_1 \boxplus f_2) \star \text{epi}(f_2)$. Since f_1 and f_2 are lower semi-continuous and $\text{dom}(f_2)$ is compact we can apply Lemma 3.1.6 and conclude that the epigraph of $f_1 \boxplus f_2$ and the epigraph of f_2 are closed.

$$\text{epi}(f_1 \boxplus f_2) \star \text{epi}(f_2) = \text{cl}(\text{epi}(f_1 \boxplus f_2)) \star \text{cl}(\text{epi}(f_2)).$$

Equation (3.1) yields

$$\text{cl}(\text{epi}(f_1 \boxplus f_2)) \star \text{cl}(\text{epi}(f_2)) = (\text{cl}(\text{epi}(f_1)) + \text{cl}(\text{epi}(f_2))) \star \text{cl}(\text{epi}(f_2)).$$

Since $\text{dom}(f_2)$ is compact, we can apply Proposition 3.1.4 to conclude

$$(\text{cl}(\text{epi}(f_1)) + \text{cl}(\text{epi}(f_2))) \star \text{cl}(\text{epi}(f_2)) = \text{cl}(\text{epi}(f_1)). \quad (3.5)$$

Again, we are using the one-to-one correspondence between a function being lower semi-continuous and epigraph of that functions being closed. Thus we can drop the closure operator of the right-hand side of the equation (3.5) and obtain

$$\text{epi}(f_1 \boxplus f_2) \star \text{epi}(f_2) = \text{epi}(f_1).$$

which completes the first part of the proof, since by Proposition 3.1.5 the left-hand side is equal to $\text{epi}((f_1 \boxplus f_2) \boxminus f_1)$.

- (2) To prove (3.4), by assumption $f_1 \boxminus f_2$ is proper and we know from Lemma 3.1.6 (2) that $f_1 \boxminus f_2$ is lower semi-continuous since f_1 is lower semi-continuous. We have by Lemma 3.1.6 (1) that $(f_1 \boxminus f_2) \boxplus f_2$ is lower semi-continuous since $\text{dom}(f_2)$ is compact and f_2 is lower semi-continuous. Thus

$$\begin{aligned} \text{epi}((f_1 \boxminus f_2) \boxplus f_2) &= \text{cl}(\text{epi}((f_1 \boxminus f_2) \boxplus f_2)) = \text{cl}(\text{epi}(f_1 \boxminus f_2)) + \text{cl}(\text{epi}(f_2)) \\ &= \text{cl}(\text{epi}(f_1) \star \text{epi}(f_2)) + \text{cl}(\text{epi}(f_2)) \\ &= (\text{cl}(\text{epi}(f_1)) \star \text{cl}(\text{epi}(f_2))) + \text{cl}(\text{epi}(f_2)). \end{aligned}$$

The last equality follows directly from the last line of Definition 3.1.3 of the star-difference, that is $\text{epi}(f_1) \star \text{epi}(f_2)$ is closed whenever $\text{epi}(f_1)$ is closed. By assumption $f_1 \boxminus f_2$ is a proper function and therefore $\text{epi}(f_1 \boxminus f_2) \neq \emptyset$. By applying Proposition 3.1.4 we complete the proof. $\square$

As an example, we calculate the inf-convolution and deconvolution of two indicator functions.

Example 3.1.8. Let $\delta_{\mathcal{B}}$ and $\delta_{\mathcal{C}}$ be the indicator functions of the non-empty sets $\mathcal{B}$ and $\mathcal{C}$. As a direct consequence of Proposition 3.1.5 we have

$$(1) \quad \delta_{\mathcal{B}} \boxplus \delta_{\mathcal{C}} = \delta_{\mathcal{B}+\mathcal{C}}.$$

$$(2) \quad \text{If additionally, } \mathcal{B} \sharp \mathcal{C} \neq \emptyset, \text{ then } \delta_{\mathcal{B}} \boxminus \delta_{\mathcal{C}} = \delta_{\mathcal{B} \sharp \mathcal{C}}.$$

A simple calculation shows:

$$\begin{aligned} \text{epi}_s(\delta_{\mathcal{B}} \boxplus \delta_{\mathcal{C}}) &= \text{epi}_s(\delta_{\mathcal{B}}) + \text{epi}_s(\delta_{\mathcal{C}}) = (\mathcal{B}, (0, +\infty)) + (\mathcal{C}, (0, +\infty)) \\ &= (\mathcal{B} + \mathcal{C}, (0, +\infty)) = \text{epi}_s(\delta_{\mathcal{B}+\mathcal{C}}), \end{aligned}$$

and

$$\begin{aligned} \text{epi}(\delta_{\mathcal{B}} \boxminus \delta_{\mathcal{C}}) &= \text{epi}(\delta_{\mathcal{B}}) \sharp \text{epi}(\delta_{\mathcal{C}}) = (\mathcal{B}, [0, +\infty)) \sharp (\mathcal{C}, [0, +\infty)) \\ &= (\mathcal{B} \sharp \mathcal{C}, [0, +\infty)) = \text{epi}(\delta_{\mathcal{B} \sharp \mathcal{C}}). \end{aligned} \quad \diamond$$

3.1.2 Subdifferentiability

In one of the main Theorems in this section, Theorem 3.2.8, we deal with the difference of monotone and convex functions. We require that the difference of two monotone and convex functions is itself monotone and convex. One way to obtain this property is to verify that the subdifferentials are negative and monotone.

Definition 3.1.9. We say a multimap $F : V \rightrightarrows V^*$ is *monotone* provided that for any $X, Y \in V$, $X^* \in F(X)$ and $Y^* \in F(Y)$,

$$\langle X^* - Y^*, X - Y \rangle \geq 0. \quad \diamond$$

Theorem 3.1.10. (Borwein and Zhu [12], Theorem 5.1.6) Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper and lower semi-continuous function on a Fréchet smooth Banach space V (see Definition 2.1.13). Suppose that $\partial_F f$ is monotone, then f is convex. $\square$

Next, we characterize the Fréchet subdifferential of the difference of two lower semi-continuous and convex functions, see Amahroq, Penot and Syam [3] for further details. For this theorem, we need the notion of gap-continuity.

Definition 3.1.11. A multimap $F : V \rightrightarrows V^*$ is said to be *gap-continuous* at $X \in V$ if

$$\inf\{d(X_0, Y_0); X_0 \in F(X), X_0 \in F(Y)\} =: \text{gap}(f(X), f(Y)) \rightarrow 0 \text{ as } Y \rightarrow X,$$

where d is the metric of V^* . $\diamond$

Thus F is gap-continuous at some point $X \in V$ if $F(X)$ and $F(Y)$ are not too far apart when Y is sufficiently close to X . Further details on gap-continuity can be found in Penot [61].

Theorem 3.1.12. (Amahroq et al. [3], Theorem 1) Let $f_1, f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semi-continuous functions and ∂f_2 be gap-continuous. Then for $f := f_1 - f_2$ we have

$$\partial_F f(X) = \partial f_1(X) \circledast \partial f_2(X) \quad \text{for all } X \in V. \quad \square$$

Finally, we state the main proposition for characterization of the difference of convex functions. If the functions are Gâteaux-differentiable, we get the classical results from finite dimensional calculus.

Proposition 3.1.13.

- (1) Let V be a Fréchet smooth Banach space and $f_1, f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semi-continuous and assume ∂f_2 is gap-continuous, then $f = f_1 - f_2$ is convex if $\partial f_1 \circledast \partial f_2$ is monotone.
- (2) Assume that f_1 and f_2 are Gâteaux-differentiable, then $f = f_1 - f_2$ is monotonically decreasing and convex if $\nabla f_1 - \nabla f_2$ is negative and monotone.

PROOF. (1) Combining Theorem 3.1.10 and Theorem 3.1.12.

- (2) Follows from the Definition 2.1.15. □

3.1.3 Dual Operations

Suppose we perform some operation on given functions f_1, f_2 such as adding or subtracting them or taking the inf-convolution or deconvolution. We want to characterize the conjugate of the resulting function in terms of f_1^* and f_2^* . It turns out that the duality correspondence, under certain conditions, converts an operation into another operation (and vice versa). Thus, the operations arrange themselves in dual pairs. As we will see, multiplication of scalars and epi-multiplication, addition and inf-convolution and subtraction and deconvolution will be these pairs.

These results are well known in convex analysis for finite dimensional spaces. See for example Rockafellar [62] for the epi-multiplication, inf-convolution and sum, and Hiriart-Urruty [40] for the deconvolution and the difference. The results can be carried out in more general settings, for example Van Tiel [74] treats the inf-convolution on normed linear space. This section will be crucial for the dual representation of linear combinations of risk measures, which we will state in the next Section 3.2. Therefore we will adjust these dual operations to our setting and prove the duality correspondence. To our knowledge the generalization of the deconvolution and the difference has not been done before.

We start with the definition of the epi-multiplication.

Definition 3.1.14. Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function. The *epi-multiple* $\gamma * f$ for a scalar $\gamma \geq 0$ is defined by

$$\begin{aligned} (\gamma * f)(X) &:= \gamma f(\gamma^{-1}X) \quad \text{for all } \gamma > 0, \\ (0 * f)(X) &:= 0. \end{aligned}$$

◇

Proposition 3.1.15. Let $f : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function and $\gamma \geq 0$. Then

$$(\gamma * f)^* = \gamma f^* \text{ and } (\gamma f)^* = \gamma * f^*.$$

PROOF. The two properties are following directly from the definition of the conjugate. For each $Z \in V^*$, we have

$$\begin{aligned} (\gamma * f)^*(Z) &= \sup_{X \in V} \{ \langle X, Z \rangle - (\gamma * f)(X) \} = \sup_{X \in V} \{ \gamma \langle \gamma^{-1}X, Z \rangle - \gamma f(\gamma^{-1}X) \} \\ &= \gamma f^*(Z), \end{aligned}$$

and

$$\begin{aligned} (\gamma f)^*(Z) &= \sup_{X \in V} \{ \langle X, Z \rangle - \gamma f(X) \} = \sup_{X \in V} \{ \gamma \langle X, \gamma^{-1}Z \rangle - \gamma f(X) \} \\ &= (\gamma * f^*)(Z), \end{aligned}$$

which completes the proof. □

Thus multiplication of scalars and epi-multiplication form a dual pair. We proceed with two propositions to show that inf-convolution and addition are dual to each other. These results are well known, see for example Rockafellar [62], Theorem 16.4 for the finite dimensional case. Especially, the result of the next statement, Proposition 3.1.16, is often used in the theory of risk measures.

Proposition 3.1.16. Let $f_1, f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper functions. Then

$$(f_1 \boxplus f_2)^* = f_1^* + f_2^*.$$

PROOF. For each $X \in V^*$, we have

$$\begin{aligned} (f_1 \boxplus f_2)^*(Z) &= \sup_{X \in V} \left\{ \langle X, Z \rangle - \inf_{\substack{X_1 + X_2 = X \\ X_1, X_2 \in V}} \{f_1(X_1) + f_2(X_2)\} \right\} \\ &= \sup_{X \in V} \left\{ \sup_{\substack{X_1 + X_2 = X \\ X_1, X_2 \in V}} \{ \langle X_1 + X_2, Z \rangle - f_1(X_1) - f_2(X_2) \} \right\} \\ &= \sup_{X_1 \in V} \left\{ \sup_{X_2 \in V} \{ (\langle X_1, Z \rangle - f_1(X_1)) + (\langle X_2, Z \rangle - f_2(X_2)) \} \right\} \\ &= f_1^*(Z) + f_2^*(Z). \end{aligned}$$

□

Proposition 3.1.17. Let $f_1, f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower semi-continuous, convex functions. Then

$$(f_1 + f_2)^* = \text{cl}(f_1^* \boxplus f_2^*). \quad (3.6)$$

If, in addition, $\text{dom}(f_2^*)$ is compact and $f_1^* \boxplus f_2^*$ is proper then the closure operator in (3.6) can be dropped, and we obtain the following representation

$$(f_1 + f_2)^* = f_1^* \boxplus f_2^*. \quad (3.7)$$

PROOF. We know from the definition of the conjugate that f_1^* and f_2^* are proper, convex and lower semi-continuous. We can apply Proposition 3.1.16 to the functions f_1^* and f_2^* and receive

$$f_1^{**} + f_2^{**} = (f_1^* \boxplus f_2^*)^*. \quad (3.8)$$

Since f_1 and f_2 are proper, lower semi-continuous and convex, we have by Theorem 2.1.5, that $f_1 = f_1^{**}$ and $f_2 = f_2^{**}$. Substituting these results in (3.8) yields

$$f_1 + f_2 = (f_1^* \boxplus f_2^*)^*. \quad (3.9)$$

Taking the conjugates for the equation (3.9), we obtain

$$(f_1 + f_2)^* = (f_1^* \boxplus f_2^*)^{**}. \quad (3.10)$$

By Theorem 2.1.6 the right-hand side of (3.10) is equal to $\text{cl}(\text{co}(f_1^* \boxplus f_2^*))$. The conjugates f_1^* and f_2^* are convex, it follows from a small calculation that $f_1^* \boxplus f_2^*$ is convex as well

$$\begin{aligned} & (f_1^* \boxplus f_2^*)(\gamma Z_1 + (1 - \gamma)Z_2) \\ &= \inf_{Y \in V^*} \{f_1^*(\gamma Z_1 + (1 - \gamma)Z_2 - Y) + f_2^*(Y)\} \\ &= \inf_{Y_1, Y_2 \in V^*} \{f_1^*(\gamma Z_1 + (1 - \gamma)Z_2 - \gamma Y_1 - (1 - \gamma)Y_2) + f_2^*(\gamma Y_1 + (1 - \gamma)Y_2)\} \\ &\leq \inf_{Y_1, Y_2 \in V^*} \{\gamma f_1^*(Z_1 - Y_1) + (1 - \gamma)f_2^*(Z_2 - Y_2) + \gamma f_2^*(Y_1) + (1 - \gamma)f_2^*(Y_2)\} \\ &= \gamma \inf_{Y_1 \in V^*} \{f_1^*(Z_1 - Y_1) + f_2^*(Y_1)\} + (1 - \gamma) \inf_{Y_2 \in V^*} \{f_1^*(Z_2 - Y_2) + f_2^*(Y_2)\} \\ &= \gamma(f_1^* \boxplus f_2^*)(Z_1) + (1 - \gamma)(f_1^* \boxplus f_2^*)(Z_2). \end{aligned}$$

Therefore we have

$$(f_1 + f_2)^* = (f_1^* \boxplus f_2^*)^{**} = \text{cl}(\text{co}(f_1^* \boxplus f_2^*)) = \text{cl}(f_1^* \boxplus f_2^*).$$

If, in addition, $f_1^* \boxplus f_2^*$ is proper and $\text{dom}(f_2^*)$ is compact, then by Lemma 3.1.6 $f_1^* \boxplus f_2^*$ is lower semi-continuous so we have $\text{cl}(f_1^* \boxplus f_2^*) = f_1^* \boxplus f_2^*$. This proves equation (3.7). $\square$

The definition of the deconvolution is given in Definition 3.1.2. Hiriart-Urruty [40] investigates the relation between subtraction and deconvolution. The deconvolution is a result of the conjugation of the difference of two convex functions. The following two propositions are generalizations of Theorem 3.1 and Corollary 3.2 of Hiriart-Urruty [40] from finite to infinite dimensional spaces.

Proposition 3.1.18. Let $f_1 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper function and $f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semi-continuous function such that $\text{dom}(f_1) \subset \text{dom}(f_2)$. Then

$$(f_1 - f_2)^* = f_1^* \boxminus f_2^*.$$

PROOF. The assumption $\text{dom}(f_1) \subset \text{dom}(f_2)$ ensures that $f_1 - f_2$ is proper. Let us consider a $\tilde{Z} \in \text{dom}(f_2^*)$,

$$\begin{aligned} (f_1 - f_2)^*(\tilde{Z}) &= \sup_{X \in \text{dom}(f_1)} \{ \langle X, \tilde{Z} \rangle - (f_1(X) - f_2(X)) \} \\ &= \sup_{Z \in \text{dom}(f_2^*)} \left\{ \sup_{X \in \text{dom}(f_1)} \{ \langle X, \tilde{Z} + Z \rangle - f_1(X) \} - (\langle X, Z \rangle - f_2(X)) \right\} \\ &\geq \sup_{Z \in \text{dom}(f_2^*)} \left\{ \sup_{X \in \text{dom}(f_1)} \{ \langle X, \tilde{Z} + Z \rangle - f_1(X) \} \right. \\ &\quad \left. - \sup_{\tilde{X} \in \text{dom}(f_2)} \{ \langle \tilde{X}, Z \rangle - f_2(\tilde{X}) \} \right\} \\ &= \sup_{Z \in \text{dom}(f_2^*)} \{ f_1^*(\tilde{Z} + Z) - f_2^*(Z) \} \\ &= (f_1^* \boxminus f_2^*)(\tilde{Z}). \end{aligned}$$

Next, we want to prove that $(f_1 - f_2)^*(\tilde{Z}) \leq (f_1^* \boxminus f_2^*)(\tilde{Z})$. In the case of $(f_1^* \boxminus f_2^*)(\tilde{Z}) = +\infty$ there is nothing to prove. For a given $\tilde{Z} \in V^*$, let $(f_1^* \boxminus f_2^*)(\tilde{Z}) = a$ for some $a \in \mathbb{R}$. Then

$$f_1^*(\tilde{Z} + Z) - f_2^*(Z) \leq \sup_{Z \in \text{dom}(f_2^*)} \{ f_1^*(\tilde{Z} + Z) - f_2^*(Z) \} = f_1^* \boxminus f_2^*(\tilde{Z}) = a,$$

for all $Z \in \text{dom}(f_2)$. It follows that $f_1^*(\tilde{Z} + Z) \leq f_2^*(Z) + a$. Taking conjugates on both sides with respect to Z yields

$$f_1^{**}(X) - \langle X, \tilde{Z} \rangle \geq f_2^{**}(X) - a.$$

Since f_2 is proper, convex and lower semi-continuous, by Theorem 2.1.5, we have $f_2^{**} = f_2$. Since $f_1^{**} \leq f_1$, we have $\langle X, \tilde{Z} \rangle - (f_1 - f_2)(X) \leq a$ for all $X \in \text{dom}(f_1)$. Hence $(f_1 - f_2)^*(\tilde{Z}) \leq (f_1^* \boxminus f_2^*)(\tilde{Z})$. This completes the proof. $\square$

Proposition 3.1.19. Let $f_1, f_2 : V \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, convex and lower semi-continuous functions such that $\text{dom}(f_1^*) \subset \text{dom}(f_2^*)$. Then

$$(f_1 \boxminus f_2)^* = \text{cl}(\text{co}(f_1^* - f_2^*)). \quad (3.11)$$

If in addition, $f_1^* - f_2^*$ is convex and lower semi-continuous, then

$$(f_1 \boxminus f_2)^* = f_1^* - f_2^*. \quad (3.12)$$

PROOF. We know from the definition of the conjugates that f_1^* and f_2^* are proper, convex and lower semi-continuous. By assumption $\text{dom}(f_1^*) \subset \text{dom}(f_2^*)$ thus we can apply Proposition 3.1.18 to the functions f_1^* and f_2^* and obtain

$$f_1^{**} \boxplus f_2^{**} = (f_1^* - f_2^*)^*.$$

Especially we have $f_1^{**} \boxplus f_2^{**} \not\equiv +\infty$ since $\text{dom}(f_1^*) \subset \text{dom}(f_2^*)$. Since we assumed that f_1 and f_2 are proper, convex and lower semi-continuous, we have

$$f_1 \boxminus f_2 = f_1^{**} \boxplus f_2^{**} = (f_1^* - f_2^*)^*.$$

Taking conjugates on both sides yields

$$(f_1 \boxminus f_2)^* = (f_1^* - f_2^*)^{**}.$$

By Theorem 2.1.6 $(f_1^* - f_2^*)^{**} = \text{cl}(\text{co}(f_1^* - f_2^*))$, this proves (3.11). It follows from $\text{dom}(f_1^*) \subset \text{dom}(f_2^*)$ that $f_1^* - f_2^*$ is proper. If we assume that $f_1^* - f_2^*$ is convex and lower semi-continuous as well, we can apply Theorem 2.1.5 and as a result $(f_1^* - f_2^*)^{**} = f_1^* - f_2^*$ follows. This proves the second statement 3.12 of the proposition. $\square$

As an example we derive the inf-convolution, sum, deconvolution and the difference of two support functions.

Example 3.1.20. Let δ_B^* and δ_C^* be the support functions of non-empty sets $\mathcal{B}$ and $\mathcal{C}$. We have:

$$(1) \quad \delta_B^* \boxplus \delta_C^* = \delta_{B \cap C}^*.$$

This statement follows from Proposition 3.1.17 and the fact that $\delta_B + \delta_C = \delta_{B \cap C}$.

$$(2) \quad \delta_B^* + \delta_C^* = \delta_{B+C}^*.$$

This is a direct consequence of Proposition 3.1.16 and Example 3.1.8 (1).

$$(3) \quad \text{If } \mathcal{C} \text{ is a convex and closed set and } \mathcal{B} \subset \mathcal{C}, \text{ then } \delta_B^* \boxminus \delta_C^* = \delta_B^*.$$

The condition $\mathcal{B} \subset \mathcal{C}$ is equivalent to $\text{dom}(\delta_B) \subset \text{dom}(\delta_C)$, thus we can apply Proposition 3.1.18. The result follows from the observation that $\delta_B - \delta_C = \delta_B$ if $\mathcal{B} \subset \mathcal{C}$.

- (4) If $\mathcal{B}$ and $\mathcal{C}$ are convex and closed sets such that $\mathcal{B} \neq \mathcal{C} \neq \emptyset$, then $\text{cl}(\text{co}(\delta_{\mathcal{B}}^* - \delta_{\mathcal{C}}^*)) = \delta_{\mathcal{B} \pm \mathcal{C}}^*$.

The combination of the results of Proposition 3.1.19 and Example 3.1.8 (2) yields the assertion. $\diamond$

3.2 Combinations and Convolutions of Risk Measures and their Dual Representation

In this section, the main results of the chapter are stated. We study linear combinations and convolutions of convex risk measures on L^p -spaces. The terminology and notation of L^p -spaces are as in Section 2.2. We apply the results of the last section to convex risk measures and give conditions such that the resulting function is itself a risk measure. Further we investigate basic properties of the new risk measures.

3.2.1 Epi-multiplication of a Risk Measure

We start with simplest example of a combination of risk measures which itself is a risk measure, the epi-multiplication. Barrieu and El Karoui [8] first characterized epi-multiplication of a risk measure, which they called dilated risk measure. Even though structure of the dilated risk measure is very simple, the important entropic risk measure, see Example 2.2.9, belongs to the class of dilated risk measures.

Theorem 3.2.1. Let ρ be a convex risk measure on $L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ and let the epi-multiplication, see Definition 3.1.14, of a scalar $\lambda > 0$ and the convex risk measure ρ be given by

$$\phi(X) := (\lambda * \rho)(X) \quad \text{for all } X \in L^p.$$

The following statements hold:

- (1) The epi-multiplication of ρ by λ is a convex risk measure and admits the dual representation

$$\phi(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_{\phi}(\mathbb{P}) \right\} \quad \text{for all } X \in L^p.$$

- (2) If ρ is a coherent risk measure, then $\phi = \rho$.

- (3) The penalty function of ϕ is given by

$$\alpha_{\phi}(\mathbb{P}) = \lambda \alpha_{\rho}(\mathbb{P}) \quad \text{for all } \mathbb{P} \in \mathcal{P}.$$

- (4) The acceptance set of ϕ satisfies

$$\mathcal{A}_\phi = \lambda \mathcal{A}_\rho. \quad (3.13)$$

PROOF. (1) All the properties which define a convex risk measure are following directly from ρ .

- (2) The proof is straightforward. By translation invariance of ρ , we have for all $\lambda > 0$ and $X \in L^p$ that

$$\phi(X) = \lambda \rho(\lambda^{-1}X) = \lambda \cdot \lambda^{-1} \rho(X) = \rho(X).$$

- (3) We recall from Theorem 2.2.5 that the penalty function of a risk measure ρ is defined by $\alpha_\rho(\mathbb{P}) := \rho^*(-d\mathbb{P}/d\mu)$ for all $\mathbb{P} \in \mathcal{P}$. It follows from Proposition 3.1.15 that the penalty function α_ϕ is given by

$$\phi^* = (\lambda * \rho)^* = \lambda \rho^*.$$

- (4) Let $X \in \mathcal{A}_\phi$, then $\lambda \rho(\lambda^{-1}X) = \phi(X) \leq 0$. It follows that $\lambda^{-1}X \in \mathcal{A}_\rho$ and therefore $X \in \lambda \mathcal{A}_\rho$. We conclude that $\mathcal{A}_\phi \subset \lambda \mathcal{A}_\rho$.

To prove the converse, let $X \in \lambda \mathcal{A}_\rho$ then $\rho(\lambda^{-1}X) \leq 0$. It follows that $\phi(X) = \lambda \rho(\lambda^{-1}X) \leq 0$ which proves $\mathcal{A}_\phi \supset \lambda \mathcal{A}_\rho$ and hence (3.13) holds. $\square$

3.2.2 Inf-convolution of Risk Measures

We review the basic properties of the inf-convolution of two convex risk measures, and the inf-convolution of a convex risk measure and a convex set. The inf-convolution of two risk measures is the most studied example of a combination of different risk measures in the literature. We state these results for the sake of completeness.

The inf-convolution of convex risk measures was introduced in Delbaen [22] and Barrieu and El Karoui [8], [9] in the L^∞ framework. Toussaint and Sircar [72] extended these results to L^2 , and Arai [4] characterized the inf-convolution of convex risk measures on Orlicz spaces.

Theorem 3.2.2. (Arai [4], Proposition 2) Let ρ_1, ρ_2 be two convex risk measures on $L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ with acceptance sets $\mathcal{A}_{\rho_1}$ and $\mathcal{A}_{\rho_2}$. We define ϕ by

$$\phi(X) := (\rho_1 \boxplus \rho_2)(X) \quad \text{for all } X \in L^p.$$

If $\text{dom}(\rho_2)$ is compact, then the following statements hold:

- (1) The inf-convolution of ρ_1 and ρ_2 is a convex risk measure and admits the dual representation

$$\phi(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_\phi(\mathbb{P}) \right\} \quad \text{for all } X \in L^p.$$

(2) If ρ_1 and ρ_2 are coherent risk measures, then so is ϕ .

(3) The penalty function of ϕ is given by

$$\alpha_\phi(\mathbb{P}) = \alpha_{\rho_1}(\mathbb{P}) + \alpha_{\rho_2}(\mathbb{P}) \quad \text{for all } \mathbb{P} \in \mathcal{P}.$$

(4) The acceptance set of ϕ satisfies

$$\mathcal{A}_\phi := \mathcal{A}_{\rho_1 \boxplus \rho_2} = \mathcal{A}_{\rho_1} + \mathcal{A}_{\rho_2}.$$

(5) The inf-convolution of ρ_1 and ρ_2 can be written as

$$(\rho_1 \boxplus \rho_2)(X) = (\rho_1 \boxplus \mathcal{A}_{\rho_2})(X) = (\mathcal{A}_{\rho_1} \boxplus \rho_2)(X) \quad \text{for all } X \in L^p. \quad \square$$

Theorem 3.2.3. (Arai [4], Proposition 1 and Propostion 3) Let ρ be a convex risk measure on $L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\mathcal{A}$ a non-empty, compact set containing 0, then we have the following properties:

(1) The inf-convolution of ρ and the set $\mathcal{A}$ given by $\rho \boxplus \mathcal{A}$ is a convex risk measure on L^p .

(2) If ρ is a coherent risk measure and $\mathcal{A}$ is a cone then $\rho \boxplus \mathcal{A}$ is a coherent risk measure.

(3) If $\mathcal{A}$ is translative and closed then

$$\rho \boxplus \rho_{\mathcal{A}} = \rho \boxplus \mathcal{A},$$

and the penalty function of $\rho \boxplus \rho_{\mathcal{A}}$ is given by

$$\alpha_{\rho \boxplus \rho_{\mathcal{A}}}(\mathbb{P}) = \alpha_\rho(\mathbb{P}) + \delta_{\mathcal{A}}^*(-d\mathbb{P}/d\mu) \quad \text{for all } \mathbb{P} \in \mathcal{P}. \quad \square$$

3.2.3 Sum of Risk Measures

We prove that the weighted sum of convex risk measures is again a convex risk measure. The proof is straightforward since a linear combination of monotone, convex and lower semi-continuous functions has again these properties. The representation result of the penalty function is more subtle.

Theorem 3.2.4. Let ρ_1, ρ_2 be two convex risk measures on $L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ with acceptance sets $\mathcal{A}_{\rho_1}$ and $\mathcal{A}_{\rho_2}$ and scalar $\lambda \in (0, 1)$. Define ϕ as

$$\phi(X) := (1 - \lambda)\rho_1(X) + \lambda\rho_2(X).$$

Then ϕ has the following properties:

- (1) ϕ is a convex risk measure and admits the dual representation

$$\phi(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_{\phi}(\mathbb{P}) \right\} \quad \text{for all } X \in L^p. \quad (3.14)$$

- (2) If ρ_1 and ρ_2 are coherent risk measures, then ϕ is coherent as well.

- (3) The penalty function of ϕ is given by

$$\alpha_{\phi}(\mathbb{P}) = \text{cl}(((1 - \lambda) * \alpha_{\rho_1}) \boxplus (\lambda * \alpha_{\rho_2}))(\mathbb{P}) \quad \text{for all } \mathbb{P} \in \mathcal{P}. \quad (3.15)$$

If $\text{dom}(\alpha_{\rho_2})$ is compact and $((1 - \lambda) * \alpha_{\rho_1}) \boxplus (\lambda * \alpha_{\rho_2})$ is proper, then

$$\begin{aligned} \alpha_{\phi}(\mathbb{P}) &= ((1 - \lambda) * \alpha_{\rho_1}) \boxplus (\lambda * \alpha_{\rho_2})(\mathbb{P}) \\ &= \inf_{\substack{(1-\lambda)\mathbb{P}_1 + \lambda\mathbb{P}_2 = \mathbb{P} \\ \mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}}} \{ (1 - \lambda)\alpha_{\rho_1}(\mathbb{P}_1) + \lambda\alpha_{\rho_2}(\mathbb{P}_2) \} \quad \text{for all } \mathbb{P} \in \mathcal{P}. \end{aligned} \quad (3.16)$$

- (4) The acceptance set of ϕ satisfies

$$\mathcal{A}_{\phi} = \mathcal{A}_{(1-\lambda)\rho_1 + \lambda\rho_2} = \bigcup_{m \in \mathbb{R}} \left\{ (\mathcal{A}_{\rho_1} + (1 - \lambda)^{-1}m) \cap (\mathcal{A}_{\rho_2} - \lambda^{-1}m) \right\}. \quad (3.17)$$

PROOF. (1) We check the properties of a convex risk measure. The linear combination of monotone functions with positive scalars is monotone. A short calculation shows that ϕ is translation invariant

$$\begin{aligned} \phi(X + m) &= (1 - \lambda)\rho_1(X + m) + \lambda\rho_2(X + m) \\ &= (1 - \lambda)\rho_1(X) + \lambda\rho_2(X) - m = \phi(X) - m \quad \text{for all } X \in L^p. \end{aligned}$$

The linear combination of convex and lower semi-continuous functions with positive scalars is convex and lower semi-continuous. Since $\rho_1(0) < +\infty$ and $\rho_2(0) < +\infty$, it follows that $\phi(0) < +\infty$. Thus ϕ satisfies the properties of a convex risk measure and by Theorem 2.1.5, ϕ admits the dual representation (3.14).

- (2) The proof is straightforward. We have for all $\gamma > 0$ and $X \in L^p$

$$\begin{aligned} \phi(\gamma X) &= (1 - \lambda)\rho_1(\gamma X) + \lambda\rho_2(\gamma X) = \gamma((1 - \lambda)\rho_1(X) + \lambda\rho_2(X)) \\ &= \gamma\phi(X). \end{aligned}$$

It is left to prove $\phi(0) = 0$. We have $\phi(0) = (1 - \lambda)\rho_1(0) + \lambda\rho_2(0) = 0$, since $\rho_1(0) = \rho_2(0) = 0$.

- (3) We notice that ρ_1 and ρ_2 are proper, convex and lower semi-continuous, we apply Proposition 3.1.15 and Proposition 3.1.17 to the penalty function of ϕ

$$\begin{aligned} \phi^* &= ((1 - \lambda)\rho_1 + \lambda\rho_2)^* = \text{cl}(((1 - \lambda)\rho_1)^* \boxplus (\lambda\rho_2)^*) \\ &= \text{cl}(((1 - \lambda) * \rho_1^*) \boxplus (\lambda * \rho_2^*)). \end{aligned} \quad (3.18)$$

The representation result (3.15) follows. If, in addition, $\text{dom}(\alpha_{\rho_2})$ is compact and $((1 - \lambda) * \alpha_{\rho_1}) \boxplus (\lambda * \alpha_{\rho_2})$ is proper then the closure operator in (3.18) can be dropped and we obtain the representation given in (3.16).

(4) Let $X \in \mathcal{A}_\phi$, then $\phi(X) = (1 - \lambda)\rho_1(X) + \lambda\rho_2(X) \leq 0$ and we have

$$(1 - \lambda)\rho_1(X - (1 - \lambda)^{-1}m) + \lambda\rho_2(X + \lambda^{-1}m) \leq 0 \quad \text{for all } m \in \mathbb{R}. \quad (3.19)$$

We choose $m = \rho_1(X)$, by translation invariance we have $(1 - \lambda)\rho_1(X - (1 - \lambda)^{-1}m) = 0$. It follows from (3.19) that $\lambda\rho_2(X + \lambda^{-1}m) \leq 0$. We obtain that $X \in \mathcal{A}_{\rho_1} + (1 - \lambda)^{-1}m$ and $X \in \mathcal{A}_{\rho_2} - \lambda^{-1}m$. This shows

$$\mathcal{A}_\phi \subset \bigcup_{m \in \mathbb{R}} \left\{ (\mathcal{A}_{\rho_1} + (1 - \lambda)^{-1}m) \cap (\mathcal{A}_{\rho_2} - \lambda^{-1}m) \right\}.$$

To prove the converse, let $X \in \bigcup_{m \in \mathbb{R}} \left\{ (\mathcal{A}_{\rho_1} + (1 - \lambda)^{-1}m) \cap (\mathcal{A}_{\rho_2} - \lambda^{-1}m) \right\}$. Thus, there exists an $\tilde{m} \in \mathbb{R}$ with $X \in \mathcal{A}_{\rho_1} + (1 - \lambda)^{-1}\tilde{m}$ and $X \in \mathcal{A}_{\rho_2} - \lambda^{-1}\tilde{m}$. It follows that $(1 - \lambda)\rho_1(X - (1 - \lambda)^{-1}\tilde{m}) \leq 0$ and $\lambda\rho_2(X + \lambda^{-1}\tilde{m}) \leq 0$, by translation invariance we have

$$\begin{aligned} 0 &\geq (1 - \lambda)\rho_1(X - (1 - \lambda)^{-1}\tilde{m}) + \lambda\rho_2(X + \lambda^{-1}\tilde{m}) \\ &= (1 - \lambda)\rho_1(X) + \lambda\rho_2(X) = \phi(X). \end{aligned}$$

Since the argument holds for any $\tilde{m} \in \mathbb{R}$ it follows that

$$\mathcal{A}_\phi \supset \bigcup_{m \in \mathbb{R}} \left\{ (\mathcal{A}_{\rho_1} + (1 - \lambda)^{-1}m) \cap (\mathcal{A}_{\rho_2} - \lambda^{-1}m) \right\}.$$

Thus equation (3.17) is satisfied. □

3.2.4 Deconvolution of Risk Measures

We have seen in Theorem 3.2.2 that the penalty function of inf-convolution of two convex risk measures is given by the sum of the penalty functions of the two risk measures and the acceptance set is given by the sum of the acceptance sets. In case of the deconvolution the penalty function is given by the difference of the penalty functions of the two risk measures. The acceptance set of the deconvolution is given by the star-difference of the acceptance sets of the two given risk measures.

Theorem 3.2.5. Let ρ_1, ρ_2 be two convex risk measures on $L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ with acceptance sets $\mathcal{A}_{\rho_1}$ and $\mathcal{A}_{\rho_2}$. We define ϕ by

$$\phi(X) := (\rho_1 \boxminus \rho_2)(X) \quad \text{for all } X \in L^p. \quad (3.20)$$

If $\rho_1 \boxminus \rho_2$ is finite at 0 (which for example is ensured whenever $\rho_2 \succcurlyeq \rho_1$), then the following statements hold:

- (1) The sup-convolution of ρ_1 and ρ_2 is a convex risk measure and admits the dual representation

$$\phi(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_{\phi}(\mathbb{P}) \right\} \quad \text{for all } X \in L^p. \quad (3.21)$$

- (2) If ρ_1 and ρ_2 are coherent risk measures, then so is ϕ . In this case, finiteness at 0 of ϕ is equivalent to $\rho_2 \succcurlyeq \rho_1$.

- (3) Let $\text{dom}(\alpha_{\rho_1}) \subset \text{dom}(\alpha_{\rho_2})$ then the penalty function of ϕ is given by

$$\alpha_{\phi}(\mathbb{P}) = \text{cl}(\text{co}(\alpha_{\rho_1}(\mathbb{P}) - \alpha_{\rho_2}(\mathbb{P}))) \quad \text{for all } \mathbb{P} \in \mathcal{P}. \quad (3.22)$$

If in addition, $\alpha_{\rho_1} - \alpha_{\rho_2}$ is convex and lower semi-continuous, then

$$\alpha_{\phi}(\mathbb{P}) = \alpha_{\rho_1}(\mathbb{P}) - \alpha_{\rho_2}(\mathbb{P}) \quad \text{for all } \mathbb{P} \in \mathcal{P}. \quad (3.23)$$

- (4) If $\text{dom}(\alpha_{\rho_1}) \subset \text{dom}(\alpha_{\rho_2})$ and $\mathcal{A}_{\rho_1} \stackrel{*}{\neq} \mathcal{A}_{\rho_2} \neq \emptyset$, then the acceptance set of ϕ satisfies

$$\mathcal{A}_{\phi} := \mathcal{A}_{\rho_1 \boxminus \rho_2} = \mathcal{A}_{\rho_1} \stackrel{*}{\neq} \mathcal{A}_{\rho_2}.$$

PROOF. (1) First, we want to prove monotonicity of ϕ . We know that ρ_1 is monotone, therefore we have for any $X_1, X_2 \in L^p$ such that $X_1 \leq X_2$ that $\rho_1(X_1 + Y) \geq \rho_1(X_2 + Y)$ for all $Y \in L^p$. Subtracting ρ_2 and taking the supremum on both sides yield the result, i.e.

$$\begin{aligned} \phi(X_1) &= \sup_{Y \in L^p} \{ \rho_1(X_1 + Y) - \rho_2(Y) \} \geq \sup_{Y \in L^p} \{ \rho_1(X_2 + Y) - \rho_2(Y) \} \\ &= \phi(X_2). \end{aligned}$$

Translation invariance follows from the translation invariance property of ρ_1 , i.e

$$\begin{aligned} \phi(X + m) &= \sup_{Y \in L^p} \{ \rho_1(X + m + Y) - \rho_2(Y) \} \\ &= \sup_{Y \in L^p} \{ \rho_1(X + Y) - \rho_2(Y) \} - m \\ &= \phi(X) - m \quad \text{for all } X \in L^p. \end{aligned}$$

The convexity of ϕ follows from the convexity of ρ_1 . A small calculation shows that for any $X_1, X_2 \in L^p$,

$$\begin{aligned}
 & \phi(\gamma X_1 + (1 - \gamma)X_2) \\
 &= \sup_{Y \in L^p} \{\rho_1(\gamma(X_1 + Y) + (1 - \gamma)(X_2 + Y)) \div \rho_2(Y)\} \\
 &\leq \sup_{Y \in L^p} \{\gamma\rho_1(X_1 + Y) + (1 - \gamma)\rho_1(X_2 + Y) \div \gamma\rho_2(Y) \div (1 - \gamma)\rho_2(Y)\} \\
 &\leq \sup_{Y_1, Y_2 \in L^p} \{\gamma\rho_1(X_1 + Y_1) + (1 - \gamma)\rho_1(X_2 + Y_2) \div \gamma\rho_2(Y_1) \div (1 - \gamma)\rho_2(Y_2)\} \\
 &= \gamma \sup_{Y_1 \in L^p} \{\rho_1(X_1 + Y_1) \div \rho_2(Y_1)\} + (1 - \gamma) \sup_{Y_2 \in L^p} \{\rho_1(X_2 + Y_2) \div \rho_2(Y_2)\} \\
 &= \gamma\phi(X_1) + (1 - \gamma)\phi(X_2).
 \end{aligned}$$

Lower semi-continuity of ϕ follows from Lemma 3.1.6. By assumption we have

$$\phi(0) = \sup_{Y \in L^p} \{\rho_1(Y) \div \rho_2(Y)\} < +\infty,$$

this can be ensured if $\rho_2 \succcurlyeq \rho_1$. We have shown that ϕ is monotone, translation invariant, convex, lower semi-continuous and assumed that ϕ is finite at 0. Therefore ϕ is a convex risk measure and by Theorem 2.1.5, ϕ admits the dual representation (3.21).

- (2) For all $\gamma > 0$ and $X \in L^p$ also $\gamma^{-1}X \in L^p$. By positive homogeneity of the risk measures ρ_1 and ρ_2 we have for any $\gamma > 0$

$$\begin{aligned}
 \phi(\gamma X) &= \sup_{Y \in L^p} \{\rho_1(\gamma X + Y) \div \rho_2(Y)\} = \gamma \sup_{Y \in L^p} \{\rho_1(X + \gamma^{-1}Y) \div \rho_2(\gamma^{-1}Y)\} \\
 &= \gamma\phi(X) \quad \text{for all } X \in L^p.
 \end{aligned}$$

Next, we prove that for coherent risk measures ρ_1 and ρ_2 , we have $\phi(0) = 0$ if and only if $\rho_2 \succcurlyeq \rho_1$. If $\rho_2 \succcurlyeq \rho_1$ then

$$\phi(0) = \sup_{Y \in L^p} \{\rho_1(Y) \div \rho_2(Y)\} \leq 0,$$

with $\rho_1(0) \div \rho_2(0) = 0$. On the other hand, assume that $\phi(0) = 0$, the positive homogeneity of ρ_1 and ρ_2 yields for any $X \in L^p$ that

$$0 = \phi(0) \geq \lim_{\gamma \rightarrow \infty} \{\rho_1(\gamma X) \div \rho_2(\gamma X)\} = \lim_{\gamma \rightarrow \infty} \gamma \cdot \{\rho_1(X) \div \rho_2(X)\}. \quad (3.24)$$

Equation (3.24) can only hold true if $\rho_1(X) \leq \rho_2(X)$ for all $X \in L^p$ which means $\rho_2 \succcurlyeq \rho_1$.

- (3) By assumption $\text{dom}(\alpha_{\rho_1}) \subset \text{dom}(\alpha_{\rho_2})$, additionally we have that ρ_1 and ρ_2 are proper, convex and lower semi-continuous. By Proposition 3.1.19 the penalty function α_ϕ is given by

$$\phi^* = (\rho_1 \boxminus \rho_2)^* = \text{cl}(\text{co}(\rho_1^* - \rho_2^*)),$$

which is the convex closure of the difference of α_{ρ_1} and α_{ρ_2} . The representation (3.22) follows.

If $\alpha_{\rho_1} - \alpha_{\rho_2}$ is convex and lower semi-continuous then the convex hull and closure operator can be omitted which gives (3.23).

- (4) We prove the assertion by a similar argument as the proof of Theorem 7 (c) of Klöppel and Schweizer [46] and Proposition 2 (d) of Arai [4]. Let $X \in \mathcal{A}_{\rho_1 \boxminus \rho_2}$ be given, we have for all $\tilde{Y} \in \mathcal{A}_{\rho_2}$

$$\begin{aligned} 0 &\geq \rho_1 \boxminus \rho_2(X) + \rho_2(\tilde{Y}) = \sup_{Y \in L^p} \{\rho_1(X + Y) - \rho_2(Y)\} + \rho_2(\tilde{Y}) \\ &\geq \rho_1(X + \tilde{Y}) - \rho_2(\tilde{Y}) + \rho_2(\tilde{Y}) = \rho_1(X + \tilde{Y}), \end{aligned}$$

which implies that $X + \tilde{Y} \in \mathcal{A}_{\rho_1}$. This result holds for all $\tilde{Y} \in \mathcal{A}_{\rho_2}$, we have by the definition of the star-difference that $X \in \mathcal{A}_{\rho_1} \stackrel{*}{\boxminus} \mathcal{A}_{\rho_2}$ and therefore $\mathcal{A}_{\rho_1 \boxminus \rho_2} \subset \mathcal{A}_{\rho_1} \stackrel{*}{\boxminus} \mathcal{A}_{\rho_2}$.

On the other hand, since $\text{dom}(\alpha_{\rho_1}) \subset \text{dom}(\alpha_{\rho_2})$ we have for any $-d\mathbb{P}/d\mu \in L^q$

$$\begin{aligned} \delta_{\mathcal{A}_\phi}^*(-d\mathbb{P}/d\mu) &= \alpha_\phi(\mathbb{P}) = \text{cl}(\text{co}(\alpha_{\rho_1}(\mathbb{P}) - \alpha_{\rho_2}(\mathbb{P}))) \\ &= \text{cl}(\text{co}(\delta_{\mathcal{A}_{\rho_1}}^*(-d\mathbb{P}/d\mu) - \delta_{\mathcal{A}_{\rho_2}}^*(-d\mathbb{P}/d\mu))). \end{aligned} \quad (3.25)$$

Further we know that the right hand-side of (3.25) is the convex closure of the difference of two conjugate functions. The support function is the conjugate of the delta function of the same set. Since acceptance sets are convex and closed and we assume $\mathcal{A}_{\rho_1} \stackrel{*}{\boxminus} \mathcal{A}_{\rho_2} \neq \emptyset$, we have by Example 3.1.20 (4) for all $-d\mathbb{P}/d\mu \in L^q$

$$\text{cl}(\text{co}(\delta_{\mathcal{A}_{\rho_1}}^*(-d\mathbb{P}/d\mu) - \delta_{\mathcal{A}_{\rho_2}}^*(-d\mathbb{P}/d\mu))) = \delta_{\mathcal{A}_{\rho_1} \stackrel{*}{\boxminus} \mathcal{A}_{\rho_2}}(-d\mathbb{P}/d\mu),$$

which is the support function of the star-difference of acceptance set of ρ_1 and ρ_2 . And finally we obtain

$$\sup_{X \in \mathcal{A}_\phi} \{\mathbb{E}[-Xd\mathbb{P}/d\mu]\} = \sup_{X \in \mathcal{A}_{\rho_1} \stackrel{*}{\boxminus} \mathcal{A}_{\rho_2}} \{\mathbb{E}[-Xd\mathbb{P}/d\mu]\} \quad \text{for all } -d\mathbb{P}/d\mu \in L^q. \quad (3.26)$$

Now, suppose there exists $\tilde{X} \in \mathcal{A}_{\rho_1} \stackrel{*}{\boxminus} \mathcal{A}_{\rho_2} \setminus \mathcal{A}_{\rho_1 \boxminus \rho_2}$. The separating hyperplane theorem implies that there exists $-d\mathbb{P}'/d\mu \in L^q$ such that

$$\sup_{X \in \mathcal{A}_\phi} \{\mathbb{E}[-Xd\mathbb{P}'/d\mu]\} < \mathbb{E}[-\tilde{X}d\mathbb{P}'/d\mu] < +\infty.$$

Since $\tilde{X} \in \mathcal{A}_{\rho_1} \circledast \mathcal{A}_{\rho_2}$, we have in fact

$$\sup_{X \in \mathcal{A}_\phi} \{\mathbb{E}[-X d\mathbb{P}'/d\mu]\} < \sup_{X \in \mathcal{A}_{\rho_1} \circledast \mathcal{A}_{\rho_2}} \{\mathbb{E}[-X d\mathbb{P}'/d\mu]\}.$$

This contradicts (3.26). Hence, there is no such $\tilde{X} \in \mathcal{A}_{\rho_1} \circledast \mathcal{A}_{\rho_2} \setminus \mathcal{A}_{\rho_1 \boxplus \rho_2}$. Therefore $\mathcal{A}_{\rho_1 \boxplus \rho_2} \supset \mathcal{A}_{\rho_1} \circledast \mathcal{A}_{\rho_2}$, which completes the proof of the assertion. $\square$

To ensure that $\phi = \rho_1 \boxplus \rho_2$ is monotone and translation invariant, it is sufficient that either ρ_1 or ρ_2 has these properties. Convexity and lower semi-continuity of ϕ is satisfied if ρ_1 is convex and lower semi-continuous. If we ensure that the function ϕ is finite at 0, then we can relax the previous theorem and replace ρ_2 by a set. We will see in Remark 3.2.7 that $\mathcal{A} \boxplus \rho$ is not a risk measure.

Theorem 3.2.6. Let ρ be a convex risk measure on $L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\mathcal{A}$ a non-empty set. Then:

- (1) If $\rho \boxplus \mathcal{A}$ is finite at 0 (which for example is ensured whenever $\mathcal{A} \subset \mathcal{A}_\rho$), then $\rho \boxplus \mathcal{A}$ is a convex risk measure on L^p .

Additionally, if $\mathcal{A}$ is convex and closed and $\text{dom}(\alpha_\rho) \subset \text{dom}(\delta_{\mathcal{A}}^*)$, then the penalty function $\rho \boxplus \mathcal{A}$ is given by

$$\alpha_{\rho \boxplus \mathcal{A}}(\mathbb{P}) = \text{cl}(\text{co}(\alpha_\rho(\mathbb{P}) - \delta_{\mathcal{A}}^*(-d\mathbb{P}/d\mu))) \quad \text{for all } \mathbb{P} \in \mathcal{P}.$$

- (2) Let $\rho \boxplus \mathcal{A}$ be finite at 0. If ρ is a coherent risk measure and $\mathcal{A}$ is a cone then $\rho \boxplus \mathcal{A}$ is a coherent risk measure. In this case finiteness at 0 of $\rho \boxplus \mathcal{A}$ is equivalent to $\mathcal{A} \subset \mathcal{A}_\rho$.
- (3) Let $\rho_1 \boxplus \mathcal{A}_{\rho_2}$ and $\rho_1 \boxplus \rho_2$ be finite at 0, $\text{dom}(\alpha_{\rho_1}) \subset \text{dom}(\alpha_{\rho_2})$ and $\text{dom}(\alpha_{\rho_1}) \subset \text{dom}(\delta_{\mathcal{A}_{\rho_2}}^*)$. The deconvolution of ρ_1 and ρ_2 can be written as

$$(\rho_1 \boxplus \rho_2)(X) = (\rho_1 \boxplus \mathcal{A}_{\rho_2})(X) \quad \text{for all } X \in L^p. \quad (3.27)$$

- (4) Let $\rho \boxplus \rho_{\mathcal{A}}$ and $\rho \boxplus \mathcal{A}$ be finite at 0, $\text{dom}(\alpha_\rho) \subset \text{dom}(\alpha_{\rho_{\mathcal{A}}})$ and $\text{dom}(\alpha_\rho) \subset \text{dom}(\delta_{\mathcal{A}}^*)$ and $\mathcal{A}$ is convex, translative and closed then

$$\rho \boxplus \rho_{\mathcal{A}} = \rho \boxplus \mathcal{A}.$$

PROOF. (1) It is clear from Theorem 3.2.5 (1) that $\rho \boxplus \mathcal{A}$ is monotone, translation invariant, convex and lower semi-continuous. Since we assume that $(\rho \boxplus \mathcal{A})(0) < +\infty$, we ensure that $\rho \boxplus \mathcal{A}$ is finite at 0. Especially, if $\mathcal{A} \subset \mathcal{A}_\rho$ we have

$$(\rho \boxplus \mathcal{A})(0) = \sup_{Y \in \mathcal{A}} \rho(Y) \leq \sup_{Y \in \mathcal{A}_\rho} \rho(Y) = 0.$$

To prove the second assertion, by assumption the acceptance set $\mathcal{A}$ is non-empty, convex and closed and therefore $\delta_{\mathcal{A}}$ is proper, convex and lower semi-continuous. Additionally, $\text{dom}(\alpha_{\rho}) \subset \text{dom}(\delta_{\mathcal{A}}^*)$, thus by Proposition 3.1.19 the penalty function of $\rho \boxminus \mathcal{A}_{\rho_1}$ is given by

$$\alpha_{\rho \boxminus \mathcal{A}_{\rho_1}}(\mathbb{P}) = \text{cl}(\text{co}(\alpha_{\rho_1}(\mathbb{P}) - \delta_{\mathcal{A}_{\rho_2}}^*(\mathbb{P}))) \quad \text{for all } \mathbb{P} \in \mathcal{P}.$$

- (2) By assumption $\rho \boxminus \mathcal{A}$ is finite at 0, thus we only need to prove that $\rho \boxminus \mathcal{A}$ is positive homogeneous. The set $\mathcal{A}$ is a cone, meaning if $X \in \mathcal{A}$ also $\gamma^{-1}X \in \mathcal{A}$ for any $\gamma > 0$. We have for any $X \in L^p$ and any $\gamma \geq 0$

$$\begin{aligned} (\rho \boxminus \mathcal{A})(\gamma X) &= \sup_{Y \in \mathcal{A}} \rho(\gamma X + Y) = \gamma \sup_{Y \in \mathcal{A}} \rho(X + \gamma^{-1}Y) = \gamma \sup_{Y \in \mathcal{A}} \rho(X + Y) \\ &= \gamma(\rho \boxminus \mathcal{A})(X). \end{aligned}$$

Consequently, $\rho \boxminus \mathcal{A}$ is positively homogeneous.

Next we prove $(\rho \boxminus \mathcal{A})(0) = 0$ if and only if $\mathcal{A} \subset \mathcal{A}_{\rho}$. Let $\mathcal{A} \subset \mathcal{A}_{\rho}$. We have

$$\sup_{Y \in \mathcal{A}} \rho(Y) \leq \sup_{Y \in \mathcal{A}_{\rho}} \rho(Y) = 0,$$

since $\rho(Y) \leq 0$ for all $Y \in \mathcal{A}_{\rho}$. We notice that $\rho(0) = 0$ and $0 \in \mathcal{A}$ since $\mathcal{A}$ is a cone. Thus

$$\sup_{Y \in \mathcal{A}} \rho(Y) \geq \rho(0) = 0.$$

Hence, $(\rho \boxminus \mathcal{A})(0) = 0$ holds. To prove the converse statement, $(\rho \boxminus \mathcal{A})(0) = 0$ implies that $\rho(Y) \leq 0$ for all $Y \in \mathcal{A}$. It follows that $\mathcal{A} \subset \mathcal{A}_{\rho}$.

- (3) We have by Proposition 2.4.3 that $\mathcal{A}_{\rho_2}$ is convex and closed, therefore by (1) we have that $\rho_1 \boxminus \mathcal{A}_{\rho_2}$ is a convex risk measure with penalty function

$$\alpha_{\rho \boxminus \mathcal{A}_{\rho_2}}(\mathbb{P}) = \text{cl}(\text{co}(\alpha_{\rho_1}(\mathbb{P}) - \delta_{\mathcal{A}_{\rho_2}}^*(\mathbb{P}))) \quad \text{for all } \mathbb{P} \in \mathcal{P}.$$

By Theorem 3.2.5 (1) and (3) it follows that $\rho_1 \boxminus \rho_2$ is convex risk measure with penalty function

$$\alpha_{\phi}(\mathbb{P}) = \text{cl}(\text{co}(\alpha_{\rho_1}(\mathbb{P}) - \alpha_{\rho_2}(\mathbb{P}))) \quad \text{for all } \mathbb{P} \in \mathcal{P}.$$

Proposition 2.4.7 (1) yields $\delta_{\mathcal{A}_{\rho_2}}^*(\mathbb{P}) = \alpha_{\rho_2}(\mathbb{P})$. Therefore representation (3.27) follows.

- (4) By assumption $\rho \boxminus \rho_{\mathcal{A}}$ and $\rho \boxminus \mathcal{A}$ are finite at 0. It is clear from Theorem 3.2.5 (1) and Theorem 3.2.6 (1) that both $\rho \boxminus \rho_{\mathcal{A}}$ and $\rho \boxminus \mathcal{A}$ are convex risk measures. We have seen in the proof of Theorem 3.2.6 (4) that $\rho \boxminus \rho_{\mathcal{A}} = \rho \boxminus \mathcal{A}_{\rho_{\mathcal{A}}}$ if $\mathcal{A}$ is convex, translative and closed. By Proposition 2.4.6 (2) we have $\mathcal{A}_{\rho_{\mathcal{A}}} = \mathcal{A}$. □

Remark 3.2.7. We have seen in Theorem 3.2.6 that under certain conditions the operator $\rho \boxminus \mathcal{A}$ is a convex risk measure. As we have seen in Definition 3.1.2, it is possible to define the deconvolution of a non-empty set $\mathcal{A}$ and a convex risk measure ρ by

$$(\mathcal{A} \boxminus \rho)(X) := \sup_{Y \in \mathcal{V}} \{\delta_{\mathcal{A}}(X + Y) \div \rho(Y)\}.$$

In this remark we will show that $\mathcal{A} \boxminus \rho$ can never be finite at 0, if ρ is a convex risk measure and therefore $\mathcal{A} \boxminus \rho$ is not a convex risk measure.

Let ρ be a convex risk measure on $L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ and $\mathcal{A}$ a non-empty set. Then $\mathcal{A} \boxminus \rho$ is not finite at 0 and therefore not a convex risk measure. By definition

$$(\mathcal{A} \boxminus \rho)(0) = \sup_{X \in L^p} \{\delta_{\mathcal{A}}(X) \div \rho(X)\}.$$

A necessary condition to ensure $(\mathcal{A} \boxminus \rho)(0) < +\infty$ is $\text{dom}(\rho) \subset \mathcal{A}$. Assume $\text{dom}(\rho) \subset \mathcal{A}$, we have that ρ is a convex risk measure and therefore finite at 0 and monotone, therefore we have for all $X \in L^p_+$

$$\rho(X) \leq \rho(0) < +\infty.$$

It follows that $L^p_+ \subset \text{dom}(\rho)$. And therefore we also have $L^p_+ \subset \mathcal{A}$. As a consequence we have $\delta_{\mathcal{A}}(m) = 0$ for all $m > 0$. Due to the translation invariance of ρ we conclude that

$$(\mathcal{A} \boxminus \rho)(0) = \sup_{X \in L^p} \{\delta_{\mathcal{A}}(X) \div \rho(X)\} \geq \sup_{m > 0} \{-\rho(m)\} = \sup_{m > 0} \{m - \rho(0)\} = +\infty.$$

It follows that $\mathcal{A} \boxminus \rho$ is not finite at 0.

3.2.5 Difference of Risk Measures

The last representation theorem deals with the difference of two convex risk measures. Rather strong restrictions are needed to ensure that the weighted difference of two convex risk measures is monotone, convex and lower semi-continuous and thus admit a dual representation. In Proposition 3.2.9 we state conditions on the subdifferentials or Gâteaux-differentials of ρ_1 and ρ_2 in order to ensure that ϕ is monotone, convex and lower semi-continuous.

Theorem 3.2.8. Let ρ_1, ρ_2 be two convex risk measures on $L^p \rightarrow \mathbb{R} \cup \{\infty\}$ and $\lambda > 0$ be a scalar. We define ϕ by

$$\phi(X) := (1 + \lambda)\rho_1(X) - \lambda\rho_2(X).$$

Assume ϕ is monotone, convex and lower semi-continuous, then the following statements hold:

- (1) ϕ is a convex risk measure and admits the dual representation

$$\phi(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_{\phi}(\mathbb{P}) \right\} \quad \text{for all } X \in L^p. \quad (3.28)$$

- (2) If ρ_1 and ρ_2 are coherent risk measures, then ϕ is coherent as well.
 (3) If $\text{dom}(\rho_1) \subset \text{dom}(\rho_2)$, then the penalty function of ϕ is given by

$$\begin{aligned} \alpha_{\phi}(\mathbb{P}) &= ((1 + \lambda) * \alpha_{\rho_1}) \boxminus (\lambda * \alpha_{\rho_2})(\mathbb{P}) \\ &= \sup_{\substack{(1+\lambda)\mathbb{P}_1 - \lambda\mathbb{P}_2 = \mathbb{P} \\ \mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}}} \left\{ (1 + \lambda)\alpha_{\rho_1}(\mathbb{P}_1) - \lambda\alpha_{\rho_2}(\mathbb{P}_2) \right\} \quad \text{for all } \mathbb{P} \in \mathcal{P}. \end{aligned} \quad (3.29)$$

- (4) The acceptance set of ϕ satisfies

$$\mathcal{A}_{\phi} = \mathcal{A}_{(1+\lambda)\rho_1 - \lambda\rho_2} = \bigcup_{m \in \mathbb{R}} \left\{ (\mathcal{A}_{\rho_1} + (1 + \lambda)^{-1}m) \cap (\overline{\mathcal{A}_{\rho_2}^c} + \lambda^{-1}m) \right\}. \quad (3.30)$$

PROOF. (1) We assume that ϕ is monotone, convex and lower semi-continuous. Since $\rho_1(0) < +\infty$ and $\rho_2(0) < +\infty$, also $\phi(0) = (1 + \lambda)\rho_1(0) - \lambda\rho_2(0) < +\infty$, so ϕ is finite. Translation invariance of ϕ follows from the translation invariance property of ρ_1 and ρ_2 , i.e.

$$\begin{aligned} \phi(X + m) &= (1 + \lambda)\rho_1(X + m) - \lambda\rho_2(X + m) \\ &= (1 + \lambda)\rho_1(X) - (1 + \lambda)m - \lambda\rho_2(X) + \lambda m \\ &= (1 + \lambda)\rho_1(X + m) - \lambda\rho_2(X) - m \\ &= \phi(X) - m. \end{aligned}$$

Thus ϕ is a convex risk measure and by Theorem 2.1.5 ϕ admits the dual representation (3.28).

- (2) The proof is straightforward. We have for all $\gamma > 0$ and $X \in L^p$

$$\phi(\gamma X) = (1 + \lambda)\rho_1(\gamma X) - \lambda\rho_2(\gamma X) = \gamma((1 + \lambda)\rho_1(X) - \lambda\rho_2(X)) = \gamma\phi(X).$$

It is left to prove $\phi(0) = 0$. We have $\phi(0) = (1 + \lambda)\rho_1(0) - \lambda\rho_2(0) = 0$, since $\rho_1(0) = \rho_2(0) = 0$.

- (3) The measure ρ_1 is proper and ρ_2 is proper, convex and lower semi-continuous while $\text{dom}(\rho_1) \subset \text{dom}(\rho_2)$. We can apply Proposition 3.1.18 and Proposition 3.1.15 to derive the penalty function of ϕ

$$\begin{aligned} \phi^* &= ((1 + \lambda)\rho_1 - \lambda\rho_2)^* = ((1 + \lambda)\rho_1)^* \boxminus (\lambda\rho_2)^* \\ &= ((1 + \lambda) * \rho_1^*) \boxminus (\lambda * \rho_2^*). \end{aligned}$$

Equation (3.29) follows from the second equation in Theorem 2.2.5, i.e. $\alpha_{\phi}(\mathbb{P}) = \phi^*(-d\mathbb{P}/d\mu)$.

- (4) First, we notice that $\overline{\mathcal{A}_{\rho_2}^c}$ is the set of all random variables X in L^p such that $\rho_2(X) \geq 0$.

Let $X \in \mathcal{A}_\phi$, then $\phi(X) = (1 + \lambda)\rho_1(X) - \lambda\rho_2(X) \leq 0$ and

$$(1 + \lambda)\rho_1(X - (1 + \lambda)^{-1}m) - \lambda\rho_2(X - \lambda^{-1}m) \leq 0 \quad \text{for all } m \in \mathbb{R}. \quad (3.31)$$

We choose $m = -\rho_2(X)$, it follows from translation invariance that $\lambda\rho_2(X - \lambda^{-1}(-\rho_2(X))) = 0$, and thus from (3.31) that $(1 + \lambda)\rho_1(X - (1 + \lambda)^{-1}(-\rho_2(X))) \leq 0$. We obtain that $X \in \mathcal{A}_{\rho_1} + (1 + \lambda)^{-1}(-\rho_2(X))$ and $X \in \overline{\mathcal{A}_{\rho_2}^c} + \lambda^{-1}(-\rho_2(X))$. This shows that

$$\mathcal{A}_\phi \subset \bigcup_{m \in \mathbb{R}} \left\{ (\mathcal{A}_{\rho_1} + (1 + \lambda)^{-1}m) \cap (\overline{\mathcal{A}_{\rho_2}^c} + \lambda^{-1}m) \right\}.$$

To prove the converse, let $X \in \bigcup_{m \in \mathbb{R}} \left\{ (\mathcal{A}_{\rho_1} + (1 + \lambda)^{-1}m) \cap (\overline{\mathcal{A}_{\rho_2}^c} + \lambda^{-1}m) \right\}$. This means for any $\tilde{m} \in \mathbb{R}$, we have that $X \in \mathcal{A}_{\rho_1} + (1 + \lambda)^{-1}\tilde{m}$ and $X \in \overline{\mathcal{A}_{\rho_2}^c} + \lambda^{-1}\tilde{m}$. It follows that $(1 + \lambda)\rho_1(X - (1 + \lambda)^{-1}\tilde{m}) \leq 0$ and $-\lambda\rho_2(X - \lambda^{-1}\tilde{m}) \leq 0$, by translation invariance we have

$$\begin{aligned} 0 &\geq (1 + \lambda)\rho_1(X - (1 + \lambda)^{-1}\tilde{m}) - \lambda\rho_2(X - \lambda^{-1}\tilde{m}) \\ &= (1 + \lambda)\rho_1(X) - \lambda\rho_2(X) = \phi(X), \end{aligned}$$

And therefore $X \in \mathcal{A}_\phi$. Since the argument holds for any $\tilde{m} \in \mathbb{R}$ we have

$$\mathcal{A}_\phi \supset \bigcup_{m \in \mathbb{R}} \left\{ (\mathcal{A}_{\rho_1} + (1 + \lambda)^{-1}m) \cap (\overline{\mathcal{A}_{\rho_2}^c} + \lambda^{-1}m) \right\}.$$

The equation (3.30) follows. □

It is difficult to check whether or not $\phi := (1 + \lambda)\rho_1 - \lambda\rho_2$ is monotone, convex and lower semi-continuous. Using the results of Proposition 3.1.13 we obtain conditions on the subdifferentials or Gâteaux-differentials of ρ_1 and ρ_2 to ensure ϕ is monotone, convex and lower semi-continuous.

Proposition 3.2.9. Let ρ_1, ρ_2 be two convex risk measures on $L^p \rightarrow \mathbb{R} \cup \{\infty\}$ and $\lambda > 0$ be a scalar. We define ϕ by

$$\phi(X) := (1 + \lambda)\rho_1(X) - \lambda\rho_2(X).$$

- (1) If $\partial\rho_2$ is gap-continuous, then ϕ is convex if $\partial\rho_1 \stackrel{*}{\rhd} \partial\rho_2$ is monotone.
- (2) If ρ_1 and ρ_2 are finite, meaning $\text{dom}(\rho_1) = \text{dom}(\rho_2) = L^p$ and $\nabla\rho_1 - \nabla\rho_2$ is negative and monotone, then ϕ is continuous, monotone and convex.

PROOF.

- (1) Is a direct consequence of Proposition 3.1.13 (1).
- (2) If ρ_1 and ρ_2 are finite then by Corollary 2.3.3 (2) they are Gâteaux-differentiable and by Theorem 2.3.1 continuous. Monotonicity and convexity follow from Proposition 3.1.13 (2).

□

3.3 Examples

In the first example of this section we calculate different combinations of Average Values at Risk. We also derive the epi-multiplication of an entropic risk measure and the inf-convolution and sup-convolution of two entropic risk measures. In these examples, the solution is again an entropic risk measure with different risk aversion parameters. Further we calculate the sum and the difference of two spectral risk measures. In the last example we derive the penalty function of various combinations and convolutions of two given coherent risk measures, assuming the conditions of Theorem 3.2.1 - Theorem 3.2.8 are satisfied.

Example 3.3.1. We study the well-known Average Value at Risk at level β , $AV@R_\beta$, on L^p , see Example 2.2.10. Average Value at Risk is a risk measure with penalty function given by

$$\alpha_{AV@R_\beta}(\mathbb{P}) = \delta_{\mathcal{P}_\beta} \left(\frac{d\mathbb{P}}{d\mu} \right)$$

and

$$\mathcal{P}_\beta := \left\{ \mathbb{P} \in \mathcal{M}_a^1 : \frac{d\mathbb{P}}{d\mu} \leq \frac{1}{\beta} \right\}.$$

Let $0 < \beta_1 < \beta_2 \leq 1$. We have the following statements:

- (1) Let $\phi(X) = (AV@R_{\beta_1} \boxplus AV@R_{\beta_2})(X)$. Then $\phi(X) = AV@R_{\beta_2}(X)$.

Indeed, we have seen in Theorem 3.2.2 (3) that the penalty function of ϕ is given by

$$\alpha_\phi = \delta_{\mathcal{P}_{\beta_1}} + \delta_{\mathcal{P}_{\beta_2}} = \delta_{\mathcal{P}_{\beta_1} \cap \mathcal{P}_{\beta_2}} = \delta_{\mathcal{P}_{\beta_2}}.$$

- (2) Let $\phi(X) = (1 - \lambda)AV@R_{\beta_1} + \lambda AV@R_{\beta_2}$. Then ϕ is a coherent risk measure with penalty function $\delta_{\mathcal{P}_{\frac{\beta_1}{1-\lambda}} + \mathcal{P}_{\frac{\beta_2}{\lambda}}}$.

We have seen in Theorem 3.2.4 (3) that the penalty function of ϕ is given by

$$\begin{aligned}\alpha_\phi &= \text{cl}(((1-\lambda) * \delta_{\mathcal{P}_{\beta_1}}) \boxplus (\lambda * \delta_{\mathcal{P}_{\beta_2}})) = \text{cl}(\delta_{\mathcal{P}_{\frac{\beta_1}{1-\lambda}}} \boxplus \delta_{\mathcal{P}_{\frac{\beta_2}{\lambda}}}) \\ &= \text{cl}(\delta_{\mathcal{P}_{\frac{\beta_1}{1-\lambda}} + \mathcal{P}_{\frac{\beta_2}{\lambda}}}).\end{aligned}$$

The second equality follows from the definition of the epi-multiplication and the last one from Proposition 3.1.8. Since $\mathcal{P}_\beta$ is compact, see Example 2.2.10, and $\delta_{\mathcal{P}_{\frac{\beta_1}{1-\lambda}} + \mathcal{P}_{\frac{\beta_2}{\lambda}}}$ is proper the closure operator can be dropped.

- (3) Let $\phi(X) = (AV @ R_{\beta_1} \boxminus AV @ R_{\beta_2})(X)$. Then $\phi(X) = AV @ R_{\beta_2}(X)$.

We notice that $\delta_{\mathcal{P}_{\beta_2}} - \delta_{\mathcal{P}_{\beta_1}} = \delta_{\mathcal{P}_{\beta_2}}$, since we used the convention that $(+\infty) - (+\infty) = +\infty$ and $\mathcal{P}_{\beta_2} \subset \mathcal{P}_{\beta_1}$. The penalty function $\alpha_{AV @ R_{\beta_2}}$ is a proper, convex and lower semi-continuous function. We have $AV @ R_{\beta_2} \succcurlyeq AV @ R_{\beta_1}$ and therefore can apply Theorem 3.2.5 (3) and obtain the following result

$$\alpha_\phi = \delta_{\mathcal{P}_{\beta_2}} - \delta_{\mathcal{P}_{\beta_1}} = \delta_{\mathcal{P}_{\beta_2}}.$$

- (4) Let $\phi(X) = (AV @ R_{\beta_2} \boxminus AV @ R_{\beta_1})(X)$. Then ϕ is not a risk measure.

Since $AV @ R_{\beta_2} \succcurlyeq AV @ R_{\beta_1}$, the deconvolution $AV @ R_{\beta_2} \boxminus AV @ R_{\beta_1}$ is not finite at 0 and therefore not a risk measure.

- (5) Let $\phi(X) = (1+\lambda)AV @ R_{\beta_1} - \lambda AV @ R_{\beta_2}$ with $\lambda \in (0, 1)$. Then ϕ is not a risk measure.

$AV @ R_\beta$ is a finite risk measure, therefore $\partial AV @ R_\beta(X) \neq \emptyset$ for a given $X \in L^p$. By Proposition 2.3.2 the subdifferential is given by the arguments for which $AV @ R_\beta(X)$ attains its maximum. In the case of average value at risk the subdifferential, which is a singleton, is given by

$$\partial AV @ R_\beta(X) = \left\{ -\frac{1}{\beta} (\mathbf{1}_{\{X < q\}} + \kappa \mathbf{1}_{\{X = q\}}) \right\},$$

where q denotes a β -quantile of X , and where κ is defined as

$$\kappa = \begin{cases} 0 & \text{if } \mathbb{P}[X = q] = 0, \\ \frac{\beta - \mathbb{P}[X < q]}{\mathbb{P}[X = q]} & \text{otherwise.} \end{cases}$$

See Föllmer and Schied [32], Theorem 4.47 and Remark 4.48 for the proof. It follows from Corollary 2.3.3 that $\partial \rho(X) = \{\nabla \rho(X)\}$.

Let $\phi(X) = (1+\lambda)AV @ R_{\beta_1} - \lambda AV @ R_{\beta_2}$, we see from the subdifferential that independently of the choice of β_1, β_2 and λ , the difference $(1+\lambda)\nabla AV @ R_{\beta_1} - \lambda \nabla AV @ R_{\beta_2}$ can never be negative and monotone. It follows from Proposition 3.2.9 (2) that ϕ is not a risk measure. $\diamond$

Example 3.3.2. Let us investigate the entropic risk measure Entr_β with risk aversion parameter $\beta > 0$ defined in Example 2.2.9. We have the following statements:

- (1) Let $\phi(X) = (\lambda * \text{Entr}_\beta)(X)$ for a scalar $\lambda > 0$. Then ϕ is an entropic risk measure with risk aversion parameter β/λ .

By Theorem 3.2.1 (3) the penalty function of ϕ is given by

$$\alpha_\phi(\mathbb{P}) = \lambda \alpha_{\text{Entr}_\beta}(\mathbb{P}) = \frac{\lambda}{\beta} \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right].$$

- (2) Let $\phi(X) = (\text{Entr}_{\beta_1} \boxplus \text{Entr}_{\beta_2})(X)$ with $0 < \beta_1 < \beta_2$. Then ϕ is an entropic risk measure with risk aversion parameter $(\beta_1\beta_2)/(\beta_1 + \beta_2)$.

We have seen in Theorem 3.2.2 (3) that the penalty function of ϕ is given by

$$\begin{aligned} \alpha_\phi(\mathbb{P}) &= \alpha_{\text{Entr}_{\beta_1}}(\mathbb{P}) + \alpha_{\text{Entr}_{\beta_2}}(\mathbb{P}) = \frac{1}{\beta_1} \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right] + \frac{1}{\beta_2} \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right] \\ &= \frac{\beta_1 + \beta_2}{\beta_1\beta_2} \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right]. \end{aligned}$$

- (3) Let $\phi(X) = (\text{Entr}_{\beta_1} \boxminus \text{Entr}_{\beta_2})(X)$ with $0 < \beta_1 < \beta_2$. Then ϕ is an entropic risk measure with risk aversion parameter $(\beta_1\beta_2)/(\beta_2 - \beta_1)$.

We check the condition of Theorem 3.2.5. Notice that $\text{Entr}_{\beta_1} \succcurlyeq \text{Entr}_{\beta_2}$. The difference of the penalty functions $\alpha_{\text{Entr}_{\beta_1}} - \alpha_{\text{Entr}_{\beta_2}}$ is a proper, convex and lower semi-continuous function since

$$\frac{1}{\beta_1} \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right] - \frac{1}{\beta_2} \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right] = \frac{\beta_2 - \beta_1}{\beta_1\beta_2} \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right],$$

with $0 < \beta_1 < \beta_2$ and the function $\mathbb{E}_\mathbb{P}[\log(d\mathbb{P}/d\mu)]$ is proper, convex and lower semi-continuous. By Theorem 3.2.5 (3), $\phi(X) = \text{Entr}_{\beta_1} \boxminus \text{Entr}_{\beta_2}(X)$ is a convex risk measure with penalty function

$$\begin{aligned} \alpha_\phi(\mathbb{P}) &= \alpha_{\text{Entr}_{\beta_1}} - \alpha_{\text{Entr}_{\beta_2}} = \frac{1}{\beta_1} \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right] - \frac{1}{\beta_2} \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right] \\ &= \frac{\beta_2 - \beta_1}{\beta_1\beta_2} \mathbb{E}_\mathbb{P} \left[\log \frac{d\mathbb{P}}{d\mu} \right]. \end{aligned}$$

We see that the deconvolution of two entropic risk measures is again an entropic risk measure. In our case ϕ is an entropic risk measure with risk aversion parameter $(\beta_2 - \beta_1)/(\beta_1\beta_2)$.

- (4) Let $\phi(X) = (\text{Entr}_{\beta_2} \boxminus \text{Entr}_{\beta_1})(X)$ with $0 < \beta_1 < \beta_2$. Then ϕ is not a risk measure.

Since $\text{Entr}_{\beta_2} \succcurlyeq \text{Entr}_{\beta_1}$, the function ϕ is not finite at 0 and therefore not a risk measure. $\diamond$

The previous example illustrates that the inf-convolution and the deconvolution are inverse operators. We have

- (1) $(\text{Entr}_{\beta_1} \boxplus \text{Entr}_{\beta_2}) \boxminus \text{Entr}_{\beta_2} = \text{Entr}_{\beta_1}$
 (2) $(\text{Entr}_{\beta_1} \boxminus \text{Entr}_{\beta_2}) \boxplus \text{Entr}_{\beta_2} = \text{Entr}_{\beta_1}$, if $\beta_2 > \beta_1$.

In the following example we derive sum and the difference of two spectral risk measures. The new risk measure is given by the linear combination of the spectra. In the case of the difference we need to ensure that the resulting function is non-negative and non-increasing. We will apply these results in Chapter 4 to give examples for the capital reserve model.

Example 3.3.3. Let ρ_{Υ} and ρ_{Φ} be two spectral risk measures with spectra Υ and Φ respectively. The following statements hold:

- (1) Let $\phi = (1 - \lambda)\rho_{\Upsilon} + \lambda\rho_{\Phi}$. Then ϕ is a spectral risk measure with spectrum $(1 - \lambda)\Upsilon + \lambda\Phi$.

We apply the representation formula of a spectral risk measure given in Definition 2.5.5 to $(1 - \lambda)\rho_{\Upsilon} + \lambda\rho_{\Phi}$ and receive for a given $X \in L^p$

$$\begin{aligned}\phi(X) &= (1 - \lambda)\rho_{\Upsilon}(X) + \lambda\rho_{\Phi}(X) \\ &= -(1 - \lambda) \int_0^1 q_X^+(t) \Upsilon(t) dt - \lambda \int_0^1 q_X^+(t) \Phi(t) dt \\ &= - \int_0^1 q_X^+(t) ((1 - \lambda)\Upsilon(t) + \lambda\Phi(t)) dt.\end{aligned}$$

The linear combination of non-negative and non-increasing functions with positive scalars is non-negative and non-increasing. Thus ϕ is a spectral risk measure with spectrum $(1 - \lambda)\Upsilon + \lambda\Phi$.

- (2) Let $\phi = (1 + \lambda)\rho_{\Upsilon} - \lambda\rho_{\Phi}$. Then ϕ is a spectral risk measure if and only if $(1 - \lambda)\Upsilon + \lambda\Phi$ is non-negative and non-increasing. The spectrum of ϕ is given by $(1 - \lambda)\Upsilon + \lambda\Phi$.

Similar to (1) we have for difference of two spectral risk measures the following derivation for a given $X \in L^P$

$$\begin{aligned}\phi(X) &= (1 + \lambda)\rho_{\Upsilon}(X) - \lambda\rho_{\Phi}(X) \\ &= -(1 + \lambda) \int_0^1 q_X^+(t)\Upsilon(t)dt + \lambda \int_0^1 q_X^+(t)\Phi(t)dt \\ &= - \int_0^1 q_X^+(t)((1 + \lambda)\Upsilon(t) - \lambda\Phi(t))dt.\end{aligned}$$

It follows that ϕ is a spectral risk measure if and only if $(1 + \lambda)\Upsilon(t) - \lambda\Phi$ is non-negative and non-increasing. $\diamond$

In the next example we derive the penalty functions of various combinations of coherent risk measures. We assume that the conditions of the Theorems in Section 3.2 are satisfied.

Example 3.3.4. Let ρ_1 and ρ_2 be two coherent risk measures with penalty functions $\alpha_{\rho_1} = \delta_{\mathcal{P}_1}$ and $\alpha_{\rho_2} = \delta_{\mathcal{P}_2}$ respectively.

- (1) Assume that $\text{dom}(\rho_2)$ is compact, then $\rho_1 \boxplus \rho_2$ is a coherent risk measure with representation

$$(\rho_1 \boxplus \rho_2)(X) = \sup_{\mathbb{P} \in \mathcal{P}_1 \cap \mathcal{P}_2} \mathbb{E}_{\mathbb{P}}[-X] \quad \text{for all } X \in L^P. \quad (3.32)$$

It follows from Theorem 3.2.2 that $\rho_1 \boxplus \rho_2$ is a coherent risk measure. The penalty function of the inf-convolution of two risk measures is $\delta_{\mathcal{P}_1} + \delta_{\mathcal{P}_2} = \delta_{\mathcal{P}_1 \cap \mathcal{P}_2}$ and representation (3.32) follows.

- (2) Assume that $\text{dom}(\alpha_{\rho_2})$ is compact, then $(1 - \lambda)\rho_1 + \lambda\rho_2$ is a coherent risk measure with representation

$$(1 - \lambda)\rho_1(X) + \lambda\rho_2(X) = \sup_{\mathbb{P} \in (1 - \lambda)\mathcal{P}_1 + \lambda\mathcal{P}_2} \mathbb{E}_{\mathbb{P}}[-X] \quad \text{for all } X \in L^P. \quad (3.33)$$

We have by Theorem 3.2.4 that $(1 - \lambda)\rho_1 + \lambda\rho_2$ is a coherent risk measure. The penalty function of the sum of two risk measures (3.33) is given by $((1 - \lambda) * \delta_{\mathcal{P}_1}) \boxplus (\lambda * \delta_{\mathcal{P}_2})$. Due to Proposition 3.1.8 (1) this expression is equal to $\delta_{(1 - \lambda)\mathcal{P}_1 + \lambda\mathcal{P}_2}$.

- (3) Let $\rho_1 \boxminus \rho_2$ be finite at 0, $\text{dom}(\alpha_{\rho_1}) \subset \text{dom}(\alpha_{\rho_2})$ and $\alpha_{\rho_1} - \alpha_{\rho_2}$ be convex and lower semi-continuous, then $\rho_1 \boxminus \rho_2$ is a coherent risk measure with

$$(\rho_1 \boxminus \rho_2)(X) = \sup_{\mathbb{P} \in \mathcal{P}_1} \mathbb{E}_{\mathbb{P}}[-X] \quad \text{for all } X \in L^P. \quad (3.34)$$

By assumption $\alpha_{\rho_1} - \alpha_{\rho_2}$ is a proper function. Since $\alpha_{\rho_1}, \alpha_{\rho_2}$ are indicator functions this means that $\text{dom}(\alpha_{\rho_1}) \subset \text{dom}(\alpha_{\rho_2})$ and therefore $\alpha_{\rho_1} - \alpha_{\rho_2} = \alpha_{\rho_1}$. The result (3.34) follows.

- (4) Assume that $(1 + \lambda)\rho_1 - \lambda\rho_2$ is monotone, convex and lower semi-continuous, then $(1 + \lambda)\rho_1 - \lambda\rho_2$ is a coherent risk measure and can be represented as

$$(1 + \lambda)\rho_1(X) - \lambda\rho_2(X) = \sup_{\mathbb{P} \in (1+\lambda)\mathcal{P}_1 \pm \lambda\mathcal{P}_2} \mathbb{E}_{\mathbb{P}}[-X] \quad \text{for all } X \in L^p. \quad (3.35)$$

It follows from Theorem 3.2.8 that $(1 + \lambda)\rho_1 - \lambda\rho_2$ is a coherent risk measure. We have seen in Theorem 3.2.8 (3) that the penalty function (3.35) is given by $((1 + \lambda) * \delta_{\mathcal{P}_1}) \boxminus (\lambda * \delta_{\mathcal{P}_2})$. Due to Proposition 3.1.8 (2) this expression is equal to $\delta_{(1+\lambda)\mathcal{P}_1 \pm \lambda\mathcal{P}_2}$. $\diamond$

This chapter completes the first part of the thesis. Linear combinations and convolutions of risk measures might arise in various applications. In the following chapters we are focusing on applications to the pricing and hedging of financial claims. In Chapter 4 we study the pricing and hedging problem for contingent claims in an incomplete market as a trade-off between traders and regulators. In this set up we derive risk measure prices where the risk measure is given by a weighted sum of the regulator's and trader's risk measures. In Chapter 5 we price a claim in a complete market when initial capital is too small to obtain perfect replication.

Part II

Applications to the Pricing and Hedging of Contingent Claims

The Capital Reserve Model

We study the pricing and hedging problem for contingent claims in an incomplete markets in terms of a trade-off between trader and regulator. Suppose a trader is pricing a claim with stochastic payoff $\bar{F}$ at time T . We refer to this as 'claim F ', where $F := e^{-rT} \bar{F}$ denotes the discounted payoff. Assume the financial market consists of risky stocks with a price process which is described by some locally bounded semimartingale $\bar{S}$ and risk free bonds $\bar{B}$ with constant interest $r \in \mathbb{R}$. Both stocks and bonds can be traded continuously in all possible quantities at all times without transaction costs. By no-arbitrage arguments alone, one can obtain pricing bounds using sub- and super-hedging $[\inf_{\mathbb{Q} \in \mathcal{M}_e^{loc}} \mathbb{E}_{\mathbb{Q}}[F], \sup_{\mathbb{Q} \in \mathcal{M}_e^{loc}} \mathbb{E}_{\mathbb{Q}}[F]]$, where $\mathcal{M}_e^{loc}$ is the set of equivalent local martingale measures. In case of a complete market $\mathcal{M}_e^{loc}$ is a singleton. For many applications this pricing bound would lead to unreasonable prices. As shown by Eberlein and Jacod [25], in the example of a call option, the super-hedging strategy is to buy and hold a stock, which is excessively expensive. In other words, the gap between the upper and lower pricing bound is too wide. The trader of the claim might be willing to take some risk to significantly reduce the price. If the drift of underlying stock $\bar{S}$ is large enough the trader might invest a little bit more into stocks than required. She does not perfectly replicate the payoff of the claim, but instead goes for an extra return while being exposed to risk. Therefore it is necessary to decide how much risk one wants to take.

In this chapter we introduce an extra bank account $\bar{Z}$ with an interest rate $\tilde{r}$ smaller than the risk free interest rate r . The extra bank account $\bar{Z}$ can be seen as a capital reserve or margin account against possible risks. The capital reserve serves as a collateral that the trader of a financial instrument has to deposit to cover some or all of the eventual losses.

For many institutions this collateral is determined by a regulator. Since the early-1990s Value at Risk, $V@R$, is the industry standard risk measure for modeling market risk capital requirements. In the recent financial crisis $V@R$ performed weakly since stressed markets can produce losses far in excess of the $V@R$. As a replacement for

$V@R$ the coherent risk measure Average Value at Risk, $AV@R$, which measures the expected value of losses above a given quantile, was chosen by the Basel Committee on Banking Supervision [19]. We will go one step further and allow more general functions to measure the capital requirement using convex risk measures.

The capital reserve model was first introduced by Minina and Vellekoop [57]. The capital reserve was modeled as a function of the portfolio's Greeks, in that case.

The use of a capital reserve reflects what is happening in real life. When a trader takes on a risky position to hedge a claim, she ties up some of the institution's capital, meaning some money must be deposited into the margin account in case the trader can not (super-)replicate the claim or more generally it has to be put aside. Since this money has to be readily available, it is kept in a bank account or in some very liquid instruments with a yield that can not possibly match the rate of return of $\bar{B}$.

The reserve bank account of our model has an interest rate smaller than the risk-free rate and is not traded, and therefore, the obvious arbitrage opportunity of borrowing from $\bar{Z}$ and investing in $\bar{B}$ is not possible. The reserve bank account contains a prescribed amount of money depending on the trader's portfolio risk. The mechanism of the capital reserve is as follows. If the risk of the trader portfolio increases, so does the amount of money set aside in the capital reserve. This means that more money is lost because of the lower interest rate of the reserve bank account.

Our aim is to price a financial claim with a fixed maturity, a European option. A part of the price of the claim is not used for hedging but to cover for the money that will be lost due to the lower interest rate of the reserve bank account. The capital reserve is determined at time zero by the risk of the trader's portfolio which is measured in our case by a convex risk measure chosen by the regulator. Thus, if the risk measure is very conservative a lot of money has to be put in the reserve bank account and therefore a lot of money is lost due to the lower interest in this account. This gives the regulator a tool against too aggressive trading.

We employ two pricing methods for the capital reserve model, risk measure pricing and risk indifference pricing. The concept of pricing and hedging a claim under a convex risk measure has been introduced by Xu [76] for static risk measures and was further developed to dynamic risk measures on L^∞ by Klöppel and Schweizer [45]. Xu [76] introduced the concept of risk measure pricing and risk indifference pricing and Arai and Fukasawa [5] extended these results to Orlicz spaces and characterized the resulting pricing operator as a convex risk measure.

We explain the concept of risk measure pricing. We are using the convention to measure the risk of discounted financial positions, since then the translation invariance property is valid, see Definition 2.2.1 and Remark 2.2.2. Let ρ be the risk measure of the trader, F a given claim, and $\mathcal{H}$ the set of 0-attainable claims, where each element $H \in \mathcal{H}$ represents a future discounted payoff which the trader can replicate with zero initial capital. If the trader sells the claim F and receives an initial payment y for such a contract,

the minimal risk of the trader is

$$\inf_{H \in \mathcal{H}} \rho(y + H - F).$$

The risk measure selling price is given by the smallest initial capital $\tilde{y}$ such that the risk of the trader becomes acceptable, that is $\inf_{H \in \mathcal{H}} \rho(\tilde{y} + H - F) \leq 0$. The idea of risk measure pricing is closely linked to the concept of utility based derivative pricing since in both cases an optimal hedging strategy is chosen based on an objective function, expected utility or a convex risk measure, respectively.

Risk indifference pricing can be seen as an extension of risk measure pricing. The trader starts with a portfolio L , i.e. with discounted payoff L . If she sells a claim F at time T and receives an initial payment y for this, then the minimal risk involved for the seller is

$$\inf_{H \in \mathcal{H}} \rho(y + H + L - F).$$

If, on the other hand, no claim is sold, and hence no initial payment is received, then the minimal risk for the trader is

$$\inf_{H \in \mathcal{H}} \rho(H + L).$$

The seller's risk indifference price is the smallest payment $\tilde{y}$ such that the trader does not increase her risk exposure by selling the claim F for $\tilde{y}$, that is $\inf_{H \in \mathcal{H}} \rho(\tilde{y} + H + L - F) \leq \inf_{H \in \mathcal{H}} \rho(H + L)$. We see that this pricing model is related to the utility indifference pricing introduced by Hodges and Neuberger [42].

As a result, we have that in the capital reserve model two risk measures are used when we employ one of the aforementioned pricing methods: the risk measure of the regulator to determine the capital reserve and the risk measure of the trader to price the claim. For both pricing methods the resulting risk measure used for pricing is a linear combination of the risk measures of the trader and the regulator. This will be proven in Theorem 4.2.5 when we use the method of risk measure pricing and in Theorem 4.3.5 for risk indifference pricing.

Additionally, we show that the risk measure price and the risk indifference price induced by the capital reserve model is a good deal valuation. The idea is to restrict the set of equivalent martingale measures that one can use to price a claim. A good deal valuation is a pricing operator which provides sharper pricing bounds than the no-arbitrage pricing bounds. It satisfies some properties similar to the ones satisfied by convex risk measures. We state in Proposition 4.2.12 conditions such that the pricing operator in the capital reserve model is a good deal valuation.

The idea of good deal bounds has been introduced by Cochrane and Saa-Requejo [18]. They measure the attractiveness of a payoff by its Sharpe ratio on a finite probability space. Černý [14] observed that the Sharpe ratio bounds can be equivalently deduced by quadratic utility and proposed to quantify the attractiveness of payoff by utility functions. This concept was further developed by Klöppel and Schweizer [46] to utility based

good deal valuation for general utility functions. There is a different approach using risk measures instead of utility functions. Jäschke and Küchler [43] proved a one-to-one correspondence between coherent risk measures and good deal valuation bounds. This result was further generalized to convex risk measures on Orlicz spaces by Arai and Fukasawa [5].

The chapter is structured as follows. In Section 4.1 we introduce the capital reserve model. We derive a pricing formula for European claims in the capital reserve model using risk measure pricing in Section 4.2, and risk indifference pricing in Section 4.3.

4.1 The Capital Reserve Model

In this section we define the market model. In this market one can invest in the risky asset $\bar{S}$ and there are two riskless accounts $\bar{B}$ and $\bar{Z}$. The interest rate for the bond $\bar{B}$ is $r \in \mathbb{R}$. The bank account $\bar{Z}$ is the reserve bank account which contains the capital reserve. The capital reserve is a buffer for the risk of the trader and is required by the regulator to contain the minimal amount of money which depends on the risk of the trader's position. It has an interest rate $\tilde{r} \in \mathbb{R}$ lower than the ordinary bank account $\bar{B}$, therefore $\tilde{r} < r$. The reserve bank account is not traded. We give the formal definitions.

The price process of the underlying asset is an $\mathbb{R}$ -valued locally bounded semimartingale $\bar{S} = (\bar{S}_t)_{t \in [0, T]}$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mu)$. We denote by $\mathcal{M}_e^{loc}$ the set of all equivalent local martingale measures on $(\Omega, \mathcal{F})$ absolutely continuous with respect to μ . The bond price process $\bar{B} = (\bar{B}_t)_{t \in [0, T]}$ is defined by $d\bar{B}_t = r\bar{B}_t dt$ with $r \in \mathbb{R}$ and $\bar{B}_0 = 1$. The reserve bank account is modeled by the differential equation $d\bar{Z}_t = \tilde{r}\bar{Z}_t dt$ with $\tilde{r} < r$ and $\bar{Z}_0 = 1$. For simplicity we consider the discounted asset price process $S_t := \bar{B}_t^{-1} \bar{S}_t$ and discounted reserve bank account $Z_t := \bar{B}_t^{-1} \bar{Z}_t$ in this chapter. We call a predictable and S -integrable $(\pi_t)_{t \in [0, T]}$ process admissible if $\int_0^t \pi_u dS_u$ is well defined for all $t \in [0, T]$ and uniformly bounded from below by a constant. Let

$$\mathcal{H} := \left\{ \int_0^T \pi_t dS_t; \pi \text{ is admissible} \right\}$$

be the set of 0-attainable claims. Each element H of $\mathcal{H}$ represents a discounted future payoff which the trader can replicate with zero initial capital. We have that $\mathcal{H}$ is a convex cone and $0 \in \mathcal{H}$. The capital reserve is given by an adapted process $(\theta_t)_{t \in [0, T]}$ such that $\int_0^T \theta_u dZ_u < +\infty$. The discounted payoff of a hedging portfolio is given by

$$Y_T^{\pi, \theta, y} := y + \int_0^T \pi_t dS_t + \int_0^T \theta_t dZ_t.$$

Here $y \in \mathbb{R}$ represents the initial capital and θ_t is the amount invested into the capital reserve at time t . To avoid arbitrage opportunities we require that $\theta_t \geq 0$ for all $t \in [0, T]$. This means the trader is not allowed to borrow money at interest rate $\tilde{r}$. We will see in Theorem 4.2.5 (3) and Theorem 4.3.5 (3) that in all models we consider this condition is imposed.

The aim is to price a financial claim. The discounted payoff of the financial claim is given by an $\mathcal{F}_T$ -measurable random variable $F \in L^p$. Assume the trader issues a claim F and receives the price $y \in \mathbb{R}$ then the discounted terminal value of the portfolio after honoring the claim is given by

$$y + \int_0^T \pi_t dS_t + \int_0^T \theta_t dZ_t - F. \quad (4.1)$$

Equation (4.1) is in the most general form. We want to further specify the capital reserve process $(\theta_t)_{t \in [0, T]}$. We take $(\theta_t)_{t \in [0, T]} = \theta$ to be positive constant and $\theta_T = 0$. This means that the regulator is evaluating the risk of the trader's portfolio just at time 0. Since the money, which is set aside in the reserve Z , is not lost, the trader receives this amount θ at the maturity date. The trader only loses money due to the lower interest of the reserve bank account. Thus

$$\int_0^T \theta_t dZ_t = (e^{(\tilde{r}-r)T} - 1)\theta. \quad (4.2)$$

We notice that $(e^{(\tilde{r}-r)T} - 1)\theta \leq 0$ since $\theta \geq 0$, meaning, a part of the possible return is lost due to risk taking. Therefore, for a given F and initial capital y we can split y into two parts

$$y = y_{hedge}^{F,y} + y_{cap}^{F,y}, \quad (4.3)$$

with $y_{hedge}^{F,y}$ the part of the initial capital y available for hedging the claim F and $y_{cap}^{F,y}$ the part used to cover the losses in interest earnings which is given by (4.2), i.e.

$$y_{cap}^{F,y} = (1 - e^{(\tilde{r}-r)T})\theta.$$

We are left with the choice of the capital reserve θ . The regulator chooses a convex risk measure ψ . The capital reserve is modeled by the risk exposure of the trader's hedging portfolio under the convex risk measure ψ

$$\theta = \psi(y_{hedge}^{F,y} + H - F),$$

with $H \in \mathcal{H}$ being a 0-attainable claim chosen by the trader. It follows that $y_{cap}^{F,y}$ is given by

$$y_{cap}^{F,y} = (1 - e^{(\tilde{r}-r)T})\psi(y_{hedge}^{F,y} + H - F). \quad (4.4)$$

For a given initial capital $y \in \mathbb{R}$, claim F and 0-attainable claim $H \in \mathcal{H}$ we can derive $y_{hedge}^{F,y}$ and $y_{cap}^{F,y}$ explicitly by using expression (4.3) and (4.4). A small calculation shows

$$y_{hedge}^{F,y} = e^{-(\tilde{r}-r)T} y - (e^{-(\tilde{r}-r)T} - 1)\psi(H - F), \quad (4.5)$$

$$y_{cap}^{F,y} = -(e^{-(\tilde{r}-r)T} - 1)y + (e^{-(\tilde{r}-r)T} - 1)\psi(H - F). \quad (4.6)$$

For a given initial capital $y \in \mathbb{R}$, claim F and 0-attainable claim $H \in \mathcal{H}$, the discounted terminal value of the portfolio can be written as

$$\begin{aligned} Y_T^{\pi,\theta,y} - F &= y + H - F + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{F,y} + H - F) \\ &= y + H - F - y_{cap}^{F,y} = y_{hedge}^{F,y} + H - F. \end{aligned}$$

To avoid arbitrage opportunities, we need to ensure that the capital reserve θ is non-negative. As mentioned earlier for all models we impose this condition. This will be shown in Theorem 4.2.5 (3) and Theorem 4.3.5 (3).

As discussed in the introduction, we see the pricing and hedging problem of a claim F as a trade-off between trader and regulator. The regulator chooses the interest rate $\tilde{r}$ of the reserve account and the risk measure ψ which she uses to measure the risk exposure of the trader's hedging portfolio. The trader issues a claim F and chooses an (optimal) initial capital for hedging $y_{hedge}^{F,y}$ and hedging strategy $H \in \mathcal{H}$ and charges the price

$$\begin{aligned} y &= y_{hedge}^{F,y} + (1 - e^{(\tilde{r}-r)T})\psi(y_{hedge}^{F,y} + H - F) \\ &= y_{hedge}^{F,y} + y_{cap}^{F,y}. \end{aligned}$$

We summarize our arguments of the capital reserve model given in this section in the following definition.

Definition 4.1.1. Let $\psi : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex risk measure chosen by the regulator. A claim F , initial capital $y \in \mathbb{R}$ and $H \in \mathcal{H}$ are given. The *capital reserve model* is structured as follows:

- (1) The terminal value of the discounted portfolio after issuing $\bar{F}$ is given by

$$\begin{aligned} Y_T^{\pi,\theta,y} - F &= y + H - F + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{F,y} + H - F) \\ &= y_{hedge}^{F,y} + H - F. \end{aligned} \quad (4.7)$$

- (2) The selling price y can be split in two parts

$$y = y_{hedge}^{F,y} + y_{cap}^{F,y},$$

with $y_{hedge}^{F,y}$ being the part of the capital y used for hedging the claim F and $y_{cap}^{F,y}$ being the part used to cover the losses in interest earnings, i.e.

$$y_{cap}^{F,y} = (1 - e^{(\tilde{r}-r)T})\psi(y_{hedge}^{F,y} + H - F).$$

(3) The explicit solution of $y_{hedge}^{F,y}$ and $y_{cap}^{F,y}$ is given by

$$\begin{aligned} y_{hedge}^{F,y} &= e^{-(\tilde{r}-r)T} y - (e^{-(\tilde{r}-r)T} - 1)\psi(H - F), \\ y_{cap}^{F,y} &= -(e^{-(\tilde{r}-r)T} - 1)y + (e^{-(\tilde{r}-r)T} - 1)\psi(H - F). \end{aligned}$$

(4) To avoid arbitrage opportunities $y_{cap}^{F,y} \geq 0$. $\diamond$

In the next two sections we will derive the optimal selling and buying price for the claim F in the capital reserve model using two different pricing methods. This means we choose an operator to derive an optimal $\tilde{y} \in \mathbb{R}$ and $\tilde{H} \in \mathcal{H}$. The two methods we use are risk measure pricing and risk indifference pricing. We are seeking for a pricing operator $SP : L^p \rightarrow \mathbb{R}$ as a function of the discounted payoff of a given claim. Like the convex risk measure given in Definition 2.2.1, a pricing operator is supposed to have the following properties:

Definition 4.1.2. Let $F, G \in L^p$ be discounted payoffs. A *pricing procedure* is a function $SP : L^p \rightarrow \mathbb{R}$ satisfying the following four properties:

(M) Monotonicity: If $F \leq G$, then $SP(F) \leq SP(G)$.

(T) Translation invariance: If $m \in \mathbb{R}$, then $SP(F + m) = SP(F) + m$.

(C) Convexity: $SP(\gamma F + (1 - \gamma)G) \leq \gamma SP(F) + (1 - \gamma)SP(G)$, for $0 \leq \gamma \leq 1$.

(N) Normality: $SP(0) = 0$. $\diamond$

We want to verify that in the capital reserve model we are able to derive a pricing procedure for the claim F . We start with the concept of risk measure pricing.

4.2 Risk Measure Pricing

In this section we derive the optimal investment $\tilde{y} \in \mathbb{R}$ and replication strategy $\tilde{H} \in \mathcal{H}$ given in Definition 4.1.1 of the capital reserve model. We choose risk measure pricing, which was first introduced by Xu [76], as the underlying method. Let ρ be the risk measure of the trader and F a given claim, then the risk measure selling price of F is given by the smallest initial capital $\tilde{y}$ such that the risk becomes acceptable to the trader, that is

$$\tilde{y} = \inf\{y \in \mathbb{R}; \inf_{H \in \mathcal{H}} \rho(y + H - F) \leq 0\},$$

while $\tilde{H}$ is the argument of the infimum, i.e. $\rho(\tilde{y} + \tilde{H} - F) = \inf_{H \in \mathcal{H}} \rho(\tilde{y} + H - F)$. Similarly we can define the buying pricing.

We will now extend the concept of risk measure pricing to the capital reserve model. Additionally, we give conditions such that the risk measure price induced by the capital

reserve model is a pricing procedure. Further we show that this pricing procedure describes the upper and lower bound of a good deal valuation, i.e. we find a pricing operator which provides sharper pricing bounds than the no-arbitrage pricing bounds.

We start with the formal definition of the selling and buying risk measure price.

Definition 4.2.1. Let $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex risk measure. The *risk measure price* of a given claim $\bar{F} \in L^p$ is defined as follows. The *selling price* of the claim F is defined

$$SP_{rmp}(F) = \inf\{y \in \mathbb{R}; \inf_{H \in \mathcal{H}} \rho(y + H - F) \leq 0\}, \quad (4.8)$$

and the *buying price* of the claim F is defined by

$$\begin{aligned} BP_{rmp}(F) &= \inf\{y \in \mathbb{R}; \inf_{H \in \mathcal{H}} \rho(-y + H + F) \leq 0\} \\ &= -SP_{rmp}(-F). \end{aligned} \quad (4.9) \quad \diamond$$

To ensure that there exists $\tilde{H} \in \mathcal{H}$ such that the infimum in (4.8) and (4.9) is attained we make the following assumptions. These assumptions will hold for the rest of the chapter.

Assumption 4.2.2. Let $\rho, \psi : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be two convex risk measures. We make the following assumptions for ρ and ψ .

- (1) It is not possible to reduce the risk infinitely by trading a 0-attainable claim. Meaning,

$$\inf_{H \in \mathcal{H}} \rho(H) \in \mathbb{R}, \quad \inf_{H \in \mathcal{H}} \psi(H) \in \mathbb{R}.$$

- (2) For every claim $F \in L^p$ there exists a hedging strategy $\tilde{H} \in \mathcal{H}$ such that the risk is finite, i.e.

$$\rho(\tilde{H} - F) < +\infty, \quad \psi(\tilde{H} - F) < +\infty. \quad \diamond$$

We recall that in our setting a convex risk measure is always lower semi-continuous. As a consequence we have the following theorem.

Theorem 4.2.3. (Xu [76], Theorem 2.6) Suppose that Assumption 4.2.2 is satisfied. Then for every convex risk measure ρ , claim F and initial capital $y \in \mathbb{R}$ there exists a $\tilde{H} \in \mathcal{H}$ such that

$$\rho(y + \tilde{H} - F) = \inf_{H \in \mathcal{H}} \rho(y + H - F). \quad \square$$

Due to Theorem 4.2.3 and the translation invariance of the risk measure ρ , we can rewrite the selling price $SP_{rmp}(F)$ and buying price $BP_{rmp}(F)$ in the risk measure pricing method in the following way

$$SP_{rmp}(F) = \inf_{H \in \mathcal{H}} \rho(H - F), \quad BP_{rmp}(F) = - \inf_{H \in \mathcal{H}} \rho(H + F).$$

Next, we give the definition of the risk measure price in the capital reserve model. The terminal value of the discounted portfolio after issuing F is given in (4.7). We have the following definition.

Definition 4.2.4. Let $\rho, \psi : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be two convex risk measures, where ρ is the risk measure of the trader and ψ is the risk measure of the regulator. The *risk measure price in the capital reserve model* of a given claim $F \in L^p$ is defined as follows. The *selling price* of the claim F is defined by

$$SP_{rmp}^{cr}(F) = \inf \left\{ y \in \mathbb{R}; \inf_{H \in \mathcal{H}} \rho(y + H - F + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{F,y} + H - F)) \leq 0 \right\} \quad (4.10)$$

and the *buying price* of the claim F is defined by

$$BP_{rmp}^{cr}(F) = \inf \left\{ y \in \mathbb{R}; \inf_{H \in \mathcal{H}} \rho(-y + H + F + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{-F,-y} + H + F)) \leq 0 \right\}. \quad \diamond \quad (4.11)$$

We need to characterize further and simplify the expressions (4.10) and (4.11). The terms $y_{hedge}^{F,y}$ and $y_{hedge}^{-F,-y}$ are given explicit in Definition 4.1.1. Additionally, we need to ensure that the capital reserve θ is non-negative. We recall that a risk measure ρ_1 is more conservative than ρ_2 and write $\rho_1 \succcurlyeq \rho_2$ if $\rho_1(X) \geq \rho_2(X)$ for all $X \in L^p$, see Definition 2.2.7.

Theorem 4.2.5. Let $\rho, \psi : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be two convex risk measures and F a given claim. Then the following statements hold:

- (1) The risk measure price in the capital reserve model is given by

$$SP_{rmp}^{cr}(F) := \inf_{H \in \mathcal{H}} \phi(H - F), \quad BP_{rmp}^{cr}(F) := - \inf_{H \in \mathcal{H}} \phi(H + F)$$

where ϕ is a convex risk measure defined by

$$\phi := e^{(\tilde{r}-r)T} \rho + (1 - e^{(\tilde{r}-r)T}) \psi.$$

The penalty function of ϕ is given by

$$\alpha_\phi(\mathbb{P}) = \text{cl}((e^{(\tilde{r}-r)T} * \alpha_\rho) \boxplus ((1 - e^{(\tilde{r}-r)T}) * \alpha_\psi))(\mathbb{P}) \quad \text{for all } \mathbb{P} \in \mathcal{P}. \quad (4.12)$$

- (2) If $\text{dom}(\alpha_\psi)$ is compact and $(e^{(\tilde{r}-r)T} * \alpha_\rho) \boxplus ((1 - e^{(\tilde{r}-r)T}) * \alpha_\psi)$ is proper then the penalty function of ϕ is given for all $\mathbb{P} \in \mathcal{P}$ by

$$\alpha_\phi(\mathbb{P}) = \inf_{\substack{e^{(\tilde{r}-r)T}\mathbb{P}_1 + (1-e^{(\tilde{r}-r)T})\mathbb{P}_2 = \mathbb{P} \\ \mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}}} \{e^{(\tilde{r}-r)T}\alpha_\rho(\mathbb{P}_1) + (1 - e^{(\tilde{r}-r)T})\alpha_\psi(\mathbb{P}_2)\}. \quad (4.13)$$

- (3) If $\psi \succcurlyeq \rho$, then the capital reserve θ is non-negative in this pricing method for all claims F .

PROOF. (1) We start with the selling price. Let $\tilde{y}$ be the solution of the optimization problem (4.10), meaning

$$\tilde{y} = \inf \{y \in \mathbb{R}; \inf_{H \in \mathcal{H}} \rho(y + H - F + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{F,y} + H - F)) \leq 0\}. \quad (4.14)$$

By Theorem 4.2.3 and the translation invariance of the risk measure ρ , we can rewrite (4.14) in the following way

$$\tilde{y} = \inf_{H \in \mathcal{H}} \rho(H - F + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{F,\tilde{y}} + H - F)). \quad (4.15)$$

We notice that the optimal solution $\tilde{y}$ given in (4.15), which is the selling price, is described via $y_{hedge}^{F,\tilde{y}}$ and is given explicit as we have seen in (4.5) by $y_{hedge}^{F,\tilde{y}} = e^{-(\tilde{r}-r)T}\tilde{y} - (e^{-(\tilde{r}-r)T} - 1)\psi(H - F)$. Thus

$$\begin{aligned} \tilde{y} &= \inf_{H \in \mathcal{H}} \rho(H - F + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{F,\tilde{y}} + H - F)) \\ &= \inf_{H \in \mathcal{H}} \{\rho(H - F) + (1 - e^{(\tilde{r}-r)T})\psi(H - F) + (e^{(\tilde{r}-r)T} - 1)y_{hedge}^{F,\tilde{y}}\} \\ &= \inf_{H \in \mathcal{H}} \{\rho(H - F) + (1 - e^{(\tilde{r}-r)T})\psi(H - F) \\ &\quad + (e^{(\tilde{r}-r)T} - 1)(y_{hedge}^{F,\tilde{y}} = e^{-(\tilde{r}-r)T}\tilde{y} - (e^{-(\tilde{r}-r)T} - 1)\psi(H - F))\} \\ &= (1 - e^{(r-\tilde{r})T})\tilde{y} + \inf_{H \in \mathcal{H}} \{\rho(H - F) + (e^{(r-\tilde{r})T} - 1)\psi(H - F)\}. \end{aligned}$$

Finally, we have

$$SP_{rmp}^{cr}(F) = \tilde{y} = \inf_{H \in \mathcal{H}} \phi(H - F).$$

By Theorem 3.2.4, $\phi := e^{(\tilde{r}-r)T}\rho + (1 - e^{(\tilde{r}-r)T})\psi$ is a convex risk measure with penalty function given by (4.12). A similar calculation can be done for the buying price $BP_{rmp}^{cr}(F)$.

- (2) Follows from Theorem 3.2.4.

- (3) We assume that ψ is more conservative than ρ , and since $y_{\text{hedge}}^{F,y} \leq \rho(H - F)$ for all $H \in \mathcal{H}$

$$\theta = \psi(y_{\text{hedge}}^{F,y} + H - F) = \psi(H - F) - \rho(H - F) \geq 0 \quad \text{for all } H \in \mathcal{H}.$$

And therefore we have that the amount invested into the capital reserve is non-negative. □

Due to the construction of the capital reserve model we have seen in the previous theorem that the risk measure price of a claim F in the capital reserve model is given by the optimal hedging strategy under a new risk measure ϕ , where ϕ is given by a linear combination of the risk measures of the trader ρ and regulator ψ .

We will prove that the risk measure price in the capital reserve model $SP_{\text{rmp}}^{\text{cr}}$ is a pricing procedure as given by Definition 4.1.2. Under the condition that $\mathcal{H}$ is compact we can further characterize the penalty function of the risk measure $\tilde{S}P(F) := SP_{\text{rmp}}^{\text{cr}}(-F)$ induced by the risk measure price $SP_{\text{rmp}}^{\text{cr}}$. We follow the idea of Arai and Fukasawa [5] and prove the following statement.

Theorem 4.2.6. Let F be a claim and $\rho, \psi : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be two normalized convex risk measures such that $\rho(H) \geq 0$ and $\psi(H) \geq 0$ for all $H \in \mathcal{H}$. Then the following statements hold:

- (1) $SP_{\text{rmp}}^{\text{cr}}(F)$ is a pricing procedure.
- (2) Assume that $\mathcal{H}$ is compact and define $\tilde{S}P(F) := SP_{\text{rmp}}^{\text{cr}}(-F) = -BP_{\text{rmp}}^{\text{cr}}(F)$ for any $F \in L^p$, then $\tilde{S}P$ is a normalized convex risk measure. The penalty function of $\tilde{S}P$ for any $\mathbb{P} \in \mathcal{P}$ is given by

$$\alpha_{\tilde{S}P}(\mathbb{P}) = \begin{cases} \alpha_{\phi}(\mathbb{P}), & \text{if } \mathbb{E}_{\mathbb{P}}[H] \leq 0 \quad \text{for all } H \in \mathcal{H}, \\ +\infty, & \text{else.} \end{cases}$$

PROOF. (1) We prove the properties of a pricing procedure. Monotonicity and translation invariance follow directly from ϕ . To prove convexity, choose $H_1, H_2 \in \mathcal{H}$ such that

$$\phi(H_1 - F_1) = \inf_{H \in \mathcal{H}} \phi(H - F_1), \quad \phi(H_2 - F_2) = \inf_{H \in \mathcal{H}} \phi(H - F_2).$$

By convexity of the set $\mathcal{H}$, we have for any $0 \leq \gamma \leq 1$ that $\gamma H_1 + (1 - \gamma)H_2 \in \mathcal{H}$, and by convexity of ϕ

$$\begin{aligned}
 SP_{rmp}^{cr}(\gamma F_1 + (1 - \gamma)F_2) &= \inf_{H \in \mathcal{H}} \phi(H - \gamma F_1 - (1 - \gamma)F_2) \\
 &\leq \inf_{H \in \mathcal{H}} \phi(\gamma H_1 + (1 - \gamma)H_2 - \gamma F_1 - (1 - \gamma)F_2) \\
 &= \inf_{H \in \mathcal{H}} \phi(\gamma(H_1 - F_1) + (1 - \gamma)(H_2 - F_2)) \\
 &\leq \gamma \phi(H_1 - F_1) + (1 - \gamma)(\phi(H_2 - F_2)) \\
 &\leq \gamma SP_{rmp}^{cr}(F_1) + (1 - \gamma)SP_{rmp}^{cr}(F_2).
 \end{aligned}$$

Normality is left to be proved. By assumption $\phi(H) = e^{(\tilde{r}-r)T} \rho(H) + (1 - e^{(\tilde{r}-r)T}) \psi(H) \geq 0$ for all $H \in \mathcal{H}$ and by normality $\phi(0) = e^{(\tilde{r}-r)T} \rho(0) + (1 - e^{(\tilde{r}-r)T}) \psi(0) = 0$ therefore we have $SP_{rmp}^{cr}(0) = 0$.

- (2) To ensure that $\tilde{S}P(F) := SP_{rmp}^{cr}(-F)$ is a convex risk measure we need to verified lower semi-continuity. We can rewrite $\tilde{S}P(F)$ as the inf-convolution of ϕ and the set $-\mathcal{H}$, i.e.

$$\begin{aligned}
 \tilde{S}P(F) &= SP_{rmp}^{cr}(-F) = \inf_{H \in \mathcal{H}} \phi(H + F) \\
 &= (\phi \boxplus -\mathcal{H})(F).
 \end{aligned}$$

Since $\mathcal{H}$ is compact we may apply Lemma 3.1.6 (1). It follows that $(\phi \boxplus -\mathcal{H})(F)$ is lower semi-continuous and therefore also $\tilde{S}P$. By Theorem 3.2.3 (3), the penalty function of $\tilde{S}P$ is given by

$$\begin{aligned}
 \alpha_{\tilde{S}P}(\mathbb{P}) &= \tilde{S}P^* \left(-\frac{d\mathbb{P}}{d\mu} \right) = (\phi \boxplus -\mathcal{H})^* \left(-\frac{d\mathbb{P}}{d\mu} \right) \\
 &= \phi^* \left(-\frac{d\mathbb{P}}{d\mu} \right) + \delta_{\mathcal{H}}^* \left(\frac{d\mathbb{P}}{d\mu} \right).
 \end{aligned}$$

We notice that $\mathcal{H}$ is a convex cone, thus $\delta_{\mathcal{H}}^*(d\mathbb{P}/d\mu) = +\infty$ if $\mathbb{E}_{\mathbb{P}}[H] > 0$ for some $H \in \mathcal{H}$, which completes the proof of the theorem. $\square$

In the set up of the capital reserve we see ρ as the risk measure (or pricing measure) of the trader, which might not be very prudent. The regulator would prefer a more conservative pricing procedure. Therefore she chooses ψ to measure the risk of the trader's portfolio. We would like to further analyze the span of solutions of the risk measure price in the capital reserve model between $BP_{rmp}^{cr}(F)$ and $SP_{rmp}^{cr}(F)$ and relate them to good deal valuation as well as the sub- and super-hedging prices. Further we analyze the dependence of the risk measure price on the choice of the interest rate of reserve bank account $\tilde{r}$ and the risk measure of the regulator ψ .

Definition 4.2.7. Define the *superhedging cost functional* ρ_0 on L^p by

$$\rho_0(F) := \inf\{x \in \mathbb{R}; \exists H \in \mathcal{H} \text{ such that } x + H + F \geq 0\}. \quad \diamond$$

Since $\rho_0(-F)$ represents the superhedging cost for a claim F , it gives the upper no-arbitrage pricing bound for F . By the same reasoning the lower arbitrage bound for $\bar{F}$ is given by $-\rho_0(F)$.

Arai and Fukasawa [5] proved that the superhedging cost functional is a positive homogeneous pricing procedure.

Lemma 4.2.8. (Arai and Fukasawa [5], Lemma 2.5) Let ρ_0 be the superhedging cost function. Define $SP(X) := \rho_0(-X)$ for any $X \in L^p$. Then SP is a $\mathbb{R} \cup \{+\infty\}$ valued positive homogeneous pricing procedure. $\square$

Definition 4.2.9. A pricing procedure SP is said to be a *good deal valuation* if

$$SP(F) \in [-\rho_0(F), \rho_0(-F)] \quad \text{for any } F \in L^p. \quad \diamond$$

The above definition is given from the seller's point of view. Nevertheless, it is equivalent to

$$BP(F) = -SP(-F) \in [-\rho_0(F), \rho_0(-F)] \quad \text{for any } F \in L^p,$$

which is from the buyer's viewpoint. We have by the normality and convexity property that $BP(F) = -SP(-F) \leq SP(F)$ for any $F \in L^p$. Next we give the definition of a good deal bound.

Definition 4.2.10. For a good deal valuation $SP(F)$ a *good deal bound* is constructed by

$$[-SP(-F), SP(F)] \quad \text{for any } F \in L^p,$$

which is a subinterval of $[-\rho_0(F), \rho_0(-F)]$. $\diamond$

We have the following condition given a pricing procedure SP to be a good deal valuation. The theorem is stated in Arai and Fukasawa [5].

Theorem 4.2.11. (Arai and Fukasawa [5], Theorem 3.4) For any pricing procedure SP the following two conditions are equivalent

- (1) SP is a good deal valuation.
- (2) $SP(\tilde{H}) \leq 0$ for any $\tilde{H} \in \mathcal{H}$. $\square$

Finally, we prove that SP_{rmp}^{cr} is a good deal valuation. We specify that the risk measure price in the capital reserve model is a subinterval of SP_{rmp} and the superhedging price; depending on the choice of the interest rate of the reserve bank account $\tilde{r}$ and the risk measure ψ we can further enforce the position of SP_{rmp}^{cr} .

Proposition 4.2.12. Let $\rho, \psi : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be two normalized risk measures such that $\rho(H) \geq 0$ and $\psi(H) \geq 0$ for all $H \in \mathcal{H}$, $\psi \succcurlyeq \rho$ and F a claim. Define $\phi := e^{(\tilde{r}-r)^T} \rho + (1 - e^{(\tilde{r}-r)^T}) \psi$.

(1) Then

$$SP_{rmp}^{cr}(F) = \inf_{H \in \mathcal{H}} \phi(H - F)$$

is a good deal valuation.

(2) We have $SP_{rmp}^{cr} \in [SP_{rmp}(F), \rho_0(-F)]$.

PROOF. (1) It follows from Theorem 4.2.6 (1) that $\bar{S}P$ is a pricing procedure. Additionally, we have $SP_{rmp}^{cr}(\tilde{H}) = \inf_{H \in \mathcal{H}} \phi(H - \tilde{H}) \leq \phi(\tilde{H} - \tilde{H}) = 0$ for all $\tilde{H} \in \mathcal{H}$, so according to the previous theorem, $SP_{rmp}^{cr}(F)$ is a good deal valuation.

(2) We have

$$SP_{rmp}^{cr}(F) = \inf_{H \in \mathcal{H}} \{e^{(\tilde{r}-r)^T} \rho(H - F) + (1 - e^{(\tilde{r}-r)^T}) \psi(H - F)\}.$$

Since $\psi \succcurlyeq \rho$ it follows that $SP_{rmp}^{cr}(F) \geq \inf_{H \in \mathcal{H}} \rho(H - F) = SP_{rmp}(F)$ with equality either if $\tilde{r} = 0$ or $\psi = \rho$. □

We see that it is possible for the regulator to enforce a more conservative pricing method, which in the extreme case might be superhedging, which is the case if the regulator chooses $\psi = \rho_0$ since we have $\lim_{\tilde{r} \rightarrow -\infty} SP_{rmp}^{cr}(F) = \rho_0(-F)$. We conclude this section with an example using two spectral risk measures.

Example 4.2.13. Let ρ and ψ be two spectral risk measures with spectra Υ_ρ and Υ_ψ respectively such that $\psi \succcurlyeq \rho$. Then risk measure price of a claim F in the capital reserve model is given by

$$SP_{rmp}^{cr}(F) := \inf_{H \in \mathcal{H}} \phi(H - F), \quad BP_{rmp}^{cr}(F) := - \inf_{H \in \mathcal{H}} \phi(H + F),$$

where the risk measure ϕ is a spectral risk measure with spectrum

$$\Upsilon_\phi = e^{(\tilde{r}-r)^T} \Upsilon_\rho + (1 - e^{(\tilde{r}-r)^T}) \Upsilon_\psi.$$

This follows directly from Example 3.3.3 and Theorem 4.2.5. ◇

4.3 Risk Indifference Pricing

In this section we consider to price a contingent claim F in the capital reserve model using risk indifference pricing with given initial portfolio L . We recall the definition of risk indifference price, which was first introduced by Xu [76].

Definition 4.3.1. Let $\rho : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be a convex risk measure. The *risk indifference price* of the claim $F \in L^p$ with given initial portfolio $L \in L^p$ is defined as follows. The *selling price* of F given the portfolio L is defined by

$$SP_{idp}(F) = \inf\{y \in \mathbb{R}; \inf_{H \in \mathcal{H}} \rho(y + H + L - F) \leq \inf_{H \in \mathcal{H}} \rho(H + L)\}, \quad (4.16)$$

and the *buying price* of the claim F with given initial portfolio L is defined by

$$BP_{idp}(F) = \inf\{y \in \mathbb{R}; \inf_{H \in \mathcal{H}} \rho(-y + H + L + F) \leq \inf_{H \in \mathcal{H}} \rho(H + L)\}. \quad (4.17)$$

◇

The idea behind the risk indifference pricing is as follows. Given an initial portfolio L , the trader sells the claim F for the price $SP_{idp}(F)$. If the trader can find a hedging strategy $\tilde{H} \in \mathcal{H}$ such that $\rho(SP_{idp}(F) + \tilde{H} + L - F) \leq \inf_{H \in \mathcal{H}} \rho(H + L)$, meaning that the trader does not increase her risk by selling the claim F for $SP_{idp}(F)$, then $SP_{idp}(F)$ is an acceptable price for her. The smallest value $SP_{idp}(F)$ such that the trader does not increase her risk is the selling price. A similar argument holds for the buying price $BP_{idp}(F)$.

Remark 4.3.2. We included an initial portfolio L at Definition 4.3.1 to distinguish the risk measure price and the risk indifferent price. It is common practice to assume that $\inf_{H \in \mathcal{H}} \rho(H) = 0$, in this case the risk measure price and the risk indifferent price would coincide if L is deterministic. ◇

By Theorem 4.2.3 and the translation invariance property of the risk measure ρ , we can rewrite the risk indifference selling (4.16) and buying (4.17) price in the following way

$$\begin{aligned} SP_{idp}(F) &= \inf_{H \in \mathcal{H}} \rho(H + L - F) - \inf_{H \in \mathcal{H}} \rho(H + L), \\ BP_{idp}(F) &= - \inf_{H \in \mathcal{H}} \rho(H + L + F) + \inf_{H \in \mathcal{H}} \rho(H + L). \end{aligned}$$

We extend the risk indifference price to the capital reserve model. Therefore we need to derive the part of the initial capital used for hedging given an initial capital $y = 0$ and initial portfolio L . We solve this problem in the following lemma.

Lemma 4.3.3. Let L be the initial portfolio. In the special case $y = 0$ we denote with $y_{hedge}^{-L,0}$ and $y_{cap}^{-L,0}$ the part of the initial capital, in this case $y = 0$, used for hedging and

to cover the losses in interest earnings in the capital reserve. It follows from (4.5) that the part of the initial capital used for hedging $y_{hedge}^{-L,0}$ with 0 initial capital is given by

$$y_{hedge}^{-L,0} = (1 - e^{-(\tilde{r}-r)T})\psi(H + L) \quad \text{for all } H \in \mathcal{H}. \quad (4.18)$$

Analog to the risk measure price given in the previous section, we first give the definition of the risk indifference price in the capital reserve model in more general terms and then characterize and simplify this expression in a theorem.

Definition 4.3.4. Let $\rho, \psi : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be two convex risk measures. The *risk indifference price in the capital reserve model* of the claim $F \in L^p$ and given initial portfolio $L \in L^p$ is defined as follows. The *selling price* of F given the portfolio L is defined by

$$\begin{aligned} SP_{idp}^{cr}(F) &= \inf \left\{ y \in \mathbb{R}; \inf_{H \in \mathcal{H}} \rho(y + H + L - F) \right. \\ &\quad \left. + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{F-L,y} + H + L - F) \right\} \\ &\leq \inf_{H \in \mathcal{H}} \rho(H + L + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{-L,0} + H + L)). \end{aligned} \quad (4.19)$$

Here, $y_{hedge}^{-L,0}$ part used for hedging when starting with 0 initial capital, which was given in Lemma 4.3.3. The *buying price* of the claim F given initial portfolio L is defined by

$$\begin{aligned} BP_{idp}^{cr}(F) &= \inf \left\{ y \in \mathbb{R}; \inf_{H \in \mathcal{H}} \rho(-y + H + L + F) \right. \\ &\quad \left. + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{-F-L,-y} + H + L + F) \right\} \\ &\leq \inf_{H \in \mathcal{H}} \rho(H + L + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{-L,0} + H + L)). \quad \diamond \end{aligned} \quad (4.20)$$

We simplify the expressions (4.19) and (4.20) and derive the dual representation risk indifference price in the capital reserve model. These results are analog to Theorem 4.2.5.

Theorem 4.3.5. Let $\rho, \psi : L^p \rightarrow \mathbb{R} \cup \{+\infty\}$ be two convex risk measures with $\psi \succcurlyeq \rho$ and $L, F \in L^p$.

- (1) Given an initial portfolio L , the risk indifference selling price in the capital reserve model $SP_{idp}^{cr}(F)$ of the claim F is given by

$$SP_{idp}^{cr}(F) = \inf_{H \in \mathcal{H}} \phi(H + L - F) - \inf_{H \in \mathcal{H}} \phi(H + L), \quad (4.21)$$

and the risk indifference buying price in the capital reserve model $BP_{idp}^{cr}(F)$ of the claim F is given by

$$BP_{idp}^{cr}(F) = - \inf_{H \in \mathcal{H}} \phi(H + L - F) + \inf_{H \in \mathcal{H}} \phi(H + L), \quad (4.22)$$

where ϕ is a convex risk measure defined by

$$\phi := e^{(\tilde{r}-r)T}\rho + (1 - e^{(\tilde{r}-r)T})\psi.$$

The penalty function of ϕ is given by

$$\alpha_\phi(\mathbb{P}) = \text{cl}((e^{(\tilde{r}-r)T} * \alpha_\rho) \boxplus ((1 - e^{(\tilde{r}-r)T}) * \alpha_\psi))(\mathbb{P}) \quad \text{for all } \mathbb{P} \in \mathcal{P}. \quad (4.23)$$

- (2) If $\text{dom}(\alpha_\psi)$ is compact and $(e^{(\tilde{r}-r)T} * \alpha_\rho) \boxplus ((1 - e^{(\tilde{r}-r)T}) * \alpha_\psi)$ is proper then the penalty function of ϕ is given for all $\mathbb{P} \in \mathcal{P}$ by

$$\alpha_\phi(\mathbb{P}) = \inf_{\substack{e^{(\tilde{r}-r)T}\mathbb{P}_1 + (1 - e^{(\tilde{r}-r)T})\mathbb{P}_2 = \mathbb{P} \\ \mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P}}} \{e^{(\tilde{r}-r)T}\alpha_\rho(\mathbb{P}_1) + (1 - e^{(\tilde{r}-r)T})\alpha_\psi(\mathbb{P}_2)\}. \quad (4.24)$$

- (3) The capital reserve θ is non-negative in this pricing method.

- (4) If ρ and ψ are normalized and $\rho(H) \geq 0$ and $\psi(H) \geq 0$ for all $H \in \mathcal{H}$ then $SP_{idp}^{cr}(F)$ is a pricing procedure.

PROOF. (1) The selling price of the risk indifference price in capital reserve model is given in (4.19). Let $\tilde{y}$ be the optimal solution of (4.19). By Theorem 4.2.3 and the translation invariance of ρ we have

$$\begin{aligned} \tilde{y} &= \inf_{H \in \mathcal{H}} \rho(y + H + L - F + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{F-L, \tilde{y}} + H + L - F)) \\ &\leq \inf_{H \in \mathcal{H}} \rho(H + L + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{-L, 0} + H + L)). \end{aligned}$$

By Lemma 4.3.3, the amount of capital used for hedging starting with 0 capital is given by $y_{hedge}^{-L, 0} = (1 - e^{-(\tilde{r}-r)T})\psi(H + L)$ and by (4.5) we have $y_{hedge}^{F-L, \tilde{y}} = e^{-(\tilde{r}-r)T}\tilde{y} - (e^{-(\tilde{r}-r)T} - 1)\psi(H + L - F)$. A small calculation shows

$$\begin{aligned} \tilde{y} &= \inf_{H \in \mathcal{H}} \rho(H + L - F + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{F-L, \tilde{y}} + H + L - F)) \\ &\quad - \inf_{H \in \mathcal{H}} \rho(H + L + (e^{(\tilde{r}-r)T} - 1)\psi(y_{hedge}^{-L, 0} + H + L)) \\ &= \inf_{H \in \mathcal{H}} \{\rho(H + L - F) + (e^{(r-\tilde{r})T} - 1)\psi(H + L - F)\} \\ &\quad + (1 - e^{(r-\tilde{r})T})\tilde{y} - \inf_{H \in \mathcal{H} + \mathcal{L}} \{\rho(H) + (e^{(\tilde{r}-r)T} - 1)\psi(H + L)\}. \end{aligned}$$

Thus we obtain

$$\begin{aligned} SP_{idp}^{cr}(F) = \tilde{y} &= \inf_{H \in \mathcal{H}} \{e^{(\tilde{r}-r)T}\rho(H + L - F) + (1 - e^{(\tilde{r}-r)T})\psi(H + L - F)\} \\ &\quad - \inf_{H \in \mathcal{H}} \{e^{(\tilde{r}-r)T}\rho(H + L) + (1 - e^{(\tilde{r}-r)T})\psi(H + L)\} \\ &= \inf_{H \in \mathcal{H}} \phi(H + L - F) - \inf_{H \in \mathcal{H}} \phi(H + L). \end{aligned}$$

The representation (4.21) follows. By Theorem 3.2.4, $\phi := e^{(\tilde{r}-r)^T} \rho + (1 - e^{(\tilde{r}-r)^T}) \psi$ is a convex risk measure with penalty function given by (4.23). Using the same argument we can derive representation of the buying price (4.22) as well.

- (2) Follows from Theorem 3.2.4.
- (3) Follows from Theorem 4.2.5 (3).
- (4) Follows from Theorem 4.2.6 (1).

□

We finalized this section with example using spectral risk measures.

Example 4.3.6. Let ρ and ψ be two spectral risk measures with spectra Υ_ρ and Υ_ψ respectively such that $\psi \succcurlyeq \rho$. Let an initial portfolio $L \in L^p$ be given. The risk indifference selling price of a claim F in the capital reserve is given by

$$SP_{idp}^{cr}(F) = \inf_{H \in \mathcal{H}} \phi(H + L - F) - \inf_{H \in \mathcal{H}} \phi(H + L),$$

and the risk indifference buying price in the capital reserve model is given by

$$BP_{idp}^{cr}(F) = - \inf_{H \in \mathcal{H}} \phi(H + L - F) + \inf_{H \in \mathcal{H}} \phi(H + L),$$

where ϕ is a spectral risk measure with spectrum

$$\Upsilon_\phi = e^{(\tilde{r}-r)^T} \Upsilon_\rho + (1 - e^{(\tilde{r}-r)^T}) \Upsilon_\psi.$$

The statement follows directly from Example 3.3.3 and Theorem 4.3.5.

◇

In this chapter we presented a new approach to the optimal pricing and hedging problem for contingent claims in an incomplete market; a trade-off between trader and regulator. We introduced an extra bank account with an interest rate smaller than the risk-free rate. This reserve bank account serves as a capital reserve or margin account against possible risks. We concluded that in the capital reserve model, the risk measure used for pricing is in both models a linear combination of the risk measures of the trader and the regulator. Especially the regulator is able to adjust the pricing method of the trader by choosing the interest rate of the reserve bank account and her risk measure. Additionally, we showed that the risk measure price and the risk indifference price induced by the capital reserve model is a good deal valuation.

In the next chapter we consider another application. The aim is to price a claim in a complete market set-up when the initial capital is too small to invest in perfect replication. This leads to an optimization problem where we try to minimize the difference between the claim and the hedging strategy. The risk is measured by a simple spectral risk measure.

Optimal Hedging under a Simple Spectral Risk Measure

In a complete market, every contingent claim F with payoff at time T can be hedged perfectly; given a sufficient large initial capital x_0 one can replicate the claim such that the value of the portfolio at time T is equal to the claim. The required initial capital x_0 defines the price of the claim and it can be computed by the discounted expectation F with respect the unique equivalent martingale measure. If the agent is unwilling or unable to invest in perfect replication but seeks to minimize her risk exposure, she might choose a partial hedging method. The aim of this chapter is to find a suitable trading strategy such that the risk of the difference of the hedging portfolio and the claim is minimized when x_0 is smaller than the initial capital required for a perfect hedging strategy.

The problem of optimal partial hedging depends highly on the selection of the specific risk function. Föllmer and Leukert [29] use a quantile hedging approach to determine a hedging strategy which maximizes the probability of the event $\{X_T^{\pi, x_0} - F \geq 0\}$. The Neyman-Pearson lemma is exploited to derive the solution, called the quantile hedging strategy, explicitly. In this setting very large losses could occur, as long as they occur with small probabilities. Therefore, Föllmer and Leukert [30] generalized their approach by studying the expected shortfall of the losses $\mathbb{E}[(F - X_T^{\pi, x_0})^+]$ and more generally $\mathbb{E}[l((F - X_T^{\pi, x_0})^+)]$ for some loss function l . Nakano [58] uses a coherent risk measure to measure the risk of losses due to shortfall in the setting of an incomplete market. In his work the objective function $\rho(-(F - X_T^{\pi, x_0})^+)$ is minimized. Rudloff [65], [66] [67] further improves the result of Nakano and introduces convex risk measures as an objective function. In her work she uses a convex duality approach; she derives the dual problem of $\rho(-(F - X_T^{\pi, x_0})^+)$ by Fenchel duality, proves strong duality and deduces the structure of the optimal solution. This approach can be seen as a generalization of the Neyman-Pearson method.

In the work cited above, the dynamic optimization problem of finding a self-financing strategy which minimizes the risk of losses due to shortfall is divided into two parts. Given the claim F , the optimal strategy consists of hedging a modified claim φF for some randomized test function φ , and replicating the modified claim φF . To derive the randomized test function φ the Neyman-Pearson lemma is exploited. The application of this lemma requires that the portfolio X_T^{π, x_0} is bounded from below and dominated by some integrable function. While it is reasonable to assume $X_T^{\pi, x_0} \geq 0$, for example to avoid bankruptcy, it is difficult to justify a suitable upper bound. If, as an objective function, the risk measure of minimal shortfall $\rho(-(F - X_T^{\pi, x_0})^+)$ is considered then X_T^{π, x_0} is dominated by F , since larger values for X_T^{π, x_0} do not contribute in the minimization. Therefore the Neyman-Pearson lemma can be used in that set-up.

In this chapter we focus on a different approach. Given an initial wealth x_0 smaller than the initial capital required for perfect hedging our aim is to derive a suitable hedging strategy such that the difference of the hedging portfolio and the claim under a risk measure ρ is minimized, meaning our objective function is given by $\rho(X_T^{\pi, x_0} - F)$. In contrast to the results mentioned above we do not bound X_T^{π, x_0} by F . In our setting we assume that the risk will be measured by a simple spectral risk measure ρ_{ν_n} , i.e. it can be represented as a finite weighted sum of Average Value at Risk measures. Thus, we consider the optimization problem of finding a strategy that minimizes $\rho_{\nu_n}(X_T^{\pi, x_0} - F)$ given initial wealth x_0 . The optimization problem can be divided as follows. First, we rewrite the simple spectral risk measure ρ_{ν_n} as a finite weighted sum of Average Value at Risk measures. Using the Fenchel-Legendre transform we can rewrite Average Value at Risk in terms of expected shortfall, see Rockafellar and Uryasev [63] for more details. Thus we obtain an upper boundary for the portfolio X_T^{π, x_0} and the Neyman-Pearson lemma can still be applied. By using this lemma we can derive an optimal modified claim φF , which we replicate, and reduce our dynamic problem to a static n -dimensional optimization problem. This allows us to give a semi-explicit solution.

In the first part of this chapter we solve the minimization of $AV@R_\lambda(X_T^{\pi, x_0} - F)$ given an initial wealth x_0 . We construct an optimal hedging portfolio explicitly in complete market models, which we illustrate by providing closed-form solutions for call and put options in the Black-Scholes model. Especially, we allow the situation when we overhedge F , that is $\mu[X_T^{\pi, x_0} > F] > 0$ for the optimal portfolio X_T^{π, x_0} where μ is the objective probability measure and we show that our optimization problem differs from minimizing the shortfall, i.e. $AV@R_\lambda(X_T^{\pi, x_0} - F) \neq AV@R_\lambda(-(F - X_T^{\pi, x_0})^+)$. In the second part we generalize these results to simple spectral risk measure.

Melnikov and Smirnov [54] use a similar approach and compute the minimum of the optimization problem $AV@R_\lambda(X_T^{\pi, x_0} - F)$. They state that the losses $F - X_T^{\pi, x_0}$ are always non-negative, thus $F \geq X_T^{\pi, x_0}$, but we show that this is not always the case. The examples of overhedging are somewhat hard to find, which may be the reason that the possibility of overhedging is overlooked in Melnikov and Smirnov [54]. A related problem was discussed in Li and Xu [50] from a portfolio optimization point of

view. They used the Fenchel-Legendre transform of Average Value at Risk to minimize $AV@R_\lambda$ when the returns are bounded.

This chapter is organized as follows. In Section 5.1 we state the problem formulation and rewrite the simple spectral risk measure to a weighted sum of Average Value at Risk which allows us to change the order when taking the infima. In Section 5.2 we solve the problem for the case that the risk measure is given by Average Value at Risk. We derive a condition such that scenario of overhedging occurs and illustrate our results by solving the problem for a call and put option in the Black-Scholes model. We extend the results to simple spectral risk measure in Section 5.3 and focus on the solution of the dynamic optimization problem and derive a solution for the call option in the Black-Scholes model.

5.1 Problem Formulation

The set-up is similar to the one in the last chapter. We have a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mu)$ with fixed time horizon $T > 0$. Additionally, we assume that the probability space is atomless. The discounted price process is described as an $\mathbb{R}$ -valued semimartingale $S = (S_t)_{t \in [0, T]}$. We assume a complete market, so there exists a unique equivalent martingale measure $\mathbb{Q}$ and we assume that $d\mu/d\mathbb{Q}$ has a continuous distribution.

Given an initial capital $X_0 = x_0 > 0$, a predictable and S -integrable process $(\pi_t)_{t \in [0, T]}$ is called 0-admissible if the corresponding value process $X_t^{\pi, x_0} := x_0 + \int_0^t \pi_u dS_u$ is well defined and bounded from below by 0 for all $t \in [0, T]$. Let

$$\Pi(x_0) := \left\{ X_T^{\pi, x_0}; X_T^{\pi, x_0} = x_0 + \int_0^t \pi_u dS_u \geq 0, \text{ for all } t \in [0, T], \pi \text{ is 0-admissible} \right\}$$

be the set of all attainable claims with initial capital x_0 and discounted payoff X_T^{π, x_0} . In a complete market with unique martingale measure $\mathbb{Q}$ the set $\Pi(x_0)$ can be expressed in terms of expectations with respect to $\mathbb{Q}$

$$\Pi(x_0) = \{X; X \geq 0, \mathbb{E}_{\mathbb{Q}}[X] \leq x_0\}. \quad (5.1)$$

Consider a contingent claim. Its discounted payoff is given by an $\mathcal{F}_T$ -measurable, random variable $F \in L^1$. We assume that F is non-negative, meaning $F \geq 0$. Using an optimal hedging strategy the claim F can be replicated given initial wealth $F_0 := \mathbb{E}_{\mathbb{Q}}[F]$. Assume that the initial capital $x_0 < F_0$; evidently a perfect replication cannot be used in this case. We search for the best hedge that minimizes the risk of the difference $X_T^{\pi, x_0} - F$. The risk will be measured by a simple spectral risk measure ρ_{ν_n} . Thus, we consider the dynamic optimization problem of finding a 0-admissible strategy that solves

$$\inf_{X \in \Pi(x_0)} \rho_{\nu_n}(X - F), \text{ with } x_0 < F_0. \quad (5.2)$$

Remark 5.1.1. In order to solve the optimization problem (5.2) we will exploit the Neyman-Pearson lemma. This approach requires that the set of attainable claims is bounded from below. In our set-up we have that $X \geq 0$ for all $X \in \Pi(x_0)$, but this boundary can be generalized to $X \geq a$ with $a \in \mathbb{R}$. For simplicity we choose $a = 0$. $\diamond$

In (5.2) the function ρ_{ν_n} is a simple spectral risk measure with spectrum $\Upsilon(t) = \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbf{1}_{[0, \lambda_i]}(t)$ and $\sum_{i=1}^n \alpha_i = 1$, see Definition 2.5.7. By Lemma 2.5.8 any simple spectral risk measure ρ_{ν_n} can be represented as a weighted sum of Average Value at Risk, meaning,

$$\rho_{\nu_n}(X - F) = \sum_{i=1}^n \alpha_i AV@R_{\lambda_i}(X - F).$$

Using the Fenchel-Legendre transform we are able to rewrite Average Value at Risk in terms of expected shortfall as we have seen in Proposition 2.2.11

$$AV@R_{\lambda}(X - F) = \inf_{y \in \mathbb{R}} \left\{ \frac{1}{\lambda} \mathbb{E}[(F + y - X)^+] - y \right\}. \quad (5.3)$$

The infimum above is attained in y^* if and only if y^* is a λ -quantile of $X - F$. The dynamic optimization problem (5.2) can be rewritten as follows

$$\begin{aligned} \inf_{X \in \Pi(x_0)} \rho_{\nu_n}(X - F) &= \inf_{X \in \Pi(x_0)} \left\{ \sum_{i=1}^n \inf_{y \in \mathbb{R}^n} \left\{ \frac{\alpha_i}{\lambda_i} \mathbb{E}[(F + y_i - X)^+] - \alpha_i y_i \right\} \right\} \quad (5.4) \\ &= \inf_{y \in \mathbb{R}^n} \left\{ \inf_{X \in \Pi(x_0)} \left\{ \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E}[(F + y_i - X)^+] \right\} - \sum_{i=1}^n \alpha_i y_i \right\}, \end{aligned}$$

with initial capital $x_0 < \mathbb{E}_{\mathbb{Q}}[F]$. The problem (5.4) can be solved stepwise. First, we solve the inner dynamic optimization problem

$$w(\mathbf{y}) := \inf_{X \in \Pi(x_0)} \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E}[(F + y_i - X)^+] \quad (5.5)$$

for given $\mathbf{y} \in \mathbb{R}^n$. We notice that for any given $y_i \in \mathbb{R}$ only $X \leq F + y_i$ are contributing in the minimization (5.5). Thus we can apply the Neyman-Pearson lemma to derive the solution of the problem (5.5). The second step is to solve the outer n -dimensional optimization problem

$$\inf_{\mathbf{y} \in \mathbb{R}^n} \left\{ w(\mathbf{y}) - \sum_{i=1}^n \alpha_i y_i \right\}. \quad (5.6)$$

This chapter is split in two parts. In Section 5.2 we will focus on the case $n = 1$, this corresponds to the minimization of Average Value at Risk. In the second part, Section 5.3, we solve the optimization problem for a simple spectral risk measure with $n > 1$.

5.2 Hedging under Average Value at Risk

In this section we will focus on minimizing Average Value at Risk at level $\lambda \in (0, 1]$. Thus the optimization problem (5.2) can be rewritten as

$$\inf_{\pi \in \Pi(x_0)} AV@R_\lambda(X - F), \text{ with } x_0 < F_0. \quad (5.7)$$

By the Fenchel-Legendre transform this problem is equivalent to

$$\inf_{y \in \mathbb{R}} \left\{ \frac{1}{\lambda} \inf_{X \in \Pi(x_0)} \mathbb{E}[(F + y - X)^+] - y \right\}.$$

In Section 5.2.1 we focus on the inner problem

$$w(y) := \inf_{X \in \Pi(x_0)} \mathbb{E}[(F + y - X)^+] \quad (5.8)$$

for a given $y \in \mathbb{R}$ following the approach of Rudloff [65] and use the Neyman-Pearson lemma.

In Section 5.2.2 we solve the outer problem

$$\inf_{y \in \mathbb{R}} f(y) = \inf_{y \in \mathbb{R}} \left\{ \frac{1}{\lambda} w(y) - y \right\}. \quad (5.9)$$

We prove that f is convex and derive the derivatives. Based on the derivative of f we can give conditions of the optimal y^* of (5.9). Of special interest is the case when y^* is positive, this corresponds to the scenario of overhedging, meaning that for the optimal solution X^* we have $\mu[X^* > F] > 0$. If the case $y^* > 0$ occurs then the two optimization problems $\inf_{\pi \in \Pi(x_0)} AV@R_\lambda(X - F)$ and $\inf_{\pi \in \Pi(x_0)} AV@R_\lambda(-(F - X)^+)$ are not the same. In Proposition 5.2.11 we state necessary and sufficient conditions such that the optimal X^* is of the form $\mu[X^* > F] > 0$.

In Section 5.2.3 we demonstrate our results by solving the problem for a call and put option in the Black-Scholes model.

5.2.1 Dynamic Optimization Problem

The aim of this section is to find an explicit expression for the following dynamic optimization problem

$$\inf_{X \in \Pi(x_0)} \mathbb{E}[(F + y - X)^+], \quad (5.10)$$

for a given $y \in \mathbb{R}$. Note that $(F + y - X)^+ = ((F + y)^+ - X)^+$, thus we can rewrite (5.10) as follows

$$\inf_{X \in \Pi(x_0)} \mathbb{E}[((F + y)^+ - X)^+]. \quad (5.11)$$

Expression (5.11) is a problem of linear shortfall minimization with respect to a contingent claim with discounted payoff $(F + y)^+$, depending on the parameter $y \in \mathbb{R}$. The solution can be derived with the help of the Neyman-Pearson lemma. See, for example Föllmer and Leukert [30] or Rudloff [65]. Corresponding to the dynamic optimization problem (5.11), we have the following static optimization problem

$$\inf_{\varphi \in \mathcal{R}(x_0)} \mathbb{E}[(1 - \varphi)(F + y)^+], \quad (5.12)$$

$$\mathcal{R}(x_0) := \{\varphi : \Omega \rightarrow [0, 1], \mathcal{F}_T\text{-measurable}, \mathbb{E}_{\mathbb{Q}}[\varphi(F + y)^+] \leq x_0\}. \quad (5.13)$$

The following theorem given in Rudloff [65] proves that the dynamic optimization problem (5.11) with the constraint (5.1) and the static optimization problem (5.12) with the constraint (5.13) are equal.

Theorem 5.2.1. (Rudloff [65], Theorem 4.1) Let $\tilde{\varphi}$ be the solution of the minimization problem (5.12) then $\tilde{\varphi}(F + y)^+ \in \Pi(x_0)$ solves the optimization problem (5.11), the optimal trading strategy $\tilde{\pi}$ is determined by the optional decomposition of the claim $\tilde{\varphi}(F + y)^+$ and it holds

$$\inf_{X \in \Pi(x_0)} \mathbb{E}[(F + y)^+ - X]^+ = \inf_{\varphi \in \mathcal{R}(x_0)} \mathbb{E}[(1 - \varphi)(F + y)^+]. \quad (5.14)$$

□

We apply Theorem (5.2.1) to the original problem (5.10) and see that it is equivalent to the following optimization problem

$$\sup_{\varphi \in \mathcal{R}(x_0)} \mathbb{E}[\varphi(F + y)^+]. \quad (5.15)$$

Let $\mathcal{R}$ denote the set of $\mathcal{F}_T$ -measurable functions from $\Omega \rightarrow [0, 1]$. The problem can be reformulated

$$\sup_{\varphi \in \mathcal{R}} \mathbb{E}_{\mu^*}[\varphi], \quad (5.16)$$

under the constraint

$$\mathbb{E}_{\mathbb{Q}^*}[\varphi] \leq \alpha_0, \quad (5.17)$$

with $\alpha_0 := x_0 / \mathbb{E}_{\mathbb{Q}}[(F + y)^+]$ and the probability measures μ^* and $\mathbb{Q}^*$ are defined by

$$\frac{d\mu^*}{d\mu} = \frac{(F + y)^+}{\mathbb{E}[(F + y)^+]}, \quad \frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \frac{(F + y)^+}{\mathbb{E}_{\mathbb{Q}}[(F + y)^+]}.$$

In the setting of a complete market, we can solve the problem explicitly by using the Neyman-Pearson lemma. In statistical test theory the Neyman-Pearson lemma provides an optimality criterion when performing a hypothesis test, see Neyman and Pearson [59] for the original work. In order to employ the lemma to the optimization problem (5.15), we state a different version of the Neyman-Pearson lemma.

Theorem 5.2.2. (Föllmer and Schied [32], Theorem A.30) Let $\mathcal{R}$ denote the set of $\mathcal{F}_T$ -measurable function from $\Omega \rightarrow [0, 1]$.

(1) Take $c \geq 0$, and suppose that φ^0 satisfies

$$\varphi^0 = \begin{cases} 1, & \text{on } \left\{ \frac{d\mathbb{P}_1}{d\mathbb{P}_2} > c \right\}, \\ 0, & \text{on } \left\{ \frac{d\mathbb{P}_1}{d\mathbb{P}_2} < c \right\}. \end{cases} \quad (5.18)$$

Then, for any $\varphi \in \mathcal{R}$,

$$\int \varphi d\mathbb{P}_2 \leq \int \varphi^0 d\mathbb{P}_2 \Rightarrow \int \varphi d\mathbb{P}_1 \leq \int \varphi^0 d\mathbb{P}_1. \quad (5.19)$$

(2) For any $\alpha_0 \in (0, 1)$ there is some $\varphi^0 \in \mathcal{R}$ of the form (5.18) such that $\int \varphi^0 d\mathbb{P}_2 = \alpha_0$. More precisely, if c is an $(1 - \alpha_0)$ -quantile of $d\mathbb{P}_1/d\mathbb{P}_2$ under $\mathbb{P}_2$, we can define φ^0 by

$$\varphi^0 = \mathbf{1}_{\left\{ \frac{d\mathbb{P}_1}{d\mathbb{P}_2} > c \right\}} + \kappa \mathbf{1}_{\left\{ \frac{d\mathbb{P}_1}{d\mathbb{P}_2} = c \right\}},$$

where κ is defined as

$$\kappa := \begin{cases} 0, & \text{if } \mathbb{P}_2\left[\frac{d\mathbb{P}_1}{d\mathbb{P}_2} = c\right] = 0, \\ \frac{\alpha_0 - \mathbb{P}_2\left[\frac{d\mathbb{P}_1}{d\mathbb{P}_2} > c\right]}{\mathbb{P}_2\left[\frac{d\mathbb{P}_1}{d\mathbb{P}_2} = c\right]}, & \text{otherwise.} \end{cases}$$

(3) Any $\varphi^0 \in \mathcal{R}$ satisfying (5.19) is of the form (5.18) for some $c \geq 0$. □

In the following proposition we employ Theorem 5.2.2 to solve the optimization problem (5.15).

Proposition 5.2.3. Let $y \in \mathbb{R}$ be fixed such that $\mathbb{E}_{\mathbb{Q}}[(F + y)^+] > x_0$. The optimal $\tilde{\varphi} \in \mathcal{R}(x)$ which solves (5.15) is given by

$$\tilde{\varphi} = \mathbf{1}_{\left\{ \frac{d\mu}{d\mathbb{Q}} > \zeta(y) \right\}} \quad (5.20)$$

and is called the *randomized test*, where $\zeta(y)$ is given as

$$\zeta(y) = \inf \left\{ \zeta \geq 0; \int_{\left\{ \frac{d\mu}{d\mathbb{Q}} > \zeta \right\}} (F + y)^+ d\mathbb{Q} = x_0 \right\}. \quad (5.21)$$

PROOF. By assumption we have for a fixed $y \in \mathbb{R}$ that $0 < x_0 < \mathbb{E}_{\mathbb{Q}}[(F + y)^+]$, otherwise there is nothing to show since we can replicate the claim, it follows that $\alpha_0 \in (0, 1)$.

Therefore we apply the Neyman-Pearson lemma, Theorem 5.2.2, in terms of $\mu^\star$ and $\mathbb{Q}^\star$ to the problem (5.16) and (5.17) and obtain

$$\tilde{\varphi} = \mathbf{1}_{\{\frac{d\mu^\star}{d\mathbb{Q}^\star} > \tilde{\vartheta}\}} + \kappa \mathbf{1}_{\{\frac{d\mu^\star}{d\mathbb{Q}^\star} = \tilde{\vartheta}\}},$$

where

$$\tilde{\vartheta} := \inf \left\{ \vartheta \geq 0; \mathbb{Q}^\star \left[\frac{d\mu^\star}{d\mathbb{Q}^\star} > \vartheta \right] \leq \alpha_0 \right\}.$$

and

$$\kappa := \begin{cases} 0, & \text{if } \mathbb{Q}^\star \left[\frac{d\mu^\star}{d\mathbb{Q}^\star} = \tilde{\vartheta} \right] = 0, \\ \frac{\alpha_0 - \mathbb{Q}^\star \left[\frac{d\mu^\star}{d\mathbb{Q}^\star} > \tilde{\vartheta} \right]}{\mathbb{Q}^\star \left[\frac{d\mu^\star}{d\mathbb{Q}^\star} = \tilde{\vartheta} \right]}, & \text{otherwise.} \end{cases}$$

The Radon-Nikodym derivatives $\frac{d\mu^\star}{d\mu}$ and $\frac{d\mathbb{Q}^\star}{d\mathbb{Q}}$ are positive on $\{F + y > 0\}$ and therefore the inverse exists. We have for all $\vartheta \geq 0$

$$\begin{aligned} \left\{ \frac{d\mu^\star}{d\mathbb{Q}^\star} > \vartheta \right\} \cap \{F + y > 0\} &= \left\{ \frac{d\mu}{d\mathbb{Q}} \frac{d\mu^\star}{d\mu} \frac{d\mathbb{Q}}{d\mathbb{Q}^\star} > \vartheta \right\} \cap \{F + y > 0\} \\ &= \left\{ \frac{d\mu}{d\mathbb{Q}} \frac{\mathbb{E}_{\mathbb{Q}}[(F + y)^+]}{\mathbb{E}_{\mathbb{Q}}[(F + y)^+]} > \vartheta \right\} \cap \{F + y > 0\}. \end{aligned}$$

Since $\mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{A\}}] = \mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{A\} \cap \{F + y > 0\}}]$ for any given $A \in \mathcal{F}_T$, it follows that

$$\begin{aligned} \tilde{\vartheta} &= \inf \left\{ \vartheta \geq 0; \mathbb{Q}^\star \left[\frac{d\mu^\star}{d\mathbb{Q}^\star} > \vartheta \right] \leq \alpha_0 \right\} \\ &= \inf \left\{ \vartheta \geq 0; \mathbb{E}_{\mathbb{Q}} \left[(F + y)^+ \mathbf{1}_{\{\frac{d\mu^\star}{d\mathbb{Q}^\star} > \vartheta\}} \right] \leq x_0 \right\} \\ &= \inf \left\{ \vartheta \geq 0; \mathbb{E}_{\mathbb{Q}} \left[(F + y)^+ \mathbf{1}_{\{\frac{d\mu^\star}{d\mathbb{Q}^\star} > \vartheta\} \cap \{F + y > 0\}} \right] \leq x_0 \right\} \\ &= \inf \left\{ \vartheta \geq 0; \mathbb{E}_{\mathbb{Q}} \left[(F + y)^+ \mathbf{1}_{\left\{ \frac{d\mu}{d\mathbb{Q}} \frac{\mathbb{E}_{\mathbb{Q}}[(F + y)^+]}{\mathbb{E}_{\mathbb{Q}}[(F + y)^+]} > \vartheta \right\} \cap \{F + y > 0\}} \right] \leq x_0 \right\} \\ &= \inf \left\{ \vartheta \geq 0; \mathbb{E}_{\mathbb{Q}} \left[(F + y)^+ \mathbf{1}_{\left\{ \frac{d\mu}{d\mathbb{Q}} \frac{\mathbb{E}_{\mathbb{Q}}[(F + y)^+]}{\mathbb{E}_{\mathbb{Q}}[(F + y)^+]} > \vartheta \right\}} \right] \leq x_0 \right\}. \end{aligned} \tag{5.22}$$

We assume in this chapter that the probability space is atomless and that $\frac{d\mu}{d\mathbb{Q}}$ has a continuous distribution, therefore it is always possible to find a ϑ such that the expectation given in (5.22) is equal to x_0 . By defining $\zeta(y) := \tilde{\vartheta} \cdot \mathbb{E}[(F + y)^+] / \mathbb{E}_{\mathbb{Q}}[(F + y)^+]$ the expression (5.21) follows.

To show (5.20), we notice that for any $\varphi \in \mathcal{R}$ we have that $\mathbb{E}[\varphi(F + y)^+] = \mathbb{E}[\varphi(F + y)^+ \mathbf{1}_{\{F + y > 0\}}]$ and therefore also $\tilde{\varphi} \mathbf{1}_{\{F + y > 0\}}$ is an optimal

solution to (5.15). We rewrite $\tilde{\varphi}\mathbf{1}_{\{F+y>0\}}$ as

$$\begin{aligned}\tilde{\varphi}\mathbf{1}_{\{F+y>0\}} &= \mathbf{1}_{\{\frac{d\mu^*}{d\mathbb{Q}} > \tilde{\vartheta}\} \cap \{F+y>0\}} + \kappa \mathbf{1}_{\{\frac{d\mu^*}{d\mathbb{Q}} = \tilde{\vartheta}\} \cap \{F+y>0\}} \\ &= \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y)\} \cap \{F+y>0\}} + \kappa \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} = \zeta(y)\} \cap \{F+y>0\}}.\end{aligned}$$

Since the probability space is atomless and $\frac{d\mu}{d\mathbb{Q}}$ has a continuous distribution. We have $\mu[\frac{d\mu}{d\mathbb{Q}} = \zeta(y)] = \mathbb{Q}[\frac{d\mu}{d\mathbb{Q}} = \zeta(y)] = 0$ and the representation (5.20) follows. $\square$

By applying Theorem 5.2.1 and Proposition 5.2.3 to the dynamic optimization problem (5.10) we can derive an explicit solution to this problem. We state the result in the following proposition.

Proposition 5.2.4. The optimal solution to the dynamic optimization problem (5.10) is given by

$$w(y) := \inf_{X \in \Pi(x_0)} \mathbb{E}[(F+y)^+ - X]^+ = \mathbb{E}[(F+y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}]. \quad (5.23)$$

with

$$\zeta(y) = \inf \left\{ \zeta \geq 0; \int_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta\}} (F+y)^+ d\mathbb{Q} = x_0 \right\}. \quad (5.24)$$

Thus, the optimal X^* has the form $(F+y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} \geq \zeta(y)\}}$.

PROOF. Follows from Theorem 5.2.1 and Proposition 5.2.3. $\square$

Proposition 5.2.4 shows that for a given $y \in \mathbb{R}$ we can solve the original dynamic optimization problem. The aim of the following section is to derive the optimal y^* using results from the theory of convex optimization.

5.2.2 Static Optimization Problem

We have seen in the previous section that the infinite-dimensional optimization problem (5.7) can be reduced to a one-dimensional problem.

$$\begin{aligned}\inf_{X \in \Pi(x_0)} AV@R_\lambda(X - F) &= \inf_{X \in \Pi(x_0)} \left\{ \inf_{y \in \mathbb{R}} \left\{ \frac{1}{\lambda} \mathbb{E}[(F+y-X)^+] - y \right\} \right\} \quad (5.25) \\ &= \inf_{y \in \mathbb{R}} \frac{1}{\lambda} \left\{ \inf_{X \in \Pi(x_0)} \left\{ \mathbb{E}[(F+y-X)^+] \right\} - y \right\} \\ &= \inf_{y \in \mathbb{R}} \left\{ \frac{1}{\lambda} \mathbb{E}[(F+y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}] - y \right\} \\ &= \inf_{y \in \mathbb{R}} f(y),\end{aligned}$$

where the function f is defined by $f(y) := \frac{1}{\lambda} \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}] - y$. The optimization problem (5.25) looks fairly simple since we have to solve an one-dimensional optimization problem in the real domain.

We demonstrate the problem in Figure 5.1. We assume here that the density $d\mu/d\mathbb{Q}$ and S_T are comonotone. For a given $y < 0$, we hedge $X^* = (F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y)\}}$ and remain with the shortfall of $(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}$. In other words we hedge the cheap states of $(F + y)^+$ until the initial capital x_0 is consumed.

Figure 5.1: Illustration of the static optimization problem (5.25). The optimal solution $X^* = (F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} \geq \zeta(y)\}}$ is illustrated by the gray area.

In this section we derive the optimal solution y^* and characterize it. We start with lemma to narrow the domain of y^* .

Lemma 5.2.5. The minimum of the problem (5.25) is given for some $y^* \geq y_{\min}$ with

$$y_{\min} := \inf\{y \in \mathbb{R}; \mathbb{E}_{\mathbb{Q}}[(F + y)^+] = x_0\}.$$

PROOF. We notice that $y_{\min} < 0$ since $x_0 < F_0$. The function f can be rewritten as

$$f(y) = \begin{cases} \frac{1}{\lambda} \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}] - y, & \text{for } y \geq y_{\min}, \\ -y, & \text{for } y < y_{\min}. \end{cases}$$

This shows that $f(y_{\min}) \leq f(y)$ for $y < y_{\min}$. The statement of the lemma follows. $\square$

Our aim is to use the following standard result from theory of convex optimization to derive the optimal $y^* \in \mathbb{R}$.

Proposition 5.2.6. (Van Tiel [74], Example 5.31 and Theorem 7)

- (1) Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a convex function. Then g is subdifferentiable at every $a \in \mathbb{R}$, and we have $\partial g(a) = [D^-g(a), D^+g(a)]$.
- (2) Let g be a proper convex function on $\mathbb{R}$. g has a global minimum at $a^* \in \mathbb{R}$ if and only if $0 \in \partial g(a^*)$. $\square$

It follows from the Proposition 5.2.6 that it is necessary to ensure that f is convex and derive its derivatives for $y \geq y_{\min}$ in order to find the minimal y^* of the optimization problem (5.25). We start by proving convexity.

Proposition 5.2.7. The function $f(y) := \frac{1}{\lambda} \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta(y)\}}] - y$ is convex for all $y \geq y_{\min}$.

PROOF. Let $y_1, y_2 \geq y_{\min}$ be given with $y_1 \leq y_2$ and set $y_0 = \gamma y_1 + (1 - \gamma)y_2$ with $\gamma \in (0, 1)$. We have to show that $f(y_0) \leq \lambda f(y_1) + (1 - \gamma)f(y_2)$. We notice that the function $k(y) := \mathbb{E}_{\mathbb{P}}[(F + y)^+ \mathbf{1}_{\{A\}}]$ is convex for any given $A \in \mathcal{F}_T$ and any probability measure $\mathbb{P}$. Thus

$$\begin{aligned} \mathbb{E}[(F + y_0)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta(y_0)\}}] &\leq \gamma \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta(y_0)\}}] \\ &\quad + (1 - \gamma) \mathbb{E}[(F + y_2)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta(y_0)\}}]. \end{aligned} \quad (5.26)$$

By Proposition 5.2.3, we have $\mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{\zeta(y) < \frac{d\mu}{dQ}\}}] = x_0$ for any $y \geq y_{\min}$. Thus by the convexity of the function k

$$\begin{aligned} &\gamma \mathbb{E}_{\mathbb{Q}}[(F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} > \zeta(y_1)\}}] + (1 - \gamma) \mathbb{E}_{\mathbb{Q}}[(F + y_2)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} > \zeta(y_2)\}}] \\ &= \mathbb{E}_{\mathbb{Q}}[(F + y_0)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} > \zeta(y_0)\}}] \leq \gamma \mathbb{E}_{\mathbb{Q}}[(F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} > \zeta(y_0)\}}] \\ &\quad + (1 - \gamma) \mathbb{E}_{\mathbb{Q}}[(F + y_2)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} > \zeta(y_0)\}}]. \end{aligned}$$

It follows that

$$\gamma \mathbb{E}_{\mathbb{Q}}[(F + y_1)^+ \mathbf{1}_{\{\zeta(y_1) < \frac{d\mu}{dQ} < \zeta(y_0)\}}] \leq (1 - \gamma) \mathbb{E}_{\mathbb{Q}}[(F + y_2)^+ \mathbf{1}_{\{\zeta(y_0) < \frac{d\mu}{dQ} < \zeta(y_2)\}}]. \quad (5.27)$$

By changing the measure we see that the left-hand side of inequality (5.27) is in fact larger than $\zeta^{-1}(y_0) \gamma \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{\zeta(y_1) < \frac{d\mu}{dQ} < \zeta(y_0)\}}]$, while the right hand-side of inequality (5.27) is smaller than $\zeta^{-1}(y_0) (1 - \gamma) \mathbb{E}[(F + y_2)^+ \mathbf{1}_{\{\zeta(y_0) < \frac{d\mu}{dQ} < \zeta(y_2)\}}]$. It follows that

$$0 \leq -\gamma \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{\zeta(y_1) < \frac{d\mu}{dQ} < \zeta(y_0)\}}] + (1 - \gamma) \mathbb{E}[(F + y_2)^+ \mathbf{1}_{\{\zeta(y_0) < \frac{d\mu}{dQ} < \zeta(y_2)\}}]. \quad (5.28)$$

Finally, by adding the left-hand side of (5.26) and (5.28) and the right-hand side of (5.26) and (5.28), we find

$$\begin{aligned}\lambda f(y_0) + y_0 &\leq \gamma(\lambda f(y_1) + y_1) + (1 - \gamma)(\lambda f(y_2) + y_2) \\ &= \lambda(\gamma f(y_1) + (1 - \gamma)f(y_2)) + y_0.\end{aligned}$$

Therefore $f(y)$ is a convex function for $y \geq y_{\min}$. $\square$

Before we derive the derivatives of f , we notice that the function ζ is an increasing function in y , where at times it can jump upward. Thus the left- and right-hand limit of ζ at a given point y exists and is denoted by $\zeta(y-)$ and $\zeta(y+)$ respectively. We start with a small lemma that there is no probability mass between $\zeta(y-)$ and $\zeta(y+)$.

Lemma 5.2.8.

(1) We have for all $\mathcal{F}_T$ -measurable F and $y \in \mathbb{R}$

$$\mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{\zeta(y-) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+)\}}] = \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\zeta(y-) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+)\}}] = 0.$$

(2) And $\mathbb{Q}[\zeta(y-) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+)] = \mu[\zeta(y-) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+)] = 0$.

PROOF. (1) Define the function $k(y) := \mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y)\}}]$. By Proposition 5.2.3 $k(y) = x_0$ for any $y \geq y_{\min}$. It follows that $\lim_{h \downarrow 0} k(y + h) = \mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y+)\}}] = x_0$ and $\lim_{h \uparrow 0} k(y + h) = \mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y-)\}}] = x_0$. By subtraction we obtain $\mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{\zeta(y-) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+)\}}] = 0$. The probability measures $\mathbb{Q}$ and μ are equivalent, which means $d\mathbb{Q}/d\mu > 0$ and therefore also $\mathbb{E}[(F + y)^+ \mathbf{1}_{\{\zeta(y-) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+)\}}] = 0$.

(2) Follows from (1), by choosing F to be a constant larger than $-y$. $\square$

We derive the right and left derivatives of the function f .

Proposition 5.2.9. The right and left derivative of the function f are given by

$$\begin{aligned}D^+ f(y) &= -1 + \frac{1}{\lambda} \mu \left[\{F + y \geq 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} < \zeta(y) \right\} \right] \\ &\quad + \frac{1}{\lambda} \zeta(y+) \mathbb{Q} \left[\{F + y \geq 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} > \zeta(y) \right\} \right] \\ &= -1 + \frac{1}{\lambda} \mathbb{E}_{\mathbb{Q}} \left[\mathbf{1}_{\{F + y \geq 0\}} \min \left\{ \frac{d\mu}{d\mathbb{Q}}, \zeta(y+) \right\} \right]\end{aligned}$$

for all $y \geq y_{\min}$.

$$\begin{aligned} D^- f(y) &= -1 + \frac{1}{\lambda} \mu \left[\{F + y > 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} < \zeta(y) \right\} \right] \\ &\quad + \frac{1}{\lambda} \zeta(y-) \mathbb{Q} \left[\{F + y > 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} > \zeta(y) \right\} \right] \\ &= -1 + \frac{1}{\lambda} \mathbb{E}_{\mathbb{Q}} \left[\mathbf{1}_{\{F+y>0\}} \min \left\{ \frac{d\mu}{d\mathbb{Q}}, \zeta(y-) \right\} \right] \end{aligned}$$

for all $y > y_{\min}$.

PROOF. We bound $(F + y + h)^+$ for all $h \geq 0$ in the following way

$$(F + y)^+ + h \mathbf{1}_{\{F+y \geq 0\}} \leq (F + y + h)^+ \leq (F + y)^+ + h \mathbf{1}_{\{F+y+h \geq 0\}}. \quad (5.29)$$

We notice that $\lim_{h \downarrow 0} \mu[F + y + h \geq 0] = \mu[F + y \geq 0]$. The right derivative of f is defined by

$$\begin{aligned} D^+ f(y) &= \lim_{h \downarrow 0} \frac{f(y + h) - f(y)}{h} \\ &= \lim_{h \downarrow 0} \frac{1}{h} \left[\left(\frac{1}{\lambda} \mathbb{E}[(F + y + h)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}] - (y + h) \right) \right. \\ &\quad \left. - \left(\frac{1}{\lambda} \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}] - y \right) \right] \\ &\leq -1 + \frac{1}{\lambda} \lim_{h \downarrow 0} \frac{1}{h} \left[\mathbb{E}[(F + y)^+ + h \mathbf{1}_{\{F+y+h \geq 0\}}] \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}} \right] \\ &\quad - \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}] \Big] \\ &= -1 + \frac{1}{\lambda} \lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\zeta(y) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}] \\ &\quad + \frac{1}{\lambda} \lim_{h \downarrow 0} \mu \left[\{F + y + h \geq 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} < \zeta(y + h) \right\} \right]. \end{aligned}$$

We notice that $\lim_{h \downarrow 0} \mu[\{F + y + h \geq 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} < \zeta(y + h)\}] = \mu[\{F + y \geq 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} < \zeta(y+)\}]$. Therefore we are left with $\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\zeta(y) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}]$. By Lemma 5.2.8 (1), we have $\mathbb{E}[(F + y)^+ \mathbf{1}_{\{\zeta(y) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}] = \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\zeta(y+) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}]$. Therefore we have

$$\begin{aligned} &\lim_{h \downarrow 0} \frac{1}{h} \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\zeta(y+) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}] \\ &\leq \lim_{h \downarrow 0} \frac{1}{h} \zeta(y + h) \mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{\zeta(y+) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}] \\ &= \lim_{h \downarrow 0} \frac{1}{h} \zeta(y + h) \left[\mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y+)\}}] - \mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y+h)\}}] \right] \end{aligned}$$

$$\begin{aligned}
&\leq \lim_{h \downarrow 0} \frac{1}{h} \zeta(y+h) \left[\mathbb{E}_{\mathbb{Q}}[(F+y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y+)\}}] - \mathbb{E}_{\mathbb{Q}}[(F+y+h)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y+h)\}}] \right. \\
&\quad \left. + h\mathbb{Q}\left[\{F+y+h \geq 0\} \cap \left\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y+h)\right\}\right] \right] \\
&= x_0 - x_0 + \zeta(y+)\mathbb{Q}\left[\{F+y \geq 0\} \cap \left\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y+)\right\}\right],
\end{aligned}$$

since by Proposition 5.2.3, $\mathbb{E}_{\mathbb{Q}}[(F+y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y+)\}}] = x_0$ for any $y \geq y_{\min}$. It follows that

$$\begin{aligned}
D^+f(y) &\leq -1 + \frac{1}{\lambda} \mu\left[\{F+y \geq 0\} \cap \left\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y+)\right\}\right] \\
&\quad + \frac{1}{\lambda} \zeta(y+)\mathbb{Q}\left[\{F+y \geq 0\} \cap \left\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y+)\right\}\right], \\
&= -1 + \frac{1}{\lambda} \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}_{\{F+y \geq 0\}} \min\left\{\frac{d\mu}{d\mathbb{Q}}, \zeta(y+)\right\}\right].
\end{aligned} \tag{5.30}$$

By using the lower bound for $(F+y+h)^+$ in (5.29), we can show that the inequality (5.30) is in fact an equality. We have

$$\begin{aligned}
D^+f(y) &= \lim_{h \downarrow 0} \frac{1}{h} \left[\left(\frac{1}{\lambda} \mathbb{E}[(F+y+h)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}] - (y+h) \right) \right. \\
&\quad \left. - \left(\frac{1}{\lambda} \mathbb{E}[(F+y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}] - y \right) \right] \\
&\geq -1 + \frac{1}{\lambda} \lim_{h \downarrow 0} \frac{1}{h} \left[\mathbb{E}[(F+y)^+ + h\mathbf{1}_{\{F+y \geq 0\}}) \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}] \right. \\
&\quad \left. - \mathbb{E}[(F+y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}] \right] \\
&= -1 + \frac{1}{\lambda} \lim_{h \downarrow 0} \frac{1}{h} \left[\mathbb{E}[(F+y)^+ \mathbf{1}_{\{\zeta(y+) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}] \right. \\
&\quad \left. + h\mu\left[\{F+y \geq 0\} \cap \left\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\right\}\right] \right] \\
&\geq -1 + \frac{1}{\lambda} \lim_{h \downarrow 0} \frac{1}{h} \left[\zeta(y+)\mathbb{E}_{\mathbb{Q}}[(F+y)^+ \mathbf{1}_{\{\zeta(y+) < \frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\}}] \right. \\
&\quad \left. + h\mu\left[\{F+y \geq 0\} \cap \left\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\right\}\right] \right] \\
&= -1 + \frac{1}{\lambda} \lim_{h \downarrow 0} \frac{1}{h} \left[\zeta(y+)h\mathbb{Q}\left[\{F+y+h \geq 0\} \cap \left\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y+h)\right\}\right] \right. \\
&\quad \left. + h\mu\left[\{F+y \geq 0\} \cap \left\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y+h)\right\}\right] \right]
\end{aligned}$$

$$\begin{aligned}
&= -1 + \frac{1}{\lambda} \zeta(y+) \mathbb{Q} \left[\{F + y \geq 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} > \zeta(y+)\right\} \right] \\
&\quad + \frac{1}{\lambda} \mu \left[\{F + y \geq 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} < \zeta(y+)\right\} \right] \\
&= -1 + \frac{1}{\lambda} \mathbb{E}_{\mathbb{Q}} \left[\mathbf{1}_{\{F+y \geq 0\}} \min \left\{ \frac{d\mu}{d\mathbb{Q}}, \zeta(y+)\right\} \right].
\end{aligned}$$

It follows that for all $y \geq y_{\min}$, we have

$$D^+ f(y) = -1 + \frac{1}{\lambda} \mathbb{E}_{\mathbb{Q}} \left[\mathbf{1}_{\{F+y \geq 0\}} \min \left\{ \frac{d\mu}{d\mathbb{Q}}, \zeta(y+)\right\} \right].$$

We are left with the left derivative. Here, we bound $(F + y - h)^+$ for all $h \geq 0$ by

$$(F + y)^+ - h \mathbf{1}_{\{F+y > 0\}} \leq (F + y - h)^+ \leq (F + y)^+ - h \mathbf{1}_{\{F+y-h > 0\}}. \quad (5.31)$$

Notice that $\lim_{h \downarrow 0} \mu[F + y - h > 0] = \mu[F + y > 0]$ and similarly we have $\lim_{h \downarrow 0} \mathbb{Q}[F + y - h > 0] = \mathbb{Q}[F + y > 0]$, which explains the strict inequality sign in the representation of the left derivative $D^- f(y)$. By using (5.31), we can derive the left derivative for $y > y_{\min}$ in the same way as the right derivative. Therefore we omit the proof. $\square$

We have seen that f is convex and derived its derivatives. Following Proposition 5.2.6 $y^* \in \mathbb{R}$ is the global minimum if and only if $0 \in [D^- f(y^*), D^+ f(y^*)]$. We state the main theorem of this section.

Theorem 5.2.10. An optimal solution y^* to the minimization problem

$$\inf_{y \geq y_{\min}} \left\{ \frac{1}{\lambda} \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}] - y \right\} = \inf_{y \geq y_{\min}} f(y)$$

is given as follows:

- (1) The minimum is attained by $y^* = y_{\min}$ if and only if we have $\lambda \leq \zeta(y_{\min}+) \mathbb{Q}[F + y_{\min} \geq 0]$.
- (2) Otherwise, an optimal solution is given by a $y^* \in (y_{\min}, \infty)$ satisfying the following two inequalities

$$\begin{aligned}
\zeta(y^*+) &\geq \frac{\lambda - \mu[\{F + y^* \geq 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} < \zeta(y^*)\}]}{\mathbb{Q}[\{F + y^* \geq 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} > \zeta(y^*)\}]}, \\
\zeta(y^*-) &\leq \frac{\lambda - \mu[\{F + y^* > 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} < \zeta(y^*)\}]}{\mathbb{Q}[\{F + y^* > 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} > \zeta(y^*)\}]}.
\end{aligned}$$

PROOF. First, we show that $y^* \in [y_{\min}, +\infty)$. We have for large enough y that

$$\begin{aligned} D^-f(y) &= -1 + \frac{1}{\lambda} \mu \left[\{F + y > 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} < \zeta(y) \right\} \right] \\ &\quad + \frac{1}{\lambda} \zeta(y-) \mathbb{Q} \left[\{F + y > 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} > \zeta(y) \right\} \right] \\ &\geq -1 + \frac{1}{\lambda} \mu \left[\{F + y > 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} < \zeta(y) \right\} \right] \\ &\geq -1 + \frac{1}{\lambda} > 0. \end{aligned}$$

The second last inequality follows from $\lim_{y \rightarrow +\infty} \mu[\{F + y > 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}] = 1$. Thus $\lim_{y \rightarrow +\infty} D^-f(y) > 0$ and it follows that $y^* < +\infty$ and by Lemma 5.2.5 we have $y^* \geq y_{\min}$.

- (1) By the definition of $y_{\min}$ we have $\mathbb{Q}[\{F + y_{\min} \geq 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} < \zeta(y_{\min})\}] = 0$, since $\mu \sim \mathbb{Q}$ also $\mu[\{F + y_{\min} \geq 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} < \zeta(y_{\min})\}] = 0$. We need to ensure that the right derivative of $f(y_{\min})$ is non-negative, so

$$\begin{aligned} D^+f(y_{\min}) &= -1 + \frac{1}{\lambda} \mu \left[\{F + y_{\min} \geq 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} < \zeta(y_{\min}) \right\} \right] \\ &\quad + \frac{1}{\lambda} \zeta(y_{\min}+) \mathbb{Q} \left[\{F + y_{\min} \geq 0\} \cap \left\{ \frac{d\mu}{d\mathbb{Q}} > \zeta(y_{\min}) \right\} \right] \\ &= -1 + \frac{1}{\lambda} \zeta(y_{\min}+) \mathbb{Q}[F + y_{\min} \geq 0] \geq 0. \end{aligned}$$

The statement follows.

- (2) We know by Proposition 5.2.7, that f is a convex function. The minimum is attained by y^* if and only if $0 \in [D^-f(y^*), D^+f(y^*)]$, the left and right derivatives are derived in Proposition 5.2.9. We need ensure that $D^+f(y^*) \geq 0$ and $D^-f(y^*) \leq 0$, or equivalently,

$$\begin{aligned} \zeta(y^*+) &\geq \frac{\lambda - \mu[\{F + y^* \geq 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} < \zeta(y^*)\}]}{\mathbb{Q}[\{F + y^* \geq 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} > \zeta(y^*)\}]}, \\ \zeta(y^*-) &\leq \frac{\lambda - \mu[\{F + y^* > 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} < \zeta(y^*)\}]}{\mathbb{Q}[\{F + y^* > 0\} \cap \{\frac{d\mu}{d\mathbb{Q}} > \zeta(y^*)\}]}. \end{aligned}$$

This completes the proof. $\square$

Next we state necessary and sufficient conditions such that it is optimal to overhedge F , meaning that $\mu[X^* > F] > 0$, where X^* is the value of the optimal hedging portfolio.

This proves that minimizing the risk of losses due to shortfall, which was proposed by Föllmer and Leukert [30] and Rudloff [67], is not equivalent to minimizing the risk, i.e.

$$\inf_{X \in \Pi(x_0)} AV@R_\lambda(X - F) \neq \inf_{X \in \Pi(x_0)} AV@R_\lambda(-(F - X)^+).$$

Proposition 5.2.11. The following statements are equivalent:

- (1) $\mathbb{E}_{\mathbb{Q}}[\min\{\frac{d\mu}{d\mathbb{Q}}, \zeta(0+)\}] < \lambda$.
- (2) It is optimal to overhedge, i.e. $y^* > 0$.

Especially if $\zeta(0+) < \lambda$ then the optimal y^* is positive.

PROOF. The condition $D^+f(0) < 0$ is equivalent to $y^* > 0$. By Proposition 5.2.9 the right derivative at 0 is given by

$$\begin{aligned} D^+f(0) &= -1 + \frac{1}{\lambda} \mathbb{E}_{\mathbb{Q}} \left[\mathbf{1}_{\{F \geq 0\}} \min \left\{ \frac{d\mu}{d\mathbb{Q}}, \zeta(0+) \right\} \right] \\ &= -1 + \frac{1}{\lambda} \mathbb{E}_{\mathbb{Q}} \left[\min \left\{ \frac{d\mu}{d\mathbb{Q}}, \zeta(0+) \right\} \right]. \end{aligned}$$

This proves that (1) and (2) are equivalent. We have the following inequality $\zeta(0) \leq \zeta(0+)$. Assume that $\zeta(0+) < \lambda$, then

$$\mathbb{E}_{\mathbb{Q}} \left[\min \left\{ \frac{d\mu}{d\mathbb{Q}}, \zeta(0+) \right\} \right] \leq \zeta(0+) < \lambda.$$

This proves the second part of the proposition. $\square$

Remark 5.2.12. In Section 5.2.3 we state requirements to λ , F and the parameters in the Black-Scholes Model such that the inequality $\zeta(0+) < \lambda$ is indeed satisfied. Thus we verify that statement (1) of Proposition 5.2.11 can hold true for some examples. $\diamond$

In the last proposition of this section we prove that it is never optimal to hedge in the region $\{\frac{d\mu}{d\mathbb{Q}} < \lambda\}$ by using basic properties of the risk measure $AV@R$. We notice that hedging in the region $\{\frac{d\mu}{d\mathbb{Q}} < \lambda\}$ corresponds to $\zeta(y^*) \geq \lambda$. As a consequence we have that for a given claim F such that $\mathbb{E}[F \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \lambda\}}] > 0$ and a sufficiently large initial capital x_0 we overhedge the claim F since we never hedge the part $F \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \lambda\}}$.

Proposition 5.2.13. The optimal hedging portfolio is of the form $X^* = X \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} \geq \lambda\}}$.

PROOF. Let a contingent claim F be given. Assume that $\mu[\frac{d\mu}{d\mathbb{Q}} < \lambda] > 0$ and $\mathbb{E}[F \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} < \lambda\}}] > 0$, otherwise there is nothing to show. Given a hedging portfolio X with $X = X_1 + X_2$. We split the initial capital x_0 into two parts such that $x_0 = x_1 + x_2$ with $x_1, x_2 > 0$. Let $X_1 > 0$ be a value process on $\{\frac{d\mu}{d\mathbb{Q}} \geq \lambda\}$ which is attainable with initial

capital x_1 , meaning, $X_1 = X \mathbf{1}_{\{\frac{d\mu}{dQ} > \lambda\}}$ and $\mathbb{E}_Q[X_1] = x_1$. Similarly, $X_2 > 0$ is a value process on $\{\frac{d\mu}{dQ} < \lambda\}$ which is attainable with initial capital x_2 . We have

$$x_2 = \mathbb{E}_Q[X_2 \mathbf{1}_{\{\frac{d\mu}{dQ} < \lambda\}}] > \frac{1}{\lambda} \mathbb{E}[X_2].$$

Let q be a λ -quantile of $X_1 + X_2 - F$. Then

$$\begin{aligned} AV@R_\lambda(X_1 - F + x_2) &= AV@R_\lambda(X_1 - F) - x_2 \\ &= \inf_{y \in \mathbb{R}} \frac{1}{\lambda} \mathbb{E}[(F + y - X_1)^+] - y - x_2 \\ &\leq \frac{1}{\lambda} \mathbb{E}[(F + q - X_1)^+] - q - x_2 \\ &< \frac{1}{\lambda} \mathbb{E}[(F + q - X_1)^+] - q + \frac{1}{\lambda} \mathbb{E}[-X_2] \\ &\leq \frac{1}{\lambda} \mathbb{E}[(F + q - X_1 - X_2)^+] - q \\ &= AV@R_\lambda(X_1 + X_2 - F). \end{aligned} \tag{5.32}$$

Inequality (5.32) shows that $X_1 + x_2$ is a better hedging portfolio than $X_1 + X_2$. Let $\mathbb{Q}[\frac{d\mu}{dQ} < \lambda] = \alpha$ which by assumption is larger than 0. We can construct $X_1^1 := X_1 + x_2 \mathbf{1}_{\{\frac{d\mu}{dQ} > \lambda\}}$ with $x_1^1 = \mathbb{E}_Q[X_1^1] = x_1 + (1 - \alpha)x_2$ and $X_2^1 := x_2 \mathbf{1}_{\{\frac{d\mu}{dQ} < \lambda\}}$ with $x_2^1 = \mathbb{E}_Q[X_2^1] = \alpha x_2$ and apply inequality (5.32) again, obtaining that $X_1^1 + x_2^1$ is a better hedging portfolio than $X_1^1 + X_2^1$. By doing this step iteratively, we see that $\mathbb{E}_Q[X_2^n] = \alpha^n x_2$ converges to 0 as $n \rightarrow +\infty$. Therefore it is never optimal to hedge in the region $\{\frac{d\mu}{dQ} < \lambda\}$. $\square$

5.2.3 Average Value at Risk Optimization in the Black-Scholes Model

In this section we derive an explicit solution of hedging problem

$$\inf_{X \in \Pi(x_0)} AV@R_\lambda(X - F) = \inf_{y \geq y_{\min}} \left\{ \frac{1}{\lambda} \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta(y)\}}] - y \right\},$$

in the Black-Scholes model if F is a discounted European call or put. We will illustrate our results numerically. Additionally, we derive explicit conditions such that the optimal y^* is positive which corresponds to a scenario of overhedging.

In the standard Black-Scholes, the underlying asset price process $(\bar{S}_t)_{t \in [0, T]}$ and bond price process $(\bar{B}_t)_{t \in [0, T]}$ are given as follows

$$\begin{aligned} d\bar{S}_t &= \chi \bar{S}_t dt + \sigma \bar{S}_t dW_t, \\ d\bar{B}_t &= r \bar{B}_t dt, \end{aligned}$$

where χ is the drift, $\sigma > 0$ the constant volatility, $(W_t)_{t \in [0, T]}$ a Brownian motion under μ and r is the riskless interest rate. We assume that $\chi > r$.

The discounted asset price process $S_t := \bar{B}_t^{-1} \bar{S}_t$ is therefore given by

$$dS_t = vS_t dt + \sigma S_t dW_t, \quad (5.33)$$

with $v = \chi - r$ and initial value $S_0 > 0$. We can solve (5.33) as

$$S_T = S_0 \exp \left(\sigma W_T + \left(v - \frac{1}{2} \sigma^2 \right) T \right).$$

The unique equivalent martingale measure is then given by

$$\frac{d\mathbb{Q}}{d\mu} = \exp \left(-\frac{v}{\sigma} W_T - \frac{1}{2} \left(\frac{v}{\sigma} \right)^2 T \right).$$

We notice that the equivalent martingale measure can be rewritten in terms of the discounted asset price

$$\frac{d\mathbb{Q}}{d\mu} = S_0^{v/\sigma^2} e^{(1/2)(v^2/\sigma^2 - v)T} S_T^{-v/\sigma^2} = \text{const} \cdot S_T^{-v/\sigma^2}. \quad (5.34)$$

European Call

Our aim is to price the European call option $(\bar{S}_T - K)^+$. The claim is an European call option with underlying $\bar{S}_T$. The strike price of the discounted option is given by Ke^{-rT} , thus $F = e^{-rT}(\bar{S}_T - K)^+ = (S_T - Ke^{-rT})^+$. The option F can be hedged perfectly if we have the initial capital

$$F_0 = \mathbb{E}_{\mathbb{Q}}[F] = \bar{S}_0 \Theta_+(Ke^{-rT}) - Ke^{-rT} \Theta_-(Ke^{-rT}),$$

with

$$\Theta_+(z) = \Phi \left(\frac{\ln S_0 - \ln z}{\sigma \sqrt{T}} + \frac{1}{2} \sigma \sqrt{T} \right), \quad \Theta_-(z) = \Phi \left(\frac{\ln S_0 - \ln z}{\sigma \sqrt{T}} - \frac{1}{2} \sigma \sqrt{T} \right), \quad (5.35)$$

where Φ denotes the cumulative distribution function of the standard normal distribution. For simplicity of notation, denote

$$\bar{\Theta}_{\pm}(z) := 1 - \Theta_{\pm}(z), \quad \Psi_{\pm}(z) := \Theta_{\pm}(e^{-vT}z), \quad \bar{\Psi}_{\pm}(z) := 1 - \Psi_{\pm}(z), \quad M := e^{-rT}K.$$

As mentioned in the beginning of the chapter, we suppose that the initial capital x_0 is smaller than the Black-Scholes price F_0 . We cannot construct a perfect hedge for the European call option F . Instead, we shall minimize Average Value at Risk over all attainable claims bounded from below by 0 with initial capital x_0 . The results of Section 5.2.1 and Section 5.2.2 shall be used to derive an explicit solution.

We have seen in Theorem 5.2.10 that the dynamic optimization problem can be reduced to

$$\inf_{y \geq y_{\min}} \left\{ \frac{1}{\lambda} \mathbb{E}[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta(y)\}}] - y \right\}.$$

with $y_{\min}$ being the unique real solution of $\mathbb{E}_Q[(F + y_{\min})^+] = x_0$ and $\zeta(y)$ the unique real solution of $\mathbb{E}_Q[(F + y)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} > \zeta(y)\}}] = x_0$ for all $y \geq y_{\min}$. We notice that

$$(F + y)^+ = ((S_T - M)^+ + y)^+ = \begin{cases} (S_T - (M - y))^+, & \text{if } y < 0, \\ (S_T - M)^+ + y, & \text{if } y \geq 0. \end{cases}$$

Since $x_0 < F_0$ it follows that $y_{\min} < 0$ and $y_{\min}$ is the real solution of $\mathbb{E}_Q[(S_T - (M - y_{\min}))^+] = x_0$, which is a European call with strike $M - y_{\min}$. Therefore we may apply the Black-Scholes formula for a European call and obtain that $y_{\min}$ is the solution of

$$S_0 \Theta_+(M - y_{\min}) - (M - y_{\min}) \Theta_-(M - y_{\min}) = x_0.$$

For a given $y \geq y_{\min}$ we can rewrite the hedging set $\{\frac{d\mu}{dQ} > \zeta(y)\}$ by using (5.34)

$$\left\{ \frac{d\mu}{dQ} > \zeta(y) \right\} = \left\{ \text{const} \cdot S_T^{v/\sigma^2} > \zeta(y) \right\} = \{S_T > b(y)\},$$

where $b(y)$ is the real root of the capital constraint $\mathbb{E}_Q[(F + y)^+ \mathbf{1}_{\{S_T > b(y)\}}] = x_0$. We have $\mathbb{E}_Q[(F + y)^+] > x_0$ for all $y \geq y_{\min}$ thus the equation $\mathbb{E}_Q[(S_T - (M - y))^+ \mathbf{1}_{\{S_T > b(y)\}}] = x_0$ can only hold true if $b(y) > M - y$ so it follows that $b(y)$ is the implicit solution of

$$\begin{aligned} x_0 &= \mathbb{E}_Q[(S_T - (M - y))^+ \mathbf{1}_{\{S_T > b(y)\}}] \\ &= \mathbb{E}_Q[(S_T - (M - y)) \mathbf{1}_{\{S_T > b(y)\}}] \\ &= S_0 \Theta_+(b(y)) - (M - y) \Theta_-(b(y)). \end{aligned} \tag{5.36}$$

Let $y \geq 0$, we show by contradiction that $b(y) > M$. Assume that $b(y) \leq M$ the capital constraint can be rewritten as follows

$$\begin{aligned} x_0 &= \mathbb{E}_Q[(S_T - M)^+ + y \mathbf{1}_{\{S_T > b(y)\}}] \\ &= \mathbb{E}_Q[(S_T - M)^+] + y \mathbb{Q}[S_T > b(y)] \\ &\geq F_0 > x_0. \end{aligned}$$

Therefore, if $y \geq 0$, the equation $\mathbb{E}_Q[(S_T - M)^+ + y \mathbf{1}_{\{S_T > b(y)\}}] = x_0$ can only be satisfied if $b(y) > M$. Hence $b(y)$ is the solution of the following equation

$$\begin{aligned} x_0 &= \mathbb{E}_Q[(S_T - M)^+ \mathbf{1}_{\{S_T > b(y)\}}] + y \mathbb{Q}[S_T > b(y)] \\ &= S_0 \Theta_+(b(y)) - M \Theta_-(b(y)) + y \Theta_-(b(y)) \\ &= S_0 \Theta_+(b(y)) - (M - y) \Theta_-(b(y)). \end{aligned}$$

We are left with the object function $f(y)$ which we seek to minimize, i.e for the European call we have

$$f(y) = \frac{1}{\lambda} \mathbb{E}[(S_T - M)^+ + y)^+ \mathbf{1}_{\{S_T < b(y)\}}] - y.$$

A small calculation shows that for $y < 0$ the function f can be rewritten as

$$\begin{aligned} f(y) &= \frac{1}{\lambda} \left(\mathbb{E}[(S_T - (M - y)) \mathbf{1}_{\{M - y < S_T < b(y)\}}] \right) - y \\ &= \frac{1}{\lambda} \left(S_0 e^{vT} (\Psi_+(M - y) - \Psi_+(b(y))) \right. \\ &\quad \left. - (M - y)(\Psi_-(M - y) - \Psi_-(b(y))) \right) - y, \end{aligned} \quad (5.37)$$

where $b(y)$ is the implicit solution of (5.36). For $y \geq 0$ we have

$$\begin{aligned} f(y) &= \frac{1}{\lambda} \left(\mathbb{E}[(S_T - M) \mathbf{1}_{\{M < S_T < b(y)\}}] + y \mu[S_T < b(y)] \right) - y \\ &= \frac{1}{\lambda} \left(S_0 e^{vT} (\Psi_+(M) - \Psi_+(b(y))) \right. \\ &\quad \left. - M(\Psi_-(M) - \Psi_-(b(y))) + y \bar{\Psi}_-(b(y)) \right) - y. \end{aligned} \quad (5.38)$$

We sum up the results.

Proposition 5.2.14. Let $x_0 < F_0$ and $F = (S_T - M)^+$ a discounted European call option, in the Black-Scholes model (5.33), the solution to the optimization problem

$$\inf_{X \in \Pi(x_0)} AV @ R_\lambda(X - F)$$

with $\lambda \in (0, 1]$ can be derived by solving the reduced optimization problem

$$\inf_{y \geq y_{\min}} f(y) = \inf_{y \geq y_{\min}} \frac{1}{\lambda} \mathbb{E}[(S_T - M)^+ + y)^+ \mathbf{1}_{\{S_T < b(y)\}}] - y. \quad (5.39)$$

The function $f(y)$ has the expression (5.37) for $y < 0$ and (5.38) for $y \geq 0$. And the function $b(y)$ is the implicit solution of (5.36). The optimal strategy is to replicate the contingent claim

$$X^* = ((S_T - M)^+ + y^*)^+ \mathbf{1}_{\{S_T > b(y^*)\}}$$

where y^* is a point of minimum of (5.39). □

Let's observe two properties of an optimal solution y^* . We will derive an equivalent condition for $y^* = y_{\min}$ and $y^* > 0$.

Proposition 5.2.15. Let $F = (S_T - M)^+$ be a discounted European call option in the Black-Scholes model (5.33). We can characterize y^* of the reduced optimization problem (5.39) in the following way

(1) The minimum is attained by $y^* = y_{\min}$ if and only if

$$\lambda \leq \frac{(M + y_{\min})^{v/\sigma^2}}{S_0^{v/\sigma^2} e^{(1/2)(v^2/\sigma^2 - v)T}} \Theta_-(M - y_{\min}).$$

(2) Overhedging occurs, if and only if

$$\begin{aligned} & \left(S_0^{v/\sigma^2} \Phi \left(\frac{\ln b(0+) - \ln S_0}{\sigma \sqrt{T}} + \left(\frac{1}{2} - \frac{v}{\sigma^2} \right) \sigma \sqrt{T} \right) + b(0+)^{v/\sigma^2} \Theta_-(b(0+)) \right) \\ & \cdot S_0^{-v/\sigma^2} e^{(-1/2)(v^2/\sigma^2 - v)T} < \lambda. \end{aligned} \quad (5.40)$$

A necessary condition for overhedging is

$$M < S_0 e^{(1/2)(v - \sigma^2)T} \lambda^{\sigma^2/v}. \quad (5.41)$$

PROOF. (1) Due to Theorem 5.2.10 the optimal is given by $y^* = y_{\min}$ if and only if $\zeta(y_{\min}+) \mathbb{Q}[F + y_{\min} \geq 0] \geq \lambda$. The relation of the equivalent martingale measure and the discounted asset price process is given in (5.34). We have

$$\begin{aligned} \{S_T > b(y_{\min}+)\} &= \left\{ \frac{d\mu}{d\mathbb{Q}} > \zeta(y_{\min}+) \right\} \\ &= \left\{ S_T > S_0 \left(\zeta(y_{\min}+) e^{(1/2)(v^2/\sigma^2 - v)T} \right)^{\sigma^2/v} \right\}. \end{aligned}$$

We notice that $b(y_{\min}+) = M + y_{\min}$ and therefore $\zeta(y_{\min}+)$ is given by

$$\zeta(y_{\min}+) = \frac{(M + y_{\min})^{v/\sigma^2}}{S_0^{v/\sigma^2} e^{(1/2)(v^2/\sigma^2 - v)T}}.$$

By Lemma 5.2.5 $y_{\min} < 0$ and therefore $\mathbb{Q}[S_T \geq M - y_{\min}] = \Theta_-(M - y_{\min})$. If we consider a European call option then the optimal y^* is equal to $y_{\min}$ if and only if

$$\begin{aligned} \lambda &\leq \zeta(y_{\min}+) \mathbb{Q}[F + y_{\min} \geq 0] \\ &= \zeta(y_{\min}+) \mathbb{Q}[S_T \geq M - y_{\min}] \\ &= \frac{(M + y_{\min})^{v/\sigma^2}}{S_0^{v/\sigma^2} e^{(1/2)(v^2/\sigma^2 - v)T}} \Theta_-(M - y_{\min}). \end{aligned}$$

- (2) By Proposition 5.2.11 we have $y^* > 0$ if and only if $\mathbb{E}_{\mathbb{Q}}[\min\{\frac{d\mu}{dQ}, \zeta(0+)\}] < \lambda$.
 Let's calculate $\mathbb{E}_{\mathbb{Q}}[\min\{\frac{d\mu}{dQ}, \zeta(0+)\}]$ in the Black-Scholes model. We have

$$\mathbb{E}_{\mathbb{Q}}\left[\min\left\{\frac{d\mu}{dQ}, \zeta(0+)\right\}\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{d\mu}{dQ} \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta(0+)\}}\right] + \zeta(0+)\mathbb{Q}\left[\frac{d\mu}{dQ} > \zeta(0+)\right]. \quad (5.42)$$

We calculate the first term, by using (5.34)

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{d\mu}{dQ} \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta(0+)\}}\right] = S_0^{-v/\sigma^2} e^{(-1/2)(v^2/\sigma^2 - v)T} \mathbb{E}_{\mathbb{Q}}\left[S_T^{v/\sigma^2} \mathbf{1}_{\{S_T < b(0+)\}}\right]. \quad (5.43)$$

The conditional expectation given in (5.43) is similar to a power option. A small calculation shows that

$$\mathbb{E}_{\mathbb{Q}}\left[S_T^{v/\sigma^2} \mathbf{1}_{\{S_T < b(0+)\}}\right] = S_0^{v/\sigma^2} \Phi\left(\frac{\ln b(0+) - \ln S_0}{\sigma\sqrt{T}} + \left(\frac{1}{2} - \frac{v}{\sigma^2}\right)\sigma\sqrt{T}\right).$$

We are left with the second term of (5.42)

$$\begin{aligned} \zeta(0+)\mathbb{Q}\left[\frac{d\mu}{dQ} > \zeta(0+)\right] &= b(0+)v/\sigma^2 S_0^{-v/\sigma^2} e^{(-1/2)(v^2/\sigma^2 - v)T} \mathbb{Q}[S_T > b(0+)] \\ &= b(0+)v/\sigma^2 S_0^{-v/\sigma^2} e^{(-1/2)(v^2/\sigma^2 - v)T} \Theta_-(b(0+)). \end{aligned}$$

It follows that $y^* > 0$ if and only if

$$\begin{aligned} &\left(S_0^{v/\sigma^2} \Phi\left(\frac{\ln b(0+) - \ln S_0}{\sigma\sqrt{T}} + \left(\frac{1}{2} - \frac{v}{\sigma^2}\right)\sigma\sqrt{T}\right) + b(0+)v/\sigma^2 \Theta_-(b(0+))\right) \\ &\quad \cdot S_0^{-v/\sigma^2} e^{(-1/2)(v^2/\sigma^2 - v)T} < \lambda. \end{aligned}$$

A necessary condition for overhedging is $\zeta(0+) < \lambda$. The relation between ζ and b where shown in (1), thus overhedging occurs if

$$\lambda > \zeta(0+) = \frac{b(0+)v/\sigma^2}{S_0^{v/\sigma^2} e^{(1/2)(v^2/\sigma^2 - v)T}},$$

which is equivalent to $b(0+) < S_0 e^{(1/2)(v - \sigma^2)T} \lambda^{\sigma^2/v}$. Due to the structure of the European call we have $b(0+) \geq M$. Therefore it is necessary that

$$M < S_0 e^{(1/2)(v - \sigma^2)T} \lambda^{\sigma^2/v}. \quad (5.44)$$

This completes the proof of the proposition. □

The condition (5.40) of the previous proposition is rather difficult to verify and therefore it is not convenient to use in practice. On the other hand, the necessary condition $M < S_0 e^{(1/2)(v-\sigma^2)T} \lambda \sigma^2/v$ illustrates how rarely overhedging occurs. If we consider an European call and assume that $v = \sigma^2$, then it is required that $M < S_0 \lambda$, which means S_0 is deep in the money. If we choose a common level $\lambda = 0.05$ overhedging will not occur in practice.

We illustrate the results of the previous two propositions numerically. Consider the Black-Scholes model with parameters $S_0 = 100$, $\chi = 0.05$, $r = 0.01$, $\sigma = 0.3$ and $T = 0.25$. For a European call with strike $K = 100$, the Black-Scholes price is $F_0 = 6.10$. We seek a hedging strategy that minimizes Average Value at Risk at level $\lambda = 0.01$, see Figure 5.2.

Figure 5.2: Value of $AV@R$ at level 0.01 and optimal y^* with different level of initial capital.

As we observe in Figure 5.2 the optimal y^* moves towards 0 from below as x_0 converges to F_0 , but the values of y^* are never positive. We have seen that overhedging can only occur if equation (5.41) of Proposition 5.2.15 holds true. If we adjust the parameters of the previous example and set $S_0 = 150$, $\chi = 0.1$ and $\lambda = 0.9$. Then we have $S_0 e^{(1/2)(v-\sigma^2)T} \lambda \sigma^2/v = 135 > 99.75 = M$. We see in Figure 5.3 that for sufficiently large initial capital the optimal y^* is positive.

Figure 5.3: Value of $AV@R$ at level 0.9 and optimal y^* with different level of initial capital.

Figure 5.3 shows that overhedging occurs. We verify that there indeed exists no optimal y^* such that $y^* \leq 0$ if we set the initial capital to $x_0 = 45$. In Figure 5.4 we plot the function f to verify that y^* is indeed positive. Additionally, we plot the optimal hedging portfolio payoff X^* . In this example $y^* = 5.76$ and $b(y^*) = 139.34$, thus $X^* = ((S_T - 99.75)^+ + 5.76)\mathbf{1}_{\{S_T > 139.34\}}$.

Figure 5.4: The function f and the optimal hedging strategy for call option using average value at risk. X^* is illustrated by gray area.

European Put

Next, we focus on the discounted European put option $F = (M - S_T)^+$. The derivation of the European put and call are very similar. The amount of initial capital F_0 necessary for a perfect hedge is

$$F_0 = \mathbb{E}_{\mathbb{Q}}[F] = M\bar{\Theta}_-(M) - S_0\bar{\Theta}_+(M).$$

We notice that

$$(F + y)^+ = ((M - S_T)^+ + y)^+ = \begin{cases} ((M + y) - S_T)^+, & \text{if } y < 0, \\ (M - S_T)^+ + y, & \text{if } y \geq 0. \end{cases}$$

Since $x_0 < F_0$ it follows that $y_{\min} < 0$ and $y_{\min}$ is the real solution of $\mathbb{E}_{\mathbb{Q}}[((M + y_{\min}) - S_T)^+] = x_0$, which is European put with strike $M + y_{\min}$, so we can apply the Black-Scholes formula and $y_{\min}$ is the solution of

$$(M + y_{\min})\bar{\Theta}_-(M + y_{\min}) - S_0\bar{\Theta}_+(M + y_{\min}) = x_0.$$

As for the call option, for a given $y \geq y_{\min}$ the hedging set $\{\frac{d\mu}{d\mathbb{Q}} > \zeta(y)\}$ can be rewritten as

$$\left\{ \frac{d\mu}{d\mathbb{Q}} > \zeta(y) \right\} = \{S_T > b(y)\},$$

where $b(y)$ is the real root of the capital constraint $\mathbb{E}_{\mathbb{Q}}[(F + y)^+ \mathbf{1}_{\{S_T > b(y)\}}] = x_0$. Consider the case $y < 0$, then $(F + y)^+ = ((M + y) - S_T)^+$. We have $\mathbb{E}_{\mathbb{Q}}[(F + y)^+] > x_0$ for all $y \geq y_{\min}$. It follows that the equation $\mathbb{E}_{\mathbb{Q}}[((M + y) - S_T)^+ \mathbf{1}_{\{S_T > b(y)\}}] = x_0$ can hold true if $b(y) < M + y$. Therefore $b(y)$ is the implicit solution of

$$\begin{aligned} x_0 &= \mathbb{E}_{\mathbb{Q}}[((M + y) - S_T) \mathbf{1}_{\{b(y) < S_T < M + y\}}] \\ &= (M + y)(\Theta_-(b(y)) - \Theta_-(M + y)) - S_0(\Theta_+(b(y)) - \Theta_+(M + y)). \end{aligned} \quad (5.45)$$

If $y \geq 0$ then $b(y)$ is the solution of the following equation

$$\begin{aligned} x_0 &= \mathbb{E}_{\mathbb{Q}}[(M - S_T) \mathbf{1}_{\{b(y) < S_T < M\}}] + y\mathbb{Q}[S_T > b(y)] \\ &= M(\Theta_-(b(y)) - \Theta_-(M)) - S_0(\Theta_+(b(y)) - \Theta_+(M)) + y\Theta_-(b(y)), \end{aligned} \quad (5.46)$$

if $b(y) < M$, and under the assumption that $b(y) \geq M$, we have

$$x_0 = y\mathbb{Q}[S_T > b(y)] = y\Theta_-(b(y)). \quad (5.47)$$

Now we are able to evaluate the function $f(y)$. Since $b(y) < M + y$ for $y < 0$ it follows that

$$\begin{aligned} f(y) &= \frac{1}{\lambda} \left(\mathbb{E}[(M + y - S_T) \mathbf{1}_{\{S_T < b(y)\}}] \right) - y \\ &= \frac{1}{\lambda} \left((M + y)\bar{\Psi}_-(b(y)) - S_0 e^{vT} \bar{\Psi}_+(b(y)) \right) - y. \end{aligned} \quad (5.48)$$

Similarly we have for a given $y \geq 0$ and with assumption $b(y) < M$ that f is given by

$$\begin{aligned} f(y) &= \frac{1}{\lambda} \left(\mathbb{E}[(M - S_T) \mathbf{1}_{\{S_T < b(y)\}}] + y\mu[S_T < b(y)] \right) - y \\ &= \frac{1}{\lambda} \left((M + y)\bar{\Psi}_-(b(y)) - S_0 e^{vT} \bar{\Psi}_+(b(y)) \right) - y. \end{aligned} \quad (5.49)$$

On the other hand if $b(y) \geq M$

$$\begin{aligned} f(y) &= \frac{1}{\lambda} \left(\mathbb{E}[(M - S_T) \mathbf{1}_{\{S_T < M\}}] + y\mu[S_T < b(y)] \right) - y \\ &= \frac{1}{\lambda} \left(M\bar{\Psi}_-(M) - S_0 e^{vT} \bar{\Psi}_+(M) + y\bar{\Psi}_-(b(y)) \right) - y. \end{aligned} \quad (5.50)$$

We sum up the results.

Proposition 5.2.16. Let $x_0 < F_0$ and $F = (M - S_T)^+$ a discounted European put option in the Black-Scholes model (5.33), then the optimization problem

$$\inf_{X \in \Pi(x_0)} AV@R_\lambda(X - F)$$

with $\lambda \in (0, 1]$ can be rewritten as

$$\inf_{y \geq y_{\min}} f(y) = \inf_{y \geq y_{\min}} \frac{1}{\lambda} \mathbb{E}[(M - S_T)^+ + y)^+ \mathbf{1}_{\{S_T > b(y)\}}] - y. \quad (5.51)$$

The functions $f(y)$ and $b(y)$ are given by (5.48) and (5.46) for $y < 0$. If $y \geq 0$ then $f(y)$ and $b(y)$ are given by (5.49) and (5.46) if $b(y) < M$, and by (5.50) and (5.47) if $b(y) \geq M$. The solution is given by y^* , which is a point of minimum of (5.51). The optimal strategy is to replicate the contingent claim

$$X^* = ((M - S_T)^+ + y^*)^+ \mathbf{1}_{\{S_T < b(y^*)\}}. \quad \square$$

Proposition 5.2.17. Let $F = (M - S_T)^+$ be a discounted European put option in the Black-Scholes model (5.33). We can characterize y^* of the reduced optimization problem (5.39) in the following way:

- (1) The minimum y^* is always larger than $y_{\min}$, i.e. $y^* > y_{\min}$.
- (2) Overhedging occurs, if and only if

$$\begin{aligned} &\left(S_0^{v/\sigma^2} \Phi \left(\frac{\ln b(0+) - \ln S_0}{\sigma \sqrt{T}} + \left(\frac{1}{2} - \frac{v}{\sigma^2} \right) \sigma \sqrt{T} \right) + b(0+)^{v/\sigma^2} \Theta_-(b(0+)) \right) \\ &\quad \cdot S_0^{-v/\sigma^2} e^{(-1/2)(v^2/\sigma^2 - v)T} < \lambda. \end{aligned}$$

A sufficient condition for overhedging is $b(0+) < S_0 e^{(1/2)(v - \sigma^2)T} \lambda^{\sigma^2/v}$.

- PROOF. (1) Due to Theorem 5.2.10 the optimal y^* is equal to $y_{\min}$ if and only if $\zeta(y_{\min}+)\mathbb{Q}[F + y_{\min} \geq 0] \geq \lambda$. Since $\zeta(y_{\min}+) = 0$ we have $y^* > y_{\min}$.
- (2) The first part follows directly from Proposition 5.2.15. By Proposition 5.2.11 we have $y^* > 0$ if $\zeta(0+) < \lambda$. As in Proposition 5.2.15, we have

$$\lambda > \zeta(0+) = \frac{b(0+)^{v/\sigma^2}}{S_0^{v/\sigma^2} e^{(1/2)(v^2/\sigma^2 - v)T}},$$

which is equivalent to $b(0+) < S_0 e^{(1/2)(v-\sigma^2)T} \lambda^{\sigma^2/v}$.

□

Notice that we always overhedge for the put option when x_0 is large enough. This is a key difference to the call. The condition $b(0+) < S_0 e^{(1/2)(v-\sigma^2)T} \lambda^{\sigma^2/v}$ of Proposition 5.2.17 (2) is always satisfied for sufficiently large initial capital x_0 since in the Black-Scholes model $b(0+) \rightarrow 0$ as $x_0 \rightarrow F_0$. Nevertheless, the initial capital needs to be very large. Let $v = \sigma^2 = 0.09$, $S_0 = 100$, $K = 100$, $r = 0$ and $T = 0.25$, then $F_0 = 5.9785$ is the Black-Scholes price of an European put. If $x_0 = 5.9784$ then $b(0+) = 49.8566$, which means overhedging would only occur if $\lambda \geq 0.5$. If we choose a common level $\lambda = 0.05$ overhedging will not occur in practice.

Figure 5.5 shows results for an at the money European put in the Black-Scholes model with parameters $S_0 = 100$, $K = 100$, $\chi = 0.05$, $r = 0.01$, $\sigma = 0.3$ and $T = 0.25$. It concerns a hedging strategy that minimizes Average Value at Risk at level $\lambda = 0.01$.

Figure 5.5: Value of $AV@R$ at level 0.01 and optimal y^* with different level of initial capital

5.3 Hedging under a Simple Spectral Risk Measure

In this section we will extend our optimization results to a simple spectral risk measure. Our aim is to solve (5.2), i.e.

$$\inf_{\pi \in \Pi(x_0)} \rho_{\nu_n}(X - F), \text{ with } x_0 < F_0, \quad (5.52)$$

where ρ_{ν_n} is a simple spectral risk measure with spectrum $\Upsilon(t) = \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbf{1}_{[0, \lambda_i]}(t)$ and $\sum_{i=1}^n \alpha_i = 1$. We have seen in Section 5.1, that this problem can be rewritten as follows

$$\inf_{X \in \Pi(x_0)} \rho_{\nu_n}(X - F) = \inf_{\mathbf{y} \in \mathbb{R}^n} \left\{ \inf_{X \in \Pi(x_0)} \left\{ \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E}[(F + y_i - X)^+] \right\} - \sum_{i=1}^n \alpha_i y_i \right\}.$$

In Section 5.3.1 we will focus on the inner dynamic optimization problem

$$w(\mathbf{y}) := \inf_{X \in \Pi(x_0)} \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E}[(F + y_i - X)^+],$$

for given $\mathbf{y} \in \mathbb{R}^n$. The solution of this problem extends the results of Proposition 5.2.4 for the one-dimensional case.

We will not further characterize the function $f(\mathbf{y}) = w(\mathbf{y}) - \sum_{i=1}^n \alpha_i y_i$. The derivation of the derivatives and the proof of convexity of the function f can not be directly generalized from the one-dimensional to the multi-dimensional case. These proofs are left as interesting directions for further research. Instead we will finalize this chapter by calculating numerically the solution for an European call option in the Black-Scholes model.

5.3.1 Dynamic Optimization Problem

Similarly to the one-dimensional problem, we first solve the inner problem

$$w(\mathbf{y}) := \inf_{X \in \Pi(x_0)} \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E}[(F + y_i)^+ - X]^+. \quad (5.53)$$

for fixed $\mathbf{y} \in \mathbb{R}^n$ and with $\alpha_i \in (0, 1)$, $\lambda_i \in (0, 1]$. Without loss of generality we assume that $\lambda_i < \lambda_j$ for all $i < j$. We have seen in (5.3) that y_i^* is a λ_i -quantile of $X - F$. Since the quantile function is monotone increasing, it follows that for the optimal $\mathbf{y}^*$ we have $y_i^* \leq y_j^*$ for all $i < j$. Thus throughout this section we assume that $y_i < y_j$ for all $i < j$. We define $\eta_i := \sum_{j=i}^n \frac{\alpha_j}{\lambda_j}$ for all $i \in \{1, \dots, n\}$. We state the main theorem of this section.

Theorem 5.3.1. The optimal solution X^* to problem (5.53) is given by

$$X^* = (F + y_1)^+ \mathbf{1}_{\{\eta_1 \frac{d\mu}{d\mathbb{Q}} > \zeta(y)\}} + \sum_{i=2}^n ((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{\eta_i \frac{d\mu}{d\mathbb{Q}} > \zeta(y)\}}, \quad (5.54)$$

with

$$\begin{aligned} \zeta(y) = \inf \left\{ \zeta \geq 0; \mathbb{E}_{\mathbb{Q}} \left[(F + y_1)^+ \mathbf{1}_{\{\eta_1 \frac{d\mu}{d\mathbb{Q}} > \zeta\}} \right. \right. \\ \left. \left. + \sum_{i=2}^n ((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{\eta_i \frac{d\mu}{d\mathbb{Q}} > \zeta\}} \right] = x_0 \right\}. \end{aligned} \quad (5.55)$$

PROOF. The proof will be divided into three parts. We start by splitting X into slices depending on the values of y_i with $i \in \{1, \dots, n\}$ such that for any slice X'_i the initial capital x_i is consumed. This decomposition allows us to rewrite the original problem (5.53) into an easier form such that we can use the Neyman-Pearson lemma. Second, we solve the reduced optimization problem for a given $\{x_1, \dots, x_n\}$ employing the Neyman-Pearson lemma. In the third part of the proof we will find the optimal values x_i for all $i \in \{1, \dots, n\}$ by verifying that a given choice is indeed the optimal solution. We start with the reduced optimization problem.

Part 1: We notice that the optimal $X \in [0, (F + y_n)^+]$ since we are trying to minimize the shortfall (5.53) and values above $(F + y_n)^+$ are not contributing. Therefore we can bound the set of solutions from above by $(F + y_n)^+$ and (5.53) corresponds to

$$\inf_{X \in \Pi'(x_0)} \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E}[(F + y_i)^+ - X]^+$$

with $\Pi'(x_0) := \{X; X \in [0, (F + y_n)^+], \mathbb{E}_{\mathbb{Q}}[X] \leq x_0\}$. Any random variable $X' \in \Pi'(x_0)$ can be split into slices in the following way

$$\begin{aligned} X'_1 &= X' \mathbf{1}_{\{X' \leq (F + y_1)^+\}} + (F + y_1)^+ \mathbf{1}_{\{X' > (F + y_1)^+\}}, \\ X'_i &= (X' - (F + y_{i-1})^+) \mathbf{1}_{\{(F + y_{i-1})^+ < X' \leq (F + y_i)^+\}} \\ &\quad + ((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{X' > (F + y_i)^+\}}, \quad \forall i \in \{2, \dots, n\}, \end{aligned} \quad (5.56)$$

with

$$\mathbb{E}_{\mathbb{Q}}[X'_i] \leq x_i \text{ and } \sum_{i=1}^n x_i = x_0.$$

It follows from the construction of the slice (5.55) that $x_1 \leq \mathbb{E}_{\mathbb{Q}}[(F + y_1)^+]$ and $x_i \leq \mathbb{E}_{\mathbb{Q}}[(F + y_i)^+ - (F + y_{i-1})^+]$ for $i \in \{2, \dots, n\}$. Define

$$\begin{aligned}\Pi'_1(x) &:= \{X; X \in [0, (F + y_1)^+], \mathbb{E}_{\mathbb{Q}}[X] \leq x \text{ with } x \leq \mathbb{E}_{\mathbb{Q}}[(F + y_1)^+]\}, \\ \Pi'_i(x) &:= \{X; X \in [0, (F + y_i)^+ - (F + y_{i-1})^+], \mathbb{E}_{\mathbb{Q}}[X] \leq x \\ &\quad \text{with } x \leq \mathbb{E}_{\mathbb{Q}}[(F + y_i)^+ - (F + y_{i-1})^+]\}, \quad i \in \{2, \dots, n\}.\end{aligned}$$

It follows from the definition that

$$\Pi'(x_0) = \bigcup_{\substack{\mathbf{x} \in \mathbb{R}^N \\ \sum_{i=1}^n x_i = x_0}} \sum_{i=1}^n \Pi'_i(x_i). \quad (5.57)$$

By using the decomposition (5.56) and representation of the set of attainable claims $\Pi(x_0)$ given in (5.57) we rewrite the inner problem (5.53) as

$$\begin{aligned}& \inf_{X \in \Pi(x_0)} \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E}[(F + y_i)^+ - X]^+ \\ &= \inf \left\{ \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E} \left[\left((F + y_i)^+ - \sum_{i=1}^n X'_i \right)^+ \right]; X \in \bigcup_{\substack{\mathbf{x} \in \mathbb{R}^N \\ \sum_{i=1}^n x_i = x_0}} \sum_{i=1}^n \Pi'_i(x_i) \right\}.\end{aligned}$$

This finalizes the first part of the proof.

Part 2: Let x_i for all $i \in \{1, \dots, n\}$ be given such that $\sum_{i=1}^n x_i = x_0$. We want to solve

$$\inf_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E} \left[\left((F + y_i)^+ - \sum_{i=1}^n X'_i \right)^+ \right]. \quad (5.58)$$

We are able to solve (5.58) componentwise for any $k \in \{1, \dots, n\}$. For a given $k \in \{1, \dots, n\}$ we want to derive the solution of the following problem

$$\inf_{X'_k \in \Pi'(x_k)} \sum_{i=k}^n \frac{\alpha_i}{\lambda_i} \mathbb{E} \left[\left((F + y_i)^+ - \sum_{\substack{j=1 \\ j \neq k}}^n X'_j - X'_k \right)^+ \right].$$

Since we only optimize X'_k , the above given problem has the same optimal solution as

$$\sup_{X'_k \in \Pi'(x_k)} \mathbb{E}[X'_k]. \quad (5.59)$$

The problem (5.59) corresponds to the dynamic optimization problem given in Section 5.2.1. Assume that $k \in \{2, \dots, n\}$ then $X'_k \in [0, (F + y_k)^+ - (F + y_{k-1})^+]$ and the optimization problem (5.59) is equivalent to

$$\sup_{\varphi \in \mathcal{R}'(x_k)} \mathbb{E}[\varphi((F + y_k)^+ - (F + y_{k-1})^+)], \quad (5.60)$$

$$\mathcal{R}'(x_k) := \{\varphi : \Omega \rightarrow [0, 1], \mathcal{F}_T\text{-measurable}, \mathbb{E}_{\mathbb{Q}}[\varphi((F + y_k)^+ - (F + y_{k-1})^+)] \leq x_k\}.$$

By changing the measure the optimization problem (5.60) can be reformulated as

$$\sup_{\varphi \in \mathcal{R}} \mathbb{E}_{\mu^*}[\varphi],$$

under the constraint

$$\mathbb{E}_{\mathbb{Q}^*}[\varphi] \leq \frac{x_k}{\mathbb{E}_{\mathbb{Q}}[(F + y_k)^+ - (F + y_{k-1})^+]},$$

where the probability measures μ^* and $\mathbb{Q}^*$ are defined by

$$\frac{d\mu^*}{d\mu} = \frac{(F + y_k)^+ - (F + y_{k-1})^+}{\mathbb{E}_{\mu}[(F + y_k)^+ - (F + y_{k-1})^+]}, \quad \frac{d\mathbb{Q}^*}{d\mathbb{Q}} = \frac{(F + y_k)^+ - (F + y_{k-1})^+}{\mathbb{E}_{\mathbb{Q}}[(F + y_k)^+ - (F + y_{k-1})^]}.$$

We notice that by construction of $\Pi'(x_k)$ we have $x_k / \mathbb{E}_{\mathbb{Q}}[(F + y_k)^+ - (F + y_{k-1})^+] \leq 1$. Thus we can solve this problem explicitly by the Neyman-Pearson lemma, see Theorem 5.2.2. The optimal solution for (5.60) is given by

$$X_k^* = ((F + y_k)^+ - (F + y_{k-1})^+) \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta_k\}},$$

where ζ_k is chosen such that the capital constraint is satisfied, i.e.

$$\zeta_k = \inf \left\{ \zeta \geq 0; \mathbb{E}_{\mathbb{Q}}[(F + y_k)^+ - (F + y_{k-1})^+] \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta\}} = x_k \right\}. \quad (5.61)$$

For $k = 1$ the calculation can be carried out in a similar manner and we receive

$$X_1^* = (F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta_1\}} \text{ with } \zeta_1 = \inf \left\{ \zeta \geq 0; \mathbb{E}_{\mathbb{Q}}[(F + y_1)^+] \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta\}} = x_1 \right\}. \quad (5.62)$$

For given $\mathbf{x}$ the optimal solution of problem (5.58) is

$$\sum_{i=1}^n X_i^* = (F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta_1\}} + \sum_{j=2}^n ((F + y_j)^+ - (F + y_{j-1})^+) \mathbf{1}_{\{\frac{d\mu}{d\mathbb{Q}} > \zeta_j\}}. \quad (5.63)$$

where the values $(\zeta_1, \dots, \zeta_n)$ are given by (5.61) and (5.62). Substituting $\sum_{i=1}^n X_i^*$ given in (5.63) as a solution to the inner optimization problem (5.53), we obtain

$$\begin{aligned}
 & \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E} \left[((F + y_i)^+ - \sum_{j=1}^n X_j^*)^+ \right] \\
 &= \sum_{j=1}^n \frac{\alpha_j}{\lambda_j} \mathbb{E} [(F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta_1\}}] \\
 & \quad + \sum_{i=2}^n \left(\sum_{j=i}^n \frac{\alpha_j}{\lambda_j} \mathbb{E} [((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta_i\}}] \right) \\
 &= \eta_1 \mathbb{E} [(F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta_1\}}] + \sum_{i=2}^n \left(\eta_i \mathbb{E} [((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta_i\}}] \right) \\
 &= g(\zeta_1, \dots, \zeta_n).
 \end{aligned} \tag{5.64}$$

Part 3: The aim of the third step of the proof is to derive the optimal $(\zeta_1^*, \dots, \zeta_n^*)$, which minimizes the inner optimization problem (5.64). An optimal choice $(\zeta_1^*, \dots, \zeta_n^*)$ is given by

$$\eta_1 \zeta_1^* = \eta_2 \zeta_2^* = \dots = \eta_n \zeta_n^* = \zeta(\mathbf{y}), \tag{5.65}$$

where $\zeta(\mathbf{y})$ is given such that constraint (5.55) is satisfied. We prove that for any other selection $(\zeta_1, \dots, \zeta_n)$ which satisfied the capital constraints we have

$$g(\zeta_1^*, \dots, \zeta_n^*) \leq g(\zeta_1, \dots, \zeta_n).$$

Let $(\zeta_1, \dots, \zeta_n)$ be given. Since the capital constraint is satisfied for $(\zeta_1, \dots, \zeta_n)$ and $(\zeta_1^*, \dots, \zeta_n^*)$, we have

$$\mathbb{E}_Q \left[(F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} > \zeta_1\}} + \sum_{i=2}^n ((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{\frac{d\mu}{dQ} > \zeta_i\}} \right] = x_0, \tag{5.66}$$

and

$$\mathbb{E}_Q \left[(F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} > \zeta_1^*\}} + \sum_{i=2}^n ((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{\frac{d\mu}{dQ} > \zeta_i^*\}} \right] = x_0. \tag{5.67}$$

Let $\mathbb{M}_1, \mathbb{M}_2 \subset \mathbb{N}$ such that $\zeta_i \geq \zeta_i^*$ for all $i \in \mathbb{M}_1$ and $\zeta_i < \zeta_i^*$ for all $i \in \mathbb{M}_2$. Without loss of generality, we assume that $1 \in \mathbb{M}_1$. We subtract (5.66) from (5.67) and obtain

$$\begin{aligned}
 & \mathbb{E}_Q [(F + y_1)^+ \mathbf{1}_{\{\zeta_1^* < \frac{d\mu}{dQ} < \zeta_1\}}] + \sum_{i \in \mathbb{M}_1 \setminus \{1\}} \mathbb{E}_Q [((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{\zeta_i^* < \frac{d\mu}{dQ} < \zeta_i\}}] \\
 &= \sum_{i \in \mathbb{M}_2} \mathbb{E}_Q [((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{\zeta_i < \frac{d\mu}{dQ} < \zeta_i^*\}}].
 \end{aligned} \tag{5.68}$$

The left-hand side of equation (5.68) is smaller than

$$(\zeta_1^*)^{-1} \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{\zeta_1^* < \frac{d\mu}{dQ} < \zeta_1\}}] \\ + \sum_{i \in \mathbb{M}_1 \setminus \{1\}} (\zeta_i^*)^{-1} \mathbb{E}[(F + y_i)^+ - (F + y_{i-1})^+ \mathbf{1}_{\{\zeta_i^* < \frac{d\mu}{dQ} < \zeta_i\}}],$$

and the right-hand side of equation (5.68) is larger than

$$\sum_{i \in \mathbb{M}_2} (\zeta_i^*)^{-1} \mathbb{E}[(F + y_i)^+ - (F + y_{i-1})^+ \mathbf{1}_{\{\zeta_i < \frac{d\mu}{dQ} < \zeta_i^*\}}].$$

Therefore multiplying both sides with $\zeta(\mathbf{y})$ gives in (5.65) we obtain

$$\eta_1 \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{\zeta_1^* < \frac{d\mu}{dQ} < \zeta_1\}}] + \sum_{i \in \mathbb{M}_1 \setminus \{1\}} \eta_i \mathbb{E}[(F + y_i)^+ - (F + y_{i-1})^+ \mathbf{1}_{\{\zeta_i^* < \frac{d\mu}{dQ} < \zeta_i\}}] \\ \geq \sum_{i \in \mathbb{M}_2} \eta_i \mathbb{E}[(F + y_i)^+ - (F + y_{i-1})^+ \mathbf{1}_{\{\zeta_i < \frac{d\mu}{dQ} < \zeta_i^*\}}],$$

which can be rewritten as

$$\eta_1 \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta_1^*\}}] + \sum_{i=2}^n \left(\eta_i \mathbb{E}[(F + y_i)^+ - (F + y_{i-1})^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta_i^*\}}] \right) \\ \leq \eta_1 \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta_1\}}] + \sum_{i=2}^n \left(\eta_i \mathbb{E}[(F + y_i)^+ - (F + y_{i-1})^+ \mathbf{1}_{\{\frac{d\mu}{dQ} < \zeta_i\}}] \right).$$

It follows that $g(\zeta_1^*, \dots, \zeta_n^*) \leq g(\zeta_1, \dots, \zeta_n)$ and we conclude that the optimal choice of $(\zeta_1, \dots, \zeta_n)$ is given by (5.65). This proves the theorem. $\square$

Theorem 5.3.1 allows us to reduce the infinite-dimensional optimization problem (5.52) into the following n-dimensional problem

$$\inf_{\pi \in \Pi(x_0)} \rho_{\nu_n}(X - F) = \inf_{\mathbf{y} \in \mathbb{R}^N} \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbb{E}[(F + y_i - X^*)^+] - \alpha_i y_i \quad (5.69) \\ = \inf_{\mathbf{y} \in \mathbb{R}^N} \eta_1 \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{\eta_1 \frac{d\mu}{dQ} < \zeta(\mathbf{y})\}}] - \alpha_1 y_1 \\ + \sum_{i=2}^N \left(\eta_i \mathbb{E}[(F + y_i)^+ - (F + y_{i-1})^+ \mathbf{1}_{\{\eta_i \frac{d\mu}{dQ} < \zeta(\mathbf{y})\}}] - \alpha_i y_i \right) \\ = \inf_{\mathbf{y} \in \mathbb{R}^N} f(\mathbf{y}).$$

We illustrate the optimal hedge X^* given by (5.54) in Figure 5.6. Under the assumption that the density $d\mu/dQ$ and S_T are comonotone. For given $y_1 < y_2 < y_3 < y_4$ we

start from the right, and first hedge the cheap states. The optimal X^* is a step function where the length of the step depends on the weights η_1, η_2, η_3 and η_4 . Unlike the previous section, there might be cases that slices are completely hedged or not hedged at all.

Figure 5.6: Illustration of the static optimization problem (5.69). The optimal solution X^* given by (5.54) is illustrated by the gray area.

We can solve the problem numerically, as we will demonstrate in the next section by hedging an European call option in the Black-Scholes model.

5.3.2 Simple Spectral Risk Measure Optimization in the Black-Scholes Model

We derive a semi-explicit solution for a call option in the framework of the Black-Scholes model using a simple spectral risk measure, see (5.52) and present results for a numerical example.

As in the one-dimensional case in Section 5.2.3, the discounted asset price process $(S_t)_{t \in [0, T]}$ is given in (5.33). Our aim is to price a European call option. The discounted payoff of the option is given by $F = (S_T - M)^+$. We assume that the initial capital x_0 is smaller than the Black-Scholes price F_0 . In summary, we want to solve the following optimization problem

$$\inf_{\pi \in \Pi(x_0)} \rho_{\nu_n}(X - F), \text{ with } x_0 < F_0,$$

where ρ_{ν_n} is a simple spectral risk measure with spectrum $\Upsilon(t) = \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbf{1}_{[0, \lambda_i]}(t)$ and $\sum_{i=1}^n \alpha_i = 1$. We have seen in Theorem 5.3.1 that this problem can be rewritten as

$$\begin{aligned} \inf_{\pi \in \Pi(x_0)} \rho_{\nu_n}(X - F) &= \inf_{y \in \mathbb{R}^N} \eta_1 \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{\eta_1 \frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}] - \alpha_1 y_1 \\ &\quad + \sum_{i=2}^N \left(\eta_i \mathbb{E}[(F + y_i)^+ - (F + y_{i-1})^+] \mathbf{1}_{\{\eta_i \frac{d\mu}{d\mathbb{Q}} < \zeta(y)\}}] - \alpha_i y_i \right). \end{aligned}$$

The derivation is analogous to the one-dimensional case in Section 5.2.3; we use the same notation and follow the same steps as in that section.

The definition of $y_{\min}$ is given by $y_{\min} = \inf\{y \in \mathbb{R}; \mathbb{E}_{\mathbb{Q}}[(F + y)^+] = x_0\}$, see Lemma 5.2.5. In the Black-Scholes model $y_{\min}$ is the solution of

$$S_0 \Theta_+(M - y_{\min}) - (M - y_{\min}) \Theta_-(M - y_{\min}) = x_0.$$

For any $i \leq n$ and $y_i \geq y_{\min}$ the hedging set $\{\eta_i \frac{d\mu}{d\mathbb{Q}} > \zeta(\mathbf{y})\}$ can be rewritten as

$$\begin{aligned} \left\{ \eta_i \frac{d\mu}{d\mathbb{Q}} > \zeta(\mathbf{y}) \right\} &= \left\{ \eta_i \cdot \text{const} \cdot S_T^{v/\sigma^2} > \zeta(\mathbf{y}) \right\} = \left\{ S_T > \eta_i^{-\sigma^2/v} b(\mathbf{y}) \right\} \\ &= \left\{ S_T > c_i(\mathbf{y}) \right\}, \end{aligned}$$

where we define $c_i(\mathbf{y}) := \eta_i^{-\sigma^2/v} b(\mathbf{y})$ and therefore $c(\mathbf{y})$ is the real root of the capital constraint given by

$$\begin{aligned} x_0 &= \mathbb{E}_{\mathbb{Q}}[X^*] \\ &= \mathbb{E}_{\mathbb{Q}} \left[(F + y_1)^+ \mathbf{1}_{\{S_T > c_1(\mathbf{y})\}} + \sum_{i=2}^n ((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{S_T > c_i(\mathbf{y})\}} \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[((S_T - M)^+ + y_1)^+ \mathbf{1}_{\{S_T > c_1(\mathbf{y})\}} \right. \\ &\quad \left. + \sum_{i=2}^n (((S_T - M)^+ + y_i)^+ - ((S_T - M)^+ + y_{i-1})^+) \mathbf{1}_{\{S_T > c_i(\mathbf{y})\}} \right]. \end{aligned} \tag{5.70}$$

To simplify equation (5.70) further we need to order $c_i(\mathbf{y})$, $(M - y_i)^+$ and $(M - y_{i-1})^+$ for $i \in \{2, \dots, n\}$ as well as $c_1(\mathbf{y})$ and $(M - y_1)^+$. We notice that $c_i(\mathbf{y}) > c_{i-1}(\mathbf{y})$ and $(M - y_{i-1})^+ \geq (M - y_i)^+ \geq M$ for all $i \in \{2, 3, \dots, n\}$ such that $y_{i-1} < 0$.

We split X^* in two parts $X^* = X_1 + X_2$ with $X_1 := (F + y_1)^+ \mathbf{1}_{\{S_T > c_1(\mathbf{y})\}}$. Assume that $y_1 < 0$, then $(F + y_1)^+ = (S_T - (M - y_1))^+$. We have that $\mathbb{E}_{\mathbb{Q}}[(F + y_1)^+ + X_2] \geq \mathbb{E}_{\mathbb{Q}}[(F + y_1)^+] \geq x_0$ for all $y_1 \geq y_{\min}$ thus the equation $x_0 = \mathbb{E}_{\mathbb{Q}}[X^*] = \mathbb{E}_{\mathbb{Q}}[(F + y_1)^+ \mathbf{1}_{\{S_T > c_1(\mathbf{y})\}} + X_2]$ can only hold true if $c_1(\mathbf{y}) > (M - y_1)^+$. If otherwise $y_1 \geq 0$ then $\mathbb{E}_{\mathbb{Q}}[(F + y_1)^+ + X_2] > F_0 + y + \mathbb{E}_{\mathbb{Q}}[X_2] > x_0$ therefore the equation $\mathbb{E}_{\mathbb{Q}}[(F + y_1)^+ \mathbf{1}_{\{S_T > c_1(\mathbf{y})\}} + X_2] = x_0$ can only be satisfied if $c_1(\mathbf{y}) > M$. It follows that $c_i(\mathbf{y}) > M$ if $y_i \geq 0$ and $c_i(\mathbf{y}) > (M - y_{i-1})^+ > (M - y_i)^+$ if $y_i < 0$ for $i \in \{2, \dots, n\}$.

Therefore $c(\mathbf{y})$ is the solution of the following equation

$$\begin{aligned}
 x_0 &= \mathbb{E}_{\mathbb{Q}} \left[(F + y_1)^+ \mathbf{1}_{\{S_T > c_1(\mathbf{y})\}} + \sum_{i=2}^n ((F + y_i)^+ - (F + y_{i-1})^+) \mathbf{1}_{\{S_T > c_i(\mathbf{y})\}} \right] \\
 &= \mathbb{E}_{\mathbb{Q}} \left[((S_T - M)^+ + y_1)^+ \mathbf{1}_{\{S_T > c_1(\mathbf{y})\}} + \sum_{i=2}^n (y_i - y_{i-1}) \mathbf{1}_{\{S_T > c_i(\mathbf{y})\}} \right] \quad (5.71) \\
 &= S_0 \Theta_+(c_1(\mathbf{y})) - (M - y_1) \Theta_-(c_1(\mathbf{y})) + \sum_{i=2}^n (y_i - y_{i-1}) \Theta_+(c_i(\mathbf{y})).
 \end{aligned}$$

We are left with the object function f which we seek to minimize. The function f is given by

$$\begin{aligned}
 \inf_{\mathbf{y} \in \mathbb{R}^N} f(\mathbf{y}) &= \inf_{\mathbf{y} \in \mathbb{R}^N} \left\{ \eta_1 \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{S_T < c_1(\mathbf{y})\}}] - \alpha_1 y_1 \right. \\
 &\quad \left. + \sum_{i=2}^n \left(\eta_i \mathbb{E}[(F + y_i)^+ - (F + y_{i-1})^+] \mathbf{1}_{\{\bar{S}_T < c_i(\mathbf{y})\}} \right) - \alpha_i y_i \right\} \\
 &= \inf_{\mathbf{y} \in \mathbb{R}^N} \left\{ \eta_1 g(y_1) - \alpha_1 y_1 + \sum_{i=2}^n \left(\eta_i h(y_i, y_{i-1}) - \alpha_i y_i \right) \right\}.
 \end{aligned}$$

We further specify the functions g and h for F being a discounted European call. Let $y_1 < 0$, then $(F + y_1)^+ = (S_T - (M - y_1))^+$ and

$$\begin{aligned}
 g(y_1) &= \mathbb{E}[(S_T - (M - y_1)) \mathbf{1}_{\{M - y_1 < S_T < c_1(\mathbf{y})\}}] \\
 &= S_0 e^{vT} (\Psi_+(M - y_1) - \Psi_+(c_1(\mathbf{y}))) - (M - y_1) (\Psi_-(M - y_1) - \Psi_-(c_1(\mathbf{y}))).
 \end{aligned}$$

On the other hand, if $y_1 \geq 0$ then $(F + y_1)^+ = (S_T - M)^+ + y_1$, and the function g is given by

$$\begin{aligned}
 g(y_1) &= \mathbb{E}[(S_T - M) \mathbf{1}_{\{M < S_T < c_1(\mathbf{y})\}}] + y_1 \mu[S_T < c_1(\mathbf{y})] \\
 &= S_0 e^{vT} (\Psi_+(M) - \Psi_+(c_1(\mathbf{y}))) - M (\Psi_-(M) - \Psi_-(c_1(\mathbf{y}))) + y_1 \bar{\Psi}_-(c_1(\mathbf{y})).
 \end{aligned}$$

To evaluate the function h we need to distinguish three cases. First, assume that $y_{i-1} < y_i < 0$, then

$$\begin{aligned}
 h(y_i, y_{i-1}) &= \mathbb{E}[(S_T - (M - y_i))^+ - (S_T - (M - y_{i-1}))^+] \mathbf{1}_{\{S_T < c_i(\mathbf{y})\}} \\
 &= \mathbb{E}[(S_T - (M - y_{i-1})) \mathbf{1}_{\{M - y_i < S_T < M - y_{i-1}\}} \\
 &\quad + (y_i - y_{i-1}) \mu[M - y_{i-1} < S_T < c_i(\mathbf{y})]] \\
 &= S_0 e^{vT} (\Psi_+(M - y_i) - \Psi_+(M - y_{i-1})) \\
 &\quad - (M - y_i) (\Psi_-(M - y_i) - \Psi_-(M - y_{i-1})) \\
 &\quad + (y_i - y_{i-1}) (\Psi_-(M - y_{i-1}) - \Psi_-(c_i(\mathbf{y}))).
 \end{aligned}$$

If $y_{i-1} < 0 \leq y_i$, we have

$$\begin{aligned}
 h(y_i, y_{i-1}) &= \mathbb{E}[(S_T - M)^+ + y_i - (S_T - (M - y_{i-1}))^+ \mathbf{1}_{\{S_T < c_i(\mathbf{y})\}}] \\
 &= \mathbb{E}[(S_T - M) \mathbf{1}_{\{M < S_T < M - y_{i-1}\}} + y_i \mu[S_T < M - y_i] \\
 &\quad + (y_i - y_{i-1}) \mu[M - y_i < S_T < c_i(\mathbf{y})]] \\
 &= S_0 e^{vT} (\Psi_+(M) - \Psi_+(M - y_{i-1})) - M (\Psi_-(M) - \Psi_-(M - y_{i-1})) \\
 &\quad + y_i \bar{\Psi}_-(M - y_i) + (y_i - y_{i-1}) (\Psi_-(M - y_i) - \Psi_-(c_i(\mathbf{y}))).
 \end{aligned}$$

The last scenario is given by $0 \leq y_{i-1} < y_i$. Then

$$\begin{aligned}
 h(y_i, y_{i-1}) &= \mathbb{E}[(S_T - M)^+ + y_i - (S_T - M)^+ - y_{i-1}) \mathbf{1}_{\{S_T < c_i(\mathbf{y})\}}] \\
 &= (y_i - y_{i-1}) \mu[S_T < c_i(\mathbf{y})] \\
 &= (y_i - y_{i-1}) \bar{\Psi}_-(c_i(\mathbf{y})).
 \end{aligned}$$

We sum up the results in the following proposition.

Proposition 5.3.2. Let $x_0 < F_0$ and $F = (S_T - M)^+$ a discounted European call option. In the Black-Scholes model (5.33) the solution to the optimization problem

$$\inf_{X \in \Pi(x_0)} \rho_{\nu_n}(X - F)$$

with spectrum $\Upsilon(t) = \sum_{i=1}^n \frac{\alpha_i}{\lambda_i} \mathbf{1}_{[0, \lambda_i]}(t)$ can be expressed by

$$\begin{aligned}
 &\inf_{X \in \Pi(x_0)} \rho_{\nu_n}(X - F) \tag{5.72} \\
 &= \inf_{\mathbf{y} \in \mathbb{R}^N} \left\{ \eta_1 \mathbb{E}[(F + y_1)^+ \mathbf{1}_{\{S_T < c_1(\mathbf{y})\}}] - \alpha_1 y_1 \right. \\
 &\quad \left. + \sum_{i=2}^n \left(\eta_i \mathbb{E}[(F + y_i)^+ - (F + y_{i-1})^+] \mathbf{1}_{\{S_T < c_i(\mathbf{y})\}}] - \alpha_i y_i \right) \right\} \\
 &= \inf_{\mathbf{y} \in \mathbb{R}^N} \left\{ \eta_1 g(y_1) - \alpha_1 y_1 + \sum_{i=2}^n \left(\eta_i h(y_i, y_{i-1}) - \alpha_i y_i \right) \right\}.
 \end{aligned}$$

The functions g and h are given by

$$g(y_1) = \begin{cases} S_0 e^{vT} (\Psi_+(M - y_1) - \Psi_+(c_1(\mathbf{y}))) \\ \quad - (M - y_1) (\Psi_-(M - y_1) - \Psi_-(c_1(\mathbf{y}))), & \text{if } y_1 < 0, \\ S_0 e^{vT} (\Psi_+(M) - \Psi_+(c_1(\mathbf{y}))) \\ \quad - M (\Psi_-(M) - \Psi_-(c_1(\mathbf{y}))) + y_1 \bar{\Psi}_-(c_1(\mathbf{y})), & \text{if } y_1 \geq 0, \end{cases}$$

and

$$h(y_i, y_{i-1}) = \begin{cases} S_0 e^{vT} (\Psi_+(M - y_i) - \Psi_+(M - y_{i-1})) \\ \quad - (M - y_i)(\Psi_-(M - y_i) - \Psi_-(M - y_{i-1})) \\ \quad + (y_i - y_{i-1})(\Psi_-(M - y_{i-1}) - \Psi_-(c_i(\mathbf{y}))), \text{ if } y_{i-1} < y_i < 0, \\ S_0 e^{vT} (\Psi_+(M) - \Psi_+(M - y_{i-1})) - M(\Psi_-(M) - \Psi_-(M - y_{i-1})) \\ \quad + y_i \bar{\Psi}_-(M - y_i) + (y_i - y_{i-1})(\Psi_-(M - y_i) - \Psi_-(c_i(\mathbf{y}))), \\ \quad \text{if } y_{i-1} < 0 \leq y_i, \\ (y_i - y_{i-1}) \bar{\Psi}_-(c_i(\mathbf{y})), \text{ if } 0 \leq y_{i-1} < y_i. \end{cases}$$

The function $c_i(\mathbf{y})$ is the implicit solution of (5.70). Let $\mathbf{y}^*$ be the point of minimum of (5.72), then the optimal strategy is to replicate the contingent claim

$$\begin{aligned} X^* &= ((S_T - M)^+ + y_1^*)^+ \mathbf{1}_{\{S_T > c_1(\mathbf{y}^*)\}} \\ &\quad + \sum_{i=2}^n (((S_T - M)^+ + y_i^*)^+ - ((S_T - M)^+ + y_{i-1}^*)^+) \mathbf{1}_{\{S_T > c_i(\mathbf{y}^*)\}}. \quad \square \end{aligned}$$

We illustrate numerically the optimal hedging set, see Figure 5.7, of a spectral risk measure with parameters $\alpha_1 = 0.025$, $\alpha_2 = 0.025$, $\alpha_3 = 0.95$ and $\lambda_1 = 0.1$, $\lambda_2 = 0.2$, $\lambda_3 = 0.3$. We consider the Black-Scholes model with $S_0 = 100$, $\chi = 0.05$, $r = 0.01$, $\sigma = 0.3$ and $T = 0.25$ and a European call with strike $K = 100$. In this example, we have $y_1^* = -20.25$, $y_2^* = -13.6$, $y_3^* = -8.2$ and $c_1(\mathbf{y}) = 133.28$, $c_2(\mathbf{y}) = 157.14$, $c_3(\mathbf{y}) = 171.45$.

Figure 5.7: The optimal hedging strategy for call option using a simple spectral risk measure. X^* is illustrated by gray area.

Due to the choice of the parameters α_1 , α_2 and α_3 in the example, the values of $c_1(\mathbf{y})$ and $c_3(\mathbf{y})$ are relatively close and therefore easy to illustrate in a picture. The ratio is given by $c_3(\mathbf{y})/c_1(\mathbf{y}) = 1.29$. If on the other hand we choose $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$, then $c_3(\mathbf{y})/c_1(\mathbf{y}) = (\eta_3/\eta_1)^{-\sigma^2/v} = 46.33$, which means the value of $c_3(\mathbf{y})$ is more than 46 times larger than $c_1(\mathbf{y})$.

In this chapter we priced a claim in a complete market set-up when initial capital is too small to invest in perfect replication. This leads to a partial hedging strategy. We derived a suitable trading strategy such that risk of the difference of the hedging portfolio and the claim is minimized.

The main contribution of this chapter is that by using the Fenchel-Legendre transform it is not necessary to bound the hedging portfolio from above by the claim. Therefore it is possible that the optimal hedging portfolio is partially larger than the claim, a scenario we called overhedging. We proved in Proposition 5.2.11 that under certain conditions it would be optimal to overhedge the claim. Nevertheless, we have seen in the Black-Scholes model that overhedging rarely occurs since a large value for the level of Average Value at Risk is required.

Conclusions and Recommendations

In this chapter we draw conclusions and give an outlook on possible directions for future research.

6.1 Conclusions

The aim of this dissertation has been to study and characterize linear combinations and convolutions of convex risk measures on L^p -spaces by a systematic application of convex analysis, in particular Fenchel duality. Dual representations have been derived and representation results via acceptance sets have been considered. Further, two applications to pricing and hedging financial claims have been studied. First, the problem of pricing and hedging has been treated as a trade-off between a trader and a regulator in an incomplete market setting. And second, a partial hedging method in a complete market has been derived. In both applications a convex risk measure has been used as an objective function for the optimization problem.

Chapter 2 provided an overview of the concept of convex risk measures on L^p -spaces. We used the tools of convex analysis to link properties of convex risk measures to the corresponding properties in the dual space. The aim of this chapter was to lay the foundation for the later chapters and therefore we studied risk functions in a systematic way. For simplicity we chose risk measures to be functions from L^p -spaces with $1 < p < \infty$ to $\mathbb{R} \cup \{+\infty\}$. Due to our heavy reliance on convex analysis we wanted to ensure that the dual representation of a convex risk measure exists. This requires that a convex risk measure is proper and lower semi-continuous function. To ensure properness of a convex risk measure we chose the property finiteness at 0. Even though normality is a more natural condition, doing nothing should not bear any risk, it is very difficult to ensure that the inf-convolution and deconvolution of two normalized risk measures are again normalized risk measures.

In Chapter 3 linear combinations and convolutions of convex risk measures were studied. We characterized them in terms of their penalty functions and acceptance sets by using the duality correspondence of operations on given functions. Furthermore, we investigated when certain combinations of risk measures belong to the class of convex risk measures, meaning that the resulting function is satisfying the properties of a convex risk measure. The characterization of the epi-multiplication of a risk measure and the weighted sum of two risk measure were straightforward. We needed to assume that deconvolution of two risk measures is finite at 0, since the supremum of the difference of two convex function is in general not finite. Rather strong assumptions had to be made to characterize the weighted difference of convex risk measures, since this function is not monotone, convex and lower semi-continuous. Therefore all these properties were assumed. However, we obtained conditions on the subdifferentials or Gâteaux-differentials to ensure these properties. We believe that these theoretical results can be used for many applications, for instance, when agents with different risk attitudes are studied or if an optimal design problem is considered since the operations we investigated are inverse pairs.

In Chapter 4 we studied the pricing and hedging problem of contingent claims in an incomplete market as a trade-off between a trader and a regulator. In this set up we derived the risk measure price and the risk indifference price. We assumed that the capital reserve process is a positive constant, therefore the risk exposure of the trader could be measured at time 0 and thus the part used to cover the losses in interest earning was explicitly calculated. This approach could not be used if the capital reserve process would be updated in time and therefore the price of a claim could not be calculated explicitly.

In Chapter 5 a suitable hedging strategy was derived such that the risk of the difference of the hedging portfolio and the claim was minimized under the condition that the initial capital is smaller than the Black-Scholes price. As a risk measure we considered Average Value at Risk and a simple spectral risk measure. Using the Fenchel-Legendre transform we rewrote Average Value at Risk in terms of expected shortfall; this approach yielded an upper boundary for the hedging portfolio. For simplicity, we assumed that the portfolio is non-negative to obtain a lower boundary. This is an usual assumption in the mathematical finance literature, however it can be generalized to any fixed lower boundary. By obtaining these boundaries, the Neyman-Pearson lemma could be exploited. We showed that this more natural optimization problem differs from the minimization of the risk of losses due to shortfall and that both problems have different solutions. We discovered that overhedging only arises in special situations, for example, when the level of Average Value at Risk is high, the current asset price is deep in the money or the initial capital is close to the Black-Scholes price. We did not solve the outer n -dimensional optimization problem. The calculation of the derivatives and proof of convexity could not be directly generalized from the one-dimensional to the n -dimensional case. These proofs are an interesting direction for further research.

6.2 Recommendations

In this section we state a list of several issues which could be of interest for future research:

- Like convex risk measures, also combined risk measures can be studied on more general function spaces. The properties which define a convex risk measure can be weakened, such as sub-additive instead or translation invariance of quasi-convexity instead of convexity. There is also the possibility of characterizing conditional or dynamic combined convex risk measures.
- The hedging problem in Chapter 4 can be seen as an optimal design problem from the perspective of the regulator. Given the risk measure of a trader, what risk measure should the regulator use in order to obtain a desired convex risk measure as a risk measure price? This problem might be of great interest, since it reflects an important question. What penalties should the financial authorities impose on market participants in order to maintain a 'safe' financial system?
- The pricing model given in Chapter 4 could be interpreted from the perspective of game theory. The regulator and trader are two rational decision makers with opposite goals, i.e. minimizing risk versus maximizing profit.
- In the capital reserve model the collateral and the hedging strategy were determined only at the beginning. A dynamic approach, which would reflect reality better, could be considered in which the capital reserve and the hedging strategy is continuously updated.
- The optimization problem given in Chapter 5 could be generalized by considering spectral risk measures instead of simple spectral risk measures. We suspect the solution of the dynamic hedging problem under a simple spectral risk measure converges to the solution under a spectral risk measure, but just if the set of all attainable claims is compact. We do not know about the explicit form of the solution.

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Summary

The analysis and interpretation of risk play a crucial role in different areas of modern finance. This includes pricing of financial products, capital allocation and derivation of economic capital. Key to this analysis is the quantification of the risk via risk measures. A promising approach is to define risk measures by a set of desirable properties. This leads to the main topic of this research; the characterization of so called *convex risk measures*.

First, we review the concept of convex risk measures on Lebesgue spaces and provide a structural basis for the following parts of the thesis by stating and proving the different characterization results and adjusting them to our definitions and notation.

A key result of this thesis is the characterization of linear combinations and convolutions of convex risk measures. We study different dual correspondences, which are induced by the Fenchel-Legendre transform. In our case we investigated three different duality correspondences, which are sum and inf-convolution, difference and deconvolution and multiplication of scalars and epi-multiplication. These results are used to characterize linear combinations and convolutions of convex risk measures. We investigate when certain combinations of risk measures belong to the class of convex risk measures and investigate the basic properties.

Furthermore, two applications based on theoretical results of the first part of the thesis are derived.

In the first application we study the pricing and hedging problem for contingent claims in an incomplete market as a trade-off between a trader and a regulator. In our model the regulator allows the trader to take some risk, but insists that the residual risk, which is not hedged away, has to be covered. To achieve this, the regulator introduces an extra bank account, which serves as a capital reserve to cover for eventual losses of the trader, and is dependent on the risk of the trader's portfolio. The risk attitudes of the trader and the regulator are reflected by different risk measures. We derive risk measure price and the risk indifference price. In both cases, the resulting risk measure is given by a weighted sum of the regulator's and trader's risk measures. This new operator is also a convex risk measure as we have proven in the first part of the thesis.

In the second application we consider the problem of partial hedging of a contingent claim. Under the assumption of a complete market, it is always possible to replicate the claim. In this case, the claim can be priced using the unique equivalent martingale measure. The question is of a different nature when the initial capital is less than the expectation under the equivalent martingale measure. Under this condition we derive a suitable hedging strategy such that the risk of the difference of the hedging portfolio and the claim is minimized. As risk measure we consider Average Value at Risk and a simple spectral risk measure. We discovered that overhedging may arise. Nevertheless this happens only in special situations, for example, when the level of Average Value at Risk is high or the initial capital is close to the value which is required for a perfect hedging strategy. The results are illustrated by solving the problem for a call and a put option in the Black-Scholes model.

Samenvatting

De analyse en interpretatie van risico speelt een cruciale rol in verschillende gebieden in de financiële wereld. Dit geldt ook voor de prijsstelling van financiële producten, allocatie van kapitaal en de afleiding van economisch kapitaal. Sleutel tot deze analyse is de kwantificering van het risico via risicomaten. Een veelbelovende aanpak om risicomaten te definiëren is een verzameling van gewenste eigenschappen. Dit leidt tot het belangrijkste onderwerp van dit onderzoek: de karakterisering van de zogenaamde *convexe risicomaten*.

Ten eerste beschrijven we het concept van de convexe risicomaten op Lebesgue ruimtes. Dit geeft een structurele basis voor de volgende onderdelen van het proefschrift door het verklaren en het bewijzen van verschillende resultaten en karakterisering. Deze zijn aangepast aan onze definities en notatie.

Een belangrijk resultaat van dit proefschrift is het karakteriseren van lineaire combinaties en convoluties van convexe risicomaten. We bestuderen verschillende duale verbanden, gegeven door de Fenchel-Legendre transformatie. In ons geval onderzoeken we drie verschillende duale verbanden, deze zijn som en inf-convolutie, verschil en de-convolutie, en vermenigvuldiging van scalair en epi- vermenigvuldiging. Deze resultaten worden gebruikt om lineaire combinaties en convoluties van convexe risicomaten te karakteriseren. We onderzoeken wanneer bepaalde combinaties van convexe risicomaten tot de klasse van convexe risicomaten behoren. Verder karakteriseren we de daaruit voortvloeiende risicomat, en onderzoeken de fundamentele eigenschappen.

Vervolgens worden twee toepassingen op basis van de theoretische resultaten van het eerste deel van het proefschrift afgeleid.

Bij de eerste toepassing bestuderen we een prijsstelling- en hedgingprobleem voor voorwaardelijke vorderingen (contingent claims) in een incompleete markt als een trade-off tussen een handelaar en een toezichthouder.

In ons model voert de toezichthouder een extra bankrekening in, waarop de handelaar een bepaald bedrag dient te storten dat afhankelijk is van het genomen risico. Dit dient als kapitaalreserve om eventuele verliezen van de handelaar op te vangen. De lage rente

op deze rekening dienst als prikkel tot een voorzichtiger risico-return afweging. De risicohouding van de handelaar en de toezichthouder zijn weergespiegeld door verschillende risicomaten. We berekenen het risicomat prijs en het risico onverschilligheid prijs. In beide gevallen wordt de daaruit voortvloeiende risicomat weergegeven door een gewogen som van de risicomaten van de toezichthouder en de handelaar. Deze nieuwe operator is ook een convexe risicomat waarvan we de representatie hebben afgeleid in het eerste deel van het proefschrift.

In de tweede toepassing beschouwen we het probleem van de gedeeltelijke afdekking van een voorwaardelijke vordering. Onder de veronderstelling van een complete markt is het altijd mogelijk om een vordering te repliceren. In dit geval kan de vordering geprijsd worden met behulp van de unieke equivalente martingaal maat. De vraag is van een andere aard als het aanvangskapitaal minder bedraagt als de verwachting onder het equivalent martingaal maat. Onder deze voorwaarde leiden we een optimale hedging strategie zodat het risico van het verschil van de hedging portefeuille en de vordering wordt geminimaliseerd. Als risicomaten gebruiken we Average Value at Risk en een eenvoudige spectrale risicomat. We ontdekken dat overhedging plaats kan vinden, maar alleen in bijzondere situaties, bijvoorbeeld wanneer het niveau van de Average Value at Risk hoog is, of het geld of het startkapitaal dicht ligt bij de waarde, die nodig is voor een perfecte replicatie. De resultaten worden geïllustreerd door het oplossen van het probleem voor een call en een put optie in het Black-Scholes model.