Turbulent thermal convection: From Rayleigh–Bénard to vertical convection

Qi Wang

Turbulent thermal convection: From Rayleigh-Bénard to vertical convection

Qi Wang

Graduation Committee:

Prof. Dr. J. L. Herek (chairman)	University of Twente
Prof. Dr. rer. nat. D. Lohse (supervisor)	University of Twente
Prof. Dr. R. Verzicco (supervisor)	University of Twente & Rome
Prof. Dr. D. Sun (supervisor)	University of Science and Technology of China
Dr. Z. Wan (co-supervisor)	University of Science and Technology of China
Prof. Dr. S. Luding	University of Twente
Prof. Dr. T. H. van der Meer	University of Twente
Prof. Dr. O. Shishkina	Max Plack Institute (MPIDS)
Dr. Y. Yang	Peking University
Dr. Z. Xia	Zhejiang University

The work in this thesis was carried out at the Physics of Fluids group of the Faculty of Science and Technology of the University of Twente and the Department of Modern Mechanics of the University of Science and Technology of China. This thesis was financially supported by the China Scholarship Council (CSC) and the Natural Science Foundation of China (NSF).

Dutch title:

Turbulente thermische convectie: van Rayleigh-Bénard tot verticale convectie

Front cover: Visualization of the vortical structures in two-dimensional vertical convection at $Ra = 10^{11}$ and $Pr = 0.01$. Back cover: Visualization of the zonal flow structures in two-dimensional Rayleigh-Bénard convection at $Ra = 10^9$ and $Pr = 10$.

Publisher:

Qi Wang, Physics of Fluids, University of Twente,
P.O. Box 217, 7500 AE Enschede, The Netherlands

Copyright © Qi Wang, Enschede, The Netherlands 2020.

All rights reserved. No part of this work may be reproduced or transmitted for commercial purposes, in any form or by any means, electronic or mechanical, including photocopying and recording, or by any information storage or retrieval system, except as expressly permitted by the publisher.

ISBN: 978-90-365-5085-7

DOI: 10.3990/1.9789036550857

TURBULENT THERMAL CONVECTION: FROM RAYLEIGH-BÉNARD TO VERTICAL CONVECTION

DISSERTATION

to obtain
the degree of doctor at the University of Twente,
on the authority of the rector magnificus,
Prof. Dr. T. T. M. Palstra,
on account of the decision of the graduation committee,
to be publicly defended
on Friday, November 06th, 2020 at 16:45

by

Qi Wang
Born on June 08th, 1994
in Gansu, China

This dissertation has been approved by the supervisors:

Prof. Dr. rer. nat. D. Lohse

Prof. Dr. R. Verzicco

Prof. Dr. D. Sun

and the co-supervisor:

Dr. Z. Wan

Contents

1	Introduction	1
1.1	Turbulent thermal convection	1
1.2	Models for turbulent thermal convection	3
1.3	Governing equations	5
1.4	A guide through this thesis	6
I	Rayleigh-Bénard convection	11
2	Penetrative turbulent Rayleigh-Bénard convection in two and three dimensions	13
2.1	Introduction	15
2.2	Numerical settings	17
2.3	Results and discussion	19
2.4	Conclusions	30
2.5	Appendix	31
3	On non-Oberbeck–Boussinesq effects in Rayleigh-Bénard convection of air for large temperature differences	35
3.1	Introduction	37
3.2	Numerical settings	39
3.3	Results and discussion	41
3.4	Conclusions	59
3.5	Appendix	60
4	Multiple states in turbulent large-aspect ratio thermal convection: What determines the number of convection rolls?	63
4.1	Introduction	65
4.2	Numerical settings	66

4.3	Results	66
4.4	Conclusions	74
4.5	Appendix	74
5	From zonal flow to convection rolls in Rayleigh-Bénard convection with free-slip plates	83
5.1	Introduction	85
5.2	Numerical settings	87
5.3	Two-dimensional simulations	89
5.4	Three-dimensional simulations	100
5.5	Conclusions	105
5.6	Appendix	107
II	Tilted convection	113
6	Flow reversals in two-dimensional thermal convection in tilted cells	115
6.1	Introduction	117
6.2	Numerical settings	119
6.3	Results and discussion	120
6.4	Conclusions	133
7	Flow organization and heat transfer in two-dimensional tilted convection with aspect ratio 1/2.	135
7.1	Introduction	137
7.2	Numerical settings	139
7.3	Results and discussion	141
7.4	Conclusions	155
8	Multiple states and heat transfer in two-dimensional tilted convection with large aspect ratios.	157
8.1	Introduction	159
8.2	Numerical settings	161
8.3	Results and discussion	162
8.4	Conclusions	171
9	Thermal convection in a tilted rectangular cell with aspect ratio 1/2.	173
9.1	Introduction	175

9.2 Numerical settings	175
9.3 Results	176
9.4 Conclusions	182
III Vertical convection	185
10 Non-Oberbeck-Boussinesq effects due to large temperature differences in a differentially heated square cavity filled with air	187
10.1 Introduction	189
10.2 Numerical settings	191
10.3 Results and discussion	195
10.4 Conclusions	210
IV Conclusions and outlook	213
11 Conclusions and outlook	215
11.1 Part I: Rayleigh-Bénard convection	215
11.2 Part II: Tilted convection	217
11.3 Part III: Vertical convection	218
11.4 Future research directions	219
Bibliography	223
Summary	249
Summary (Dutch)	251
Scientific output	253
Acknowledgements	255
About the author	261

Chapter 1

Introduction

1.1 Turbulent thermal convection

Turbulent thermal convection driven by a temperature difference is omnipresent in nature and it plays an important role in our daily life. Figure 1.1 shows several examples of thermal convection which happens on Earth and other planets or stars.

In the interior of the Earth, thermal convection happens in the liquid outer core and mantle as shown in figure 1.1(*a*). The Earth's magnetic field is important since it protects the planet from dangerous solar winds [1]. However, the origin of the Earth's magnetic field is still unknown. According to Ref. [2], Albert Einstein considered understanding the origin of the Earth's magnetic field as being one of the five most important unresolved problems in physics. Although this problem is not well understood, it is believed that convective fluid motions of the liquid metal in the Earth's outer core plays an important role in the genesis of the Earth's magnetic field [3]. At the same time, the rotation effect is also important. Thus the numerical studies of the Earth's outer core dynamics like geomagnetic field reversals were usually performed in rotating magnetoconvection in a spherical shell [4–7]. Above the Earth's outer core is the mantle where convection also happens [8–10]. The convective motion in the mantle can significantly influence the motion of plate tectonics [11–13] and the eruption of volcanos [14, 15].

Convection also happens in the atmosphere of the Earth as seen from figure 1.1(*b*). The heated air near the equator rises, whereas the cooled air near the poles sinks, this will generate a circulation called Hadley cell [16]. However, in real life, this “single-roll” circulation cannot be sustained over long distances

Figure 1.1: Examples of turbulent thermal convection in nature. (a) Convection in the Earth’s outer core and mantle [17]. (b) Convection rolls in the atmosphere [18]. (c) Convection in the ocean [19]. (d) Convection in the interior of the Sun [20]. (e) Zonal flow in the Jupiter’s atmosphere [21]. (f) Polygonal convection cells on the surface of Pluto [22].

between the equator and the poles. As a result, this single circulation breaks into three cells in each hemisphere as shown in figure 1.1(b).

Turbulent thermal convection in the ocean plays an important role in the transport of heat and salinity. Temperature and salinity gradients can drive the global circulation in the sea, which is called “thermohaline circulation” [23, 24]. This circulation delivers an enormous amount of tropical heat to the northern Atlantic. The record contained in ice and sediment shows that this circulation has not run steadily, but jumped from one mode of operation to another [25–27]. These jumps have significant impact on the Earth’s climate which was shown to undergo frequent large and abrupt global changes during the last glacial period [28].

If we go far beyond the Earth, we can see that turbulent convection also occurs in other stars and planets. Figure 1.1(d) shows the structure of the Sun. The so called penetrative convection [29], where convection in a thermally unstable fluid layer penetrates into adjacent stable layers, is an important process in the interior of the Sun. Between the convective zone and the radiative zone

lies a stably stratified region called tachocline. Cold plumes from the outer convective zone of the Sun penetrate into the upper layers of the tachocline and generate internal gravity waves, this process is thought to play an important role in the turbulent transport of momentum in the tachocline [30].

The atmosphere of the Jupiter is characterized by the banded zonal flow [21, 31, 32] as shown in figure 1.1(e). Such kind of zonal flow also exists in the other three Jovian planets [33–36]. Even though such alternating zonal winds on the Jupiter have been accurately measured for several decades [37, 38], their depth of penetration into its interior and the origin of the winds are still uncertain [32]. Generally, there are two different models on the origin of zonal winds of the Jupiter. The first one is the shallow convection model which assumes that the cloud level zonal winds are shallow and confined within a thin, stably-stratified weather layer about 70 km thick at cloud top [39, 40]. The other one is the deep convection model which assumes that the zonal flow originates from the penetrative convection that takes place in the deep hydrogen–helium interior [41, 42]

Another example of convection in other planets is the convective cell pattern on the surface of Pluto. Figure 1.1(f) shows the high-resolution picture of the polygonal convection cell structures on Pluto’s surface obtained by the New Horizons spacecraft [22]. Pluto’s surface is very young and geologically active [43]. One of its youngest terrains is the near-equatorial region named Sputnik Planum. This region is divided into irregular polygons about 20–30 kilometres in diameter, whose centers rise tens of meters above their sides. A recent study showed that this region is vigorously convecting and Rayleigh–Bénard convection (RBC) is the mostly likely explanation for these polygons [22]. A later study further revealed that the internal heat generation also plays an important role on the formation of polygonal cells [44].

Apart from geophysical and astrophysical settings mentioned above, thermal convection is also important in various industrial applications. Examples include heat exchangers and passive cooling in electrical devices [45], indoor ventilation [46–48], nuclear reactors [49, 50], crystal growth [51], biological convection [52], and so on.

1.2 Models for turbulent thermal convection

To have a better understanding of the various complex turbulent convection phenomena mentioned in §1.1, several different model problems have been adopted to study the physics of turbulent thermal convection problems. These

Figure 1.2: Instantaneous temperature fields for two-dimensional (a) Rayleigh-Bénard convection, (b) tilted convection, and (c) vertical convection for $Ra = 10^{10}$, $Pr = 10$, $\Gamma = 1$. Rayleigh-Bénard convection is studied in part I, tilted convection in part II, and vertical convection in part III of this thesis. The sidewalls are usually no-slip for the velocity and insulating for the temperature, only in §4 and §5, periodic conditions are used at these locations. Most of the simulations in this thesis are two-dimensional. For some chapters, we also offered three-dimensional simulations for comparison.

include RBC [53–56] where a fluid layer in a cavity heated from below and cooled from above (see figure 5.1a), and vertical convection (VC) [57,58] where the fluid layer is heated/cooled at the sides (see figure 5.1c). Both RBC and VC can be viewed as extreme cases of tilted convection [59] (see figure 5.1b) where the tilt angle is $\beta = 0^\circ$ for RBC and $\beta = 90^\circ$ for VC. These three model problems will be the focus of this thesis.

RBC is perhaps the most common form of thermal convection in nature and the most thoroughly studied model for thermal convection. It has been investigated for more than one hundred years since the pioneering work of Bénard [60] and Rayleigh [61]. It also played a crucial role in the development of stability theory in hydrodynamics [62,63], and had been a paradigm in the study of pattern formation and spatial-temporal chaos [64,65]. Over the past thirty years or so, turbulent RBC has received significant attention, and much effort has been devoted to study heat transport, large scale dynamics [53,55,56] and small scale turbulence [54]. We will address some aspects of turbulent RBC in part I of this thesis.

In RBC, the gravity is aligned with the temperature gradient. However, in some situations, there is a certain angle between the gravity and the tem-

perature gradient [66]. Such configuration can be found, for example, in the convection along hillside. Another relevant example is the process of crystal growth from melts in tilted ampoules with axial heating, where larger mass transport rates are typically obtained by slightly tilting the horizontal Bridgman configuration. In the same way, the heat transfer in heat exchangers and thermosiphons can be enhanced by selecting an optimal tilt angle [66]. All these cases can be approximated by tilting the RBC cell. We will explore tilted convection in part II of this thesis.

When the tilt angle reaches 90° . The problem becomes VC. VC finds many applications in engineering: thermal insulation using double-pane windows or double walls, horizontal transport in water bodies, crystal growth procedures, nuclear reactors, ventilation of rooms, and cooling of electronic devices to name but a few. Moreover, it has been served as a model to study atmospherical circulation [16, 66]. We will further study VC in part III of this thesis.

Apart from the three models mentioned above, there are also various other models for thermal convection, including horizontal convection [67–71] where the heating and cooling are applied at a horizontal surface, double-diffusive convection [72, 73] where the convection is driven by two scalar components, internally heated convection where the convection is driven by volume internal heating [74–76] and so on. All these model problems have significant relevance in geo- and astrophysical convection flows, while these model problems will not be addressed in this thesis.

1.3 Governing equations

The convection problem is usually simplified by the so-called Oberbeck-Boussinesq (OB) approximation [77, 78] in which the fluid density ρ is assumed to depend linearly on the temperature T :

$$\rho = \rho_0[1 - \alpha(T - T_0)], \quad (1.1)$$

where α is the thermal expansion coefficient, T_0 the reference temperature, and ρ_0 the reference density. In addition, it is assumed that the material properties of the fluid such as α , the viscosity ν , and the thermal diffusivity κ do not depend on the temperature. Any deviations of these assumptions are commonly referred to as the non-Oberbeck-Boussinesq (NOB) effects.

The governing equations of the three model problems shown in figure 1.2

are then the Oberbeck-Boussinesq equations:

$$\nabla \cdot \mathbf{u} = 0, \quad (1.2)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho_0} \nabla p + \nu \nabla^2 \mathbf{u} + \alpha g (T - T_0) \vec{e}, \quad (1.3)$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T, \quad (1.4)$$

for the velocity field $\mathbf{u}(\mathbf{x}, t)$, the kinematic pressure field $p(\mathbf{x}, t)$, and the temperature field $T(\mathbf{x}, t)$. $\vec{e}$ is a unit vector defined by $\vec{e} = (\sin(\beta), \cos(\beta))$. The bottom plate has high temperature T_h and the top plate has low temperature T_c .

For a given cell geometry, which is characterized by the width-to-height aspect ratio Γ , the problem is governed by two dimensionless parameters, i.e., the Rayleigh number Ra and the Prandtl number Pr :

$$Ra = \frac{g \alpha \Delta H^3}{\nu \kappa}, Pr = \frac{\nu}{\kappa}, \quad (1.5)$$

where g is the gravitational acceleration, and $\Delta = T_h - T_c$ the temperature difference between the two plates. Ra is the ratio of the buoyancy and viscous force, and it quantifies the strength of the thermal driving. Pr characterizes the properties of the working fluid.

The response of the system is characterized by the Nusselt number Nu and the Reynolds number Re :

$$Nu = \frac{QH}{k\Delta}, Re = \frac{UH}{\nu}, \quad (1.6)$$

where Q is the heat flux into the system, k the thermal conductivity of the fluid, and U a characteristic velocity amplitude. Nu quantifies the heat transfer efficiency and Re denotes the flow strength.

Both two-dimensional (2D) and three-dimensional (3D) direct numerical simulations (DNS) were performed in this thesis. Results were obtained either using the lMn2d [79] or the AFiD [80] code.

1.4 A guide through this thesis

This thesis contains three main parts. The main contents for each part are as follows:

- In part I, we study RBC. We first focus on NOB effects either due to the density maximum of cold water near 4°C (§2) or due to large temperature differences in air (§3). We then study two important phenomena in turbulent flows, namely multistability (§4) and metastability (§5), using turbulent RBC as the model problem.
- In part II, we explore tilted convection over a wide range of parameter space spanned by (Ra, Pr, Γ, β) . In §6 we first report an interesting finding that tilting can promote flow reversals of the large-scale circulation in a 2D cell with $\Gamma = 2$, which is different from the $\Gamma = 1$ case, where flow reversals are significantly suppressed by tilting. We further show that tilting can also promote flow reversals in a 2D slender cell with $\Gamma = 0.5$ (§7). Much larger-aspect-ratio (up to $\Gamma = 16$) cases are studied in §8, where we report multistability phenomena in 2D tilted convection with large Γ . We further investigate flow organization and heat transport properties in 3D tilted convection with $\Gamma = 0.5$ in §9.
- In part III, we focus on VC. We study NOB effects caused by large temperature differences in §10.

Below we give a more comprehensive introduction to the three parts in this thesis.

1.4.1 Part I: Rayleigh-Bénard convection

In §2, we study penetrative turbulent RBC which depends on the density maximum of cold water near 4°C. An important control parameter for this problem is the density inversion parameter θ_m . Several universal properties are revealed in this system. We show that the ratio of the top and bottom thermal boundary layer thickness ($F_\lambda = \lambda_t^\theta / \lambda_b^\theta$), the dimensionless center temperature θ_c , the normalized Nusselt number $Nu(\theta_m)/Nu(0)$, and the normalized Reynolds number $Re(\theta_m)/Re(0)$ all have universal relationships with the density inversion parameter θ_m , which seem to be independent of Ra .

In §3, we present DNS of NOB RBC due to large temperature differences using perfect air as the working fluid. It is found that due to top-bottom symmetry breaking under NOB conditions, an increase of the center temperature T_c is found compared to the arithmetic mean temperature T_m of the top and bottom plates, and the shifts of T_c are strongly dependent on Ra and the temperature differential ϵ . Both Nu and Re are insensitive to NOB effects. Furthermore, the extended laminar boundary layer equations are developed

based on the low-Mach-number Navier-Stokes equations, which qualitatively predict the NOB effects on velocity profiles.

In §4, we analyze the multiple states in large-aspect-ratio 2D turbulent RBC with no-slip plates and horizontally periodic boundary conditions. By making use of a generalized Friedrichs inequality, we show that the elliptical shape of the rolls and viscous damping determine the Γ_r -window (Γ_r is the mean aspect ratio of a convection roll) for the realizable turbulent states. The theoretical results are in excellent agreement with our numerical finding $2/3 \leq \Gamma_r \leq 4/3$, where the lower threshold is approached for the larger Ra . Finally, we show that the theoretical approach to frame Γ_r also works for free-slip boundary conditions.

In §5, we investigate RBC with free-slip plates and horizontally periodic boundary conditions. Two configurations are considered, one is 2D RBC and the other one 3D RBC with a rotating axis parallel to the plate. We show that zonal flow, which was observed, for example, by Ref. [81] for aspect ratio $\Gamma = 2$, is only stable when Γ is smaller than a critical value, which depends on Ra and Pr . With increasing Γ , we find a second regime in which both zonal flow and different convection roll states can be statistically stable. For even larger Γ , in a third regime, only convection roll states are statistically stable and zonal flow is not sustained. For the 3D simulations, we demonstrate that with increasing the aspect ratio Γ , zonal flow, which was observed for small $\Gamma = 2\pi$ by Ref. [82], also disappears for $\Gamma = 16$. For such large Γ only convection roll states are statistically stable. In between, here for medium aspect ratio $\Gamma = 8$, the convection roll state and the zonal flow state are both statistically stable.

1.4.2 Part II: Tilted convection

In §6, the influence of tilt on flow reversals in 2D thermal convection is addressed for $\Gamma = 1$ and $\Gamma = 2$. For $\Gamma = 1$, tilt tends to suppress flow reversals. However, we find that flow reversals characterized by two main rolls are promoted by tilt for $\Gamma = 2$, which are even observed for some small Pr and large tilt angles β cases. The mode decomposition analysis shows that the appearance of the intermediate single-roll mode is crucial for reversals, and flow reversals in a tilted cell with $\Gamma = 2$ can be viewed as a mode competition process between single-roll mode and horizontally adjacent double-roll mode. They can only occur in a limited range of β where the two modes have comparable strength.

In §7, we focus on a slender 2D tilted cell with $\Gamma = 0.5$. Different flow states are identified over the β - Pr parameter space. We find that the flow tends to

organize itself in a stable vertically stacked double-roll state (DRS) for small Pr and small β , while this DRS becomes unstable and flow reversals happen with increasing β . This finding is similar to the phenomenon reported in a $\Gamma = 2$ cell in §6.

In §8, we study multiple states and heat transport in 2D tilted convection with large aspect ratios. Multiple states are identified for small β with $\Gamma \geq 2$. Moreover, for large $\Gamma \geq 8$, when β is larger than a critical value, flow mode transitions between different roll states are observed. We find that for $\Gamma \leq 4$, Nu first increases with increasing β , and then declines after reaching its local maximum. However, Nu decreases monotonically with increasing β for $\Gamma \geq 8$ cases.

In §9, we further study 3D tilted convection with $\Gamma = 1/2$ both for OB cases and considering NOB effects. For a relatively small tilt angle β , depending on the initial conditions, the large-scale-circulation (LSC) can either orient along one of the vertical diagonal planes (denoted by M_d mode) or orient parallel to the front wall (denoted by M_p mode). Which of the two modes transports heat more efficiently depends on Ra . By investigating NOB effects in tilted convection for fixed $Ra = 10^6$, we find that the influence of NOB effects on Nusselt number Nu , Reynolds number Re and central temperature T_c is different for different β range. NOB effects can either increase or decrease Nu , Re and T_c when β is varied.

1.4.3 Part III: Vertical convection

In §10, we discuss NOB effects in 2D VC. The working fluid is air and Pr is 0.71 for the reference state. We study the NOB effects on the temperature and velocity fields. The center temperature is found to be increased in comparison with the OB cases. We also demonstrate the robustness of Nu and Re to NOB effects even for very large temperature difference up to 360K.

The thesis concludes with a summary of the results and an outlook for future work.

Part I

Rayleigh-Bénard convection

Chapter 2

Penetrative turbulent Rayleigh-Bénard convection in two and three dimensions^o

Penetrative turbulent Rayleigh-Bénard convection which depends on the density maximum of water near 4°C is studied using two-dimensional (2D) and three-dimensional (3D) direct numerical simulations (DNS). The working fluid is water near 4°C with Prandtl number $Pr = 11.57$. The considered Rayleigh numbers Ra range from 10^7 to 10^{10} . The density inversion parameter θ_m varies from 0 to 0.9. It is found that the ratio of the top and bottom thermal boundary-layer thickness ($F_\lambda = \lambda_t^\theta / \lambda_b^\theta$) increases with increasing θ_m , and the relationship between F_λ and θ_m seems to be independent of Ra . The center temperature θ_c is enhanced compared to that of Oberbeck-Boussinesq (OB) cases, as θ_c is related to F_λ with $1/\theta_c = 1/F_\lambda + 1$, θ_c is also found to have a universal relationship with θ_m , which is independent of Ra . Both the Nusselt number Nu and the Reynolds number Re decrease with increasing θ_m , the normalized Nusselt number $Nu(\theta_m)/Nu(0)$ and Reynolds number $Re(\theta_m)/Re(0)$ also have universal relationships with θ_m , which seem to be independent of both Ra and the aspect ratio Γ . The scaling exponents of $Nu \sim Ra^\alpha$ and $Re \sim Ra^\beta$ are found to be insensitive to θ_m despite of the remarkable change of the flow organizations.

^oBased on: **Qi Wang**, Quan Zhou, Zhen-Hua Wan, De-Jun Sun, *Penetrative turbulent Rayleigh-Bénard convection in two and three dimensions*, J. Fluid Mech. 870, 718-734 (2019).

2.1 Introduction

Penetrative convection refers to the phenomena whenever convection in a thermally unstable fluid layer penetrates into adjacent stable layers. The unstable fluid layers are often bounded by rigid boundaries in most laboratory experiments on convection. However, stellar convection zones are bounded by stably-stratified regions and the understanding of penetration of convection across the interface between stable and unstable layers is of astrophysical importance. For instance, cold plumes from the outer convective zone of the sun penetrate into the upper layers of the tachocline and generate internal gravity waves, this process is thought to play an important role in the turbulent transport of momentum in the tachocline [30]. The observable motions in the outer stable regions of the sun presumably arise from the solar convective zone which is just below them [83]. The origin of Jupiter's zonal winds remains a puzzle and there are two different views: the shallow convection model and the deep convection model [32]. The deep convection model thinks that the zonal flow originates from the penetrative convection that takes place in the deep hydrogen-helium interior [41, 42]. Although the Earth's liquid core is convectively unstable to convection, some studies show that the outermost part of the Earth's core may be stably-stratified [84]. The extent of the penetration of convection into the stable layers is important since anomalous depletion of lithium in late stars might be explained if convection penetrates deep enough for the Li-He reaction to occur [85].

Penetrative convection is usually studied in simplified model problems. Ref. [86] considered an internally heated fluid layer which is confined between top and bottom plates of equal temperatures. The unstably-stratified upper region drives convection that penetrates into the stably-stratified lower region. Another commonly used model to study penetrative convection is convection of water near 4°C . Consider a Rayleigh-Bénard convection (RBC) [53–55] system using water as the working fluid, the temperature of the bottom plate is higher than 4°C while the temperature of the top plate is lower than 4°C . The fluid in the lower layer above the 4°C level is convectively unstable while the upper layer below the 4°C level is gravitationally stable, any convection in the lower, unstable region will penetrate into the stable layer above, and this will lead to the so-called penetrative convection. Convection of water near 4°C was often used as a model problem to study penetrative convection in the past.

The first theoretical work on the subject of penetrative convection considered a layer of water in which the bottom boundary is maintained at 0°C and

the top boundary is kept at a temperature larger than 4°C [29]. A criteria is built for the onset of instability, and it is demonstrated that the system can become unstable to a finite amplitude disturbance at values of the Rayleigh number Ra less than the critical value of the infinitesimal stability theory. This nonlinear behaviour was later studied by Ref. [87]. Ref. [88] used numerical simulations to extend the small-amplitude solution of penetrative convection to higher Ra . Ref. [89] experimentally investigate penetrative RBC, they found that the conduction-convection transition is hysteretic in nature, and the flow undergoes secondary transitions to either hexagonal cellular or longitudinal roll states at higher Ra . Ref. [90] investigated RBC of water near 4°C in a cubical cavity with different thermal boundaries on the sidewalls and found that there are multiple flow pattern coexistence and hysteresis phenomenon during the flow pattern transition.

Most of the previous works are conducted at relatively low Ra where the flow is laminar. There are less works devoted to turbulent penetrative convection. Ref. [91] studied internal gravity wave excitation by convection of water near 4°C at $Ra = 5.8 \times 10^7$ using two-dimensional (2D) simulations. A recent work [92] investigated 2D penetrative convection at relatively high Rayleigh numbers $10^6 \leq Ra \leq 10^8$ using lattice Boltzmann method, they adopted periodical conditions in the horizontal direction and the Prandtl numbers are 1 and 11.6. More recently, Ref. [93] numerically investigated 2D penetrative convection with horizontal periodical conditions and found that a periodic, oscillating mean flow spontaneously develops from turbulently generated internal waves. The non-Oberbeck-Boussinesq (NOB) nature of water near 4°C will cause a top-down symmetry breaking, leading to a shift of center temperature θ_c , i.e., θ_c will deviate from that of the OB case. This deviation has been investigated in RBC with NOB effects caused by large temperature differences [94–102]. However, there is still a lack of systematic studies on the center temperature θ_c in turbulent penetrative RBC of water near 4°C . We try to fill this gap and report universal relationship of θ_c with the density inversion parameter θ_m . The ratio of the top and bottom thermal boundary-layer (BL) thicknesses $F_\lambda = \lambda_t^\theta / \lambda_b^\theta$, the normalized Nusselt number $Nu(\theta_m)/Nu(0)$ and Reynolds number $Re(\theta_m)/Re(0)$ are also found to have universal relationships with θ_m which seem to be independent of Ra .

The rest of this chapter is organized as follows. We first briefly describe the governing equations and numerical methods in §2.2. Section 2.3 presents and discusses the universal relationships of F_λ , θ_c , normalised Nusselt number $Nu(\theta_m)/Nu(0)$ and normalised Reynolds number $Re(\theta_m)/Re(0)$ with respect

Figure 2.1: Sketch of (a) 2D and (b) 3D configurations for penetrative Rayleigh-Bénard convection

to θ_m . Finally, we summarize our findings in §2.4, and the simulation details are tabulated in §2.5.

2.2 Numerical settings

The problem is sketched in figure 2.1, the temperature of the bottom plate $\hat{T}_b$ (quantities marked with a circumflex are dimensional) is higher than 4°C while the temperature of the top plate is lower than 4°C . The sidewalls are insulated. We assume thermophysical properties to be constant except the density in the buoyancy term. The nonlinear relationship of density with temperature for cold water near 4° is described as $\hat{\rho} = \hat{\rho}_m(1 - \hat{\alpha}|\hat{T} - \hat{T}_m|^q)$ [103], where $\hat{\rho}_m \approx 1000\text{kg/m}^3$ is the maximum density at the temperature $\hat{T}_m = 4^\circ\text{C}$, the isobaric thermal expansion coefficient $\hat{\alpha} = 9.30 \times 10^{-6}(\text{K})^{-q}$, where $q = 1.895$. The dimensionless governing equations of this problem read:

$$\nabla \cdot \mathbf{u} = 0, \quad (2.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \sqrt{\frac{Pr}{Ra}} \nabla^2 \mathbf{u} + |\theta - \theta_m|^q \vec{e}_z, \quad (2.2)$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \frac{1}{\sqrt{RaPr}} \nabla^2 \theta, \quad (2.3)$$

where $\mathbf{u}$, θ and p are the velocity, temperature and pressure, respectively.

For non-dimensionalization, we choose $\hat{H}$ and $\hat{U} = (\hat{g}\hat{\alpha}\Delta\hat{T}^q\hat{H})^{1/2}$ as reference length and velocity, where $\hat{g}$ is the gravitational acceleration, $\Delta\hat{T} = \hat{T}_b - \hat{T}_t$ is the temperature difference. The reference time is free-fall time $\hat{t}_f = \hat{H}/\hat{U}$. Temperature is nondimensionalized as $\theta = (\hat{T} - \hat{T}_t)/(\hat{T}_b - \hat{T}_t)$. Three main control parameters are the Prandtl number Pr , the Rayleigh number Ra , the density inversion parameter θ_m , and they are defined as

$$Pr = \frac{\hat{\nu}}{\hat{\kappa}}, \quad Ra = \frac{\hat{g}\hat{\alpha}(\hat{T}_b - \hat{T}_t)^q\hat{H}^3}{\hat{\nu}\hat{\kappa}}, \quad \theta_m = \frac{\hat{T}_m - \hat{T}_t}{\hat{T}_b - \hat{T}_t}, \quad (2.4)$$

where $\hat{\kappa}$ is the thermal diffusivity, and $\hat{\nu}$ the kinematic viscosity. θ_m is an important parameter that describes density maximum effect, quantifying the location of $\hat{T}_m$ with respect to wall temperature $\hat{T}_t$ and $\hat{T}_b$. For $\theta_m = 0$, the top plate is 4°C, and the situation is quite similar to OB approximation. For $\theta_m = 1$, the bottom plate is 4°C, and the flow is pure conduction state for any Ra . Another control parameter is the aspect ratio. For the 2D configuration, the aspect ratio is defined as $\Gamma = \hat{L}/\hat{H}$. For the 3D configuration, the dimensionless length is fixed to 1 and the aspect ratio is defined as $\Gamma_3 = \hat{W}/\hat{H}$.

The governing equations are solved numerically by an in-house code lMn2d/3d [79]. Since our code has been validated and used to study various convection problem before [104–109], we only give its main features here. All spatial terms are discretised using a second-order central difference scheme. Non-uniform grids with clustered points near walls are used. Time integration is accomplished using an Adams-Bashforth scheme for the nonlinear terms and Crank-Nicolson scheme for the viscous and diffusion terms. Multi-grid strategy is used to solve the pressure Poisson equation.

The Pr is fixed to 11.57 corresponding to the water at 4°C. We simulate 2D cases in the range of $10^7 \leq Ra \leq 10^{10}$ and 3D cases for $Ra = 10^8$ and 10^9 . Grids which satisfy resolution requirement of DNS [110] are used. The details of the main simulations are tabulated in the Appendix. For all the simulations, we generally perform at least 1600 free fall time units. We waited at least 800 free-fall time units before starting to average in order to ensure all transients have been dissipated, and the Nu and Re are averaged for at least 800 free-fall time units. For large θ_m close to 0.9, longer-time simulations are needed to reach statistically steady state [92]. For example, for the 2D case with $Ra = 5 \times 10^8$, $\theta_m = 0.9$, $\Gamma = 1$, we even preformed 30 000 free-fall time units for the flow to become statistically steady, and another 10 000 free-fall time units are simulated for time average. The results are also compared with

OB cases where we set $q = 1$ and $\theta_m = 0$. Note that for the penetrative case with $\theta_m = 0$, the flow is similar to OB cases, but it cannot fully recover to OB case since $q = 1.895$, and the top-bottom symmetry is still broken. For the OB case, the initial conditions are conduction states superposed with a small perturbation added to the temperature, while for the penetrative cases, we generally use fully-developed flow fields for the OB cases as initial conditions.

2.3 Results and discussion

2.3.1 Flow organisation and center temperature

Figure 2.2 shows instantaneous temperature (top row)/time-averaged kinetic energy $0.5(v^2 + w^2)$ (middle row) fields superimposed with velocity vectors for 2D cases with $Ra = 10^9, \Gamma = 1$, and instantaneous temperature fields for 3D cases with $Ra = 10^9, \Gamma_3 = 1/2$ (bottom row). It is seen that a large-scale circulation (LSC) which spans the whole size of the convection cell exists for the OB cases as shown in figures 2.2(a), 2.2(d) and 2.2(g). For the penetrative cases, due to the penetration of the lower hotter unstable fluid layer into the upper stably-stratified layer, the temperature in the bulk increases compared to that of the OB cases. For relatively small θ_m , the fluid in the lower layer can penetrate into the whole part of the upper layer as shown in figures 2.2(b), 2.2(e) and 2.2(h). However, for large θ_m , vigorous convection cannot happen in the whole cell, and the flow is stably-stratified near the top plate as depicted in figures 2.2(c), 2.2(f) and 2.2(i). The convection is very weak near the top plate, and heat is transferred mainly by thermal conduction in this region. This stably-stratified temperature field is somewhat similar to the stably-stratified angular velocity field in counter-rotating Taylor-Couette flow where Rayleigh-stable zones exist near the outer cylinder [111–113]. The black lines in figures 2.2(b) and 2.2(c) denote the location where the temperature is 4°C . Some fluctuations are observed due to the interaction between lower convection region and upper stably-stratified region. The white lines in figures 2.2(e) and 2.2(f) also depict the location at which the temperature is 4°C , but in time-averaged sense. It is clearly shown that the fluctuations disappear after time average. In figures 2.2(d–f), one can also observe that the velocity magnitude decreases with increasing θ_m .

We now quantitatively investigate velocity profiles. Figure 2.3(a) shows time-averaged horizontal velocity v profile at the line $y = 0.5$ for 2D cases with $Ra = 10^9, \Gamma = 1$. The profile is symmetrical for the OB case. For $\theta_m = 0$,

Figure 2.2: (a-c) Instantaneous temperature fields superimposed with velocity vectors for $Ra = 10^9$ for 2D cases with $\Gamma = 1$. (d-f) Time-averaged kinetic energy $1/2(v^2 + w^2)$ distribution superimposed with velocity vectors for $Ra = 10^9$ for 2D cases with $\Gamma = 1$. (g-h) Instantaneous temperature fields for $Ra = 10^9$ for 3D cases with $\Gamma_3 = 1/2$. (a,d,g) OB. (b,e,h) $\theta_m = 0.4$. (c,f,i) $\theta_m = 0.8$. The black lines in (b, c) and white lines in (e, f) denote the location where the temperature is 4° .

Figure 2.3: Time-averaged velocity profiles for $Ra = 10^9$ for 2D cases with $\Gamma = 1$ (a) v profile at $y = 0.5$ for different θ_m . (b) w profile at $z = 0.5$ for different θ_m .

the profile is very close to the OB case, however, we can see that the peak horizontal velocity near the bottom plate is slightly increased compared to the OB case, while it decreases a little near the top plate. As θ_m increases, the fluid motion is obviously weakened, and the horizontal velocity peak decreases monotonically with increasing θ_m , the velocity boundary layers that form near the top plate can still be clearly identified for not too large θ_m . As θ_m increases further to 0.7, 0.8, there is a region where v is around 0 near the top plate. This corresponds to the stably-stratified flow structure for relative large θ_m . For $\theta_m = 0.9$, there exist flow reversals of the LSC [114], thus we only average counterclockwise circulation for this case. It is seen that convection only happens at the lower layer with z roughly smaller than 0.7. Figure 2.3(b) shows time-averaged vertical velocity w profile at the line $z = 0.5$. For $\theta_m = 0$, the profile is also asymmetrical, and the velocity peak is larger than the OB case near the left plate. The vertical velocity peak is also found to decrease with increasing θ_m .

We now investigate thermal boundary layers and center temperature. In Figure 2.4(a) we show time-averaged temperature profiles at the location of $y = 0.5$ for the 2D case with $Ra = 10^9, \Gamma = 1$. For the OB case, the temperature at the center θ_c is the arithmetic mean of the top- and bottom-plate temperatures. For $\theta_m=0$, it is clearly seen that θ_c is enhanced compared to that of the OB case. With increasing θ_m , the thermal BLs become thicker and θ_c becomes larger. It is also evident that the top BL is thicker than that of the bottom BL. Figure 2.4(b) shows time-averaged temperature profiles at the

Figure 2.4: Time-averaged temperature profiles at $y = 0.5$ for 2D cases with $\Gamma = 1$ for (a) $Ra = 10^9$ with different θ_m and (b) $\theta_m = 0.4$ with different Ra .

location of $y = 0.5$ for the 2D cases with $\theta_m = 0.4, \Gamma = 1$. An interesting feature is that although the thermal BLs become thinner with increasing Ra , the bulk temperature still keeps nearly unchanged for all Ra .

We then quantify the asymmetrical feature of the two thermal BLs. Figure 2.5(a) illustrates top and bottom thermal BL thicknesses versus θ_m for both 2D cases with $\Gamma = 1$ and 3D cases with $\Gamma_3 = 1/2$. The thickness of top thermal BL λ_t^θ and bottom thermal BL λ_b^θ are defined as Ref. [100]

$$\lambda_t^\theta = \frac{\theta_t - \theta_c}{\partial_z \langle \theta \rangle_{A,t} |_{z=1}}, \quad \lambda_b^\theta = \frac{\theta_c - \theta_b}{\partial_z \langle \theta \rangle_{A,t} |_{z=0}}, \quad (2.5)$$

where $\theta_t = 0$, $\theta_b = 1$ are the temperatures at the top/bottom plates. $\langle \rangle_{A,t}$ denotes average over any horizontal plane (3D)/line (2D) and in time. One can clearly see that both the two quantities increase with increasing θ_m , and λ_t^θ increases more rapidly. Figure 2.5(b) shows the ratio of the top and bottom thermal BL thicknesses $F_\lambda = \lambda_t^\theta / \lambda_b^\theta$. It can be seen that F_λ is an increasing function of θ_m , and it rapidly increases for large θ_m near 0.9, due to the fact that the stably-stratified flow structures appear for large θ_m , as shown in figures 2.2(c) and 2.2(f), leading to a much thicker thermal BL near the top plate as seen in Figure 2.5(a). One noteworthy feature is that all data sets collapse well on top of each other. It is surprising that F_λ has a universal relationship with θ_m which seems to be independent of Ra , and the universality also holds for the 3D case. Note that in the present study we only consider the chaotic cases. For relatively small Ra and large θ_m , chaotic motions disappear, the flow is periodical/quasi-periodical or steady. A pure conduction state without BLs is even obtained for $Ra = 10^7$ and 5×10^7 with $\theta_m = 0.9$. Previous studies

Figure 2.5: (a) Bottom thermal boundary layer thickness λ_b^θ (solid lines) and top thermal boundary layer thickness λ_t^θ (dashed lines) as a function of θ_m for 2D cases with $\Gamma = 1$ and 3D cases with $\Gamma_3 = 1/2$. (b) The ratio of the top and bottom thermal BL thicknesses $F_\lambda = \lambda_t^\theta / \lambda_b^\theta$ as a function of θ_m for 2D cases with $\Gamma = 1$ and 3D cases with $\Gamma_3 = 1/2$.

on NOB convection caused by large temperature differences also found that F_λ is practically independent of Ra , but increases with increasing temperature differences [100]. Our results demonstrate that this finding also holds for the penetrative RBC, suggesting some similarity between penetrative convection and NOB convection with large temperature differences.

The asymmetrical feature of the thermal BLs will lead to a shift of θ_c compared to the OB cases. In previous NOB studies, it was found that θ_c increases for water [95, 99] but decreases for gaseous ethane [96] compared to that of the OB cases. Figure 2.6 shows θ_c as a function of θ_m for different Ra for both 2D cases and 3D cases. An interesting finding is that all the data points can collapse well onto a single curve. As the thermal conductivity is assumed to be constant, the temperature gradients at the bottom and top walls should be equal, and the Nu can be simply expressed as $Nu = -\partial_z \langle \theta \rangle_{A,t} |_{z=1} = -\partial_z \langle \theta \rangle_{A,t} |_{z=0}$. Based on equation (2.5), one can obtain

$$\theta_c = \frac{\lambda_t^\theta}{\lambda_t^\theta + \lambda_b^\theta}. \quad (2.6)$$

Thus $1/\theta_c = 1/F_\lambda + 1$. The solid red line in figure 2.6 shows line segment of $\lambda_t^\theta / (\lambda_t^\theta + \lambda_b^\theta)$ as a function of θ_m calculated at different θ_m for $Ra = 10^{10}$, the data sets for different Ra can well collapse onto this curve. The dashed black curve is a polynomial fitting which is expressed as $\theta_c = 0.530 - 0.133\theta_m + 0.858\theta_m^2 - 0.237\theta_m^3$, the data sets for different Ra can well fall onto this curve.

Figure 2.6: center temperature θ_c as a function of θ_m for different Ra . Hollow symbols for 2D simulations with $\Gamma = 1$, filled symbols for 3D simulations. The solid line is line segment of $\lambda_t^\theta / (\lambda_t^\theta + \lambda_b^\theta)$ calculated at different θ_m for $Ra = 10^{10}$. The dashed lines is a polynomial fitting which is expressed as $\theta_c = 0.530 - 0.133\theta_m + 0.858\theta_m^2 - 0.237\theta_m^3$.

2.3.2 Nusselt number and Reynolds number

We now turn to the heat-transfer properties measured by the Nusselt number Nu , and the flow intensity which can be expressed in terms of the Reynolds number Re . The Nu and Re are expressed as:

$$Nu = \left\langle \sqrt{RaPr} u_z \theta - \partial_z \theta \right\rangle_{A,t}, Re = \sqrt{Ra/Pr} U, \quad (2.7)$$

where $\langle \rangle_{A,t}$ means average over any horizontal plane(3D)/line(2D) and time, $U = \sqrt{(\mathbf{u} \cdot \mathbf{u})_{y,z,t}}$. The Nu studied here is averaged over the bottom plate. In figure 2.7, we show the absolute and normalised Nu and Re as a function of θ_m for different Ra for 2D cases with $\Gamma = 1$. It is seen from figure 2.7(a) that Nu drops monotonically with increasing θ_m . This is because that with increasing θ_m , the stably-stratified flow regimes gradually appear and grow near the top plate, where the flow is mainly dominated by the thermal conduction. For the extreme limit $\theta_m = 1$, the flow should be totally within the conduction regime with $Nu = 1$. Figure 2.7(b) shows the normalized Nusselt number $Nu(\theta_m)/Nu(0)$ as a function of θ_m for various Ra . We can see that all data sets collapse well onto a single curve. From equation (2.5), one can obtain $Nu(\lambda_t^\theta + \lambda_b^\theta) = 1$, and thus

Figure 2.7: Absolute and normalized Nusselt numbers and Reynolds numbers as a function of θ_m for 2D cases with different Ra and fixed $\Gamma = 1$ (a) Absolute Nusselt numbers Nu (b) Normalized Nusselt numbers $Nu(\theta_m)/Nu(0)$, the red solid line denotes line segment of $(\lambda_t^\theta(0) + \lambda_b^\theta(0))/(\lambda_t^\theta(\theta_m) + \lambda_b^\theta(\theta_m))$ calculated for $Ra = 10^{10}$ at different θ_m , the black dashed line denotes a polynomial fitting of $Nu(\theta_m)/Nu(0) = 0.992 - 0.170\theta_m - 2.244\theta_m^2 + 1.397\theta_m^3$ (c) Absolute Reynolds numbers Re (d) Normalized Reynolds numbers $Re(\theta_m)/Re(0)$, the black dashed line denotes a linear fitting of $Re(\theta_m)/Re(0) = 0.98 - 1.046\theta_m$.

$$\frac{Nu(\theta_m)}{Nu(0)} = \frac{\lambda_t^\theta(0) + \lambda_b^\theta(0)}{\lambda_t^\theta(\theta_m) + \lambda_b^\theta(\theta_m)}. \quad (2.8)$$

We note that this relationship is analytically exact (see also Ref. [95, 100]). The red solid line in figure 2.7(b) shows line segment of $(\lambda_t^\theta(0) + \lambda_b^\theta(0))/(\lambda_t^\theta(\theta_m) + \lambda_b^\theta(\theta_m))$ as a function of θ_m calculated at different θ_m for $Ra = 10^{10}$. One can see that the data points for other Ra all nicely fall onto this curve. The black dashed curve is a polynomial fitting denoted by $Nu(\theta_m)/Nu(0) = 0.992 - 0.170\theta_m - 2.244\theta_m^2 + 1.397\theta_m^3$, the data sets for different Ra can well collapse on top of this fitted curve. Figure 2.7(c) shows the absolute Re as a function of θ_m for different Ra . It is seen that Re also decreases monotonically with increasing θ_m . The reduction of Re can be also qualitatively understood, in view that the appearance of stably-stratified flow structures weakens the fluid motion, and in turn drops considerably the magnitude of Re . Figure 2.7(d) plots the normalised Reynolds number $Re(\theta_m)/Re(0)$ as a function of θ_m . One can see that $Re(\theta_m)/Re(0)$ roughly drops linearly with increasing θ_m , and the data sets can be well described by a linear fitting of $Re(\theta_m)/Re(0) = 0.980 - 1.046\theta_m$. The curve is fitted for the $Ra = 10^{10}$ data, and only the data sets for relatively low Ra (i.e., $Ra = 10^7$ and 5×10^7) show some deviations from this fitting.

We then investigate the aspect ratio Γ dependence of Nu and Re . Figure 2.8 shows the absolute and normalised Nu and Re as a function of θ_m for different Γ and fixed $Ra = 10^8$ for 2D simulations. For large Γ , it was already found that multiple states with different numbers of convection rolls may exist [106], and this will lead to different heat transfer properties. To avoid such possible multiple states from complicating the present problem, we gradually increase θ_m using the flow field for smaller θ_m as initial conditions. The Nu still reduces monotonically with increasing θ_m and again the normalized Nusselt numbers for different Γ can collapse well on top of each other, implying the insensitivity of this dependence on Γ . From figure 2.8(c), one sees that for all considered Γ , Re decreases linearly with increasing θ_m , and drops with decreasing Γ due to the geometric confinement of the sidewalls. The normalized Reynolds number $Re(\theta_m)/Re(0)$ for different Γ also collapse well onto the fitted curve $Re(\theta_m)/Re(0) = 0.980 - 1.046\theta_m$ as shown in figure 2.8(d), suggesting an aspect ratio insensitive feature.

We also show the 3D results in figure 2.9 in order to find whether the universal behaviour for normalised Nusselt number and Reynolds number found in 2D can be extended to 3D cases. For $Ra = 10^8$, we only consider the cases

Figure 2.8: Absolute and normalized Nusselt numbers and Reynolds numbers as a function of θ_m for different Γ and fixed $Ra = 10^8$ for 2D simulations. (a) Absolute Nusselt numbers Nu . (b) Normalized Nusselt numbers $Nu(\theta_m)/Nu(0)$. (c) Absolute Reynolds numbers Re . (d) Normalized Reynolds numbers $Re(\theta_m)/Re(0)$. The dashed lines are the same as figure 2.7.

Figure 2.9: Absolute and normalized Nusselt numbers and Reynolds numbers as a function of θ_m for 3D cases with different Γ_3 . (a) Absolute Nusselt numbers Nu (b) Normalized Nusselt numbers $Nu(\theta_m)/Nu(0)$. (c) Absolute Reynolds numbers Re (d) Normalized Reynolds numbers $Re(\theta_m)/Re(0)$. The dashed lines are the same as figure 2.7.

Figure 2.10: (a) Nusselt numbers Nu and (b) Reynolds numbers Re as a function of Ra for different θ_m for 2D simulations.

with $\Gamma_3 \geq 1/4$. For smaller Γ_3 where the flow in the confined [115–118] or severely confined regime [119], the flow is laminar for a wide range of θ_m , and it is not the focus of the present study. We can clearly see in figure 2.9(a) that the absolute Nu is less influenced by Γ_3 for all considered θ_m , despite the fact that the flow strength is weakened when reducing Γ_3 as illustrated in figure 2.9(c). After normalisation, the data for both Nu and Re again fall onto a single curve as seen in figures 2.9(b) and 2.9(d). Thus, we can conclude that the universal behaviour for normalised Nu and Re can also be found in 3D configurations.

We finally discuss the scalings of the Nu and Re versus Ra for different θ_m . As mentioned above $Nu(\theta_m)/Nu(0)$ and $Re(\theta_m)/Re(0)$ both have a universal relationship with θ_m which are independent of Ra , and thus the scaling exponents of $Nu \sim aRa^\alpha$ and $Re \sim bRa^\beta$ for different θ_m should also be expected to be independent of θ_m . Figure 5.8(a) shows a log-log plot of Nu versus Ra for the OB and penetrative cases with different θ_m for 2D simulations. Detailed exponents and prefactors for all the θ_m considered in this work are listed in table 2.1. For the OB case, the scaling exponent 0.29 is quite close to those obtained in the previous 2D studies. For example, Ref. [120] obtained 0.285 for periodical cells with $Pr = 1$, Ref. [121] obtained 0.29 for $Pr = 10$ for no-slip sidewalls similar to the present study. Ref. [122] obtained 0.3 ± 0.02 for $Pr = 0.71$ and 5.3 also for no-slip sidewalls. For penetrative cases all the exponents vary between 0.28 and 0.3 for different θ_m , i.e., the scaling exponent does not change much for the penetrative cases. Figure 5.8(b) shows a log-log plot of Re vs. Ra for 2D simulations. For the OB case, the scaling exponent 0.59 is also quite close to previous 2D studies (0.62 for $Pr = 4.3$ [99] and 0.6

for $Pr = 0.71$ and 5.3 [122].) For penetrative cases as listed in table 2.1, one can see that the scaling exponent for Re only deviates from the OB case for large θ_m close to 0.9 . According to the data listed in table 2.1, one may come to the conclusion that the scaling exponents almost keep unchanged for the penetrative cases for not too large θ_m . It is interesting that the scaling exponents are so robust to this density maximum effect despite of the remarkable change of the flow organisations as mentioned before.

θ_m	a	α	b	β
OB	0.131	0.290	5.43e-3	0.590
0	0.131	0.291	5.53e-3	0.591
0.1	0.123	0.291	4.67e-3	0.594
0.2	0.113	0.291	3.94e-3	0.596
0.3	0.128	0.280	3.17e-3	0.599
0.4	0.086	0.291	2.26e-3	0.606
0.5	0.058	0.299	1.59e-3	0.612
0.6	0.036	0.305	1.01e-3	0.621
0.7	0.028	0.297	5.62e-4	0.632
0.8	0.013	0.302	1.60e-4	0.667
0.9	6.18e-3	0.288	1.77e-5	0.722

Table 2.1: The coefficients and exponents of the fitted scalings $Nu \sim aRa^\alpha$ and $Re \sim bRa^\beta$

2.4 Conclusions

In summary, we have studied penetrative turbulent RBC of water near 4°C using 2D and 3D DNSs. In the present system, the flow dynamics and heat transport highly depend on an important parameter, i.e., the density inversion parameter θ_m . Several universal properties are identified and are found to hold over a wide range of parameters for both 2D and 3D. The ratio of the top and bottom thermal BL thickness F_λ increases with increasing θ_m , and the relationship is found to be independent of Ra . Similar to the previous studies for NOB convection caused by large temperature differences, the center temperature θ_c is particularly investigated. θ_c is found to be increased compared to that of the OB cases. As θ_c is related to F_λ with $1/\theta_c = 1/F_\lambda + 1$, θ_c also has a universal relationship with θ_m which is valid for both 2D and 3D cases. With increasing θ_m , both the heat transfer and the flow intensity

are substantially weakened. The normalized Nusselt number $Nu(\theta_m)/Nu(0)$ and Reynolds number $Re(\theta_m)/Re(0)$ also have universal dependence on θ_m which seems to be independent of both Ra and the aspect ratio Γ . Finally, the scaling laws of Nu and Re versus Ra are investigated. It is interesting to find that the scaling exponents of $Nu \sim Ra^\alpha$ and $Re \sim Ra^\beta$, i.e., $\alpha \approx 0.29$ and $\beta \approx 0.59$, are insensitive to θ_m , despite the drastic changes in the flow organizations.

2.5 Appendix

The details of the main 2D simulations are listed in table 2.2 and the 3D simulations in table 2.3. In order not to make the tables too long, we only list some of the main cases for 2D simulations. It can be seen that the differences between Nu at the hot and cold plates are generally within 1%.

Ra	θ_m	$Ny \times Nz$	N_b	N_t	Nu_b	Re	θ_c	$100\Delta Nu$	t_{avg}
10^7	OB	128×128	10	10	13.83	68.00	0.4995	0.29	6000
10^7	0	128×128	9	10	13.92	71.86	0.5277	0.12	6000
10^7	0.1	128×128	9	11	13.13	61.69	0.5320	0.11	6000
10^7	0.2	128×128	9	11	12.11	52.86	0.5425	0.17	6000
10^7	0.3	128×128	11	13	10.85	43.47	0.5571	0.17	6000
10^7	0.4	128×128	11	15	9.21	34.77	0.5880	0.03	6000
10^7	0.5	128×128	13	20	7.05	27.13	0.6488	0.17	6000
10^7	0.6	128×128	14	29	4.89	20.78	0.7195	0.15	6000
10^7	0.7	128×128	15	39	3.39	14.89	0.7923	0.14	5000
10^8	OB	256×256	10	10	27.70	302.06	0.5032	0.13	3000
10^8	0	256×256	10	11	27.86	310.17	0.5279	0.14	3000
10^8	0.1	256×256	10	11	26.27	280.26	0.5345	0.24	3000
10^8	0.2	256×256	11	12	24.37	247.35	0.5415	0.14	3000
10^8	0.3	256×256	11	14	21.99	209.11	0.5609	0.26	3000
10^8	0.4	256×256	12	17	18.51	174.06	0.5988	0.54	3000
10^8	0.5	256×256	13	23	14.22	136.34	0.6559	0.14	3000
10^8	0.6	256×256	15	34	9.96	102.07	0.7224	0.04	3000
10^8	0.7	256×256	17	51	6.37	66.52	0.7909	0.19	3000
10^8	0.8	256×256	21	81	3.43	33.38	0.8606	0.64	3000
10^9	OB	512×512	11	11	53.31	1110.99	0.4994	0.12	1000
10^9	0	512×512	11	12	53.77	1152.52	0.5287	0.35	1000

Ra	θ_m	$Ny \times Nz$	N_b	N_t	Nu_b	Re	θ_c	$100\Delta Nu$	t_{avg}
10^9	0.1	512×512	11	12	50.41	1040.75	0.5342	0.15	1000
10^9	0.2	512×512	11	14	46.94	898.78	0.5443	0.14	1000
10^9	0.3	512×512	12	16	42.23	785.55	0.5643	0.07	1600
10^9	0.4	512×512	13	19	36.11	651.50	0.5967	0.39	1600
10^9	0.5	512×512	14	26	28.28	523.61	0.6516	0.47	1600
10^9	0.6	512×512	16	37	20.42	406.66	0.7159	0.62	1600
10^9	0.7	512×512	19	59	12.93	291.71	0.7870	0.31	2000
10^9	0.8	512×512	23	101	7.02	179.57	0.5287	0.05	2000
10^9	0.9	256×256	15	107	2.45	55.09	0.9313	0.51	5000
10^{10}	OB	1024×1024	12	12	104.49	4344.77	0.5045	0.89	800
10^{10}	0	1024×1024	11	12	106.01	4481.58	0.5261	0.44	800
10^{10}	0.1	1024×1024	12	13	99.09	4010.30	0.5290	0.88	800
10^{10}	0.2	1024×1024	12	14	91.61	3504.57	0.5414	0.97	800
10^{10}	0.3	1024×1024	13	16	82.50	2931.49	0.5598	0.22	800
10^{10}	0.4	1024×1024	14	20	69.35	2466.07	0.5951	0.23	800
10^{10}	0.5	1024×1024	16	27	55.20	1971.33	0.6426	0.09	800
10^{10}	0.6	1024×1024	17	40	40.17	1541.42	0.7078	0.17	800
10^{10}	0.7	1024×1024	20	66	25.72	1112.56	0.7810	0.67	1600
10^{10}	0.8	512×512	12	60	13.82	691.90	0.8560	0.22	3200
10^{10}	0.9	512×512	17	138	4.77	286.17	0.9281	0.42	2400

Table 2.2: 2D simulation parameters. The columns from left to right indicate Rayleigh number Ra , density inversion parameter θ_m (OB cases are also included for comparison), grid resolutions, number of grid points in the bottom (N_b) and top (N_t) thermal boundary layers, Nusselt number at the bottom plate Nu_b , Reynolds number Re , center temperature θ_c , the difference between Nusselt number at the bottom and top plates ($\Delta Nu = |Nu_b - Nu_t|/Nu_b$), and the averaging time for the simulations t_{avg} . Note that not all the 2D simulations in this work are listed in this table.

Ra	Γ_3	θ_m	$Nx \times Ny \times Nz$	N_b	N_t	Nu_b	Re	θ_c	$100\Delta Nu$	t_{avg}
10^8	1/4	OB	$64 \times 256 \times 256$	9	9	31.64	135.63	0.4987	0.31	1000
10^8	1/4	0	$64 \times 256 \times 256$	9	9	31.86	139.23	0.5229	0.10	1000
10^8	1/4	0.1	$64 \times 256 \times 256$	9	10	29.94	124.77	0.5275	0.40	1000
10^8	1/4	0.2	$64 \times 256 \times 256$	10	11	27.75	109.95	0.5345	0.07	1000
10^8	1/4	0.3	$64 \times 256 \times 256$	10	12	25.08	95.16	0.5497	0.10	1000

Ra	Γ_3	θ_m	$Nx \times Ny \times Nz$	N_b	N_t	Nu_b	Re	θ_c	$100\Delta Nu$	t_{avg}
10^8	1/4	0.4	$64 \times 256 \times 256$	11	15	21.02	77.90	0.5814	0.17	1000
10^8	1/4	0.5	$64 \times 256 \times 256$	12	21	16.13	63.14	0.6458	0.19	1000
10^8	1/4	0.6	$64 \times 256 \times 256$	14	31	11.36	50.03	0.5202	0.36	1000
10^8	1/4	0.7	$64 \times 256 \times 256$	15	46	7.34	36.55	0.7949	0.27	2000
10^8	1/2	OB	$128 \times 256 \times 256$	9	9	31.21	165.85	0.5036	0.01	1000
10^8	1/2	0	$128 \times 256 \times 256$	9	10	31.75	170.99	0.5277	0.23	1000
10^8	1/2	0.1	$128 \times 256 \times 256$	9	10	29.77	154.09	0.5285	0.50	1000
10^8	1/2	0.2	$128 \times 256 \times 256$	9	11	27.53	136.12	0.5444	0.09	1000
10^8	1/2	0.3	$128 \times 256 \times 256$	10	13	24.89	116.55	0.5576	0.26	1000
10^8	1/2	0.4	$128 \times 256 \times 256$	11	15	21.38	95.72	0.5939	0.44	1000
10^8	1/2	0.5	$128 \times 256 \times 256$	12	20	16.36	76.28	0.6373	0.07	1000
10^8	1/2	0.6	$128 \times 256 \times 256$	14	30	11.41	58.62	0.7180	0.55	1000
10^8	1/2	0.7	$128 \times 256 \times 256$	16	44	7.51	44.66	0.7817	0.99	1000
10^8	1	OB	$256 \times 256 \times 256$	9	9	31.85	197.17	0.4985	0.10	1000
10^8	1	0	$256 \times 256 \times 256$	9	9	32.00	200.52	0.5250	0.12	1000
10^8	1	0.1	$256 \times 256 \times 256$	9	10	30.19	181.41	0.5332	0.24	1000
10^8	1	0.2	$256 \times 256 \times 256$	9	11	27.95	162.68	0.5424	0.55	1000
10^8	1	0.3	$256 \times 256 \times 256$	10	12	25.46	137.66	0.5567	0.99	1000
10^8	1	0.4	$256 \times 256 \times 256$	11	15	21.80	116.22	0.5956	0.60	1000
10^8	1	0.5	$256 \times 256 \times 256$	12	20	16.82	91.53	0.6463	0.75	1000
10^8	1	0.6	$256 \times 256 \times 256$	13	30	11.90	71.11	0.7167	0.32	1000
10^8	1	0.7	$256 \times 256 \times 256$	15	44	7.71	52.23	0.7895	0.91	1000
10^8	1	0.8	$256 \times 256 \times 256$	9	36	4.18	34.66	0.8604	0.38	1000
10^9	1/2	OB	$256 \times 512 \times 512$	10	10	62.84	595.02	0.5011	0.28	800
10^9	1/2	0	$256 \times 512 \times 512$	9	10	63.41	610.08	0.5286	0.44	800
10^9	1/2	0.1	$256 \times 512 \times 512$	9	11	59.49	551.32	0.5333	0.04	800
10^9	1/2	0.2	$256 \times 512 \times 512$	10	12	54.81	482.42	0.5428	0.70	800
10^9	1/2	0.3	$256 \times 512 \times 512$	10	13	49.54	421.28	0.5622	0.18	800
10^9	1/2	0.4	$256 \times 512 \times 512$	11	16	42.52	351.36	0.5962	0.08	800
10^9	1/2	0.5	$256 \times 512 \times 512$	12	22	33.28	282.92	0.6475	0.10	800
10^9	1/2	0.6	$256 \times 512 \times 512$	14	33	23.49	220.98	0.7149	0.06	800
10^9	1/2	0.7	$256 \times 512 \times 512$	16	53	15.13	155.12	0.7884	0.48	800
10^9	1/2	0.8	$128 \times 256 \times 256$	10	46	7.97	100.01	0.8585	0.44	1000
10^9	1/2	0.9	$128 \times 256 \times 256$	14	100	2.80	42.69	0.9312	0.17	1000

Ra	Γ_3	θ_m	$Nx \times Ny \times Nz$	N_b	N_t	Nu_b	Re	θ_c	$100\Delta Nu$	t_{avg}
------	------------	------------	--------------------------	-------	-------	--------	------	------------	----------------	-----------

Table 2.3: 3D simulation parameters. The columns from left to right indicate Rayleigh number Ra , density inversion parameter θ_m (OB cases are also included for comparison), grid resolutions, number of grid points in the bottom (N_b) and top (N_t) thermal boundary layers, Nusselt number at the bottom plate Nu_b , Reynolds number Re , center temperature θ_c , the difference between Nusselt number at the bottom and top plates ($\Delta Nu = |Nu_b - Nu_t|/Nu_b$), and the averaging time for the simulations t_{avg} .

Chapter 3

On non-Oberbeck–Boussinesq effects in Rayleigh–Bénard convection of air for large temperature differences[◦]

We present direct numerical simulations (DNS) of non-Oberbeck-Boussinesq (NOB) Rayleigh–Bénard (RB) convection due to large temperature differences in two-dimensional (2D) and three-dimensional (3D) cells. Perfect air is chosen as the operating fluid and the Prandtl number (Pr) is fixed to 0.71 for the reference state $\hat{T}_0 = 300$ K. In the present system, we consider large temperature differences ranging from 60 K to 240 K, and relatively strong NOB effects are induced at moderate Rayleigh number (Ra) in the range $3 \times 10^6 \leq Ra \leq 5 \times 10^9$, which also leads to the turbulence system with large density variations. Due to top-down symmetry breaking under NOB conditions, an increase of the center temperature T_c is found compared to the arithmetic mean temperature T_m of the top and bottom plates, and the shifts of T_c are strongly dependent on Rayleigh number Ra and temperature differential ϵ . The NOB effects on the Nusselt number (Nu) are quite small ($\lesssim 2\%$). The power-law scalings of Nu versus Ra are robust against NOB effects, even for the extremely large temperature difference 240 K, which has never been reached in previous experiments. The Reynolds numbers Re , as well as the scalings of Re versus Ra , are also insensitive to NOB effects.

[◦]Based on: Zhen-Hua Wan, **Qi Wang**, Ben Wang, Shu-Ning Xia, Quan Zhou, De-Jun Sun, *On non-Oberbeck–Boussinesq effects in Rayleigh–Bénard convection of air for large temperature differences*, J. Fluid Mech. 889, A10 (2020).

It is noteworthy that the influence of NOB effects on Nu and Re in 3D RB flow are weaker than its 2D counterpart. Furthermore, the extended laminar boundary layer (BL) equations are developed based on the low-Mach-number Navier-Stokes equations, which qualitatively predicts the NOB effects on velocity profiles. DNS results indicate that the top and bottom thermal BLs can compensate each other much better than the velocity BLs under NOB conditions, which contribute to the robustness of Nu .

3.1 Introduction

Turbulent Rayleigh-Bénard (RB) convection, in which the fluid is heated from the below and cooled from the above (see, e.g., Refs. [53–56, 65]), by now has been studied extensively. Most of these studies strive to stay in a limit described by the Oberbeck-Boussinesq (OB) approximation, which relies on following main assumptions: The density is regarded a constant except in the buoyancy term, where the density ρ is assumed to be dependent on temperature T linearly; All fluid properties (e.g., thermal conductivity k , dynamic viscosity μ) are considered constants.

The OB approximation is found to be reasonably satisfactory if the temperature difference is below a certain threshold which is different for different fluids [123]. However, large temperature differences are ubiquitous in many practical applications, where non-Oberbeck-Boussinesq (NOB) effects become significant. For instance, in nuclear reactors where the typical temperature differences of thermal insulation systems can reach up to several hundred degrees, the variations of ρ , k and μ must be taken into account simultaneously. In foundry processes and some astrophysical flows, NOB effects also play an important role.

There is by now a large body of literature on NOB effects in RB convection [53, 55] for various fluids, such as gaseous helium [97, 124], glycerol [94, 100, 125, 126], gaseous ethane [96, 127], water [95, 99, 101, 128], SF₆ [129, 130] and air [79, 105]. In general, NOB effects in turbulent RB convection can be induced by two paths [55]. One is related to the convection beyond the incompressible limit, which mainly occurs in gases [96], in which the strong compressibility effects and large variations of the fluid properties can arise. The other is usually found in liquids, where NOB effects are almost induced solely from the temperature dependent material properties [95]. In early experiments of Ref. [94], an important issue relevant for NOB effects in RB convection was confirmed that the flow structures become asymmetric due to a top-down symmetry breaking. With the asymmetry of top and bottom thermal boundary layers (BL), the temperature at the cell center will deviate from the arithmetic average of the temperatures at the top and bottom plates. For the convection in water, Ref. [95] experimentally found that the Nusselt number Nu and Reynolds number Re are rather insensitive against the NOB effects, which only result in a small reduction of Nu ($\lesssim 2\%$) under NOB conditions. No obvious modification of Re is found within the experimental resolution. The center temperature is increased due to NOB effects.

While for gaseous ethane, Ref. [96] found a decrease of the center temperature and an increase of Nu which are of opposite and greater magnitude than those for NOB effects in water. Direct numerical simulations (DNS) were performed by Ref. [98] to disentangle the importance of different material parameters for the NOB effects on heat transport and flow structures. More recently, Ref. [131] performed experimental study of the whole velocity field under NOB conditions. The prior studies indicate one fact that NOB effects depend very sensitively on the particular working fluid. The NOB effects are especially relevant to high- Ra convection where the temperature difference is usually large, however, the NOB effects in various fluids at high Ra are still poorly understood [55].

In RB convection of air with large temperature differences, the flow reversals [79] and instabilities [105] have been studied recently, however, the heat and momentum transports for relatively high Ra have rarely been studied. In order to fill this gap, we here investigate NOB effects for compressible air via DNS in two-dimensional (2D) and three-dimensional (3D) cells. To induce NOB effects, we consider large temperature differences ranging from 60 K to 240 K. With large variations in fluid properties, especially the large density variations, we try to understand such NOB effects on heat, momentum transport as well as flow structures quantitatively. There are three major purposes for choosing this model. First, we can study strong NOB effects on heat transport induced by large temperature differences in a moderate range of Rayleigh numbers. Secondly, the relationships between fluid properties and temperature are well described by Sutherland's law, which permits us to investigate the influences of variations in μ and k in an accurate way. Finally, flow compressibility with a rather large density variation is appropriately taken into account by employing low-Mach-number Navier-Stokes equations with acoustic waves filtered [132]. In short, this study is favourable for inferring the possible influence of large temperature differences and fluid compressibility on heat transport and flow structures in some high Ra experiments that usually encounter large temperature differences. To the best of our knowledge, the extremely large temperature difference (e.g., $\Delta\hat{T} = 240$ K) has never been reached in previous experiments and simulations, which is several times that of the largest one used in previous studies. In addition, the low-Mach-number Navier-Stokes equations are also firstly employed to treat 3D fully turbulence at relatively high Ra with large density variations [133].

The reminder of this chapter is organized as follows. In §3.2, we describe the detailed numerical procedures. Section 3.3 shows the major results for the

Figure 3.1: The sketch of the present Rayleigh-Bénard convection in a (a) 2D and (b) 3D cell with the working fluid of air, where $\hat{D}/\hat{W} = 1$ and $\hat{W}/\hat{H} = 1$.

OB and NOB cases, including the flow organizations, heat and momentum transport, and BL profiles, etc. Finally, we summarize our findings and conclude in §3.4.

3.2 Numerical settings

Figure 3.1 shows the configuration of the present system, where the working fluid is air in a cell of width $\hat{W}$, height $\hat{H}$ and depth $\hat{D}$ (for 3D cases). ($\hat{\cdot}$ denotes dimensional quantity). $\hat{H}$ is chosen as the reference length. For all cases, $\hat{D}/\hat{W} = 1$ and $\hat{W}/\hat{H} = 1$. The bottom and top walls are fixed at temperatures of $\hat{T}_H$ and $\hat{T}_C$, with $\hat{T}_H > \hat{T}_C$, while the lateral walls are thermally insulated. No-slip and non-penetrative boundary conditions are applied at all rigid walls. To treat NOB effects in air due to large temperature difference, we employ the low-Mach-number Navier-Stokes equations [132]. The reference temperature $\hat{T}_0 = (\hat{T}_H + \hat{T}_C)/2$ is chosen to be 300K, and the reference quantities such as $\hat{\mu}_0, \hat{k}_0, \hat{c}_{p0}$ are determined at this temperature [79]. The dimensional temperature difference is $\Delta\hat{T} = \hat{T}_H - \hat{T}_C = 2\epsilon\hat{T}_0$, where ϵ is the temperature differential, quantifying the intensity of NOB effects, with $\epsilon \leq 0.4$ corresponding to $\Delta\hat{T} \leq 240\text{K}$. The dimensionless temperatures T at the hot and cold walls are given by $1 + \epsilon$ and $1 - \epsilon$, respectively. The free-fall velocity $\hat{u}_0 = (2\epsilon\hat{g}\hat{H})^{1/2}$ is used as the reference velocity, where $\hat{g}$ is the gravitational acceleration, and thus the reference time $\hat{t}_0 = \hat{H}/\hat{u}_0$. The hydrodynamic pressure π is non-dimensionalized by $\hat{\rho}_0\hat{U}^2$, and the thermodynamic pressure p is non-dimensionalized by $\hat{\rho}_0\hat{R}\hat{T}_0$, where $\hat{R}$ is the gas constant. Finally, the di-

mensionless form of low-Mach-number Navier-Stokes equations with acoustic waves filtered can be written as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \quad (3.1)$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial \pi}{\partial x_j} + \sqrt{\frac{Pr}{Ra}} \frac{\partial \tau_{ij}}{\partial x_j} + \frac{1}{2\epsilon} (\rho - 1) n_i, \quad (3.2)$$

$$\rho c_p \left(\frac{\partial T}{\partial t} + u_j \frac{\partial T}{\partial x_j} \right) = \frac{1}{\sqrt{Ra Pr}} \frac{\partial}{\partial x_j} k \frac{\partial T}{\partial x_j} + \Gamma \frac{dp}{dt}, \quad (3.3)$$

$$p = \rho T, \quad (3.4)$$

where n_i is the unit vector in the direction of gravity, k the thermal conductivity. $\gamma = 1.4$ is the ratio of specific heats. $\Gamma = (\gamma - 1)/\gamma$ is a measure of the resilience of the fluid. The isobaric specific heat c_p is fixed to 1. π is the hydrodynamic pressure. p is the thermostatic pressure, the more detailed definition can be found in [79, 105]. The viscous stress tensor τ_{ij} is expressed as $\tau_{ij} = \mu (\partial u_i / \partial x_j + \partial u_j / \partial x_i) - (2/3) \delta_{ij} \mu \partial u_k / \partial x_k$, where δ_{ij} is the Kronecker delta function. The four control dimensionless parameters of this problem are the temperature differential ϵ , Rayleigh number Ra , Prandtl number Pr and aspect ratio A , defined as:

$$\epsilon \equiv \frac{\Delta \hat{T}}{2\hat{T}_0}, \quad Ra \equiv \frac{2\epsilon \hat{c}_{p0} \hat{\rho}_0^2 \hat{g} \hat{H}^3}{\hat{\mu}_0 \hat{k}_0}, \quad Pr \equiv \frac{\hat{c}_{p0} \hat{\mu}_0}{\hat{k}_0}, \quad A \equiv \frac{\hat{W}}{\hat{H}}. \quad (3.5)$$

We fixed the reference Pr to 0.71 and A to 1. Dimensionless thermal conductivity k and dynamic viscosity μ are determined by the Sutherland's law:

$$k = T^{1.5} (1 + S_k) / (T + S_k), \quad \mu = T^{1.5} (1 + S_\mu) / (T + S_\mu). \quad (3.6)$$

For air, the dimensionless Sutherland constants $S_k = \hat{S}_k / \hat{T}_0 = 0.648$ and $S_\mu = \hat{S}_\mu / \hat{T}_0 = 0.368$ for the reference state $\hat{T}_0 = 300$ K [134–136]. The Sutherland's law is developed based on the kinetic theory of ideal gases and an idealized intermolecular-force potential [137], which is commonly used in fully compressible flows [138–140]. Fairly accurate results can be obtained with an error less than a few percent over a wide range of temperatures. OB cases are also investigated in purpose of making a comparison. We set $\epsilon = 0.005$, $\mu = k = p = 1$, $\partial u_j / \partial x_j = 0$ and $\rho = 1$ except in the buoyancy term $\rho = \frac{1}{T} = \frac{1}{1+\delta T} \approx 1 - \delta T$, so the equations (3.1)-(3.4) can be reduced to the classical OB equations [80, 107].

The governing equations are solved numerically by our in-house code lMn2d/3d. In this solver, all spatial terms are discretized by a second-order central difference scheme. To avoid pressure-velocity decoupling, the staggered grid is utilized for temperature and velocities, and pressure is also staggered in time with all other variables. The fractional-step method is used to solve the equations [141]. A multi-grid strategy [142] is adopted to solve the pressure Poisson equation. For time advancement, the wall-normal viscous terms are semi-implicitly treated with Crank-Nicolson scheme, while all other terms are all discretized by third order Runge-Kutta scheme. The numerical details and validations of the code have been elaborated in our previous works [79,105,109]. The grid sizes and some other simulation parameters are given in table 3.3 and table 3.4 in the appendix. For all simulations, the Nu and Re are averaged for at least 400 free-fall time units after all transients have been dissipated. The grid is chosen to satisfy the resolution requirement which can resolve the smallest scales of the problem, i.e., the Kolmogorov scale η_K and the Batchelor scale η_B [110]. In addition, non-uniform grid is adopted with more grid points clustered near walls in order to resolve small scales inside BLs. For all cases, there are at least 10 grid points inside the thermal BLs.

3.3 Results and discussion

3.3.1 Flow organizations

The velocity fields

Figure 3.2(a-d) shows the instantaneous velocity and reduced temperature fields at $Ra = 5 \times 10^8$ and 8×10^8 for OB and $\epsilon = 0.2$. Here, the reduced temperature is defined as

$$\Theta = (T - 1)/2\epsilon. \quad (3.7)$$

In 2D simulations, the velocity fields are very complex, even when we miss 3D flow modes. As shown in figure 3.2(a,b), we can find a relatively stable large scale circulation (LSC) at $Ra = 5 \times 10^8$, the size of which is comparable to the box size H in both OB and NOB cases [99,143,144]. As Ra increases, the instantaneous velocity fields that the corner rolls become more unstable and the relatively stable LSC is broken. For the case $Ra = 8 \times 10^8$ shown in figure 3.2(c,d), some relatively large-scale vortices detach from the corners and convect along the main wind. Movies at different Ra are provided in the supplementary materials. It should be mentioned that the LSC can still be

Figure 3.2: Snapshots of the 2D velocity (arrows) and reduced temperature Θ (colour) fields for $Pr = 0.71$ with different ϵ and Ra . Upper panel: $Ra = 5 \times 10^8$. (a) OB, (b) $\epsilon = 0.2$; Middle panel: $Ra = 8 \times 10^8$. (c) OB, (d) $\epsilon = 0.2$. Bottom panel: volume rendering of 3D temperature fields (T) for $Ra = 10^7$: (e) OB, (f) $\epsilon = 0.2$.

observed from the mean flow field in this case, but the mean velocity distribution is changed, which will be further discussed in the following. Figure 3.2 (e,f) show the volume rendering of 3D temperature fields. Vigorous sheetlike plumes are emitted from the bottom and top plates, while no significant differences in plume structures between the OB and NOB cases. Overall, for air, we cannot observe obvious qualitative differences from the snapshots between OB and NOB cases for the same Ra , which is different from the NOB cases in water [99].

To better show NOB effects on the velocity field, we present in figure 3.3 the mean horizontal velocity profiles $\bar{v}(z)$ at $y = 0.5$ and mean vertical velocity profiles $\bar{w}(y)$ at $z = 0.5$ for $Ra = 10^8$ and 10^9 in OB and various ϵ in 2D cases, respectively. Due to the inherent nature of top-down symmetry-breaking, the flow structures will be asymmetric as stated in Ref. [94]. From figure 3.3, we can see that the profiles of horizontal velocity $\bar{v}(z)$ and vertical velocity $\bar{w}(y)$ become asymmetric to a certain degree. It should be mentioned that the profiles at $Ra = 10^9$ shown in figure 3.3(b) are different from those at $Ra = 10^8$ shown in figure 3.3(a), due to the change of flow pattern as illustrated in figure 3.2. At $Ra = 10^8$, there is a relatively stable LSC, giving rise to the peaks of the profile near the edge of the top/bottom viscous BL. However, at $Ra = 10^9$, the stable LSC is broken and there are large vortices detached from corner rolls, and the convection of those large vortices has led the peaks of the profile occurring in the bulk (e.g., $z \simeq 0.2$ at the bottom) rather than the edge of the viscous BL. Interestingly, even without stable LSC in this case, we can still find the BL structures and a relatively sharp transition of the velocity profiles occurring near the edge of the viscous BL. In §3.3.4, we will further discuss the NOB effects on the profiles of the viscous BLs, which can be described qualitatively by laminar BL theory.

In order to quantify the intensity of velocity asymmetry, for 2D cases, the absolute value of the ratio of horizontal maximum mean velocity $\bar{v}_{max}$ and horizontal minimum mean velocity $\bar{v}_{min}$ is calculated, namely, $\bar{v}_r = |\bar{v}_{max}|/|\bar{v}_{min}|$, and the similar ratio is also obtained for vertical velocity $\bar{w}_r = |\bar{w}_{max}|/|\bar{w}_{min}|$. Figure 3.4 shows $\bar{v}_r$ and $\bar{w}_r$ as a function of Ra for different ϵ . It is well known that the velocity ratios should equal to 1 within OB approximation due to inherent symmetry of the system, while they deviate from 1 under NOB conditions. Here, we can see that $\bar{v}_r$ or $\bar{w}_r$ is a non-monotonic function of Ra for different ϵ . For $Ra \lesssim 8 \times 10^8$, $\bar{v}_r$ shows a strong deviation under NOB conditions, and the maximum deviation is around 8% at $Ra = 3 \times 10^8$ with $\epsilon = 0.4$. Similar behaviors are found for $\bar{w}_r$. Furthermore, the most interesting

Figure 3.3: Mean velocity profiles of $\bar{v}(z)$ for 2D cases in the plane of $y = 0.5$ (upper panel) and $\bar{w}(y)$ in the plane of $z = 0.5$ (bottom panel) for (a,c) $Ra = 10^8$; (b,d) $Ra = 10^9$.

Figure 3.4: $\bar{v}_r$ (a) and $\bar{w}_r$ (b) versus Ra for different ϵ for 2D cases.

feature is that the Ra dependence of $\bar{v}_r$ or $\bar{w}_r$ is not monotonous, similar to the convection of air with NOB effects in a differentially heated cell [109]. Based on the present data, it seems that the asymmetric feature of velocity is weakened for high Ra , which should also be attributed to the change of flow pattern when $Ra \gtrsim 8 \times 10^8$, as illustrated in figure 3.2.

The temperature profiles and shift of center temperature

Apart from the velocity distribution, the NOB effects also exert a significant influence on the temperature field. Here, we are committed to study the influence of NOB effects on temperature profiles and center temperature quantitatively. Within OB approximation, the center temperature T_c should be equal to 1 (or $\Theta_c = 0$). However, the T_c measured under NOB conditions will deviate from this mean value, due to the asymmetry between top and bottom thermal BLs [96, 99, 100, 102]. For instance, with much smaller μ/k , the thicknesses of viscous/thermal BLs near the the cold wall become much thinner in the present system [79].

Figure 3.5 shows the profiles of time averaged reduced temperature $\langle \Theta \rangle_t$ on the line at $y = 0.5$ in 2D OB and NOB cases with different ϵ and Ra . For OB cases, Θ in the bulk is nearly zero, and slight overshoots are observed near the edge of the thermal BL for $Ra = 10^8$, which is influenced by the relatively large size corner rolls. Such overshoots can also be observed in NOB cases. At $Ra = 10^9$, the overshoots disappear since that these corner rolls are smaller and detach from the corner frequently. For NOB cases, Θ in the bulk become positive for various ϵ and Ra . This finding is similar to that in strongly turbulent RB convection in liquids [95], which is opposed to the previous finding for ethane gas close to its critical point [96]. It should be

Figure 3.5: The profiles of time averaged reduced temperature $\langle \Theta \rangle_t$ for 2D cases at $y = 0.5$ for (a) $Ra = 10^8$, (b) $Ra = 10^9$.

Figure 3.6: (a) The profiles of time and plane averaged reduced temperature $\langle \Theta \rangle_{t,S}$ for 3D case at $Ra = 10^8$. (b) and (c) are the local enlargement of (a).

Figure 3.7: (a) Variations of k with Θ . (b) The profiles of Θ for OB and various ϵ cases at the motionless state.

mentioned that the prediction of the central temperature is still a tough task because of the nonlinear temperature dependence of material properties. In the present system, the thermal conductivity increases with the increase of the temperature, similar to that in liquids instead of gaseous ethane close to its critical point, which might play a dominant role. Figure 3.6 shows the profiles of time and plane averaged reduced temperature $\langle \Theta \rangle_{t,S}$ for the 3D case at $Ra = 10^8$ for various ϵ . The asymmetry of top and bottom thermal boundary layers is still presented under NOB conditions, but the shift of T_c in the bulk becomes much smaller. At $z/H = 0.5$, the center reduced temperature Θ_c is of order 10^{-3} , which is almost close to the level of statistical uncertainties. The NOB effects on the shift of T_c are greatly reduced in 3D cases, which has not been fully understood. We guess that it might be attributed to better fluid mixing for air in the bulk with the flow motion in the third dimension.

In the present cases, we conjecture that the positive shift of T_c is mainly attributed to the temperature-dependent thermal conductivity k . In other word, the sign of Θ_c can be judged beforehand based on the solution of motionless state as follows. It is well known that the motionless conductive state is always a solution of the system before convection onset, which can be used as the base flow for studying onset of instability [105]. With the fixed temperatures at the top and bottom boundaries, the profiles of temperature T for different ϵ can be obtained by solving the nonlinear equation

$$\frac{d}{dz} \left(k \frac{dT}{dz} \right) = 0 \quad (3.8)$$

with given boundary conditions at the top and bottom plates:

$$T|_{z=H} = 1 - \epsilon, \quad T|_{z=0} = 1 + \epsilon$$

Figure 3.8: (a) The relative deviations versus ϵ for air for various Ra in 2D cases. (b) The relative deviations as a function of Ra for various ϵ in 2D cases.

Figure 3.7(a) shows the thermal conductivity k as a function of Θ , while the profiles of temperature in the conduction state in the OB and NOB cases are presented in figure 3.7(b). It is seen that T_c will always be higher than 1 in present cases, which is qualitatively in consistence with the present DNS results shown in figure 3.5. The shifts of T_c at the cell center under motionless states are around 0.022, 0.044 and 0.087 for $\epsilon = 0.1, 0.2$ and 0.4 , respectively, which will be compared with DNS results with turbulent convection at relatively high Ra .

As mentioned before, the shift of T_c is greatly reduced in 3D cases, thus in figure 3.8(a) we only show the relative deviation $(T_c - T_m)/2\epsilon$ at the cell center versus ϵ at various Ra in 2D cases. We can see that all deviations are positive, in qualitative agreement with predictions under motionless states as shown in figure 3.7. This is similar to the finding for water [99]. Moreover, the T_c is strongly dependent on Ra , and the maximum relative deviation is about 5%, occurring at $Ra = 3 \times 10^8$, but all the relative deviations are below the values of motionless states. Previously, for convection in water under NOB conditions, Ref. [99] found that the relative deviation is rather independent of Ra as $Ra \gtrsim 10^5$ and it almost increases linearly with increasing the temperature difference. However, for 2D RB convection of air with large temperature difference, the data points of relative deviations are quite scattered, and such a linear relationship seems to be no longer valid. Furthermore, for RB convection in water with NOB effects, the T_c can even be reasonably predicted by the extended Prandtl-Blasius BL theory [95] despite the large deviations in the temperature profiles. Nevertheless, due to the strong scatter and Ra

Figure 3.9: The time averaged temperature $(T_c - T_m)/2\epsilon$ fields at (a) $Ra = 5 \times 10^8$ and (b) $Ra = 10^9$ under NOB conditions. The mean velocity vectors are also shown.

dependence of our data, we cannot directly borrow the extended BL theory to predict T_c in present cases. Figure 3.8(b) shows relative deviations versus ϵ for different Ra . Clearly, the largest deviations occur at $\epsilon = 0.4$ for all Ra numbers. Similar to $\bar{v}_r$ or $\bar{w}_r$, the Ra dependence of T_c is not monotonous. Interestingly, for $Ra \gtrsim 5 \times 10^8$, the value of relative deviation will not be enhanced, but reduced slightly for all ϵ as Ra increases, which might be attributed to the altered flow organizations. Figure 3.9 shows the mean temperature fields $(T_c - T_m)/2\epsilon$ at $Ra = 5 \times 10^8$ and 10^9 . It is clear that the size of corner roll is smaller at $Ra = 10^9$ than that at $Ra = 5 \times 10^8$, and the former case has smaller value of $(T_c - T_m)/2\epsilon$. The mean flow organization at $Ra = 10^9$ for case $\epsilon = 0.1$ is quite similar to that for case $\epsilon = 0.2$, and then we can see their values of $(T_c - T_m)/2\epsilon$ are very close, as shown in figure 3.8(b). The case $\epsilon = 0.4$ at $Ra = 10^9$ has a larger hot left-bottom corner roll, which produces a larger relative deviation at the cell center. Thus, except for the boundary layers, we think the flow organization also plays a certain role in determining the relative deviation at the cell center.

3.3.2 Global heat transport

Case	OB	$\epsilon = 0.1$	$\epsilon = 0.2$	$\epsilon = 0.4$
2D	$Nu \simeq 0.109Ra^{0.295}$	$Nu \simeq 0.110Ra^{0.295}$	$Nu \simeq 0.116Ra^{0.292}$	$Nu \simeq 0.107Ra^{0.296}$
3D	$Nu \simeq 0.164Ra^{0.285}$	$Nu \simeq 0.169Ra^{0.283}$	$Nu \simeq 0.167Ra^{0.284}$	$Nu \simeq 0.168Ra^{0.282}$

Table 3.1: The best power-law fit of $Nu \sim Ra^\alpha$ for OB and various ϵ cases in 2D and 3D cases.

We now pay attention to the global heat transport with/without NOB effects, which is measured by the Nusselt number Nu , defined as:

$$Nu = \frac{Q}{k2\epsilon/H}, \quad (3.9)$$

where Q is the heat flux across any horizontal plane. Figures 3.10(a) and 3.10(c) show the log-log plot of Nu versus Ra for OB and NOB cases for various ϵ . It is clear that the change of Nu caused by NOB effects is quite small, which cannot be distinguished directly in a log-log plot. Therefore, we list the raw data of Nu for all cases in tables 3.3 and 3.4. It is well known that in RB convection for a fixed Pr , the relationship between Nu and Ra can usually be expressed by a power-law scaling, i.e., $Nu \sim Ra^\alpha$. For 2D RB convection within OB approximation, Ref. [122] examined Ra scalings of Nu for $Pr = 0.7$ and $Pr = 5.3$, which yields $Nu \sim Ra^{0.30 \pm 0.02}$ for both Pr . Presently, table 3.1 lists the best power-law fitting to Nu versus Ra

Figure 3.10: The Nusselt number Nu versus Ra for OB and various ϵ for (a) 2D cases and (c) 3D cases. The dashed lines are power-law fitting for OB data. Nu_{NOB}/Nu_{OB} versus ϵ for various Ra for (b) 2D cases and (d) 3D cases.

for OB and various ϵ cases. Clearly, for current 2D cases in the Ra number range $5 \times 10^7 \leq Ra \leq 5 \times 10^9$, the Nu versus Ra for OB cases yields $Nu \sim Ra^{0.295}$, which is very close to that in Refs. [120, 122, 145, 146]. It should be emphasized that the Ra -scaling exponents are also insensitive to NOB effects in RB convection despite of the changes in flow organizations caused by top-down symmetry breaking, similar to the differentially heated cavity [109]. For present 3D cases in the Ra number range $3 \times 10^6 \leq Ra \leq 10^8$, the Nu versus Ra for OB cases yields $Nu \sim Ra^{0.285}$, which is also close to its 2D counterpart.

In addition, to quantitatively show the NOB effects on Nu , the Nusselt number ratio $Nu_{\text{NOB}}/Nu_{\text{OB}}$ versus ϵ for all Ra is shown in figures 3.10(b,d). As aforementioned, the flow organizations are changed due to NOB effects, including the mean velocity and temperature profiles. However, the Nusselt number Nu seems to be insensitive to the change of ϵ , even up to $\epsilon = 0.4$, corresponding to the dimensional temperature difference 240 K. For 2D cases, the maximum deviation of $Nu_{\text{NOB}}/Nu_{\text{OB}}$ from 1 for all Ra is less than 2%. It is also found that the maximum deviation for 3D cases ($\lesssim 1\%$) is even smaller than that for 2D cases. Interestingly, similar to Ra dependence of T_c , the Nusselt number ratios here display an obvious dependence on Ra despite of the limited deviations, while they are only weakly dependent on Ra for water with NOB effects [99].

3.3.3 Reynolds numbers

In RB convection, the Reynolds number Re is an important response parameter, which is defined as:

$$Re = \frac{\hat{U} \hat{H}}{\hat{\nu}_0} = \sqrt{\frac{Ra}{Pr}} U \quad (3.10)$$

where $\hat{U}$ and U are the dimensional and dimensionless characteristic velocity.

Here, we choose the root mean square (r.m.s.) velocities as the characteristic velocity [122, 147–149], i.e., $U = U^{\text{rms}} = \sqrt{\langle \mathbf{u} \cdot \mathbf{u} \rangle_V}$, where $\langle \cdot \rangle_V$ denotes space average in two and three dimensions, and $\mathbf{u}$ is the velocity vector. Figure 3.11(a,c) shows Re based on r.m.s. of velocities as a function of Ra for OB and NOB for various ϵ . Similar to Nu , the Reynolds number can also be expressed by a power-law scaling, i.e., $Re \sim Ra^\beta$. It is clear that the influence of NOB effects on Re are quite limited, which almost has no influence on the Ra scaling exponents. Based on U^{rms} , the power-law fitting to the data of OB cases yields the scaling of $Re \sim Ra^{0.617}$ in 2D cases and $Re \sim Ra^{0.491}$ in 3D cases. For 2D cases, the scaling exponent 0.617 is in excellent agreement with the exponent 0.62 found for OB case of water by [99], and is also close to 0.6 reported by [122, 146] recently. For 3D cases, the scaling exponent is notably smaller than that for 2D RB flows, which agrees with previous findings that this exponent ranges from 0.42 to 0.5 for 3D RB flows with various working fluids based on one- and multiple-point measurement [150–153]. To show the differences quantitatively due to NOB effects, the Reynolds number ratio $Re_{\text{NOB}}/Re_{\text{OB}}$ versus ϵ for all Ra is presented in figure 3.11(b,d). We note that the maximum deviation of

Figure 3.11: The Reynolds number Re^{rms} based on the root mean square (rms) velocities versus Ra for OB and NOB with various ϵ in 2D (a) cases and 3D cases (c). The error bars are smaller than the symbol sizes. The dashed lines are the best power-law fit for OB data. Re_{NOB}/Re_{OB} versus ϵ at various Ra numbers for 2D cases (b) and 3D cases (d).

$Re_{\text{NOB}}/Re_{\text{OB}}$ from 1 for all Ra numbers is about 3% for 2D cases and about 1% for 3D cases. In the scope of present data, this deviation does not show an obvious growth trend with increasing Ra or ϵ , so we guess that the NOB effects at some higher Ra experiments might also be limited at least for air.

Case	OB	$\epsilon = 0.1$	$\epsilon = 0.2$	$\epsilon = 0.4$
2D	$Re \simeq 0.042Ra^{0.617}$	$Re \simeq 0.044Ra^{0.615}$	$Re \simeq 0.045Ra^{0.613}$	$Re \simeq 0.042Ra^{0.618}$
3D	$Re \simeq 0.236Ra^{0.491}$	$Re \simeq 0.238Ra^{0.491}$	$Re \simeq 0.235Ra^{0.492}$	$Re \simeq 0.253Ra^{0.487}$

Table 3.2: The best power-law fittings for Re versus Ra for OB and various ϵ in 2D and 3D cases

3.3.4 The viscous and thermal boundary layers

Here, we derive the laminar BL equations starting from the non-dimensionalized low-Mach-number Navier-Stokes equations. For derivation of a steady BL solution, we assume time derivatives to be zero ($\partial(\cdot)/\partial t \equiv 0$) in equations (10.1)-(10.3). Furthermore, we consider a two-dimensional BL in the $x-z$ plane with zero-pressure gradient, and the buoyancy term is neglected. The temperature is assumed to be advected passively. Finally, the equations of flow motion are simplified to

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z} = 0, \quad (3.11)$$

$$\rho u \frac{\partial u}{\partial x} + \rho w \frac{\partial u}{\partial z} = \sqrt{\frac{Pr}{Ra}} \frac{\partial}{\partial z} \left(\mu \frac{\partial u}{\partial z} \right), \quad (3.12)$$

$$\rho C_p \left(u \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial z} \right) = \frac{1}{\sqrt{RaPr}} \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right), \quad (3.13)$$

Here we further assume $\rho T = 1$ which is valid for open system. Furthermore, introduce a stream function:

$$\rho u = \frac{\partial \Psi}{\partial z}, \quad \rho w = -\frac{\partial \Psi}{\partial x}. \quad (3.14)$$

Next, refer to Refs. [95, 96], we further introduce a self-similar variable $\tilde{Z} = z/L$, while $\tilde{\Psi} = \Psi/LU$, such that $L = (Pr/Ra)^{1/4} \sqrt{x/U}$. So the velocity components become

$$u = \frac{U}{\rho} \tilde{\Psi}', \quad w = \sqrt{\frac{Pr}{Ra}} \frac{1}{2L} (\tilde{Z} \tilde{\Psi}' - \tilde{\Psi}). \quad (3.15)$$

Then, the momentum equation becomes:

$$\mu \tilde{\Psi}''' + \left(\frac{\tilde{\Psi}}{2} - \mu \frac{2\rho'}{\rho} + \mu' \right) \tilde{\Psi}'' + \left[\mu \left(\frac{2\rho'\rho'}{\rho^2} - \frac{\rho''}{\rho} \right) - \mu' \frac{\rho'}{\rho} - \frac{\tilde{\Psi}}{2} \frac{\rho'}{\rho} \right] \tilde{\Psi}' = 0 \quad (3.16)$$

and the temperature equation becomes:

$$kT'' + (k' + \frac{1}{2}C_p \text{Pr } \tilde{\Psi})T' = 0, \quad (3.17)$$

From the equation of state $\rho = 1/T$, we can get:

$$\mu\tilde{\Psi}''' + (\frac{\tilde{\Psi}}{2} + \mu\frac{2T'}{T} + \mu')\tilde{\Psi}'' + (\mu\frac{T''}{T} + \mu'\frac{T'}{T} + \frac{1}{2}\tilde{\Psi}\frac{T'}{T})\tilde{\Psi}' = 0, \quad (3.18)$$

$$kT'' + (k' + \frac{1}{2}C_p \text{Pr } \tilde{\Psi})T' = 0, \quad (3.19)$$

where

$$\mu' = (1 + S_\mu)(-\frac{T^{1.5}}{(T + S_\mu)^2} + \frac{1.5T^{0.5}}{(T + S_\mu)})T', \quad (3.20)$$

$$k' = (1 + S_k)(-\frac{T^{1.5}}{(T + S_k)^2} + \frac{1.5T^{0.5}}{(T + S_k)})T', \quad (3.21)$$

and the boundary conditions are:

$$\Psi(0) = 0, \quad \Psi'(0) = 0, \quad \Psi'(\infty) = \rho_\infty, \quad (3.22)$$

$$T(0) = T_w, \quad T(\infty) = T_\infty. \quad (3.23)$$

In this paper, we use the density and temperature at the cell center from DNS results to define the boundary conditions for the BL equations at infinity (ρ_∞, T_∞). T_w is the temperature at the wall as sketched in figure 3.1, i.e., $T_w = 1 + \epsilon$ (hot plate) or $T_w = 1 - \epsilon$ (cold plate). Then, the new BL equations can be solved by shooting method, the results of which are compared to the DNS results in §(3.3.4).

The viscous/thermal boundary layer

Here, in some cases, we cannot find the maximum velocity near the BL edge as shown in figure 3.3(b). Therefore, we use a new definition of the thickness of viscous BL (δ_v) as illustrated in figure 3.12, which is slightly different from that in prior studies [154, 155]. According to BL theory, the viscous effects mainly come from the term containing the secondary derivative of velocity. In order to find the edge of viscous BL, we calculate the secondary derivative of velocity with respect to z , i.e., $v''(z)$, and set the location of 3% of the peak value of $|v''(z)|$ to be the edge of viscous BL, ensuring that the viscous effects become small at this location and beyond. Then, the velocity at the edge of viscous BL is denoted as v_e . The thickness of viscous BL δ_v is defined as the distance from the wall at which the extrapolation of the linear part of the horizontal velocity profile meets the horizontal line passing through the velocity v_e . We examined that for $Ra = 10^8$ this difference of δ_v between current and prior definitions is less than 1%. The thickness of thermal BL (δ_T) can also be defined using the center temperature T_c in a similar way [156].

Figure 3.12: The definition of the thickness of viscous BL for DNS results, where the OB case at $Ra = 10^9$ is used. The edge of BL (red dot) with velocity v_e is defined at the position of 3% of the peak value of secondary derivative of velocity $|v''(z)|$.

Figure 3.13: The comparison of normalized velocity profiles of current BL equations and DNS results for OB and NOB cases with different Ra on the line of $y = 0.5$. (a, b) $Ra = 10^8$, (c, d) $Ra = 10^9$. Left panel: the BL profiles on the bottom plate. Right panel: the BL profiles on the top plate. δ_v is the thickness of viscous BL. u_e is the velocity at the edge of the BL.

Figure 3.14: The comparison of normalized temperature profiles of BL theory and DNS results as a function of z/δ_T for OB and NOB cases with different Ra . (a, b) $Ra = 10^8$, (c,d) $Ra = 10^9$. Left panel: the velocity profiles on the bottom plate. Right panel: the velocity profiles on the top plate. δ_T is the thickness of thermal BL.

Figure 3.15: The compensation of viscous and thermal BLs. F_{δ_v} (a) and F_{δ_T} (b) versus Ra for various ϵ for 2D cases on the line at $y = 0.5$.

Figure 3.13 shows the profiles of normalized velocity $v(z)/v_e$ as a function of z/δ_v for $Ra = 10^8, 10^9$ for OB and NOB cases on the line of $y = 0.5$ in 2D cases. For each DNS case, the combination of v_e and δ_v can be uniquely determined. Due to symmetry breaking under NOB conditions, the velocity profile on the bottom (top) plate is shown on the left (right) panel in figure 3.13. There are some deviations of velocity profile between the solutions of present BL equations and DNS which are also found in experiments by Ref. [155], and DNS by Ref. [157]. However, we find similar trend of viscous BL profiles due to NOB effects between the extended BL solutions and DNSs. Clearly, we find that the changes of the normalized velocity profiles on the bottom plate and the top plate are opposite. The deviations could result from the generation of plumes, pressure gradient and time variation of BL, since that the laminar BL theory relies on a series of assumptions. However, in general, our extended BL equations can reasonably predict the influence of NOB effects on the velocity profiles, regardless of the Ra . Figure 3.14 further shows the profiles of Θ as a function of z/δ_T for $Ra = 10^8, 10^9$ for both OB and NOB cases. The value of δ_T for each DNS case is determined according to δ_v , since that the equations (3.16) and (3.17) are coupled. Again, the temperature profiles obtained by extended BL equations agree qualitatively with the DNS results, and we still observe a certain degree of deviations near the edge of thermal BLs.

As shown in figure 3.13, we easily find that the thickness of viscous/thermal BL is increased near the hot plate due to NOB effects, and decreased near the cold plate. In figure 3.15, we compare the sum of thicknesses of top and bottom viscous/thermal BL for NOB case to OB case by calculating the ratios F_{δ_v} and F_{δ_T} at $y = 0.5$ for 2D cases, which are defined as

$$F_{\delta_v} = 2\delta_v^{\text{OB}}/(\delta_v^t + \delta_v^b)^{\text{NOB}}, \quad F_{\delta_T} = 2\delta_T^{\text{OB}}/(\delta_T^t + \delta_T^b)^{\text{NOB}}. \quad (3.24)$$

From figure 3.15(a), it is seen that there are relatively strong deviations from 1 for F_{δ_v} (at most around 6%), which are also Ra dependent. For $Ra \lesssim 8 \times 10^8$, the NOB effects on velocity fields are relatively stronger. For higher Ra , the LSC is broken, the corner rolls become unstable and the velocity fields are dominated by smaller vortex structures, which weaken the impact of NOB effects. In contrast to F_{δ_v} , the NOB effects have less impact on F_{δ_T} . The maximum deviation from 1 for F_{δ_T} is at most around 3% due to cancellations between δ_T^t and δ_T^b , close to the deviations of $Nu_{\text{NOB}}/Nu_{\text{OB}}$, but no tendency can be seen. In water under NOB conditions, Ref. [95] also attributed the robustness of Nu towards NOB effects to compensation effect of top and bottom thermal BLs. Present data for air even with large temperature differences seems to support this statement, suggesting that this mechanism holds in different system under NOB conditions. As aforementioned, the flow structure is reorganized around $Ra \simeq 8 \times 10^8$ in 2D cases, but one cannot observe any clear transition from the data of F_{δ_T} .

3.4 Conclusions

To summarize, we have systematically studied 2D and 3D RB convection of air under NOB conditions due to large temperature differences by DNS. We consider the turbulence system with large temperature differences (60 K-240 K), which gives rise in large density variations. The extremely large temperature difference case with $\Delta\hat{T} = 240$ K is also considered, which has not been reached in previous experiments. The strong NOB effects are induced in a moderate Ra range $3 \times 10^6 \leq Ra \leq 5 \times 10^9$. To the best of our knowledge, the low-Mach-number Navier-Stokes equations are firstly employed in order to treat 3D turbulence with large density variations and temperature-dependent fluid properties in Rayleigh-Bénard convection at relatively high Ra [133].

First, for air with large density variations, the top-down symmetry is broken with NOB effects which induces some changes in flow organizations, such as the shift of center temperature T_c and the asymmetry of the velocity/thermal BLs. The T_c is increased under NOB conditions compared to the arithmetic mean temperature of the top and bottom plates T_m , which is different from ethane gas close to its critical point [96]. The maximum relative deviation $(T_c - T_m)/2\epsilon$ is around 5% in 2D cases, while it becomes quite small in 3D cases. As reported by Ref. [95, 99], the T_c for water under NOB conditions are weakly dependent on Ra , which can be generally predicted by an extended Prandtl-Blasius theory. However, this theory fails in the present system due to the fact that present results of T_c are scattered and strongly dependent on Ra . Combined with previous experimental and numerical results, we can conclude that the deviation of the center temperature is sensitively dependent on the particular working fluid.

Moreover, for all Ra and ϵ explored, we find that the NOB effects have a very limited influence on Nu . Nu_{NOB} will differ from Nu_{OB} by 2% or less regardless of the existence of stable LSC. The Nusselt number ratios $Nu_{\text{NOB}}/Nu_{\text{OB}}$ are Ra and ϵ dependent, while they are only weakly dependent on Ra for water with NOB effects [99, 128]. More importantly, we also find that the power-law scaling of $Nu \sim Ra^\alpha$ are rather insensitive to the NOB effects. The maximum influence of NOB effects on Re is around 3% for 2D cases and 1% for 3D cases, which are also Ra and ϵ dependent. In general, the NOB effects on Nu and Re are much weaker in 3D cases. This finding is also different from that for the low temperature gaseous helium [97] in 3D cylindrical cell. Besides, unlike the NOB effects in water, the extremely large temperature differences induce large density variations, which is the most significant feature of the present system, however, the effects of large density variations seem to play a very limited role on global heat and momentum transport in the scope of present study.

Finally, the extended BL equations based on low-Mach number Navier-Stokes equations are developed, involving the strong variation of density, as well as the temperature-dependent fluid properties. The NOB effects on BL velocity profiles shown by DNS are predicted qualitatively by the extended BL theory despite of the large deviations in the z -dependence of horizontal mean velocity profiles on the

line of $y = 0.5$ for 2D cases. With increasing ϵ , the relative velocity inside BLs near the hot plate is increased, while it decreases near the cold plate. From the data of F_{δ_v} and F_{δ_T} , we interestingly find that the NOB effects have a much stronger impact on the viscous BLs than the thermal BLs. For all Ra explored, the data of F_{δ_T} are scattered but their deviations from 1 are at most around 3%, close to the deviations of Nu_{NOB}/Nu_{OB} . In the present system, the data also support that the compensation effect between the top and bottom thermal BL leads to the robustness of Nu against the NOB effects, suggesting that this mechanism holds in many different systems under NOB conditions.

In a word, this study fills the gap of numerical study of NOB effects in air with large temperature differences at relatively high Ra in both 2D and 3D systems. The results show that the power-law scalings for Nu and Re versus Ra are rather insensitive to the NOB effects for air, in particular for 3D RB flows, which implies that the NOB effects may play a very limited role for Nu and Re at some high Ra experiments with large temperature differences in air or other perfect gases.

3.5 Appendix

The detailed simulation parameters and results are presented in table 3.3 and table 3.4.

Case	Ra	$N_x \times N_z$	t_{avg}	N_{BL}^b	N_{BL}^t	Nu	Nu_V	Re
OB	5×10^7	256×256	1500	14	14	19.77 ± 0.06	19.71	2385 ± 6
$\epsilon = 0.1$		384×384	1500	20	22	19.80 ± 0.10	-	2406 ± 10
$\epsilon = 0.2$		384×384	1500	19	23	20.12 ± 0.09	-	2452 ± 8
$\epsilon = 0.4$		384×384	1500	16	26	19.60 ± 0.08	-	2430 ± 3
OB	8×10^7	384×384	1500	18	18	23.30 ± 0.09	23.28	3153 ± 7
$\epsilon = 0.1$		512×512	1500	30	33	23.42 ± 0.07	-	3153 ± 4
$\epsilon = 0.2$		512×512	1500	28	34	23.53 ± 0.12	-	3162 ± 5
$\epsilon = 0.4$		512×512	1500	23	37	23.19 ± 0.08	-	3116 ± 8
OB	1×10^8	384×384	1000	17	17	25.31 ± 0.10	25.21	3623 ± 8
$\epsilon = 0.1$		512×512	1000	28	30	25.16 ± 0.07	-	3591 ± 9
$\epsilon = 0.2$		512×512	1000	26	32	25.17 ± 0.15	-	3550 ± 6
$\epsilon = 0.4$		512×512	1000	22	35	24.91 ± 0.16	-	3519 ± 6
OB	3×10^8	512×512	1000	17	17	35.79 ± 0.12	35.63	6793 ± 8
$\epsilon = 0.1$		512×512	1000	21	22	35.74 ± 0.05	-	6805 ± 17
$\epsilon = 0.2$		512×512	1200	20	23	35.75 ± 0.05	-	6768 ± 27
$\epsilon = 0.4$		512×512	1600	17	25	35.67 ± 0.09	-	6597 ± 18
OB	5×10^8	512×512	1000	14	14	42.32 ± 0.07	42.26	9298 ± 25
$\epsilon = 0.1$		512×512	1000	14	19	42.28 ± 0.13	-	9266 ± 17
$\epsilon = 0.2$		512×512	1000	13	20	42.35 ± 0.11	-	9409 ± 17
$\epsilon = 0.4$		512×512	1600	11	22	42.43 ± 0.13	-	9372 ± 16
OB	8×10^8	768×768	800	20	20	46.94 ± 0.45	47.31	13896 ± 47

Case	Ra	$N_x \times N_z$	t_{avg}	N_{BL}^b	N_{BL}^t	Nu	Nu_V	Re
$\epsilon = 0.1$		768×768	800	19	21	47.06 ± 0.47	-	13768 ± 91
$\epsilon = 0.2$		768×768	1000	16	22	46.93 ± 0.17	-	13689 ± 92
$\epsilon = 0.4$		768×768	900	14	22	46.92 ± 0.17	-	14084 ± 59
OB	1×10^9	768×768	800	21	21	50.18 ± 0.25	50.01	15327 ± 123
$\epsilon = 0.1$		768×768	800	20	22	49.70 ± 0.24	-	15499 ± 41
$\epsilon = 0.2$		768×768	1000	19	23	49.83 ± 0.36	-	15494 ± 134
$\epsilon = 0.4$		768×768	800	17	24	49.53 ± 0.36	-	15301 ± 55
OB	5×10^9	1152×1152	600	21	21	76.58 ± 0.63	76.49	39691 ± 520
$\epsilon = 0.1$		1152×1152	800	20	21	76.76 ± 0.81	-	39218 ± 365
$\epsilon = 0.2$		1152×1152	1000	19	22	76.68 ± 0.96	-	39187 ± 538
$\epsilon = 0.4$		1152×1152	800	17	23	76.03 ± 0.96	-	39520 ± 412

Table 3.3: Simulation parameters for 2D cases. The columns from left to right indicate the following: OB and various ϵ cases (NOB), Rayleigh number Ra , grid size ($N_x \times N_z$), the averaging time for calculating statistics t_{avg} , the number of grid points within the bottom thermal BL N_{BL}^b and top thermal BL N_{BL}^t , Nusselt numbers Nu measured at the bottom plate, Nusselt numbers $Nu_V = \sqrt{RaPr\langle\epsilon_u\rangle_{V,t}} + 1$ measured by the globally averaged kinetic energy dissipation rates and Reynolds numbers $Re = \sqrt{Ra/Pr}U^{rms}$. The Prandtl number Pr is fixed to 0.71.

Case	Ra	$N_x \times N_z$	t_{avg}	N_{BL}^b	N_{BL}^t	Nu	Nu_V	Re
OB	3×10^6	$128 \times 128 \times 128$	500	13	13	11.50 ± 0.08	11.47	357.8 ± 2.7
$\epsilon = 0.1$		$128 \times 128 \times 128$	500	12	12	11.56 ± 0.02	-	358.4 ± 3.6
$\epsilon = 0.2$		$128 \times 128 \times 128$	500	12	11	11.49 ± 0.04	-	357.4 ± 1.8
$\epsilon = 0.4$		$128 \times 128 \times 128$	500	11	10	11.49 ± 0.08	-	359.9 ± 1.6
OB	1×10^7	$192 \times 192 \times 192$	400	19	19	16.25 ± 0.05	16.23	649.6 ± 2.5
$\epsilon = 0.1$		$192 \times 192 \times 192$	400	20	18	16.20 ± 0.03	-	652.7 ± 2.9
$\epsilon = 0.2$		$192 \times 192 \times 192$	400	21	17	16.24 ± 0.07	-	652.6 ± 3.6
$\epsilon = 0.4$		$192 \times 192 \times 192$	400	22	15	16.28 ± 0.13	-	653.0 ± 1.0
OB	3×10^7	$256 \times 256 \times 288$	400	21	21	22.14 ± 0.04	22.15	1112.9 ± 1.5
$\epsilon = 0.1$		$256 \times 256 \times 288$	400	22	20	22.07 ± 0.02	-	1116.9 ± 3.1
$\epsilon = 0.2$		$256 \times 256 \times 288$	400	23	19	22.09 ± 0.02	-	1114.7 ± 2.1
$\epsilon = 0.4$		$256 \times 256 \times 288$	400	25	17	21.96 ± 0.03	-	1107.8 ± 0.7
OB	1×10^8	$384 \times 384 \times 392$	400	28	28	31.29 ± 0.02	31.39	2004.1 ± 4.6
$\epsilon = 0.1$		$384 \times 384 \times 392$	400	29	26	31.24 ± 0.08	-	2004.7 ± 1.1
$\epsilon = 0.2$		$384 \times 384 \times 392$	400	31	25	31.12 ± 0.03	-	2005.6 ± 3.3
$\epsilon = 0.4$		$384 \times 384 \times 392$	400	33	22	31.00 ± 0.03	-	1989.6 ± 4.9

Case	Ra	$N_x \times N_z$	t_{avg}	N_{BL}^b	N_{BL}^t	Nu	Nu_V	Re
------	------	------------------	-----------	------------	------------	------	--------	------

Table 3.4: Simulation parameters for 3D cases. The columns from left to right indicate the following: OB and various ϵ cases (NOB), Rayleigh number Ra , grid size ($N_x \times N_y \times N_z$), the averaging time for calculating statistics t_{avg} , the number of grid points within the bottom thermal BL N_{BL}^b and top thermal BL N_{BL}^t , Nusselt numbers Nu measured at the bottom plate, Nusselt numbers $Nu_V = \sqrt{RaPr\langle\epsilon_u\rangle_{V,t}} + 1$ measured by the globally averaged kinetic energy dissipation rates and Reynolds numbers $Re = \sqrt{Ra/Pr}U^{\text{rms}}$. The Prandtl number Pr is fixed to 0.71.

Chapter 4

Multiple states in turbulent large-aspect ratio thermal convection: What determines the number of convection rolls?[◦]

Wall-bounded turbulent flows can take different statistically stationary turbulent states, with different transport properties, even for the very same values of the control parameters. What state the system takes depends on the initial conditions. Here we analyze the multiple states in large-aspect ratio (Γ) two-dimensional turbulent Rayleigh–Bénard flow with no-slip plates and horizontally periodic boundary conditions as model system. We determine the number n of convection rolls, their mean aspect ratios $\Gamma_r = \Gamma/n$, and the corresponding transport properties of the flow (i.e., the Nusselt number Nu), as function of the control parameters Rayleigh (Ra) and Prandtl number. The effective scaling exponent β in $Nu \sim Ra^\beta$ is found to depend on the realized state and thus Γ_r , with a larger value for the smaller Γ_r . By making use of a generalized Friedrichs inequality, we show that the elliptical shape of the rolls and viscous damping determine the Γ_r -window for the realizable turbulent states. The theoretical results are in excellent agreement with our numerical finding $2/3 \leq \Gamma_r \leq 4/3$, where the lower threshold is approached for the larger Ra . Finally, we show that the theoretical approach to frame Γ_r also works for free-slip boundary conditions.

[◦]Based on: **Qi Wang**, Roberto Verzicco, Detlef Lohse, Olga Shishkina, *Multiple states in turbulent large-aspect ratio thermal convection: What determines the number of convection rolls?* Phys. Rev. Lett. **125**, 07451 (2020)

4.1 Introduction

For laminar flows, flow transitions can often be calculated from linear stability analysis. Such an analysis not only gives the critical value of the control parameter at which the instability sets in, but also the wavelength of the emerging structure. Famous classical examples for linearly unstable wall-bounded flows are the onset of convection rolls in Rayleigh–Bénard convection or the onset of Taylor rolls in Taylor–Couette flow [62]. In both cases, the rolls of the most unstable mode have a certain wavelength which follows from the linear stability analysis. With increasing flow driving strength, more and more modes become unstable, and in the fully turbulent case the base flow is unstable to basically any perturbation.

What then sets the size of the flow structures in such *fully turbulent* wall-bounded flow? Several findings have suggested that wall-bounded turbulent flows can take *different* statistically stationary turbulent states, with different length scale of the flow structures and with different transport properties, even for the very same values of the control parameters. Examples for the coexistence of such multiple turbulent states include turbulent (rotating) Rayleigh–Bénard convection [107, 145, 158–162], Taylor–Couette turbulence [163–165], double-diffusive convection turbulence [166], von Karman flow [167–170], rotating spherical Couette flow [171], Couette flow with spanwise rotation [172], but also geophysical flows [173, 174] such as in ocean circulation [25, 27, 175], in the liquid metal core of Earth [4, 6, 176, 177], or in the atmosphere [178, 179].

The occurrence of multiple states in fully turbulent flows can be considered unexpected since, according to Kolmogorov [180], for strong enough turbulence, the fluctuations should become so strong that the whole highly dimensional phase-space is explored. Of course, one could argue that in the above given cases and examples, the turbulence driving has not yet been strong enough to reach that state and that the occurrence of multiple turbulent states in wall-bounded turbulence may be a finite size effect, but in any case even then it remains open what sets the range of allowed sizes of the flow structures in such turbulent flows.

To illuminate this question, as a model system we picked two-dimensional (2D) Rayleigh–Bénard (RB) turbulence, the flow in a closed container heated from below and cooled from above [53–55]. The control parameters are the Rayleigh number Ra , which is the dimensionless temperature difference between the plates, the Prandtl number Pr , which is the ratio between kinematic viscosity (ν) and thermal diffusivity (κ), and the aspect ratio $\Gamma = W/H$, i.e., the ratio between horizontal (W) and vertical (H) extension of the system. The response parameters are the Nusselt number $Nu = QH/(k\Delta)$ and the Reynolds number $Re = UH/\nu$, which indicate the dimensionless heat transport and flow strength in the system. Here Q is the heat flux crossing the system, k the thermal conductivity, Δ the temperature difference at the plates, and $U = \sqrt{\langle \mathbf{u}^2 \rangle_{V,t}}$ the time and volume-averaged velocity.

The flow dynamics is given by the Boussinesq approximation of the Navier–Stokes and the advection-diffusion equation, with the corresponding boundary conditions

(BCs) for the temperature and velocity fields. For the latter at the plates we will first apply no-slip BCs, but later also examine free-slip BCs – a difference which will turn out to be major for the range of allowed states. Periodic BCs are used in the horizontal direction.

We are very much aware of the differences between 2D and 3D RB flow [181], but in particular for large $Pr \geq 1$ there are extremely close similarities between 2D and 3D RB flows, and we wanted to pick a model system for which (i) we can reasonably explore the considerable parameter space for a large enough number of initial flow conditions and (ii) we have the chance to obtain exact analytical results for the range of allowed flow structures.

The parameter range we will explore is for large Prandtl numbers in the range $1 \leq Pr \leq 100$, for Rayleigh numbers in the range $10^7 \leq Ra \leq 10^{10}$ and for large Γ up to $\Gamma = 32$. Note that in 2D RB, multiple coexisting turbulent states had been found before for $Ra = 10^7$, $Pr = 0.7$ and $\Gamma \approx 0.64$ (i.e., in an extremely limited range of the parameter space) [159], but not for such large Γ and Ra as we explore here, as the range of chosen initial flow conditions was not large enough [145], and clearly not as general and omnipresent as we will find here.

4.2 Numerical settings

The direct numerical simulations were done with an advanced finite difference code (AFiD [80]) with the criteria for the grid resolution, as found to be required in Ref. [110]. The code has extensively been tested and benchmarked against other codes [182] and applied in 2D RB even up to very large $Ra = 4.64 \times 10^{14}$ [183, 184]. More simulation details for all explored cases can be found in the supplementary material. In order to trigger the possible convection roll state, we use different initial roll states generated by a Fourier basis: $u(x, z) = \sin(n^{(i)}\pi x/\Gamma)\cos(\pi z)$, $w(x, z) = -\cos(n^{(i)}\pi x/\Gamma)\sin(\pi z)$, where $n^{(i)}$ indicates the initial number of rolls in the horizontal direction. The initial temperature has a linear profile with random perturbations.

4.3 Results

In figure 4.1(a) and 4.1(b) we show the temporal evolution of some flow characteristics for six different initial flow conditions for the case of $Ra = 10^{10}$, $Pr = 10$, and $\Gamma = 8$. We vary the initial number $n^{(i)}$ of rolls from $n^{(i)} = 4$ to $n^{(i)} = 14$, implying aspect ratios of the initial rolls from $\Gamma_r^{(i)} = \Gamma/n^{(i)} = 2$ to $\Gamma_r^{(i)} = 4/7$. As flow characteristics we picked the Reynolds number ratio Re_z/Re_x and the Nusselt number Nu . Here $Re_z = \sqrt{\langle w^2 \rangle_V} H/\nu$ is the volume averaged vertical Reynolds number and $Re_x = \sqrt{\langle u^2 \rangle_V} H/\nu$ the horizontal one, where $w(t)$ and $u(t)$ are the respective velocities. As one can see in figure 4.1(a, b), depending on the six initial conditions, over the very long time of more than 10^4 free-fall time units, the system evolves to either of three

Figure 4.1: (a, b) Time evolution of (a) Re_z/Re_x and (b) Nu for different initial roll states, $Ra = 10^{10}$, $Pr = 10$, $\Gamma = 8$. $\Gamma_r^{(i)} = \Gamma/n^{(i)}$ is the initial and $\Gamma_r = \Gamma/n$ the final mean aspect ratio of the rolls; $n^{(i)}$ is the initial and n the final number of the rolls. Note the logarithmic scales of the time axes. (c) Re_z/Re_x as a function of Γ_r , for all studied combinations of Ra , Pr and $\Gamma = 8$. The dashed curve follows Γ_r^{-1} . (d) Snapshots of the temperature fields for the three statistically stable turbulent states for $\Gamma_r = 1$ ($n = 8$), $\Gamma_r = 4/5$ ($n = 10$) and $\Gamma_r = 2/3$ ($n = 12$) and the corresponding Nusselt numbers, for $Ra = 10^{10}$, $Pr = 10$, and $\Gamma = 8$. All sub-figures are for no-slip BCs.

different final turbulent states, all with different Reynolds number ratio Re_z/Re_x and Nusselt number Nu . The smaller the final mean aspect ratio Γ_r of the rolls, the larger the Reynolds number ratio Re_z/Re_x and Nu , due to more plume-ejecting regions which have strong vertical motion.

The time evolution of some of the six different initial states ($n^{(i)} = 6, 14$) analyzed in figure 4.1(a, b) can be seen in the supplementary movies [185], displaying roll merging and splitting events. The states with large initial rolls (corresponding to $n^{(i)} = 4, 6$) break up quickly, while those with small initial rolls ($n^{(i)} = 14$) first undergo a transition into an unstable twelve-roll state (with smaller Re_z/Re_x than the stable one) as seen in figure 4.1(a), followed by merging events of two neighboring convection rolls. Though both, the Reynolds number ratio and the Nusselt number, keep on fluctuating in time, reflecting the turbulent nature of the states, the three final statistically stable turbulent states are clearly distinguished. We characterize them by the final aspect ratio of their rolls, namely $\Gamma_r = 1$, $\Gamma_r = 4/5$, and $\Gamma_r = 2/3$, corresponding to $n = \Gamma/\Gamma_r = 8, 10$, and 12 rolls, respectively. Snapshots of these states and their corresponding Nusselt numbers are shown in figure 4.1(c). As one can see, the larger the number of rolls, the better the (heat) transport properties of the system, a characteristics which was found in Taylor–Couette flow before [163] and which can intuitively be understood, due to the larger number of emitted plumes at the interfaces between the rolls.

Consequently, when the cell aspect ratio Γ is stretched, the stretching of the mean aspect ratio $\Gamma_r = \Gamma/n$ of the corresponding individual rolls is accompanied with a decrease of the corresponding Nusselt number, as seen in figure 4.2(a) and 4.2(b). Though this behavior has been seen before [145], in figure 4.2(a) and 4.2(b) we clearly see the coexistence of the different turbulent states. The determining relevance of the final mean aspect ratio $\Gamma_r = \Gamma/n$ of the individual rolls for the Nusselt number Nu and Reynolds number Re in the statistically stationary case is nicely demonstrated in figure 4.2(c) and 4.2(d), where we show that the dependences $Nu(\Gamma_r)$ and $Re(\Gamma_r)$ are universal and irrespective of the individual values of Γ or n .

Remarkably, not only the absolute value of the Nusselt number depends on Γ_r , but even the effective *scaling* behavior of Nu with Ra , as can be seen in figure 4.3(a). The same holds for the Reynolds number, figure 4.3(b). In both cases the effective scaling exponent is larger for turbulent states with smaller mean aspect ratio Γ_r of the rolls (see the values given in Fig 4.3(a,b)), i.e., when the system can accommodate a larger number $n = \Gamma/\Gamma_r$ of rolls, presumably reflecting the larger disorder and the larger number of emitted plumes for those states.

From figure 4.3(a, b) we also see that turbulent states with a too large aspect ratio Γ_r of their rolls cease to exist with increasing Ra . Which turbulent states – as characterized by the mean aspect ratio Γ_r of their rolls – are statistically stable for given Ra and Pr can be seen from the phase diagrams in figure 4.4. For fixed $Pr = 10$, all statistically stable turbulent states in the no-slip case have an aspect ratio Γ_r in the range $2/3 \leq \Gamma_r \leq 4/3$, in the Ra -range analyzed in this chapter. With increasing Ra , we see the range moving towards smaller values of Γ_r ; e.g. with $1 \leq \Gamma_r \leq 4/3$ for

Figure 4.2: (a) Nu and (b) final aspect ratio $\Gamma_r = \Gamma/n$ of individual rolls, as function of Γ , for different turbulent states. The numbers and colors in the legend denote the number n of convection rolls of that state. (c) Nu and (d) Re , as functions of $\Gamma_r = \Gamma/n$ for three different values of Γ . In this figure $Ra = 10^9$ and $Pr = 10$ and the BCs are no-slip.

Figure 4.3: (a) Compensated Nusselt number $Nu/Ra^{1/3}$ and (b) compensated Reynolds number $Re/Ra^{2/3}$, as functions of Ra , for four different turbulent states as characterized by Γ_r , for $Pr = 10$, $\Gamma = 8$, and no-slip BCs. The effective scaling exponents β and γ in the scaling relations $Nu \sim Ra^\beta$ and $Re \sim Ra^\gamma$ are shown next to the curves in the respective color of the state and curve.

$Ra = 10^8$ and $2/3 \leq \Gamma_r \leq 1$ for $Ra = 10^{10}$, see figure 4.4(a). For $Ra = 10^9$, we find $2/3 \leq \Gamma_r \leq 1$ for all Pr analysed in this chapter, see figure 4.4(b).

We now set out to mathematically understand the range of Γ_r the system can take for given control parameters. First, we recall that the roll size in 3D flows [186] is restricted by the elliptical instability [186–189]. Similarly, the strain–vorticity balance in 2D flows is determined by the elliptical shape of the rolls. Thus we assume that the essence of the flow is a set of elliptical rolls, each of which can be described by a stream function $\Psi(x, z) = (\xi + \eta)\frac{z^2}{2} + (\xi - \eta)\frac{x^2}{2}$ with $\xi \geq \eta \geq 0$.

The aspect ratio of the rolls, $\bar{\Gamma}_r$, is directly related to the strain η and (half of) the vorticity ξ through the relation $\bar{\Gamma}_r = \sqrt{(\xi + \eta)/(\xi - \eta)}$, corresponding to $\eta/\xi = (\bar{\Gamma}_r^2 - 1)/(\bar{\Gamma}_r^2 + 1)$. Averaging u^2 and w^2 over the area $[-\bar{\Gamma}_r H/2, \bar{\Gamma}_r H/2] \times [-H/2, H/2]$, where $u(x, z) = \partial\Psi/\partial z$ and $w(x, z) = -\partial\Psi/\partial x$, we obtain $Re_x = \bar{\Gamma}_r Re_z$, which is in agreement with the simulations, see supplementary material. The Reynolds number of the roll, Re_r , satisfies

$$Re^2 \geq Re_r^2 = Re_x^2 + Re_z^2 = \xi(\xi + \eta)H^4/(6\nu^2). \quad (4.1)$$

To form a roll, the strain must be bounded by the vorticity, $\eta \leq \xi$, which gives $Re^2 \geq \eta^2 H^4/(3\nu^2)$. We now make use of the exact global balance for the total enstrophy Ω^2 and the mean kinetic energy dissipation rate ϵ_u [190], namely, $4\nu\xi^2 = \nu\Omega^2 = \epsilon_u = \nu^3 H^{-4}(Nu - 1)RaPr^{-2}$. With the definition

$$B \equiv Re^2 Pr^2 Ra^{-1}(Nu - 1)^{-1} \quad (4.2)$$

(which can be seen as non-dimensionalized ratio between the kinetic energy and the

Figure 4.4: Phase diagram in the (a, c) $Ra - \Gamma_r$ and (b, d) $Pr - \Gamma_r$ parameter space for (a, b) no-slip BCs and (c, d) free-slip BCs: (a) $Pr = 10$, $\Gamma = 8$; (b) $Ra = 10^9$, $\Gamma = 8$; (c) $Pr = 10$, $\Gamma = 16$; (d) $Ra = 10^8$, $\Gamma = 16$. Black circles denote that the corresponding roll state is stable, whereas red crosses mean that it is not stable. The theoretical estimates for the transitions between the regimes are shown as solid lines.

Figure 4.5: $B = \frac{Re^2 Pr^2}{Ra(Nu-1)}$, as functions of (a, c) Ra and (b, d) Pr , for (a, b) no-slip BCs and (c, d) free-slip BCs. The solid lines are fits to the data (see supplementary material). Note that the values of B in the no-slip case (a, b) are much smaller than those in the free-slip case (c, d).

energy dissipation rate), from the relation between η/ξ and Γ_r and the inequalities (4.1) and $\eta \leq \xi$ we obtain $B \geq (\Gamma_r^2 - 1)^2 / (\Gamma_r^2 + 1)^2 / 12$, which gives an estimate of the maximal size of the rolls as

$$\Gamma_r \leq \sqrt{1 + 2\sqrt{3B}} / \sqrt{1 - 2\sqrt{3B}}. \quad (4.3)$$

Figs. 4.5(a, b) show the Ra - and Pr -dependences of B , obtained numerically for the no-slip BCs, as well as their data fits (see supplementary material). As seen from the phase diagrams in Figs. 4.4(a, b), using these fits in the upper bound (4.3) gives quite reasonable estimates for the maximal mean roll size of statistically stable turbulent states, see upper lines in Figs. 4.4(a, b).

We now come to the lower bound of the window of allowed Γ_r . First note that Γ_r cannot be infinitesimally small, because, in order to form the rolls, the following should hold: $\Gamma_r \geq \sqrt{1 - 2\sqrt{3B}} / \sqrt{1 + 2\sqrt{3B}}$. One can obtain a more accurate estimate when considering a rectangular region that frames a particular roll. Under the assumption that the velocity components achieve their maximum at the boundaries of this rectangle or vanish there, there exists a certain constant $c > 2$ such that $c^2 \bar{\mathbf{u}}^2 \leq \Gamma_r^2 H^2 (\nabla \mathbf{u})^2$. We call this relation the generalized Friedrichs inequality and derive it in the supplementary material. This inequality gives an estimate for the lower bound of Γ_r ,

$$\Gamma_r \geq c\sqrt{B}, \quad (4.4)$$

which is plotted as lower lines in figure 4.4(a, b) for $c = 9$. As can be seen, the theoretical slopes reflect the general tendency of the numerical results. Note however that at this point we cannot calculate the absolute value of the transition, i.e., the value of c . We also remark that in the case when the container is too slender to keep 2 rolls (with the size, according to (4.4)), i.e., when $\Gamma < 2c\sqrt{B}$, the flow can take the form of a zonal flow only. Thus, in the no-slip case, the zonal flow configuration [81,191] is possible, but only in small-aspect ratio containers.

Our approach also leads to reasonable estimates for the window of allowed statistically stable states in the *free-slip case*, which was numerically analyzed in Ref. [192]: Taking the dependences $Nu(Ra, Pr)$ and $Re(Ra, Pr)$ for the various roll states from that work, with the definition (4.2) we obtain $B(Ra, Pr)$, see Figs. 4.5(c, d). In this case, the values of B are significantly larger, allowing much larger rolls than in the no-slip case. Furthermore, when the inequality (4.1) holds for any η , also when the strain is as strong as the vorticity, $\eta = \xi$, the rolls can be of any size $1 \leq \Gamma_r \leq \Gamma/2$ and the zonal flows are possible as well. This situation happens when $B \geq 1/12$. Thus, very large- Γ_r states are possible, while particular state realizations depend on the initial conditions. Indeed, this is consistent with the numerical findings in Ref. [192], which are shown in Figs. 4.4(c, d). Note in particular that in Ref. [192] an as large stable state as $\Gamma_r = 64$ (for $\Gamma = 128$, $Ra = 10^8$, $Pr = 10$) was found.

What about the lower bound for allowed Γ_r in the free-slip case? In contrast to the upper bound, it always exists and just as in the no-slip case, by arguments again based on the generalized Friedrich inequality (see supplementary material), we can

find it. In figure 4.4(c, d) the result for the smallest Γ_r is plotted. It is based on the estimate (4.4) and uses the fits from figure 4.5(c, d) for the smallest values of B with $c = 7$. Again, the theoretical slopes in the $\Gamma_r - Ra$ and $\Gamma_r - Pr$ phase diagrams reflect the general trend of the numerical results.

4.4 Conclusions

In conclusion, we have numerically shown the coexistence of multiple statistically stable states in turbulent RB convection with no-slip BCs, with different mean aspect ratios of their turbulent rolls and different transport properties, even scaling-wise. We then theoretically illuminated what principles determine the allowed window of the mean size of the turbulent convection rolls (and thus their absolute number), occurring from the ellipticity of the convection rolls. These criteria also work for the free-slip case.

Even though a 2D model may seem somehow artificial, there are various cases in which the flow dynamics is mostly 2D, e.g. because of geometrical confinement, stratification or background rotation. Therefore our model in itself is relevant, but the main ideas of our approach can also be generalized to other wall-bounded turbulent flows, such as rotating Rayleigh–Bénard flow, Taylor–Couette flow, Couette flow with span-wise rotation, double diffusive convection, etc., and also to geophysical flows such as those mentioned in the introduction. They may also give guidance for turbulence flow control, in order to predict which turbulent states are feasible to be realizable.

4.5 Appendix

4.5.1 Derivation of the Generalized Friedrichs Inequality

In the chapter we have used the following theorem: Assume that the components u and w of the velocity $\mathbf{u} = (u, w)$ attain their maximum or vanish at the boundaries of a rectangle $\Omega = [-\Gamma_r H/2, \Gamma_r H/2] \times [-H/2, H/2]$ that frames a convection roll and that u and w equal zero if averaged over Ω . Then there exists a certain constant $c > 2$, such that

$$c^2 \int_{\Omega} \mathbf{u}^2 d\Omega \leq \Gamma_r^2 H^2 \int_{\Omega} (\nabla \mathbf{u})^2 d\Omega. \quad (4.5)$$

We call this relation the generalized Friedrichs inequality. Here we will proof this theorem.

To do this, we will use three auxiliary relations:

1. (Friedrichs inequality in 1D) For a smooth function v in an interval $[a, b]$, such that $v(a) = v(b) = 0$ it holds:

$$\int_a^b v^2 dx \leq \lambda^{-1} \int_a^b (v')^2 dx, \quad (4.6)$$

where λ is the lowest eigenvalue of the Laplace operator in $[a, b]$, which is

$$\lambda = \min_{m \in \mathbb{N}} \frac{\pi^2 m^2}{(b-a)^2} = \frac{\pi^2}{(b-a)^2}.$$

2. For a smooth function v in an interval $[a, b]$, which achieves maximum of its absolute value at the boundary, $\max_{x \in [a, b]} |v| = |v(a)|$, and which equals zero in mean, $\int_a^b v \, dx = 0$, it holds the following:

$$\int_a^b v^2 \, dx \leq \frac{(b-a)^2}{4} \int_a^b (v')^2 \, dx. \quad (4.7)$$

This can be derived from the Cauchy–Schwarz inequality. Indeed,

$$(v(x) - v(a))^2 = \left(\int_a^x v' \, dx \right)^2 \leq (x-a) \int_a^x (v')^2 \, dx.$$

Integrating and using the above assumptions, one obtains:

$$2 \int_a^b v^2 \, dx \leq \int_a^b (v(x) - v(a))^2 \, dx \leq \frac{(b-a)^2}{2} \int_a^b (v')^2 \, dx.$$

3. (Friedrichs inequality in 2D) For a smooth function v in Ω that vanishes at the boundary $\partial\Omega$, it holds:

$$\int_{\Omega} v^2 \, d\Omega \leq \Lambda^{-1} \int_{\Omega} (\nabla v)^2 \, d\Omega, \quad (4.8)$$

with

$$\Lambda = \min_{m, n \in \mathbb{N}} \left(\frac{\pi^2 m^2}{(\Gamma_r H)^2} + \frac{\pi^2 n^2}{H^2} \right) = \frac{\pi^2 (\Gamma_r^2 + 1)}{\Gamma_r^2 H^2}$$

being the lowest eigenvalue of the Laplace operator in Ω .

We start our consideration with the vertical component of the velocity, w . For both, the no-slip and free-slip BCs, it holds:

$$w|_{z=-\frac{H}{2}} = w|_{z=\frac{H}{2}} = 0.$$

We can further assume that for any z , $z \in [-H/2, H/2]$, it holds

$$\max_{x \in [-\frac{\Gamma_r H}{2}, \frac{\Gamma_r H}{2}]} |w(x, z)| = |w(-\Gamma_r H/2, z)|,$$

i.e. the strongest vertical fluid motion occurs at the vertical boundary of the frame. Finally, for any z , $z \in [-H/2, H/2]$, it holds

$$\int_{-\frac{\Gamma_r H}{2}}^{\frac{\Gamma_r H}{2}} w(x, z) \, dx = 0.$$

Using (4.7), we then obtain

$$\int_{\Omega} w^2 d\Omega = \int_{-\frac{H}{2}}^{\frac{H}{2}} dz \int_{-\frac{\Gamma_r H}{2}}^{\frac{\Gamma_r H}{2}} w^2 dx \leq \frac{\Gamma_r^2 H^2}{4} \int_{\Omega} \left(\frac{\partial w}{\partial x} \right)^2 d\Omega. \quad (4.9)$$

On the other hand, from (4.6) we have

$$\int_{\Omega} w^2 d\Omega = \int_{-\frac{\Gamma_r H}{2}}^{\frac{\Gamma_r H}{2}} dx \int_{-\frac{H}{2}}^{\frac{H}{2}} w^2 dz \leq \frac{H^2}{\pi^2} \int_{\Omega} \left(\frac{\partial w}{\partial z} \right)^2 d\Omega. \quad (4.10)$$

Multiplying relation (4.9) by $\frac{4}{4 + \Gamma_r^2 \pi^2}$ and relation (4.10) by $\frac{\Gamma_r^2 \pi^2}{4 + \Gamma_r^2 \pi^2}$ and then summing them up, we obtain

$$\int_{\Omega} w^2 d\Omega \leq \frac{\Gamma_r^2 H^2}{4 + \Gamma_r^2 \pi^2} \int_{\Omega} (\nabla w)^2 d\Omega. \quad (4.11)$$

Although for the vertical velocity in the no-slip and free-slip cases the same relation (4.11) holds, to derive the corresponding inequalities for the horizontal velocity u , we need to consider these cases *separately*. In the no-slip case it holds

$$u|_{z=-\frac{H}{2}} = u|_{z=\frac{H}{2}} = 0$$

due to the boundary conditions. We also can take

$$u|_{x=-\frac{\Gamma_r H}{2}} = u|_{x=\frac{\Gamma_r H}{2}} = 0, \quad (4.12)$$

assuming that the flow goes strictly up or down at $x = \pm \frac{\Gamma_r H}{2}$. With these assumptions the 2D Friedrichs inequality (4.8) holds, i.e.,

$$\int_{\Omega} u^2 d\Omega \leq \frac{\Gamma_r^2 H^2}{\pi^2 (\Gamma_r^2 + 1)} \int_{\Omega} (\nabla u)^2 d\Omega. \quad (4.13)$$

Since the prefactor in (4.11) is larger than that in (4.13), from the relations (4.11) and (4.13) we obtain

$$\int_{\Omega} \mathbf{u}^2 d\Omega \leq \frac{\Gamma_r^2 H^2}{4 + \Gamma_r^2 \pi^2} \int_{\Omega} (\nabla \mathbf{u})^2 d\Omega < \frac{\Gamma_r^2 H^2}{4} \int_{\Omega} (\nabla \mathbf{u})^2 d\Omega,$$

which means that (4.5) is fulfilled in the no-slip case, for c being larger than 2, for any Γ_r .

Finally we consider the horizontal velocity in the free-slip case. Similarly to the no-slip case, relation (4.12) holds. We can further assume that for any x , $x \in [-\Gamma_r H/2, \Gamma_r H/2]$, it holds

$$\max_{z \in [-\frac{H}{2}, \frac{H}{2}]} |u(x, z)| = |u(x, -H/2)|,$$

i.e. the strongest horizontal fluid motion occurs at the horizontal boundaries of the frame, and that for any x , $x \in [-\Gamma_r H/2, \Gamma_r H/2]$, it holds

$$\int_{-\frac{H}{2}}^{\frac{H}{2}} u(x, z) dz = 0.$$

Using (4.7), we then obtain

$$\int_{\Omega} u^2 d\Omega = \int_{-\frac{\Gamma_r H}{2}}^{\frac{\Gamma_r H}{2}} dx \int_{-\frac{H}{2}}^{\frac{H}{2}} u^2 dz \leq \frac{H^2}{4} \int_{\Omega} \left(\frac{\partial u}{\partial z} \right)^2 d\Omega. \quad (4.14)$$

On the other hand, from (4.6) we have

$$\int_{\Omega} u^2 d\Omega = \int_{-\frac{H}{2}}^{\frac{H}{2}} dz \int_{-\frac{\Gamma_r H}{2}}^{\frac{\Gamma_r H}{2}} u^2 dx \leq \frac{\Gamma_r^2 H^2}{\pi^2} \int_{\Omega} \left(\frac{\partial u}{\partial x} \right)^2 d\Omega. \quad (4.15)$$

Multiplying relation (4.14) by $\frac{4\Gamma_r}{\pi^2 + 4\Gamma_r^2}$ and relation (4.15) by $\frac{\pi^2}{\pi^2 + 4\Gamma_r^2}$ and then summing them up, we obtain

$$\int_{\Omega} u^2 d\Omega \leq \frac{\Gamma_r^2 H^2}{\pi^2 + 4\Gamma_r^2} \int_{\Omega} (\nabla u)^2 d\Omega. \quad (4.16)$$

When $\Gamma_r < 1$, the prefactor in (4.11) is larger than that in (4.16); when $\Gamma_r > 1$, the situation is opposite. Thus, from (4.11) and (4.16) we obtain

$$\int_{\Omega} \mathbf{u}^2 d\Omega \leq \max \left\{ \frac{\Gamma_r^2 H^2}{4 + \Gamma_r^2 \pi^2}; \frac{\Gamma_r^2 H^2}{\pi^2 + 4\Gamma_r^2} \right\} \int_{\Omega} (\nabla \mathbf{u})^2 d\Omega < \frac{\Gamma_r^2 H^2}{4} \int_{\Omega} (\nabla \mathbf{u})^2 d\Omega,$$

which means that (4.5) is fulfilled in the free-slip case as well, for c being larger than 2, for any Γ_r .

4.5.2 Fits of functions B in figure 5

In figure 4.5 of the chapter we presented how B (see equation (2) of the paper) depends on Ra and Pr . The numerical results were fitted with quadratic curves in the double-logarithmic representations of figures 5a, b, c, and d. More specifically, in figure 5(a) the data are fitted for $\Gamma_r = 4/3$ and 1; in figure 5(b) for $\Gamma_r = 4/5$ and 1; in figure 5(c) for $\Gamma_r = 8/5$; in figure 5(d) for the smallest values of B at each Pr . In the free-slip case (figure 5(c,d)) only the smallest values of B are considered for the fits, since they are used for the lower bounds only.

The quadratic fits in double-logarithmic representation read

$$B = A_1 Ra^{a_1 + b_1 \log_{10} Ra},$$

$$B = A_2 Pr^{a_2 + b_2 \log_{10} Pr}.$$

We obtain the following fitting functions:

- (a) $B = 0.75 Ra^{-0.38 + 1.61 \times 10^{-2} \log_{10}(Ra)}$;
- (b) $B = 4.65 \times 10^{-3} Pr^{2.22 \times 10^{-2} + 5.08 \times 10^{-2} \log_{10}(Pr)}$;
- (c) $B = 5.46 \times 10^{-2} Ra^{-1.69 \times 10^{-2} + 1.65 \times 10^{-3} \log_{10}(Ra)}$;
- (d) $B = 7.22 \times 10^{-2} Pr^{-0.22 + 5.84 \times 10^{-2} \log_{10}(Pr)}$.

As seen from the solid curves in figures 5a, b, c, d, they very well describes the data.

4.5.3 List of simulation details

Table 4.1: Simulation details for cases with fixed $\Gamma = 8$ and no-slip boundary conditions. The first seven columns from left to right indicate the control and input parameters, namely the Rayleigh number Ra , the Prandtl number Pr , the aspect ratio Γ , the grid resolutions, the number of initial convection rolls $n^{(i)}$. Furthermore, “IC type” means the type of initial convection roll states, where “F” means initial roll states generated by Fourier basis: $u(x, z) = \sin(n^{(i)} \pi x / \Gamma) \cos(\pi z)$, $w(x, z) = -\cos(n^{(i)} \pi x / \Gamma) \sin(\pi z)$; $Ra = \dots$ means initial conditions with fully-developed roll states for the corresponding Ra . – The further columns from left to right indicate the output parameters, namely the number of final convection rolls n , the size of the final stable convection rolls ($\Gamma_r = \Gamma / n$), the Nusselt number Nu , the global Reynolds number Re , the horizontal Reynolds number Re_x , the vertical Reynolds number Re_z , and the ratio $B = \frac{Re^2 Pr^2}{Ra(Nu-1)}$ (see equation (2) of the paper). The last column shows the time used to average Nu and Re (t_{avg}). Initial conditions denoted in blue mean that the corresponding roll states are not stable and will undergo transitions to other states. These cases are stopped after the transition happens and the response parameters are not sampled.

Ra	Pr	Γ	$N_x \times N_z$	$n^{(i)}$	IC type	n	Γ_r	Nu	Re	Re_x	Re_z	B	t_{avg}
10^9	10	8	4096×512	2	F	10	0.80						
10^{10}	10	8	8192×1024	2	F	10	0.80						
10^7	10	8	2048×256	4	F	6	1.33						
10^8	10	8	2048×256	4	F	8	1.00						
10^9	10	8	4096×512	4	F	10	0.80						
10^{10}	10	8	8192×1024	4	F	10	0.80						
10^7	10	8	1024 × 128	6	F	6	1.33	14.33	113.48	89.30	70.03	9.66e-3	59103
2×10^7	10	8	2048 × 256	6	$Ra = 10^8$	6	1.33	17.16	166.57	131.76	101.90	8.58e-3	8000
3×10^7	10	8	2048 × 256	6	$Ra = 10^8$	6	1.33	19.07	208.91	165.70	127.22	8.06e-3	9953
4×10^7	10	8	2048 × 256	6	$Ra = 10^8$	6	1.33	20.56	245.86	195.45	149.14	3.86e-3	8000

Ra	Pr	Γ	$N_x \times N_z$	$n^{(i)}$	IC type	n	Γ_r	Nu	Re	Re_x	Re_z	B	t_{avg}
5×10^7	10	8	2048×256	6	$Ra = 10^8$	6	1.33	21.80	279.06	222.20	168.82	7.72e-3	8000
7×10^7	10	8	2048×256	6	$Ra = 10^8$	6	1.33	23.82	339.49	270.96	204.52	7.22e-3	8000
10^8	10	8	2048×256	6	F	6	1.33	26.17	417.09	333.55	250.43	6.92e-3	8703
3×10^8	10	8	4096×512	6	F	10	0.80						
10^9	10	8	4096×512	6	F	10	0.80						
3×10^9	10	88192	$\times 1024$	6	F	12	0.67						
10^{10}	10	88192	$\times 1024$	6	F	10r	0.80						
10^7	10	8	2048×256	8	F	6	1.33						
2×10^7	10	8	2048×256	8	$Ra = 10^8$	6	1.33						
3×10^7	10	8	2048×256	8	$Ra = 10^8$	8	1.00	19.89	213.66	152.70	149.44	8.06e-3	49875
5×10^7	10	8	2048×256	8	$Ra = 10^8$	8	1.00	22.83	285.02	204.03	199.01	7.44e-3	4000
10^8	10	8	2048×256	8	F	8	1.00	27.69	423.00	302.99	295.17	6.70e-3	8577
2×10^8	10	8	4096×512	8	$Ra = 10^9$	8	1.00	33.22	633.88	454.59	441.76	6.24e-3	2000
3×10^8	10	8	4096×512	8	$Ra = 10^9$	8	1.00	36.96	810.14	581.33	564.25	6.08e-3	4511
5×10^8	10	8	4096×512	8	$Ra = 10^9$	8	1.00	42.29	1099.44	789.70	764.95	5.86e-3	4000
10^9	10	8	4096×512	8	F	8	1.00	50.77	1645.92	1182.62	1144.76	5.44e-3	18000
2×10^9	10	88192	$\times 1024$	8	$Ra = 10^{10}$	8	1.00	61.23	2453.21	1762.69	1706.21	5.00e-3	1000
3×10^9	10	88192	$\times 1024$	8	$Ra = 10^{10}$	8	1.00	68.50	3116.16	2239.67	2166.65	4.80e-3	1000
5×10^9	10	88192	$\times 1024$	8	$Ra = 10^{10}$	8	1.00	79.40	4226.16	3036.88	2939.00	4.56e-3	1000
10^{10}	10	88192	$\times 1024$	8	F	8	1.00	96.80	6570.72	4721.01	4570.18	4.50e-3	3100000
10^7	10	8	2048×256	10	F	6	1.33						
3×10^7	10	8	2048×256	10	$Ra = 10^9$	8	1.00						
10^8	10	8	2048×256	10	F	8	1.00						
2×10^8	10	8	4096×512	10	$Ra = 10^9$	8	1.00						
3×10^8	10	8	4096×512	10	$Ra = 10^9$	10	0.80	38.52	848.11	547.33	647.86	6.40e-3	9207
5×10^8	10	8	4096×512	10	$Ra = 10^9$	10	0.80	44.29	1141.20	733.59	874.16	6.02e-3	4000
10^9	10	8	4096×512	10	F	10	0.80	53.05	1742.10	1120.35	1334.06	5.84e-3	18000
2×10^9	10	88192	$\times 1024$	10	$Ra = 10^{10}$	10	0.80	63.53	2634.48	1695.76	2016.13	5.54e-3	1000
3×10^9	10	88192	$\times 1024$	10	$Ra = 10^{10}$	10	0.80	71.06	3271.96	2099.30	2509.72	5.10e-3	2000
5×10^9	10	88192	$\times 1024$	10	$Ra = 10^{10}$	10	0.80	82.72	4417.86	2825.70	3396.01	4.78e-3	1000
10^{10}	10	88192	$\times 1024$	10	F	10	0.80	101.53	6680.37	4275.11	5133.29	4.44e-3	10000
10^7	10	8	2048×256	12	F	6	1.33						
3×10^7	10	8	2048×256	12	F	8	1.00						
10^8	10	8	2048×256	12	F	8	1.00						
3×10^8	10	8	4096×512	12	F	10	0.80						
10^9	10	8	4096×512	12	F	10	0.80						
2×10^9	10	88192	$\times 1024$	12	$Ra = 10^{10}$	12	0.67	66.09	2670.59	1537.40	2183.69	5.48e-3	4500
3×10^9	10	88192	$\times 1024$	12	$Ra = 10^{10}$	12	0.67	73.64	3378.75	1939.55	2766.57	5.24e-3	2000
4×10^9	10	88192	$\times 1024$	12	$Ra = 10^{10}$	12	0.67	79.89	4005.32	2295.89	3281.97	5.08e-3	1000
5×10^9	10	88192	$\times 1024$	12	$Ra = 10^{10}$	12	0.67	85.27	4578.96	2623.13	3753.12	4.98e-3	1000
7×10^9	10	88192	$\times 1024$	12	$Ra = 10^{10}$	12	0.67	94.27	5598.02	3204.58	4590.03	4.80e-3	1000
10^{10}	10	88192	$\times 1024$	12	F	12	0.67	105.17	6950.16	3970.16	5704.60	4.64e-3	10000
10^9	1	8	4096×512	6	F	10	0.80						
10^9	1	8	4096×512	8	F	8	1.00	48.26	14629.64	10453.78	10232.72	4.52e-3	15000
10^9	1	8	4096×512	10	F	10	0.80	49.53	15463.32	10226.11	11599.17	4.92e-3	15000
10^9	1	8	4096×512	12	F	10	0.80						
10^9	3	8	4096×512	6	F	10	0.80						

Ra	Pr	Γ	$N_x \times N_z$	$n^{(i)}$	IC type	n	Γ_r	Nu	Re	Re_x	Re_z	B	t_{avg}
10^9	3	8	4096×512	8	F	8	1.00	49.32	5018.66	3606.08	3490.41	4.68e-3	15000
10^9	3	8	4096×512	10	F	10	0.80	51.95	5212.07	3334.76	4005.62	4.80e-3	15000
10^9	3	8	4096×512	12	F	12	0.67	54.08	5450.34	3118.03	4470.35	5.04e-3	15000
10^9	3	8	4096×512	14	F	10	0.80						
10^9	30	8	4096×512	6	F	10	0.80						
10^9	30	8	4096×512	8	F	8	1.00	51.67	597.46	428.68	416.17	6.34e-3	10000
10^9	30	8	4096×512	10	F	10	0.80	53.05	620.07	400.61	473.28	6.66e-3	10000
10^9	30	8	4096×512	12	F	10	0.80						
10^9	50	8	4096×512	6	F	8	1.00						
10^9	50	8	4096×512	8	F	8	1.00	51.33	377.11	270.22	263.04	7.08e-3	1000
10^9	50	8	4096×512	10	F	8	1.00						
10^9	50	8	4096×512	12	F	8	1.00						
10^9	1008		4096×512	6	F	8	1.00						
10^9	1008		4096×512	8	F	8	1.00	50.46	201.11	144.08	140.31	8.18e-3	1000
10^9	1008		4096×512	10	F	8	1.00						
10^9	1008		4096×512	12	F	8	1.00						

Table 4.2: Simulation details for cases with fixed $Ra = 10^9$, $Pr = 10$, and no-slip boundary conditions for various different aspect ratios Γ . The caption is the same as in table 8.1, except that all the simulations here use the Fourier basis to generate the initial roll states. For small $\Gamma \leq 1$, the flow can become “zonal”, which we then remark in the table, rather than the final number n of rolls.

Ra	Pr	Γ	$N_x \times N_z$	$n^{(i)}$	n	Γ_r	Nu	Re	Re_x	Re_z	B	t_{avg}
10^9	10	0.5	1024×512	2	zonal							
10^9	10	1	1024×512	2	zonal							
10^9	10	1	1024×512	2r	zonal							
10^9	10	1.33	1024×512	2	2	0.67	54.01	1896.67	1120.14	1530.57	6.78e-3	39000
10^9	10	1.5	1024×512	2	2	0.75	53.09	1833.70	1148.86	1429.19	6.46e-3	5500
10^9	10	2.0	1024×512	2	2	1.00	50.62	1712.57	1237.62	1183.72	5.92e-3	5500
10^9	10	2.2	1024×512	2	2	1.10	49.89	1648.28	1237.23	1089.07	5.56e-3	5500
10^9	10	2.5	1024×512	2	2	1.25	48.89	1623.90	1278.46	1001.25	5.50e-3	13000
10^9	10	3	2048×512	2	4	0.75						
10^9	10	2.3	2048×512	4	2	1.15						
10^9	10	2.5	2048×512	4	2	1.50						
10^9	10	2.67	2048×512	4	4	0.67	54.26	1874.32	1104.29	1514.47	6.60e-3	15000
10^9	10	3.0	2048×512	4	4	0.75	53.37	1814.95	1133.62	1417.37	6.30e-3	3500
10^9	10	3.5	2048×512	4	4	0.88	51.92	1738.58	1175.65	1280.82	5.92e-3	3500
10^9	10	4	2048×512	4	4	1.00	50.71	1653.55	1189.61	1148.50	5.50e-3	3500
10^9	10	4.5	2048×512	4	4	1.13	49.73	1625.69	1229.24	1063.85	5.42e-3	8000
10^9	10	5	2048×512	4r	6r	0.83						

Ra	Pr	Γ	$N_x \times N_z$	$n^{(i)}$	n	Γ_r	Nu	Re	Re_x	Re_z	B	t_{avg}
10^9	10	3.5	4096×512	6	4	0.88						
10^9	10	3.7	4096×512	6	4	0.93						
10^9	10	4	4096×512	6	4	1.00						
10^9	10	4.5	4096×512	6	6	0.75	53.35	1808.64	1129.18	1412.84	6.24e-3	10050
10^9	10	5	4096×512	6	6	0.83	52.47	1749.43	1152.64	1316.03	5.94e-3	2500
10^9	10	5.5	4096×512	6	6	0.92	51.53	1690.11	1166.10	1223.38	5.66e-3	2500
10^9	10	6	4096×512	6	6	1.00	50.66	1645.31	1182.59	1143.90	5.46e-3	5305
10^9	10	6.5	4096×512	6	6	1.08	49.94	1629.75	1213.18	1088.25	5.42e-3	10500
10^9	10	7	4096×512	6	10	0.70						
10^9	10	7.5	4096×512	6	10	0.75						
10^9	10	5.5	4096×512	8	6	0.91						
10^9	10	5.33	4096×512	8	6	0.88						
10^9	10	6	4096×512	8	8	0.75	53.58	1776.71	1105.27	1391.07	6.00e-3	12240
10^9	10	7	4096×512	8	8	0.88	52.08	1696.64	1143.17	1253.69	5.64e-3	2500
10^9	10	8	4096×512	8	8	1.00	50.78	1649.46	1185.35	1147.02	5.46e-3	2500
10^9	10	8.5	4096×512	8	8	1.06	50.17	1638.99	1210.34	1105.16	5.46e-3	2500
10^9	10	9.0	4096×512	8	8	1.13	49.72	1625.20	1228.71	1063.75	5.42e-3	13101
10^9	10	9.5	4096×512	8	10	0.95						
10^9	10	5.5	4096×512	10	8	0.69						
10^9	10	6	4096×512	10	8	0.75						
10^9	10	6.67	4096×512	10	8	0.83						
10^9	10	7	6144×512	10	10	0.70	54.42	1773.25	1058.25	1422.86	5.88e-3	9304
10^9	10	8	6144×512	10	10	0.80	53.11	1709.79	1096.41	1311.97	5.62e-3	1000
10^9	10	9	6144×512	10	10	0.90	51.80	1684.96	1150.86	1230.70	5.58e-3	1000
10^9	10	10	6144×512	10	10	1.00	50.71	1652.47	1187.86	1148.76	5.50e-3	4671
10^9	10	10.5	6144×512	10	10	1.05	50.26	1636.54	1202.12	1110.49	5.44e-3	10000
10^9	10	11	6144×512	10	14	0.79						
10^9	10	11.5	6144×512	10	14	0.82						
10^9	10	6.5	6144×512	12	10	0.65						
10^9	10	7	6144×512	12	10	0.70						
10^9	10	7.5	6144×512	12	8	0.94						
10^9	10	8	6144×512	12	10	0.80						
10^9	10	9	6144×512	12	12	0.75	53.75	1750.22	1084.53	1373.71	5.80e-3	9400
10^9	10	10	6144×512	12	12	0.83	52.52	1728.21	1136.19	1302.23	5.80e-3	1000
10^9	10	11	6144×512	12	12	0.92	51.67	1668.81	1149.55	1209.73	5.50e-3	1000
10^9	10	12	6144×512	12	12	1.00	50.82	1651.71	1186.92	1148.65	5.48e-3	1000
10^9	10	12.5	6144×512	12	12	1.04	50.32	1634.22	1196.30	1113.35	5.42e-3	1000
10^9	10	13	6144×512	12	12	1.08	50.02	1630.85	1213.91	1089.08	5.42e-3	10000
10^9	10	14	6144×512	14r	14r	1.00						
10^9	10	9	6144×512	14	12	0.75						
10^9	10	9.33	6144×512	14	12	0.78						
10^9	10	10	6144×512	14	12	0.83						
10^9	10	11	6144×512	14	14	0.79	53.28	1739.52	1106.78	1342.01	5.78e-3	10000
10^9	10	12	6144×512	14	14	0.86	52.41	1700.93	1133.01	1268.64	5.64e-3	1000
10^9	10	13	6144×512	14	14	0.93	51.63	1668.42	1157.07	1202.00	5.50e-3	1000

Ra	Pr	Γ	$N_x \times N_z$	$n^{(i)}$	n	Γ_r	Nu	Re	Re_x	Re_z	B	t_{avg}
10^9	10	14	6144×512	14	14	1.00	50.73	1647.59	1183.91	1145.82	5.46e-3	10000
10^9	10	15	6144×512	14	18	0.83						
10^9	10	16	6144×512	14	18	0.89						
10^9	10	10	8192×512	16	14	0.71						
10^9	10	10.5	8192×512	16	14	0.75						
10^9	10	10.72	8192×512	16	14	0.77						
10^9	10	11	8192×512	16	14	0.79						
10^9	10	12	8192×512	16	16	0.75	53.75	1750.56	1085.19	1373.61	5.80e-3	10000
10^9	10	13	8192×512	16	16	0.81	52.84	1729.48	1121.18	1316.84	5.76e-3	1000
10^9	10	14	8192×512	16	16	0.88	52.10	1693.04	1140.22	1251.51	5.60e-3	1000
10^9	10	15	8192×512	16	16	0.94	51.32	1668.87	1163.05	1196.85	5.54e-3	1000
10^9	10	16	8192×512	16	16	1.00	50.77	1643.40	1180.71	1143.11	5.44e-3	1000
10^9	10	17	8192×512	16	16	1.06	50.22	1631.48	1204.23	1100.71	5.40e-3	1000
10^9	10	18	8192×512	16	16	1.13	49.71	1623.65	1227.41	1062.88	5.42e-3	11334
10^9	10	19	8192×512	16	20	0.95						
10^9	10	20	8192×512	16	20	1.00						
10^9	10	16	8192×512	12	18	0.89						
10^9	10	16	8192×512	14	18	0.89						
10^9	10	16	8192×512	16	16	1.00	50.77	1649.07	1185.03	1146.79	5.46e-3	7000
10^9	10	16	8192×512	18	18	0.89	51.99	1685.56	1143.74	1238.13	5.58e-3	7000
10^9	10	16	8192×512	20	20	0.80	53.13	1730.93	1111.65	1326.78	5.74e-3	7000
10^9	10	16	8192×512	22	22	0.73	54.21	1756.90	1069.79	1393.64	5.80e-3	7000
10^9	10	16	8192×512	24	22	0.73						
10^9	10	32	16384×512	30	32	0.94						
10^9	10	32	16384×512	32	32	1.00	50.76	1647.44	1183.77	1145.75	5.46e-3	5000
10^9	10	32	16384×512	34	34	0.94	51.38	1664.81	1162.40	1191.81	5.50e-3	7000
10^9	10	32	16384×512	36	36	0.89	52.01	1685.14	1143.57	1237.73	5.56e-3	5000
10^9	10	32	16384×512	38	38	0.84	52.56	1711.66	1130.00	1285.64	5.68e-3	5000
10^9	10	32	16384×512	40	40	0.80	53.15	1727.12	1109.02	1324.01	5.72e-3	4000
10^9	10	32	16384×512	42	42	0.76	53.69	1740.48	1087.31	1359.05	5.76e-3	4000
10^9	10	32	16384×512	44	42	0.76						
10^9	10	32	16384×512	44	42	0.76						

Chapter 5

From zonal flow to convection rolls in Rayleigh-Bénard convection with free-slip plates^o

Rayleigh-Bénard (RB) convection with free-slip plates and horizontally periodic boundary conditions is investigated using direct numerical simulations. Two configurations are considered, one is two-dimension (2D) RB convection and the other one three-dimension (3D) RB convection with a rotating axis parallel to the plate, which for strong rotation mimics 2D RB convection. For the 2D simulations, we explore the parameter range of Rayleigh numbers Ra from 10^7 to 10^9 and Prandtl numbers Pr from 1 to 100. The effect of the width-to-height aspect ratio Γ is investigated for $1 \leq \Gamma \leq 128$. We show that zonal flow, which was observed, for example, by Ref. [81] for $\Gamma = 2$, is only stable when Γ is smaller than a critical value, which depends on Ra and Pr . The regime in which only zonal flow can exist is called the first regime in this study. With increasing Γ , we find a second regime in which both zonal flow and different convection roll states can be statistically stable. For even larger Γ , in a third regime, only convection roll states are statistically stable and zonal flow is not sustained. How many convection rolls form (or in other words, what the mean aspect ratio of an individual roll is), depends on the initial conditions and on Ra and Pr . For instance, for $Ra = 10^8$ and $Pr = 10$, the aspect ratio Γ_r of an individual, statistically stable convection roll can vary in a large range between 16/11 and 64. A convection roll with an as large aspect ratio of $\Gamma_r = 64$ or more generally already with $\Gamma_r \gg 10$ can be seen as “localized” zonal flow, and indeed

^oBased on: **Q. Wang**, K. L. Chong, R. J. A. M. Stevens, R. Verzicco, and D. Lohse, *From zonal flow to convection rolls in Rayleigh-Bénard convection with free-slip plates*. J. Fluid Mech. (In press)

carries over various properties of the global zonal flow. For the 3D simulations, we fix $Ra = 10^7$ and $Pr = 0.71$, and compare the flow for $\Gamma = 8$ and $\Gamma = 16$. We first show that with increasing rotation rate both the flow structures and global quantities like the Nusselt number Nu and the Reynolds number Re increasingly behave like in the 2D case. We then demonstrate that with increasing aspect ratio Γ , zonal flow, which was observed for small $\Gamma = 2\pi$ by Ref. [82], completely disappears for $\Gamma = 16$. For such large Γ only convection roll states are statistically stable. In between, here for medium aspect ratio $\Gamma = 8$, the convection roll state and the zonal flow state are both statistically stable. What state is taken depends on the initial conditions, similarly as we found for the 2D case.

5.1 Introduction

Large scale so-called zonal flows, which display strong horizontal winds, can be observed in many buoyancy-driven flows. Typical examples include zonal flow in the atmosphere of Jupiter [21, 31, 32] and other three Jovian planets [33–36], in the oceans [193–195], and possibly in the Earth’s outer core [196]. In toroidal tokamak devices, zonal flows in the poloidal direction are crucial to confine plasmas magnetically [197].

How to study such flows? In general, Rayleigh-Bénard (RB) convection [53–56], i.e., a fluid in a container heated from below and cooled from above, is the paradigmatic model system for buoyancy-driven flows. The key control parameters are the Rayleigh number $Ra = g\alpha\Delta H^3/\nu\kappa$ and the Prandtl number $Pr = \nu/\kappa$. Here, g is the gravitational acceleration, α the thermal expansion coefficient, H the height of the fluid sample, $\Delta = T_b - T_t$ the temperature difference between the hot bottom and the cold top plate, κ the thermal diffusivity, and ν the kinematic viscosity of the fluid. The third control parameter is the aspect ratio Γ , which is defined as the ratio of the width to the height of the container. The response of the system is characterized by the Nusselt number $Nu = QH/(k\Delta)$ and the Reynolds number $Re = UH/\nu$, which indicate the non-dimensional heat transport and flow strength in the system, respectively. Here Q is the heat flux crossing the system and $U = \sqrt{\langle \mathbf{u} \cdot \mathbf{u} \rangle_{V,t}}$ the characteristic velocity, where $\langle \rangle_{V,t}$ indicates volume and time averaging. Indeed, to study zonal flow, RB convection with free-slip plates and horizontally periodic boundary conditions has commonly served as model system [81, 82, 191, 198]. Here the free-slip at the plates is very crucial to enable the zonal flow; for no-slip boundary conditions, zonal flow is significantly suppressed and it only exists for Γ smaller than roughly 1.2 [191]. In our recent extensive numerical simulations using no-slip boundary conditions, we found that zonal flow appears when Γ is smaller than approximately 4/3 [199]. Note that these values depend on Ra and Pr .

In the 2D version of the RB system with free-slip plates and horizontally periodic boundary conditions, indeed, for small $\Gamma = 2$, zonal flow develops readily since the horizontal periodicity allows for a horizontal mean flow, while the free-slip boundaries apply no shear stresses to slow down the fluid. In addition, the two-dimensionality precludes transverse perturbations that could disrupt the mean flow [81, 191]. Such zonal flow in 2D RB convection has attracted quite some attention because of its relevance to thermal convection in the atmosphere [174, 200, 201]. For free-slip boundary conditions at the plates, $Pr = 1$, $Ra \geq 10^7$, and a small $\Gamma = 2$, Ref. [191] found that a flow topology consisting of two shear layers with a predominant horizontal motion is formed. The flow in the lower half of the domain moves in the opposite direction as that in the top part. Most of the time, the heat transfer of this flow is $Nu \approx 1$, while there are intermittent bursts in which $Nu \gg 1$. Ref. [81] studied the characteristics of such 2D zonal flows in a periodic $\Gamma = 2$ cell for an extended parameter range $10^3 \leq Ra \leq 10^{10}$ and $1 \leq Pr \leq 10$. They found that for $Pr \leq 2$, the zonal flow undergoes strong global oscillations on long timescales. Also intermittent bursts

in the heat transport as in Ref. [191] are observed. For $Pr \geq 3$, the zonal flow is sustained at all times without bursts, and the Nusselt number Nu is always much larger than 1.

To what degree can 2D simulations mimic the dynamics in three-dimensional (3D) flows? Actually many 3D geophysical and astrophysical flows exhibit certain 2D properties when anisotropic effects, such as geometrical confinement, rapid rotation, stratification, or magnetic fields, are imposed. We will show in this study how the 2D flow structures arise with increasing rotation rate for RB convection rotating about a horizontal axis. Such flow will be called span-wise rotating RB convection in this chapter. Note that the 2D flow structures mentioned here are very similar to those observed in 2D RB convection, however, quite different from the 2D Taylor column structures occurring for RB convection rotating about the vertical axis with large angular velocity [202,203]. 2D simulations, which are computationally more accessible than 3D simulations, have also been widely used to study thermal convection with no-slip plates in recent years [109,114,120,121,183,204–210]. In Ref. [181] 2D and 3D simulations are compared in detail, and many similarities are found for $Pr \geq 1$.

In contrast to the 2D configuration, zonal flow has not been reported in horizontally isotropic 3D simulations of RB convection with free-slip plates [211,212]. It seems that in 3D convection zonal flow only appears when an anisotropy is added. For example, Ref. [82] found that a strong zero-wavenumber wind (i.e., zonal flow) can arise in 3D RB convection if the horizontal isotropy is broken by strong enough uniform rotation about a horizontal axis. Indeed, according to the Taylor-Proudman theorem, the flow can become 2D-like when the rotation is sufficiently fast. Recently, Ref. [198] further generalized the situation by varying the tilting angle of the rotation axis with respect to gravity. This configuration mimics the flow at different latitudes in a rotating fluid shell. A large-scale cyclonic vortex tilted along the rotation axis is identified for ϕ between 45° and 90° , where ϕ is the angle between the rotation axis and the horizontal plane. At moderate latitudes the calculations of Ref. [198] suggest the possible coexistence of zonal jets and tilted-vortex solutions.

Even though flows in geophysics, astrophysics, and plasma-physics often occur in large-aspect ratio systems, most of the previous zonal flow studies with free-slip conditions at the plates were performed for horizontally periodic *small*-aspect ratio cells, typically $\Gamma = 2$ (2D) or $\Gamma = 2\pi$ (3D). However, recent studies on *large*-aspect ratio 3D RB convection with no-slip plates revealed the existence of superstructures with horizontal extent of six to seven times the height of the domain [213–218]. These findings motivated us to study zonal flow at much larger Γ (up to 128) than hitherto done, in order to test whether zonal flow will sustain at these much larger Γ , or whether some other large-scale structures evolve, which are not captured in simulations with small Γ .

We will find that for free-slip plates and periodic boundary conditions the aspect ratio indeed has a very strong influence on the flow phenomena in 2D RB convection and in 3D RB convection with span-wise rotation. In particular, we will show that zonal flow is only stable when the aspect ratio of the system is smaller than a

critical value, which depends on Ra and Pr ; it disappears in large-aspect ratio flow configurations.

This chapter is organized as follows. We first describe the simulation details in §5.2. The 2D results are presented and analysed in §5.3, which is divided into three parts. Section 5.3.1 demonstrates the disappearance of zonal flow with increasing the aspect ratio Γ . Section 5.3.2 studies the coexistence of multiple convection roll states. The effective scaling relations for $Nu(Ra, Pr)$ and $Re(Ra, Pr)$ for different convection roll states are discussed in §5.3.3. The 3D RB convection with increasing rotation strength about an axis parallel to the plate (i.e., increasing two-dimensionalization according to Taylor-Proudman theorem) is discussed in §5.4, where we also show the transition from zonal flow to convection roll states with increasing Γ . We summarize our findings in §5.5.

Ra	Pr	Γ	N_z^r	N_z^z	N_{BL}^r	N_{BL}^z
10^7	0.71	16	256	-	12	-
10^7	10	[1,64]	256	256	13	21
3×10^7	10	[1,64]	256	256	11	20
10^8	[1,100]	[1,128]	384	256,384	15	20
3×10^8	10	[1,64]	512	256,384	17	24
10^9	10	2,16	768	768	20	46

Table 5.1: Overview of the run simulations in 2D. The first three columns indicate the Ra , Pr , and Γ range of the simulations. N_z^r and N_z^z indicate the number of grid points in the vertical direction for the simulations with initial conditions of roll states and shear flow, respectively. N_{BL}^r and N_{BL}^z indicate the minimum number of grid points in the thermal boundary layer for convection roll states and zonal flow state, respectively. We note that the number of grid points in the boundary layer is always higher than that given by the recommendation of [110] for the no-slip case, which is about 5 to 9 for this Ra range, to ensure that the boundary layers are resolved. The number of grid points in the horizontal direction is generally equal or larger than $N_x = N_z \times \Gamma$. For $Ra = 10^8$ and 3×10^8 , $N_z^z = 256$ is used only for the large Γ cases (for example, $Ra = 3 \times 10^8$, $\Gamma = 32$ and 64) where very long simulations are performed, in order to test whether the zonal flow state can stably exist.

5.2 Numerical settings

The configurations and the coordinate systems used in this work are shown in figure 5.1. We performed direct numerical simulations using the second-order staggered finite difference code AFiD. Details about the numerical method can be found in Refs. [80, 141, 219]. The governing equations in dimensionless form read

Figure 5.1: Sketch of (a) 2D RB convection and (b) 3D RB convection with span-wise rotation for free-slip plates and horizontally periodical conditions.

$$\nabla \cdot \mathbf{u} = 0, \quad (5.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \sqrt{\frac{Pr}{Ra}} \nabla^2 \mathbf{u} - \frac{1}{Ro} \vec{e}_y \times \mathbf{u} + \theta \vec{e}_z, \quad (5.2)$$

$$\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = \frac{1}{\sqrt{RaPr}} \nabla^2 \theta. \quad (5.3)$$

Here $\vec{e}_y$ and $\vec{e}_z$ are the unit vector in the y and z direction, respectively. $\mathbf{u}$, t , p , θ are velocity, time, pressure and temperature, respectively. The length and velocity are non-dimensionalized using the height of the convection cell H and the free-fall velocity $U = (g\alpha\Delta H)^{1/2}$, respectively. This implies as reference time the free-fall time $t_f = H/U$. For the 3D simulations also the Rossby number $Ro = U/(2\Omega H)$ is used, where Ω is the angular velocity. Non-uniform grids with points clustered near the top and bottom plates are employed.

How to choose the initial conditions to trigger the different flow states? For the zonal flow simulations we used a linear shear-flow velocity profile $u(z) = 2z - 1, w = 0$ in combination with a linear temperature profile $\theta(z) = 1 - z$ as initial conditions. Moreover, random perturbations were added to the initial temperature fields. The perturbation had a value uniformly distributed between -0.01Δ and 0.01Δ . Note that the perturbations were added to each grid point in the volume, while the temperature at the plates still has the constant value. In addition, different convection roll states were generated using a Fourier basis: $u(x, z) = \sin(n^{(i)}\pi x/\Gamma)\cos(\pi z), w(x, z) = -\cos(n^{(i)}\pi x/\Gamma)\sin(\pi z)$ where $n^{(i)}$ indicates the number of initial rolls in the horizontal direction, while the initial temperature is the same as zonal flow simulations. A similar Fourier basis was also used to study heat transport [118] and flow reversals [107, 108, 220, 221]. An overview of the 2D simulations and the used grid resolutions are given in table 5.1. Note that we only used even-number roll states as initial conditions, as odd-number roll states cannot satisfy the horizontally periodic boundary

conditions. Table 5.3 and 5.4 in the Appendix list 2D simulation details for the main cases where Nu and Re are discussed. The 3D simulation details are given in table 5.5, where the 2D simulations for the corresponding parameters are also listed for comparison.

5.3 Two-dimensional simulations

5.3.1 Disappearance of zonal flow with increasing Γ

We first show what will happen to zonal flow with increasing Γ . From Ref. [81], it is known that zonal flow exists for $Ra = 10^8$, $Pr = 10$, $\Gamma = 2$. With increasing Γ , we find that for this Ra and Pr , zonal flow can stably exist for $\Gamma \leq 12$. In figure 5.2(a) we show that zonal flow is statistically stable at least up to 200 000 free-fall time units for $\Gamma = 4$ and at least up to 100 000 free-fall time units for $\Gamma = 12$. Here, we used “statistically stable” to denote that the corresponding chaotic flow state is always sustained in our long-time simulations. Temperature snapshot of the zonal flow for $\Gamma = 12$ in figure 5.2(b) demonstrates that hot plumes drift leftwards, and cold plumes drift rightwards. This produces a strong horizontal shear in which however the vertical heat transport is low.

We now explore even larger aspect ratio domains. Figure 5.2(c) shows the time evolution of Nu for three separate simulations with random perturbations added to the initial temperature field for $Ra = 10^8$ and $Pr = 10$ in a $\Gamma = 64$ cell. For all the three simulations, the zonal flow eventually evolves to a convection roll state. The time at which the transition occurs is very different for each simulation. The reason is that the flow is susceptible to small differences in the initial conditions, which are different for each simulation due to the random perturbations to the initial temperature field. Such sensitivity to the initial conditions is typical for chaotic systems and makes it impossible to predict when the transition happens.

Figure 5.2(d) displays three temperature snapshots at different time instants indicated by the red dashed lines in figure 5.2(c). A complementary movie, showing how the zonal flow undergoes a transition towards a convection roll state, is given in the supplementary material. Initially, the hot plumes travel leftwards and the cold plumes to the right. The transition starts when some local hot plumes are strong enough to deviate upwards and cross the whole fluid layer up to the collision with the upper cold plate. This prevents the further rightward motion of the neighbouring cold plumes, which instead start to move downwards. This process generates a local large-scale circulation, as is observed in the temperature field at $t = 4172$. The circulation grows over time until two stable convection rolls of equal size are formed as seen in the temperature field at $t = 5000$. Figure 5.2(e) shows that we also obtain a two-roll state for $\Gamma = 128$, such that the horizontal extent of the convection roll is 64 times the height of the convection cell, and this state is stable for more than 10 000 free-fall time units.

The cases in figure 5.2 use shear flow as the initial condition. If we use convection

Figure 5.2: (a) Time evolution of Nu for the zonal flow state for $Ra = 10^8$ and $Pr = 10$ with $\Gamma = 4$ (black line) and $\Gamma = 12$ (red line). (b) Temperature snapshot for the zonal flow state for $Ra = 10^8$, $Pr = 10$, $\Gamma = 12$. (c) Time evolution of Nu for $Ra = 10^8$, $Pr = 10$, $\Gamma = 64$. The three curves correspond to three separate simulations with random perturbations added to the initial temperature field. In all the cases, the flow undergoes a transition from zonal flow to convection roll states, for which Nu is larger. (d) Temperature snapshots at different times denoted by the red dashed lines for the simulation indicated by the red curve in panel (c). At $t = 2000$, there is zonal flow, whereas later it features an increasing number of turbulent convection rolls. (e) The final two-roll state for $Ra = 10^8$, $Pr = 10$, $\Gamma = 128$, and the zoom in of the two plume-ejecting regions. For all these simulations the initial velocity had a linear shear flow profile $u(z) = 2z - 1$, $w = 0$, in order to trigger a zonal flow state.

Figure 5.3: Temperature snapshots of different roll states for $Ra = 10^8$ and $Pr = 10$ in a $\Gamma = 12$ periodic cell. (a) Two-roll state. (b) Four-roll state. (c) Six-roll state. (d) Eight-roll state. The different roll states are from initial conditions with different number of initial rolls.

roll states as initial conditions, we can also obtain statistically stable convection roll states as indicated in figure 5.3 for different roll states for $Ra = 10^8$, $Pr = 10$, $\Gamma = 12$. So for this case, both zonal flow and different convection roll states can stably exist, depending on the initial conditions.

We now explore the phase diagram for the different flow states in the parameter space spanned by Ra , Pr , and Γ , see figure 5.4 for the simulated cases. We find that in small-aspect ratio cells, only zonal flow is stable, while in large-aspect ratio cells, only convection roll states are stable. For intermediate aspect ratios, we find a regime in which both zonal flow and convection roll states are stable, depending on the initial conditions mentioned in §5.2. We call this regime the bi-stable one. In order to map out an accurate phase diagram, we performed long simulations for the bi-stable cases with largest Γ to conclude that the corresponding zonal flow can stably exist and does not evolve to a convection roll state. Overall, we ran the simulations at least 50 000 free-fall time units for these cases, which corresponds to at least 5 viscous diffusive time units (H^2/ν) or 0.5 thermal diffusive time units (H^2/κ). From figure 5.4(a) it can be seen that, when Ra is increased, the bi-stable regime exists in an increasing Γ range. This is also consistent with the finding that zonal flow develops more readily for higher Ra for $\Gamma = 2$ [81]. Figure 5.4(b) demonstrates that the Γ range for the bi-stable state also depends on Pr . For $Ra = 10^8$ the largest range of bi-stable state exists at $Pr \sim 30$, namely between $\Gamma = 3$ and $\Gamma = 24$.

We have already shown that zonal flow can not be sustained, and only convection roll states are observed, when Γ is larger than a critical value, which depends on Ra and Pr . A related question is how many convection rolls (in other words, what is the mean aspect ratio of individual convection roll) can develop for a specific (Ra , Pr , Γ). In the next subsection, we will explore the possible convection roll states using different initial roll states generated by different Fourier basis, as explained in §5.2.

Figure 5.4: Phase diagram in the (a) $Ra - \Gamma$ parameter space for $Pr = 10$ and in the (b) $Pr - \Gamma$ parameter space for $Ra = 10^8$. Black circles (●) correspond to only zonal flow, red squares (■) denote coexistence of zonal flow and convection rolls, and blue diamonds (◆) indicate that only convection roll states are stable. The black hollow circles mark the cases shown in figure 5.2. Note that we also performed simulations for $Ra = 10^9$, $Pr = 10$ for convection roll states with $\Gamma = 16$ and zonal flow state for $\Gamma = 2$, to obtain $Nu \sim Ra^\alpha$ and $Re \sim Ra^\beta$ effective scaling relations for Nu and Re .

5.3.2 Coexistence of multiple convection roll states at large Γ

In this subsection, we study the coexistence of multiple convection roll states in large-aspect ratio domains, all being statistically stable states once achieved. Figure 5.5 shows that for $Ra = 10^8$, $Pr = 10$, and free-slip at the plates, in a $\Gamma = 16$ system convection rolls with a mean dimensionless horizontal size of $1.6 \leq \Gamma_r \leq 8$ are all statistically stable. The heat transport considerably increases with decreasing mean aspect ratio Γ_r of an individual convection roll. For example, the heat transport for the $\Gamma_r = 1.6$ state is almost twice as high as that for $\Gamma_r = 8$. Although it had been observed before that convection rolls with smaller Γ_r imply a higher heat transport – e.g. for 2D RB convection with no-slip plates [106, 145, 199], for RB convection in an annulus convection cell [161], and also for Taylor-Couette flow [163] – in those cases the observed increase in the transport is typically a couple of percents, and by far not as large as observed for RB with free-slip plates and large aspect ratio cells as studied here. This difference is due to different plume dynamics and the associated spatial dependence of the local Nusselt number, $Nu(x)$, as we will discuss later.

We also tested initial conditions with 12, 14, 16 rolls for $Ra = 10^8$, $Pr = 10$, $\Gamma = 16$. These states with smaller rolls are not stable and will finally undergo a transition to the ten-roll state by merging of convection rolls. From figure 5.6(a) it is seen that the vertical Reynolds number Re_z has a sudden decrease during merging of rolls, because the strong vertical motion is concentrated near the plume-ejecting regions between two

Figure 5.5: Temperature snapshots of different roll states for $Ra = 10^8$ and $Pr = 10$ in a $\Gamma = 16$ periodic cell. The dimensionless mean horizontal size of the convection roll Γ_r (i.e., the mean aspect ratio of one individual roll) and the Nusselt number Nu for each state are indicated. The different roll states are from initial conditions with different number of initial rolls. All these states can stably exist for long time (see table 5.3 in Appendix) without undergoing a transition to other states.

neighbouring rolls. The decrease of Nu during merging events can also be observed in figure 5.6(b), which is related to the decreased vertical motion. Figure 5.6(c) shows how the flow undergoes a transition from the initial sixteen-roll state to the final ten-roll state by successive merging of convection rolls. The transition happens when the balance of the roll state is broken by horizontal motion of local hot/cold plumes: In the second snapshot at $t = 85$ the system is still in the initial sixteen roll state. However, one can already see that a local hot plume moves leftwards while its neighbouring cold plume moves rightwards (marked by red arrows). In the third snapshot at $t = 87$ two hot plumes merge to a single one and so do two cold ones, thus annihilating two rolls. The resulting fourteen-roll state is shown in the fourth snapshot taken at $t = 141$. At later times the horizontal motion of the plumes and their further merging let the fourteen-roll state evolve to a twelve-roll state ($t = 182$, III), and finally to a ten-roll state ($t = 300$, IV). We also performed very long simulations as indicated in figure 5.6(d), from which we conclude that the ten-roll state can statistically stably exist for a very long time without undergoing any further transition to yet another state.

Figure 5.7 displays phase diagrams for all the possible convection roll states in the $Ra - \Gamma_r$ and $Pr - \Gamma_r$ parameter spaces. The stable roll states can last for several thousand free-fall time units without undergoing a transition to other states (see Appendix). Figure 5.7(a) shows a weak dependence of Γ_r on Ra . One can observe the same stable roll states for the considered Ra range. In contrast, a pronounced

Figure 5.6: Time evolution of (a) Re and (b) Nu for $Ra = 10^8$, $Pr = 10$, $\Gamma = 16$ with an initial sixteen-roll state. $Re_x = \sqrt{(Ra/Pr)}\sqrt{(\langle u^2 \rangle_V)}$ is horizontal Reynolds number and $Re_z = \sqrt{(Ra/Pr)}\sqrt{(\langle w^2 \rangle_V)}$ the vertical one. (c) Temperature snapshots at different times. The roll merging can be seen, namely the flow undergoes a transition from initial sixteen-roll state (I), to a fourteen-roll state (II), to a twelve-roll state (III), and then to the final ten-roll state (IV). The figure has the same colour scale as figure 5.5. (d) Time evolution of Re for much longer time (on a log-scale) to show that the final ten-roll state is stable without undergoing a transition to an other roll state.

Figure 5.7: Phase diagram for different roll states for (a) $Pr = 10$, $\Gamma = 16$ and (b) $Ra = 10^8$, $\Gamma = 16$. Circles denote that the corresponding roll state with the mean aspect ratio Γ_r of an individual roll is stable, while crosses denote that the roll state is not stable. The solid line in (b) connects the minimal mean aspect ratio $\Gamma_{r,min}$ of an individual convection roll for different Pr for $\Gamma = 16$, while the dashed line connects $\Gamma_{r,min}$ for different Pr for $\Gamma = 32$.

dependence of Γ_r on Pr is observed in figure 5.7(b), where convection roll states with the smallest Γ_r are observed for intermediate $Pr \approx 10$. The minimum $\Gamma_r = 16/11$ occurs for $\Gamma = 32$, which means that the horizontal extent of a stable convection roll is always larger than the height of the system, also for smaller $\Gamma = 8$, we found that the smallest roll size ($\Gamma_r = 4/3$) is always larger than 1. This explains why convection rolls cannot be supported for small $\Gamma \approx 2$, where indeed only zonal flow was obtained. From figure 5.7(b) it can also be concluded that these results are independent of the aspect ratio of the system once it is large enough, as we obtained almost the same result for $\Gamma = 16$ and $\Gamma = 32$ domains.

5.3.3 Nusselt number and Reynolds number

We now discuss the effective scaling relations of Nu and Re as function of Ra and Pr . These relationships are usually expressed with effective scaling laws $Nu \sim Ra^{\gamma_{Nu}} Pr^{\alpha_{Nu}}$ and $Re \sim Ra^{\gamma_{Re}} Pr^{\alpha_{Re}}$ [53]. The effective scaling laws have been widely discussed for no-slip cases for both 2D and 3D convection [53]. For the 2D horizontally periodic cases with no-slip plates, $Nu \sim Ra^{0.29}$ is found with $Pr = 1$, $Ra \leq 10^{10}$ [120, 183]. For 2D RB convection with no-slip plates and sidewalls with unit aspect ratio, several studies have shown that $Nu \sim Ra^{0.3}$ and $Re \sim Ra^{0.6}$ [99, 122, 146]. However, how these effective scaling relations will change for free-slip plates has not been explored, especially not for convection roll states, which are only present in large enough domain size.

Flow state	γ_{Nu}	γ_{Re}	α_{Re}
$\Gamma_r = 8$	0.321	0.675	-1.043

Flow state	γ_{Nu}	γ_{Re}	α_{Re}
$\Gamma_r = 4$	0.320	0.663	-1.065
$\Gamma_r = 8/3$	0.318	0.665	-1.077
$\Gamma_r = 2$	0.318	0.667	-1.082
$\Gamma_r = 1.6$	0.302	0.657	-1.078
Zonal flow	0.170	0.603	-

Table 5.2: The effective scaling exponents for fitted effective scaling relations $Nu \sim Ra^{\gamma_{Nu}}$, $Re \sim Ra^{\gamma_{Re}}$ and $Re \sim Pr^{\alpha_{Re}}$ for different roll states for $Pr = 10$ obtained in an aspect ratio $\Gamma = 16$ domain. The zonal flow data is for $\Gamma = 2$.

Figure 5.8(a) and 5.8(b) show Nu and Re as function of Ra for both zonal flow ($\Gamma = 2$) and convection roll states ($\Gamma = 16$) for $Pr = 10$. Figure 5.8(a) reveals that the heat transfer in the convection roll state is much higher than that of the zonal flow state. Detailed information about the obtained scaling exponents is listed in table 5.2. For the convection roll states we find that the effective scaling exponent γ_{Nu} in $Nu \sim Ra^{\gamma_{Nu}}$ is approximately 0.3. It increases with increasing Γ_r , reaching about 1/3 for the largest $\Gamma_r = 8$, which is the value predicated by the Grossmann-Lohse (GL) theory for the $I_\infty^\leq$ and III_∞ regimes [222–224] for the no-slip case. For zonal flow γ_{Nu} is much smaller, namely only 0.17. The effective scaling exponent γ_{Re} in $Re \sim Ra^{\gamma_{Re}}$ is about 0.6 for zonal flow and about 0.67 for the convection roll state, which is also close to the GL predication of 2/3 for the $I_\infty^\leq$ and III_∞ regimes [223, 224] for no-slip cases, while it is larger than 0.6 that has been reported for 2D RB convection with no-slip plates [99, 122, 146].

Next, we will discuss the Prandtl number dependence of the Nusselt number, $Nu(Pr)$. For no-slip plates in 3D cases, it is known that the Nu is maximal around $Pr \sim 2 - 3$, and after that it declines with increasing Pr [225–227]. This remarkable maximum in $Nu(Pr)$ had in fact been predicted before by the GL theory [222, 223]. In contrast, for the 2D cases, [121] showed that $Nu(Pr)$ has a minimum, rather than maximum as in the 3D case, namely at $Pr \sim 2 - 3$ for moderate Ra . This anomalous relation is caused by counter-gradient heat transport in 2D cases.

How does the $Nu(Pr)$ dependence look like for the 2D RB case with free-slip plates? For the zonal flow state [81] already showed that Nu is an increasing function of Pr in the range $1 \leq Pr \leq 10$. Figure 5.8(c) and 5.8(d) show the relations for $Nu(Pr)$ and $Re(Pr)$ for all states with free-slip plates (i.e., both for zonal flow and for various convection roll states). From figure 5.8(c) it can be seen that the $Nu(Pr)$ trend shown by [81] is also valid for the wider range of Pr analysed in this chapter. The reason why Nu is much smaller for small Pr is that zonal flow features intermittent bursts whereas most of the time Nu is around 1 [81]. For large Pr , the flow does not burst and convective heat transport with $Nu \gg 1$ is sustained at all times, thus the corresponding Nu is larger than that of the small Pr cases.

For the convection roll states, Figure 5.8(c) shows that Nu increases monotonically with increasing Pr for large mean convection roll size $\Gamma_r = 16$ (see the inset). This

Figure 5.8: (a) Nu and (b) Re as function of Ra for different convection roll states (see legend in panel (d)) for $Pr = 10$, $\Gamma = 16$ and the zonal flow state (see orange stars on solid orange line) for $Pr = 10$, $\Gamma = 2$. (c) Nu and (d) Re as function of Pr for different roll states for $Ra = 10^8$, $\Gamma = 16$ and for the zonal flow state, for which we put $\Gamma = 2$. Again, see the legend in panel (d). The inset in (c) shows Nu as function of Γ_r for $Ra = 10^8$ and different Pr obtained for $\Gamma = 16$ (hollow symbols) and $\Gamma = 32$ (solid symbols). The solid symbols often overshadow the hollow ones as the symbol sizes are the same.

can be interpreted as that the flow for the large $\Gamma_r = 16$ cases can be viewed as “localised” zonal flow, thus the $Nu(Pr)$ follows the similar trend as that of zonal flow. In contrast, for small Γ_r , Nu decreases with increasing Pr . For $Re(Pr)$, figure 5.8(d) shows that the Re follows $Re \sim Pr^{-1}$ for convection roll states (see table 5.2), the exponent -1 is the same as that of the GL predication for the $I_\infty^<$ and III_∞ regimes for no-slip cases.

In order to understand different $Nu(Pr)$ dependence for large and small Γ_r as shown in figure 5.8(c), we first look at the flow organizations for convection roll states for different Pr . Figure 5.9(a) gives the time-averaged temperature fields for the $\Gamma_r = 16$ roll state for different Pr . The flow near the bottom plate can be divided into plume-ejecting region, plume-impacting region, and between them there is wind-shearing region which occupies large fraction of the domain. In the ejecting region, thermal plumes are emitted, while in the impacting region, the boundary layer is impinged by the plumes from the opposite plate. The wind-shearing region is sheared by the large-scale circulation. The impacting regions on the top plate are the opposite of the ejecting regions of the bottom plate and vice versa. This kind of division is also adopted in periodic 2D RB convection with no-slip plates [204]. A remarkable observation is the stable-stratification near the plume impacting region. Figure 5.9(c) shows a zoom-in of the regions where hot plumes are ejected for $Pr = 1$ and 10. When hot fluid impinges the cold plate, it does not have sufficient time to cool down, before it moves horizontally. The consequence is that the temperature of the fluid between the top boundary layer and the bulk is higher than that of the bulk fluid, thus implying a stable-stratification. This behavior is even observed at the centre line of the hot plume (the vertical line at the horizontal location where the local bottom wall heat flux is minimal) as is shown in temperature profiles in figure 5.9(d), where for $Pr > 1$ stable-stratification near the cold plate is observed. The stable-stratification has also been observed at the axis in cylindrical RB convection [228–230], and in 2D RB convection with no-slip plates and sidewalls for unit aspect ratio in the central region near the plates [206]. The instantaneous temperature fields shown in figure 5.9(b) for $Pr = 100$ reveals the “localised” zonal flow structures. It can be seen that plumes are ejected everywhere while they can only move vertically and impinge the cold plate in the central region.

Next, we focus on the local properties of the wall heat flux. The local wall heat flux is expressed through the local wall Nusselt number $Nu(x)|_{z=0,1} = \partial \langle \theta \rangle_t / \partial z|_{z=0,1}$. Figure 5.9(e) and 5.9(f) show the spatial $Nu(x)$ dependence at the plates for $Ra = 10^8$ and $\Gamma = 32$ for $\Gamma_r = 16$ and $\Gamma_r = 4$, respectively. For small $Pr = 1$, one sees from figure 5.9(a) the accumulation of hot fluid in the ejecting region near the bottom plate, which cause small temperature gradient (see figure 5.9d), and correspondingly small local Nu (see figure 5.9e). So the centre of the ejecting region can be denoted as the point where local wall heat flux is minimal. In contrast, for the impacting region ($x/H \approx 0$ at the bottom plate) where cold fluid directly impinges the hot plate, there is a sharp temperature gradient and thus large local Nu (see figure 5.9e). Similar Nu behavior is also observed in the ejecting/impacting regions near the top plate (dashed

Figure 5.9: (a) Time-averaged temperature fields for the $\Gamma_r = 16$ roll state for different Pr at $Ra = 10^8$ obtained in a $\Gamma = 32$ domain. (b) Instantaneous temperature field for $Pr = 100$ for the $\Gamma_r = 16$ state with $Ra = 10^8$, $\Gamma = 32$. (c) Zoom-in for the time-averaged temperature fields in (a) for $Pr = 1$ and 10. (d) Temperature profiles for the different Pr at the centre point of plume-ejecting regions ($x/H \approx 16$) where the local Nu is smallest. Panels (e) and (f) show the spatial dependence of $Nu(x)$ at the hot plate at $z = 0$ (solid lines) and the cold plate at $z = 1$ (dashed lines) for different Pr for the (e) $\Gamma_r = 16$ and the (f) $\Gamma_r = 4$ roll states. Note that all curves are shifted such that the minimum local Nu at the hot plate is located at $x/H = 16$.

lines in figure 5.9e). The physical interpretation is as follows: The heat is ejected into the system through the bottom plate mainly at the plume-impacting regions where local temperature gradient is large, and then it is advected by large-scale circulation to the plume-ejecting regions, where conductive heat flux is low on the wall while convective heat flux is high above the wall. The heat is mainly removed from the system when hot plume impinges the cold plate.

For $\Gamma_r = 16$, there is only one impacting region near the bottom plate, and the heat input is still dominated by the wind-shearing region, which occupies a large fraction of the domain. As the wind-shearing region is like “localised” zonal flow where Nu increases with increasing Pr , the global Nu thus also increases with increasing Pr . In contrast, for smaller Γ_r , there are more impacting regions on the bottom plate, and these impacting regions contribute significantly to the global heat input. As the heat flux at the impacting region increases with decreasing Pr , we thus see that the global Nu also increases with decreasing Pr .

5.4 Three-dimensional simulations

We have already shown that the zonal flow observed in 2D RB convection with free-slip plates and horizontally periodic boundary conditions for $\Gamma = 2$ [81, 191] eventually disappears with increasing Γ . What about in 3D, under the same conditions? For the 3D RB convection with free-slip plates, previous studies have not reported zonal flow [211, 212]. However, if we introduce *span-wise rotation* as illustrated in figure 5.1(b) where the rotating axis is parallel to the y axis, the flow will become 2D-like at sufficiently fast rotation, due to the Taylor-Proudman theorem. In this way we may observe zonal flow at certain parameters, as indeed was already reported in Ref. [82]. Therefore here we will study span-wise rotating RB convection, focusing on the transition from zonal flow to convection roll states with increasing aspect ratio Γ of the container.

We first show that both the global transport properties like the Nusselt number Nu and the Reynolds number Re , as well as the flow organization increasingly behave like 2D cases with increasing rotation rate. We fix the Rayleigh number to $Ra = 10^7$ and the Prandtl number to $Pr = 0.71$. To be on the safe side, we choose a large domain with $\Gamma = 16$, as previous studies showed that large aspect ratios are needed in order to capture the superstructures which have a horizontal size of 6-7 times the height of the domain for 3D RB convection with no-slip plates [215–218]. The initial conditions have zero velocity and a linear temperature profile for these simulations. For RB convection rotating about a vertical axis for small $Pr = 0.71$ with no-slip plates, Nu initially does not change much with increasing the rotation rate (denoted by the inverse Rossby number $1/Ro$), until after $1/Ro \geq 1$, Nu drops monotonically with increasing $1/Ro$ [231]. Figure 5.10(a) shows that for span-wise rotating RB convection Nu also initially does not change much for $1/Ro \leq 1$. After that Nu drops monotonically until reaching its minimum at $1/Ro \approx 10$, and then it increases monotonically towards the 2D value for a two-roll state. Similar non-monotonic behavior of Nu with the control parameter

Figure 5.10: 3D RB convection with span-wise rotation: (a) Nu and (b) Re as function of $1/Ro$ for $Ra = 10^7, Pr = 0.71, \Gamma = 16$ (black circles). For orientation with respect to the Nusselt number, the data for non-rotation ($1/Ro = 0$, red diamond) and the 2D cases with the same control parameters ($Ra = 10^7, Pr = 0.71, \Gamma = 16$) for different roll aspect ratios Γ_r (blue squares) are also shown; for these data points the value at the $1/Ro$ axis has no meaning. The Reynolds number Re for the $\Gamma_r = 8/3$ (7702.85) and $\Gamma_r = 2$ (7726.20) states are close to each other and can not be differentiated in the figure.

was also found in sheared RB convection, where Nu also non-monotonically depends on the wall shear Reynolds number [232]. For span-wise rotating RB convection, Re monotonically increases from the 3D value towards the 2D value with increasing rotation rate, as shown in figure 5.10(b). Note that this behavior is very different from RB convection rotating around the *vertical axis*, where Re decreases monotonically with increasing $1/Ro$ [117].

We now connect the global transport properties with the flow organization. Figure 5.11 shows instantaneous temperature fields at the midheight (top row) and boundary layer height close to the bottom plate (bottom row) for different $1/Ro$ with $Ra = 10^7, Pr = 0.71, \Gamma = 16$. We can clearly see the connection between large-scale thermal structure at midheight and boundary layer height for different $1/Ro$, which has also been shown in 3D RB convection with no-slip plates [216, 218]. With increasing rotation rate, one sees the increasing two-dimensionlization of the flow. For the non-rotation case ($1/Ro = 0$), figure 5.11(a) shows qualitatively similar superstructures as the no-slip case [215, 216]. When $1/Ro$ increases to 1, a meandering large-scale convection roll state develops, as can be seen from figure 5.11(b). Interestingly, similar meandering structures have also been observed in many shear-driven flows when the horizontal isotropy is broken, such as plane Couette flow [233], wavy Taylor rolls in Taylor-Couette flow [234], and also in sheared RB convection [232]. Figure 5.10(a) shows that this meandering structure at $1/Ro = 1$ has still similar Nu as in the non-rotation case. For medium rotation rates $1/Ro = 3.75$ and 10, a two-roll state evolves; interestingly, the cyclonic circulation has a larger size than the anticyclonic one. The

Figure 5.11: 3D RB convection with span-wise rotation (the rotating axis is parallel to y axis): top view snapshots of temperature fields at midheight ($z = H/2$, top row) and thermal boundary layer height ($z = z_{bl} = H/(2Nu)$, bottom row) for $Ra = 10^7$, $Pr = 0.71$, $\Gamma = 16$ with different rotation rates. (a) $1/Ro = 0$. (b) $1/Ro = 1$. (c) $1/Ro = 3.75$. (d) $1/Ro = 10$. (e) $1/Ro = 50$.

smaller Nu for these two-roll states is related to the decreased plume emission area. With further increasing rotation rate, the flow increasingly behaves like 2D cases. For the largest $1/Ro = 50$ as shown in figure 5.11(e), a two-roll state with equal size of each roll has developed, with small span-wise variation in temperature.

After we have shown the increasing two-dimensionlization of the flow with increasing rotation rate for span-wise rotating RB convection, next we will study the transition from zonal flow to the convection roll states with increasing aspect ratio Γ , similarly as we have already done for the 2D case. Ref. [82] studied span-wise rotating RB convection for $Ra = 10^7$, $Pr = 0.71$ with fixed $\Gamma = 2\pi$. They observe strong zonal flow that is perpendicular to both rotation vector and gravity vector. Both the cyclonic zonal flow and the anticyclonic one have been obtained using different initial conditions. These two kinds of zonal flow are symmetric for 2D, while they are not in 3D with span-wise rotation, as the Coriolis force depends on the direction of velocity and thus it breaks the symmetry between the two kinds of zonal flow, which have an opposite flow direction. The main difference between the two kinds of zonal flow is that intermittent bursts exist for anticyclonic zonal flow, similar to what is observed in 2D cases with small $Pr \leq 2$, while these bursts are absent for the cyclonic zonal flow [82]. We note that, as in Ref. [82], the dimensionless angular velocity $\Omega' = \Omega\tau_{th}$ is used to quantify the rotation velocity, where $\tau_{th} = H^2/\kappa$ is thermal diffusive time. The dimensionless angular velocity is related to the Rossby number by $Ro = \sqrt{RaPr}/(2\Omega')$ [198].

We performed simulations for $Ra = 10^7$, $Pr = 0.71$, $\Gamma = 8$ and 16 with $1/Ro =$

Figure 5.12: 3D RB convection with span-wise rotation (the rotating axis is parallel to y axis): time evolution of Nu for $Ra = 10^7$, $Pr = 0.71$, $1/Ro = 3.75$ with three different initial conditions for (a) $\Gamma = 8$ and (b) $\Gamma = 16$. IC_0 means initial condition with zero velocity. IC_c / IC_a denote cyclonic/anticyclonic shear flow as initial condition. (c-h) Side view temperature snapshots at mid span-wise length ($y/H = \Gamma/2$) at different times denoted by the dashed lines in (a) and (b).

3.75, which corresponds to $2\Omega' = 10000$ in [82]. Three different initial conditions were used to trigger different possible states, namely:

- IC_0 with zero initial velocity,
- IC_c with cyclonic shear flow $u(z) = 2z - 1$, $v = 0$, $w = 0$, to trigger possible cyclonic zonal flow, and
- IC_a with anticyclonic shear flow $u(z) = 1 - 2z$, $v = 0$, $w = 0$, to trigger possible anticyclonic zonal flow.

We first report the results for $\Gamma = 8$: Figure 5.12(a) illustrates that for the initial conditions IC_0 , the flow quickly develops into a two-roll state as indicated in figure 5.12(c). We note that the cyclonic circulation is again larger than the anticyclonic one. For IC_c as initial conditions, the cyclonic zonal flow shown in figure 5.12(d) can statistically stably exist for more than 3000 free-fall time units. Finally, the initial conditions IC_a can trigger an anticyclonic zonal flow with burst phenomenon, which is consistent with the findings of Ref. [82]. However, here this feature only lasts for about 380 free-fall time units and then the system undergoes a transition to a two-roll state. We thus conclude that for $\Gamma = 8$ the system can again display a bi-stability behavior, in which both zonal flow and convection roll states are statistically stable.

We now come to the case of $\Gamma = 16$. From figure 5.12(b) we conclude that for three different initial conditions, the flow eventually evolves to the very same final state, namely the convection roll state. The cyclonic zonal flow initially seen for short time for the initial conditions IC_c quickly undergoes a transition to the convection roll state. The snapshots in figure 5.12(b–d) reveal similar transition processes as we had already observed in 2D. Again, the final two-roll state has a larger cyclonic circulation. For the final convection state the horizontal scale of the flow reaches the domain size of 16, which is much larger than the typical horizontal scale of superstructures observed in 3D RB convection with no-slip plates [215, 216]. Such large-scale structures can not be captured in small domains, which is the reason why for small domains only zonal flow states can be realized [82].

To summarize our results from our 3D simulations with free-slip plates and span-wise rotation (with fixed $Ra = 10^7$, $Pr = 0.71$, and relative strong rotation $1/Ro = 3.75$): We have revealed a similar physical picture as we had done before for 2D RB convection with free-slip plates:

- (i) For small aspect ratio $\Gamma = 2\pi$, the flow is zonal [82].
- (ii) With increasing Γ up to at least $\Gamma = 8$, the convection roll state and the zonal flow state coexist in phase space and which one is taken depends on the initial conditions.
- (iii) For larger $\Gamma = 16$, we always obtain the convection roll state, independently on of what kind of initial conditions were employed.

5.5 Conclusions

In summary, we have studied 2D RB convection and span-wise rotating RB convection with free-slip plates in horizontally periodic domains using direct numerical simulations. Based on the present data, we find that the zonal flow, which was previously observed in a $\Gamma = 2$ cell [81, 191], cannot be sustained and will undergo transitions to convection roll states when the aspect ratio Γ is larger than a critical value, which depends on the Rayleigh number Ra and Prandtl number Pr .

We reveal three regimes:

- (i) For small Γ (typically $\Gamma \leq 1 - 3$, depending on Ra and Pr), only zonal flow can be observed.
- (ii) With increasing Γ , we first find a bi-stable regime in which, depending on the initial conditions, both zonal flow and convection roll states can be stable .
- (iii) For even larger-aspect ratio systems, only convection roll states can be sustained.

How many convection rolls develop in the convection roll states (in other words, what is the mean aspect ratio Γ_r of an individual roll) depends on the initial conditions. For instance, for $Ra = 10^8$ and $Pr = 10$ the horizontal extent of the stable convection rolls varies between 16/11 and 64 times the height of the convection cell. A convection roll with an as large aspect ratio of $\Gamma_r = 64$ or more generally already with $\Gamma_r \gg 10$ can be seen as “localized” zonal flow.

The heat transfer in the system increases significantly when the horizontal extent of the convection roll is reduced. It is found that the Prandtl number dependence of the Nusselt number $Nu(Pr)$ has very different trends for large and small Γ_r : For large Γ_r (like $\Gamma_r = 16$ for a $\Gamma = 32$ cell), the flow behaves like a “localized” zonal flow state, and Nu increases with increasing Pr , similarly as we found for the zonal flow state. In contrast, for small Γ_r , the heat flux into the system is dominated by the plume-impacting regions on the bottom plate, in which the local heat flux is very high and increases with decreasing Pr , implying that the global heat flux Nu also increases with decreasing Pr .

For span-wise rotating 3D RB convection, we find that with increasing rotation rate $1/Ro$, both the transport properties (like the Nusselt number Nu and the Reynolds number Re) and the flow organization increasingly behave like in the corresponding 2D cases. In particular, just as in 2D, the zonal flow observed in a small periodic cell with $\Gamma = 2\pi$ [82], disappears in larger cells with $\Gamma = 16$. For intermediate $\Gamma = 8$, bistability is observed, again similarly as observed in 2D RB convection.

Finally, an interesting but still open question is the final fate of the aspect ratio dependence of the zonal flow for higher Ra , i.e., is there a finite Ra above which zonal flow exists for all Γ ? On the one hand, for fixed Ra within our explored range $Ra \leq 3 \times 10^8$, zonal flow always seems to disappear when Γ is sufficiently large; on the other hand, for fixed Γ , zonal flow always seems to stably exist when Ra is large enough. We have to conclude that in spite of our efforts it still is an open question

whether zonal flow can exist for all Γ for a large enough but finite Rayleigh number. Due to the chaotic nature of the flow, mapping out the parameter regime where zonal flow can be found is not easy, especially not for high Ra and large Γ .

From a broader perspective, our study underlines the importance of having large enough aspect ratios in numerical simulations of wall-bounded turbulent flows, even when one employs periodic boundary conditions. We had seen this before in 3D RB convection with no-slip velocity boundary conditions at the plates [215–218], but apparently this conclusion is much more general.

5.6 Appendix

5.6.1 Simulation details

Ra	Pr	Γ	$N_x \times N_z$	$n^{(i)}$	n	Γ_r	Nu	Re	Re_x	Re_z	t_{tot}	t_{avg}
10^7	10	16	4096×256	2	2	8	25.67	413.19	405.27	80.52	7000	5000
10^7	10	16	4096×256	4	4	4	30.95	448.47	427.38	135.91	7000	5000
10^7	10	16	4096×256	6	6	2.67	40.84	493.15	453.37	194.05	7000	5000
10^7	10	16	4096×256	8	8	2	41.94	474.28	417.81	224.44	7000	4000
10^7	10	16	4096×256	10	10	1.6	48.53	488.77	409.76	266.45	7000	3000
3×10^7	10	16	6144×384	2	2	8	36.16	881.17	864.21	172.06	6500	4500
3×10^7	10	16	6144×384	4	4	4	46.15	956.10	910.58	291.51	6500	4500
3×10^7	10	16	6144×384	6	6	2.67	55.61	997.30	917.21	391.60	11500	5500
3×10^7	10	16	6144×384	8	8	2	64.98	1033.58	910.66	488.87	11500	7500
3×10^7	10	16	6144×384	10	10	1.6	69.43	1016.46	852.94	552.91	6500	4500
10^8	10	16	6144×384	2	2	8	53.30	1995.38	1955.07	399.06	6000	4000
10^8	10	16	6144×384	4	4	4	67.66	2112.39	2011.37	645.40	6000	4000
10^8	10	16	6144×384	6	6	2.67	86.49	2296.92	2112.33	902.13	8000	6000
10^8	10	16	6144×384	8	8	2	94.10	2281.70	2010.51	1078.91	6000	3000
10^8	10	16	6144×384	10	10	1.6	96.27	2200.71	1846.23	1197.73	6000	3000
10^8	10	16	6144×384	12	10	1.6	96.20	2198.65	1844.19	1197.09	6000	3000
10^8	10	16	6144×384	14	10	1.6	96.81	2202.08	1846.89	1199.23	2456	1456
10^8	10	16	6144×384	16	10	1.6	96.20	2200.23	1845.68	1197.70	12000	10000
3×10^8	10	16	8192×512	2	2	8	75.85	4154.56	4068.12	843.14	5500	3500
3×10^8	10	16	8192×512	4	4	4	92.95	4298.72	4092.94	1314.09	5500	3500
3×10^8	10	16	8192×512	6	6	2.67	121.79	4738.24	4357.01	1862.08	5500	3500
3×10^8	10	16	8192×512	8	8	2	129.09	4675.29	4121.31	2207.51	5500	3500
3×10^8	10	16	8192×512	10	10	1.6	134.00	4552.41	3821.76	2473.57	5500	3500
10^9	10	16	12288×768	2	2	8	112.36	9258.76	9061.98	1898.78	1600	1100
10^9	10	16	12288×768	4	4	4	137.80	9622.01	9160.21	2945.08	5300	1300
10^9	10	16	12288×768	6	6	2.67	171.87	10380.02	9548.89	4069.56	5046	2046
10^9	10	16	12288×768	8	8	2	186.55	10373.03	9148.56	4889.31	4742	2742
10^9	10	16	12288×768	10	10	1.6	198.01	10135.10	8504.77	5512.29	3700	1700

Ra	Pr	Γ	$N_x \times N_z$	$n^{(i)}$	n	Γ_r	Nu	Re	Re_x	Re_z	t_{tot}	t_{avg}
------	------	----------	------------------	-----------	-----	------------	------	------	--------	--------	-----------	-----------

Table 5.3: Simulation details for all cases shown in figure 5.7(a). The columns from left to right indicate Ra , Pr , Γ , grid resolutions $N_x \times N_z$, the number of initial rolls $n^{(i)}$, the number of final convection rolls n , the mean aspect ratio of the convection rolls $\Gamma_r = \Gamma/n$, the Nusselt number Nu , the Reynolds number Re based on root-mean-square of the global velocity, the horizontal Reynolds number Re_x based on root-mean-square of the horizontal velocity, the vertical Reynolds number Re_z , the total simulation time t_{tot} , and the time t_{avg} used to average Nu and Re .

Ra	Pr	Γ	$N_x \times N_z$	$n^{(i)}$	n	Γ_r	Nu	Re	Re_x	Re_z	t_{tot}	t_{avg}
10^8	1	16	6144×384	2	2	8	55.04	20633.77	20190.00	4261.16	4000	s
10^8	1	16	6144×384	4	4	4	81.63	24070.00	22920.00	7357.00	6000	s
10^8	3	16	6144×384	2	2	8	52.18	6679.56	6536.21	1376.25	10000	2000
10^8	3	16	6144×384	4	4	4	78.27	7822.00	7448.00	2391.00	5434	1434
10^8	3	16	6144×384	6	6	2.67	93.06	8156.64	7502.60	3200.30	5345	2345
10^8	3	16	6144×384	8	8	2	102.51	8188.54	7220.01	3863.14	4600	1600
10^8	30	16	6144×384	2	2	8	56.69	646.20	634.14	124.28	6000	4000
10^8	30	16	6144×384	4	4	4	64.17	670.83	639.73	201.89	6000	4000
10^8	30	16	6144×384	6	6	2.67	77.54	697.39	641.45	273.67	4311	3311
10^8	30	16	6144×384	8	8	2	84.60	688.69	606.92	325.50	6000	4000
10^8	30	16	6144×384	10	10	1.6	89.96	673.49	565.05	366.48	6000	4000
10^8	30	16	6144×384	12	10	1.6	90.13	674.42	565.86	366.94	3000	2000
10^8	30	16	6144×384	14	10	1.6	89.99	673.58	565.10	366.58	4000	3000
10^8	30	16	6144×384	16	10	1.6	89.94	673.42	565.20	366.13	4000	3000
10^8	100	16	6144×384	2	2	8	57.98	163.70	160.24	33.45	5000	2000
10^8	100	16	6144×384	4	4	4	62.53	179.11	171.18	52.71	6000	5000
10^8	100	16	6144×384	6	6	2.67	68.65	187.57	172.86	72.82	6000	5000
10^8	100	16	6144×384	8	8	2	74.72	184.90	163.07	87.16	6000	5000
10^8	1	32	12288×384	2	2	16	35.69	16990.00	16820.00	2421.00	6000	s
10^8	1	32	12288×384	4	4	8	55.06	20656.18	20210.0	4264.00	5000	s
10^8	1	32	12288×384	6	6	5.33	69.79	22760.00	21989.82	5886.63	5278	s
10^8	1	32	12288×384	8	8	4	81.62	24070.44	22918.82	7356.22	4177	s
10^8	3	32	12288×384	2	2	16	44.84	6337.77	6274.55	892.89	6500	4500
10^8	3	32	12288×384	4	4	8	51.70	6647.40	6504.83	1369.35	6500	4500
10^8	3	32	12288×384	6	6	5.33	69.52	7526.00	7270.00	1947.00	3678	s
10^8	3	32	12288×384	8	8	4	78.26	7820.16	7446.00	2390.36	4032	2032
10^8	3	32	12288×384	10	10	3.2	84.72	7960.42	7456.85	2786.39	3440	1440
10^8	3	32	12288×384	12	12	2.67	93.09	8154.74	7500.63	3200.02	5205	3205
10^8	3	32	12288×384	14	14	2	99.91	8236.24	7422.50	3569.63	3738	1738
10^8	10	32	12288×384	2	2	16	52.96	2037.02	2017.49	281.43	6000	4000
10^8	10	32	12288×384	4	4	16	53.25	1994.84	1954.55	398.86	6000	4000
10^8	10	32	12288×384	6	6	5.33	62.34	2090.82	2020.23	538.71	6000	4000
10^8	10	32	12288×384	8	8	4	67.62	2111.59	2010.61	645.18	6000	4000
10^8	10	32	12288×384	10	10	3.2	71.35	2121.42	1987.41	742.14	6000	4000
10^8	10	32	12288×384	12	12	2.67	87.73	2313.32	2127.41	908.67	6000	4000
10^8	10	32	12288×384	14	14	2.29	91.47	2307.20	2078.88	1000.70	6000	3000

Ra	Pr	Γ	$N_x \times N_z$	$n^{(i)}$	n	Γ_r	Nu	Re	Re_x	Re_z	t_{tot}	t_{avg}
10^8	10	32	12288×384	16	16	2	95.01	2291.69	2019.13	1083.97	6000	3000
10^8	10	32	12288×384	18	18	1.78	97.25	2261.39	1945.47	1152.82	6000	3000
10^8	10	32	12288×384	20	20	1.6	96.13	2199.85	1845.37	1197.48	6000	3000
10^8	10	32	12288×384	22	22	1.45	97.22	2155.22	1761.17	1242.27	6000	3000
10^8	10	32	12288×384	24	22	1.45	97.53	2158.19	1763.30	1244.42	2587	1087
10^8	30	32	12288×384	2	2	16	56.36	670.65	664.57	90.07	5800	4000
10^8	30	32	12288×384	4	4	8	56.76	646.62	634.56	124.33	6000	4000
10^8	30	32	12288×384	6	6	5.33	59.03	651.68	631.49	160.97	5800	4000
10^8	30	32	12288×384	8	8	4	64.22	671.01	639.90	201.94	6000	4000
10^8	30	32	12288×384	10	10	3.2	70.29	683.07	640.19	238.19	6000	4000
10^8	30	32	12288×384	12	12	2.67	77.55	697.21	641.28	273.60	6000	4000
10^8	30	32	12288×384	14	14	2.29	81.51	695.09	626.40	301.29	6000	4000
10^8	30	32	12288×384	16	16	2	84.53	688.59	606.83	325.45	6000	4000
10^8	30	32	12288×384	18	18	1.78	88.06	684.88	589.31	348.96	6000	4000
10^8	30	32	12288×384	20	20	1.6	89.93	673.55	565.16	366.41	6000	4000
10^8	100	32	12288×384	2	2	16	56.69	161.43	159.12	27.17	5500	3500
10^8	100	32	12288×384	4	4	8	58.09	163.88	160.43	33.47	6000	4000
10^8	100	32	12288×384	6	6	5.33	60.23	173.48	167.99	43.28	5800	4000
10^8	100	32	12288×384	8	8	4	62.53	178.89	170.97	52.62	6000	4000
10^8	100	32	12288×384	10	10	3.2	65.79	186.90	175.65	63.88	6000	4000
10^8	100	32	12288×384	12	12	2.67	68.81	187.90	173.15	72.98	6000	4000
10^8	100	32	12288×384	14	14	2.29	71.83	186.70	168.44	80.53	6000	4000

Table 5.4: Simulation details for the main cases in figure 5.7(b) for $\Gamma = 16$ and 32. The table head is the same as table 5.3. “s” denotes that the final flow state is steady, which means that both Nu and Re are independent on time once the final state is achieved.

Ra	Pr	$1/Ro$	Γ	$N_x \times N_y \times N_z$	IC	Nu	Re	Re_x	Re_y	Re_z	t_{tot}	t_{avg}
10^7	0.71	3.75	8	$1024 \times 1024 \times 128$	IC ₀	22.32	2282.55	2052.21	641.90	765.78	950	550
10^7	0.71	3.75	8	$1024 \times 1024 \times 128$	IC _c	10.32	1994.96	1929.91	424.65	273.82	3200	2700
10^7	0.71	0	16	$2048 \times 2048 \times 128$	IC ₀	25.39	1184.12	707.09	733.01	604.04	700	400
10^7	0.71	0.1	16	$2048 \times 2048 \times 128$	IC ₀	25.42	1180.84	702.13	728.91	608.35	500	300
10^7	0.71	0.3	16	$2048 \times 2048 \times 128$	IC ₀	25.66	1195.08	756.55	677.92	629.49	500	300
10^7	0.71	1	16	$2048 \times 2048 \times 128$	IC ₀	25.20	1409.11	1030.76	712.38	644.72	700	300
10^7	0.71	3.75	16	$2048 \times 2048 \times 128$	IC ₀	19.19	2395.52	2217.79	680.81	596.99	550	250
10^7	0.71	10	16	$2048 \times 2048 \times 128$	IC ₀	18.98	3357.81	3184.53	741.39	764.23	600	300
10^7	0.71	20	16	$2048 \times 2048 \times 128$	IC ₀	19.97	3713.41	3527.17	784.23	856.41	600	200
10^7	0.71	50	16	$2048 \times 2048 \times 128$	IC ₀	23.44	5436.73	5283.16	532.86	1167.17	1200	600
10^7	0.71	-(2D)	16	4096×256	$n^{(i)} = 2$	25.26	6270.74	6137.35	-	1286.55	3700	s
10^7	0.71	-(2D)	16	4096×256	$n^{(i)} = 4$	37.57	7326.16	6977.15	-	2234.26	3700	s
10^7	0.71	-(2D)	16	4096×256	$n^{(i)} = 6$	45.67	7702.85	7086.30	-	3019.64	3700	s
10^7	0.71	-(2D)	16	4096×256	$n^{(i)} = 8$	51.02	7726.20	6809.66	-	3650.03	3700	s

Table 5.5: Simulations details for 3D RB convection with span-wise rotation. The corresponding 2D simulations are also included for comparison. The table head is similar to that of table 5.3 and 5.4, apart from the inverse Rossby number $1/Ro$ for the 3D cases. IC₀ means initial conditions with zero velocity and conductive temperature profile with random perturbations. IC_c means initial conditions with cyclonic shear flow $u(z) = 2z - 1$, $v = 0$, $w = 0$ and conductive temperature profile with random perturbations.

Figure 5.13: (a) Reynolds number ratio Re_z/Re_x as function of Γ for the zonal flow state and (b) Reynolds number ratio Re_z/Re_x as function of Γ_r for the convection roll state for $Ra = 10^8$ with different Pr . Solid symbols in (b) for $\Gamma = 16$ and hollow symbols for $\Gamma = 32$. The solid symbols can hardly be seen as they are mostly hidden by the hollow symbols. The data in (b) can be well described by the effective scaling relation $Re_z/Re_x = 0.86\Gamma_r^{-0.68}$.

5.6.2 Reynolds number ratio

Figure 5.13(a) shows the Reynolds number ratio Re_z/Re_x as function of Γ for the zonal flow state. The ratio Re_z/Re_x increases with increasing Pr for $Pr \geq 10$, which has a similar trend as Nu discussed before. Figure 5.13(b) shows Reynolds number ratio Re_z/Re_x as function of Γ_r for convection roll states for $Ra = 10^8$ with different Pr . It is remarkable that Re_z/Re_x seems to have a universal dependence on Γ_r for different Pr , despite that Nu has a different trend on Pr for large and small Γ_r . The data can be well represented by the effective scaling relation $Re_z/Re_x = 0.86\Gamma_r^{-0.68}$.

Part II

Tilted convection

Chapter 6

Flow reversals in two-dimensional thermal convection in tilted cells^o

The influence of tilt on flow reversals in two-dimensional (2D) thermal convection in rectangular cells with two typical aspect ratios ($\Gamma = \text{width/height}$), i.e. $\Gamma = 1, 2$, are investigated by means of direct numerical simulations (DNS). For $\Gamma = 1$, tilt tends to suppress flow reversals. However, it is found that flow reversals characterized by two main rolls are promoted by tilt for $\Gamma = 2$, which are even observed for some small Prandtl numbers (Pr) and large tilt angles (β) cases. Different from level cases where the four corner rolls all have opportunities to grow and trigger a flow reversal, the reversals in a counterclockwise tilted cell with $\Gamma = 2$ are always led by the growth of the bottom-right or the top-left corner roll. Tilt is favorable for the growth of the bottom-right or the top-left corner roll and thus breaking the balance between the two main rolls and trigger a flow reversal. The mode decomposition analysis shows that the appearance of the intermediate single-roll mode is crucial for reversals, and flow reversals in a tilted cell with $\Gamma = 2$ can be viewed as a mode competition process between single-roll mode and horizontally adjacent double-roll mode. They can only occur in a limited range of β where the two modes have the comparative strength. Furthermore, the Nusselt numbers at the hot plate Nu_h and at the cold plate Nu_c behave differently during a flow reversal for $\Gamma = 2$ due to the preference of single corner roll growth.

^oBased on: **Qi Wang**, Shu-Ning Xia, Bo-Fu Wang, De-Jun Sun, Quan Zhou, Zhen-Hua Wan, *Penetrative turbulent Rayleigh-Bénard convection in two and three dimensions*, J. Fluid Mech. **870**, 718-734 (2019).

6.1 Introduction

Spontaneous reversal phenomena are common in many buoyancy driven fluid dynamical systems, including reversals of geomagnetic field of the Earth [4], the large scale fluid motion in the sea and the atmosphere [235]. There also exist flow reversals of the large scale circulation (LSC) in a model problem, i.e., Rayleigh-Bénard convection (RBC) where a fluid layer is heated from the below and cooled from the above (see Refs. [53, 56, 236] and references therein). The study of flow reversals in the model system like RBC may enable us to gain more insights into the general reversal phenomena in nature [237].

In cylindrical cells, there are two categories of flow reversals [238]: one is rotation-led reversals, which are caused by incessant azimuthal meandering of the circulating plane. The other is cessation-led reversals in which the LSC slows to a stop and restarts in a random new orientation. Flow reversals in cylindrical cells have been studied both experimentally [158, 238–243] and numerically [244]. Several models either of stochastic ones [245, 246] or of deterministic ones [247] are proposed to account for the flow reversals. The model proposed by Ref. [246] consists of two stochastic ordinary differential equations, and it agrees well with many experimental results for the LSC dynamics. A recent experimental work of Ref. [243] found that the second mode and the remaining higher-order modes play important roles in the dynamical process of the reversal and the Nusselt number Nu drops to its minimum during a flow reversal. It is once thought that cessation-led reversals presumably would prevail in systems where the rotational invariance is broken [238]. However, the cessations of the LSC had not been observed in a cubic cell where the rotational invariance is broken [248]. It is still ambiguous why cessations are absent in a cubic cell.

Without complicated three-dimensional (3D) structures, flow reversals in two-dimensional (2D) [79, 114, 144, 220, 249–254] or quasi-2D [237, 255, 256] square cavities have attracted much attention in recent years. Ref. [114] studied flow reversals in a square cavity with unit aspect ratio by means of 2D DNS combined with the experiments performed in a quasi-2D cell, they found that the results of experiments and simulations are in very good agreement. They also showed that the growth of corner rolls both in kinetic energy and in size plays key roles in triggering flow reversals. They concluded that flow reversals in a 2D/quasi 2D cell occur only in a limited parameter space, that is to say the reversals do not exist for too large or too small Pr as well as too large or too small Ra . Refs. [144, 220] used a mode decomposition method to study flow reversals, they showed that flow reversals are caused by a vortex reconnection of two attracting rolls and the appearance of $(2, 2)$ mode is critical to trigger flow reversals. Refs. [250, 253] used proper orthogonal decomposition to study flow reversals in a 2D square cell and proposed a 3D model which is based on the interaction of the LSC, the quadrupolar flow and horizontal rolls. Ref. [252] found that the reversal cycle is composed of three successive phases: acceleration, accumulation and release including a rebound event. Ref. [256] experimentally found the frequency of flow reversals increases drastically in smaller Γ cells. It should be noted

that 2D/quasi-2D convection promotes reversals particularly for topological reasons. In 3D cubic cells, reversals are very rare [248]. Previous works are almost restricted to a unit aspect ratio cell where there exists only one LSC that spans the size of the convection cell. It has already been realized by Ref. [114] that Γ has a strong effect on the reversal dynamics. For a $\Gamma = 2$ cell, the flow is characterized by two main rolls and flow reversals also exist [220, 251]. However, much less works have been devoted to flow reversals in a $\Gamma = 2$ cell so far.

There are several differences between flow reversals in 2D/quasi-2D and 3D geometries. Firstly, the mean time between successive reversals has a strong Ra dependence in the former [114] while it has a weak Ra dependence in the latter [239, 241]. Secondly, flow reversals in 2D/quasi-2D cells occur in a limited parameter space, and there are no reversals for Ra larger than 5×10^8 (2D) or 1.6×10^9 (quasi-2D) [114]. However, flow reversals in 3D cylindrical cells can occur at larger Ra ($Ra > 10^{10}$) [238, 239, 243] and high Pr ($Pr = 19.4$) [257]. Whether 2D reversals and cessation-led reversals correspond to the same phenomenon is still an open question [252].

Almost all previous studies are conducted in a symmetrical system. However, flow reversals in real world usually take place in an asymmetrical system. One example is that the heat flux on the core-mantle boundary in earth is non-uniform [258] and this has great influence on the frequency of geomagnetic reversals [5]. Another is the solar convection zone that has both torsional oscillations and periodic reversals [259]. Therefore, it is important to study the influence of symmetry breaking on flow reversals. Refs. [239, 260] found that flow reversals in a $\Gamma = 1$ cylindrical container are suppressed significantly even for very small tilt angles. Ref. [261] also found that in a cylindrical container with $\Gamma = 0.5$, a minor tilt of 2° will lock the LSC in the azimuthal plane along the tilted direction, and reversals are suppressed, similar results were also found in a rectangular cell with $\Gamma = 0.5$ [104]. Ref. [79] studied non-Oberbeck-Boussinesq (NOB) effects on flow reversals, they found interestingly, when considering NOB effects, flow reversals can be observed for relatively small $Pr = 0.71$. Ref. [237] studied the influence of thermal boundary conditions on flow reversals. They claimed that the reversal of the LSC can be viewed as a process for the system to restore its symmetry in a statistical way.

It is generally accepted that tilt will suppress flow reversals in a $\Gamma = 1$ cell. However, how will cell tilt influence flow reversals in larger aspect ratio cells where multiple convection rolls may exist is still an open question. In this chapter, we try to fill this gap by studying the influence of cell tilt on flow reversals in a 2D rectangular cell with $\Gamma = 2$ where two convection rolls may develop, the results are also compared with those for $\Gamma = 1$ cases. The reason why we choose $\Gamma = 2$ is that it has been found previously that flow reversals of double roll state exist in a $\Gamma = 2$ cell [220, 251], while whether flow reversals exist for $\Gamma > 2$ is still unknown. There are two advantages on 2D DNS: (i) the numerical effort required for 2D simulations is much smaller and thus we can study flow reversals in a considerable fraction of the Rayleigh number Ra /Prandtl number Pr versus tilt angle β parameter space for different Γ . (ii) The flow visualization is much easier, and detailed flow reversal scenario can be obtained.

In recent years, 2D DNS has been used widely to study turbulent convection, e.g. it was used to study turbulent heat transfer [121, 145, 159, 191, 262], to study the boundary layer (BL) dynamics [122, 204, 263, 264], to reveal the non-Oberbeck-Boussinesq effects on the flow organization [99], to probe the effect of roughness on the heat transfer [148, 265–268], to investigate statistics of kinetic and thermal energy dissipation rates in RBC [269] and more recently to explore the transition to the ultimate regime [183].

The reminder of this chapter is organized as follows. In §6.2, we introduce the governing equations and numerical methods. The main results are presented in §6.3, which is divided into three parts. Section 6.3.1 studies the influence of tilt on flow reversals in $\Gamma = 1$ and $\Gamma = 2$ cells. In §6.3.2, we use a mode decomposition method to reveal the inherent mechanism for the reversals in a tilted $\Gamma = 2$ cell. Section 6.3.3 discusses the heat transfer properties during the reversals. Finally, a short summary is presented in §6.4.

6.2 Numerical settings

We studied flow reversals in a tilted cell with width $\hat{W}$ and height $\hat{H}$ (hatted quantities are dimensional), the bottom plate has a constant high temperature $\hat{T}_H$ while the top plate has a constant low temperature $\hat{T}_C$. Here, with Oberbeck-Boussinesq approximation the dimensionless governing equations read

$$\nabla \cdot \mathbf{u} = 0, \quad (6.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \sqrt{\frac{Pr}{Ra}} \nabla^2 \mathbf{u} + T \vec{e}, \quad (6.2)$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{1}{\sqrt{RaPr}} \nabla^2 T, \quad (6.3)$$

where $\mathbf{u} = (v, w)$ and $\mathbf{x} = (y, z)$ are the velocity components and coordinates in the horizontal and vertical directions, p is the reduced kinetic pressure, T is the temperature, $\vec{e}$ is a unit vector defined by $\vec{e} = (\sin(\beta), \cos(\beta))$. β is the tilt angle which is positive when the cell rotates counterclockwise. For non-dimensionalization, the free fall velocity $\hat{U} = (\hat{g}\hat{\alpha}\Delta\hat{T}\hat{H})^{1/2}$ and the height of the container $\hat{H}$ are used, where $\hat{g}$ is the gravitational acceleration, $\hat{\alpha}$ is the isobaric thermal expansion coefficient, $\Delta\hat{T} = \hat{T}_H - \hat{T}_C$ is the temperature difference. the reference time is free-fall time $\hat{t}_f = \hat{H}/\hat{U}$, the dimensionless temperature is defined by $T = (\hat{T} - \hat{T}_C)/\Delta\hat{T}$. This problem is governed by three dimensionless parameters, i.e., the Prandtl number Pr , the Rayleigh number Ra and the aspect ratio Γ :

$$Pr = \frac{\hat{\nu}}{\hat{\kappa}}, Ra = \frac{\hat{g}\hat{\alpha}\Delta\hat{T}\hat{H}^3}{\hat{\nu}\hat{\kappa}}, \Gamma = \frac{\hat{W}}{\hat{H}}, \quad (6.4)$$

where $\hat{\kappa}$ is the thermal diffusivity, and $\hat{\nu}$ the kinematic viscosity. No-slip and impermeability conditions are imposed at all walls and the sidewalls are insulated.

The governing equations are solved numerically by a well-validated in-house code lMn2d. All spatial terms are discretised using second-order central difference scheme. A non-uniform grid with clustered grid points near walls is used to resolve small scales inside boundary layers. Pressure and velocities are staggered in space in order to avoid pressure-velocity decoupling. Time integration is accomplished by using the Adams-Bashforth scheme for the nonlinear terms and Crank-Nicolson scheme for the viscous and diffusion terms. Detailed numerical schemes and validations of this code can be found in our previous work [79].

The simulation parameters of this chapter are listed in table 6.1. The initial field for cases with $\beta = 0^\circ$ is conductive state with small perturbations added to the temperature field, while for tilted cases we use the flow field of $\beta = 0^\circ$ as the initial condition. In all cases, the simulations run for a long time to determine whether flow reversals exist. Moreover, in order to obtain better statistics of the flow reversal frequencies, the longest simulation is performed for 249,000 free-fall time units.

Γ	Ra	Pr	β	$N_x \times N_z$
1	$[10^7, 10^8]$	2	$[0^\circ, 20^\circ]$	256×256
1	5×10^7	$[1.5, 5]$	$[0^\circ, 20^\circ]$	256×256
2	10^7	$[0.05, 4.38]$	$[0^\circ, 90^\circ]$	512×256
2	$[10^7, 10^8]$	0.3	$[0^\circ, 60^\circ]$	512×256

Table 6.1: Simulation parameters. The columns from left to right indicate the following: Γ , Ra , Pr , β and mesh resolutions.

6.3 Results and discussion

6.3.1 Flow reversals in two-dimensional tilted cells

We start by studying the influence of tilt on flow reversals for a typical aspect ratio $\Gamma = 1$. Figure 6.1 shows snapshots of the temperature and velocity fields during one of the reversals at $Ra = 5 \times 10^7$, $Pr = 2$ and $\beta = 6^\circ$. Although the symmetry of the system is broken, it is found that the reversal process is similar to the level case [114]. The two corner rolls grow simultaneously in kinetic energy and also in size due to the plume detachment from the boundary layers, until they merge as a new LSC with the opposite direction, and a flow reversal happen. It is well known that flow reversals occur in a certain $Ra - Pr$ parameter space for $\Gamma = 1$. For a system with only one main roll, the reversal process can be nicely indicated by a global quantity, i.e., the net angular momentum:

$$L(t) = \langle -(z - 1/2)v(\mathbf{x}, t) + (y - 1/2)w(\mathbf{x}, t) \rangle_V, \quad (6.5)$$

where $\langle \cdot \rangle_V$ represents averaging over the total area. Figure 6.2 shows time evolution of L for the level case with $\beta = 0^\circ$ and the tilted case with $\beta = 6^\circ$ at $Ra = 5 \times 10^7$

Figure 6.1: Three snapshots of the temperature field superposed with velocity vectors during one of flow reversals for $Ra = 5 \times 10^7$, $Pr = 2$, $\beta = 6^\circ$ at different times. (a) $t = 500$. (b) $t = 655$. (c) $t = 705$.

and $Pr = 2$. A sign change of L can reveal a reversal, the frequent flow reversals in a level cell can be observed in figure 6.2(a). From figure 6.2(b), it is evident that the symmetry breaking induced by tilt greatly suppress flow reversals for $\Gamma = 1$. One sees that the clockwise main roll with negative L stays for quite a short time before reversing its orientation. This is because for the clockwise LSC, as shown in figure 6.1(c), tilt introduces a horizontal buoyancy force along the y -direction so that the hot plumes are more likely to move up to the right-bottom corner roll and the cold plumes tend to move down to the left-top corner roll. The two corner rolls have more opportunities to grow and trigger a reversal. Besides, the tilt may weaken the clockwise LSC since along the hot and cold plate the horizontal buoyancy force is in the opposite direction of the flow velocity of LSC. Thus the clockwise LSC is less stable and is easier to reverse its orientation, and it always stays for a much shorter time as shown in figure 6.2(b). Previous studies have shown that the main roll will be locked in the plane along the tilted direction both in cylindrical cells [59, 143, 239] and in rectangular cells [270, 271] with $\Gamma = 1$. Thus the suppression of flow reversals by tilt for $\Gamma = 1$ is within expectation.

We then map out phase diagrams for $\Gamma = 1$ over $\beta - Pr$ parameter space with $Ra = 5 \times 10^7$ and over $\beta - Ra$ parameter space with $Pr = 2$ in figure 6.3. We run at least 5 000 large eddy turnover time t_E to detect whether there exist flow reversals. t_E is defined by $t_E := 4\pi / \langle |\omega_c(t)| \rangle$, where ω_c denotes vorticity in the center of the main roll. For some cases (for example, $Ra = 10^8$, $Pr = 2$, $\Gamma = 1$, $\beta = 3^\circ$), we even calculate more than 10 000 t_E to conclude there are indeed no reversals. As shown in figure 6.3(a), the critical angle β_c displays a non-monotonic relation with Pr . For larger Pr , the β_c is smaller, for $Pr = 5$, a minor tilt of 1° will completely suppress flow reversals due to the fact that the thickness of thermal boundary layer (BL) is much smaller than the thickness of kinetic BL and thermal coupling between the corner roll and the thermal BL is weakened, as stated in prior study [114]. For small Pr , the thermal diffusion becomes dominant and suppresses the growth of the corner roll. Therefore,

Figure 6.2: Time evolution of the angular momentum L for $\Gamma = 1$, $Ra = 5 \times 10^7$, and $Pr = 2$. (a) $\beta = 0^\circ$. (b) $\beta = 6^\circ$. The positive and negative signs indicate counterclockwise and clockwise circulations, respectively.

flow reversals in the tilted cell are also easier to occur at intermediate Pr , and the corresponding β_c is also larger. In figure 6.3(b), the β_c also displays a non-monotonic relation with Ra . For intermediate Ra , the corresponding β_c is larger. The reason lies in the non-monotonic dependence of the flow reversal frequency on Ra . It can be seen in figure 3 of [114] that flow reversals are most frequent for intermediate Ra (about 7×10^7 for $Pr = 4.3$), any increase or decrease of Ra will reduce flow reversal frequency. For the $Pr = 2$ considered in our work, the reversals are most frequent for $Ra \sim 4 \times 10^7$ thus the corresponding β_c is larger, and any increase or decrease of Ra will reduce β_c . The reversals occur less and less frequently with increasing Ra and reversals disappear when $Ra \geq 2 \times 10^8$ for $Pr = 2$ [114]. Thus the main roll here is more sensitive to tilt at larger Ra , and a minor tilt of the cell will suppress flow reversals at relatively high Ra .

Compared with $\Gamma = 1$, there are less studies devoted to horizontally adjacent double-roll reversals in a $\Gamma = 2$ cell [220, 251]. Here, we first explore the flow reversals in a $\Gamma = 2$ cell at a fixed $Ra = 10^7$, but varied Pr and β , as listed in table 6.1. The Ra dependence for a fixed $Pr = 0.3$ is also studied. Figure 6.4 shows flow reversal scenarios for a level case with $Ra = 10^7$, $Pr = 0.71$ and $\Gamma = 2$, it is seen that the single corner roll growth leads to a flow reversal, which is qualitatively the same as that of [251]. Due to the inherent symmetry of the system, the two bottom corner rolls in figure 6.4(a) and the two top corner rolls in figure 6.4(f) all have opportunities to grow and trigger flow reversals. However, when the symmetry of the system is broken by tilt, the two corner rolls will no longer play equivalent roles.

Figure 6.5 shows snapshots for two successive flow reversals at $Ra = 10^7$, $Pr =$

Figure 6.3: (a) Phase diagrams in the $\beta - Pr$ plane for $Ra = 5 \times 10^7, \Gamma = 1$. (b) Phase diagrams in the $\beta - Ra$ plane for $Pr = 2, \Gamma = 1$. Red filled diamonds ($\blacklozenge$) correspond to detected flow reversals, green squares ($\blacksquare$) to a single roll state without reversals.

Figure 6.4: Six snapshots of the temperature field superposed with velocity vectors during one of flow reversals for $Ra = 10^7, Pr = 0.71, \beta = 0^\circ$ at different times. (a) $t = 12283$. (b) $t = 12338$. (c) $t = 12349$. (d) $t = 12354$. (e) $t = 12361$. (f) $t = 12377$.

Figure 6.5: Nine snapshots of the temperature field superposed with velocity vectors during two of flow reversals for $Ra = 10^7$, $Pr = 0.71$, $\beta = 1^\circ$ at different times. (a) $t = 1684.5$. (b) $t = 1701$. (c) $t = 1704$. (d) $t = 1710$. (e) $t = 1711.5$. (f) $t = 1725$. (g) $t = 2860.5$. (h) $t = 2869.5$. (i) $t = 2890.5$.

0.71 and $\beta = 1^\circ$. As shown in figure 6.5(a), the flow field is characterized by two main rolls and two bottom corner rolls. Since tilt will introduce a horizontal buoyancy force along the y -direction, the hot plumes detached from the lower BL will bear a top-right buoyancy force and be apt to feed the bottom-right corner roll. This is favorable for the growth of the bottom-right corner roll as illustrated in figure 6.5(b). The bottom-left corner roll, however, almost remains unchanged. The subsequent reversal process shown in figure 6.5(c) - 6.5(f) is similar to that in a level case. Similarly, for the next reversal cycle, due to the y -direction buoyancy force introduced by tilting the cell, the cold plumes detached from the upper BL are more likely to enter the top-left corner roll. Thus the top-left corner roll growth will lead to a flow reversal, as shown in figure 6.5(g) - 6.5(i). It should also be noted that the horizontal buoyancy force induced by tilt near the top and bottom walls is always in the opposite direction of the flow velocity of one of the two main rolls. Thus the effect of tilt on the main rolls may always play some roles in flow reversals in a tilted $\Gamma = 2$ cell, and the unstable clockwise main roll will always first reverse its direction. Due to the symmetry breaking, only the bottom-right corner roll in figure 6.5(a) and the top-left corner roll in figure 6.5(f) have opportunities to grow and lead to a flow reversal, this is different from a level case where the four corner rolls all have opportunities to grow and trigger a flow reversal. This argument can be quantitatively demonstrated by investigating the long time evolution of L as shown in figure 6.6. Note the L is

Figure 6.6: Time evolution of angular momentum L for $Ra = 10^7$, $Pr = 0.71$ and $\Gamma = 2$. (a) $\beta = 0^\circ$. (b) $\beta = 0.5^\circ$. (c) $\beta = 1^\circ$

Figure 6.7: Flow reversal frequency f as a function of β . Red circles ($\circ$) for $Ra = 5 \times 10^7$, $Pr = 2$, $\Gamma = 1$, blue diamonds ($\diamond$) for $Ra = 10^7$, $Pr = 0.71$, $\Gamma = 2$.

calculated with respect to the center point (1,0.5) here. The flow is dominated by the double-roll mode with $L \simeq 0$ most of the time due to compensation of two main rolls. When the flow reversal occurs, the clockwise/counterclockwise intermediate single-roll mode appears (see figure 6.4(c) and figure 6.5(c)), thus large negative/positive L will appear, indicated by sharp spikes in figure 6.6. The large positive L indicates that it is the growth of the bottom-right corner roll shown in figure 6.5(a) or the top-left corner roll shown in figure 6.5(f) that leads to flow reversals, while the large negative L indicates that it is the growth of the bottom-left corner roll shown in figure 6.4(a) or the top-right corner roll shown in figure 6.4(f) that leads to a successful flow reversal. It is seen that both positive and negative spikes exist for the level case while only positive spikes exist for the tilted case. Thus only the growth of the bottom-right corner roll or the top-left corner roll will lead to a flow reversal in a tilted cell. From figure 6.6, we can also see that there are much more flow reversals in tilted cells, demonstrating that tilt will promote flow reversals for $\Gamma = 2$. To further quantify the promoting effect of cell tilt on the flow reversal, the reversal frequencies f defined by reversal numbers per free-fall time unit are calculated, which are plotted in figure 6.7. Clearly, for $\Gamma = 1$, f does not drop much for small $\beta \leq 2^\circ$, while for $\beta > 2^\circ$, f drops rapidly. On the contrary, f increases monotonically with β for $\Gamma = 2$, it even increases more than an order from $\beta = 0^\circ$ to 1° . Thus a minor tilt will greatly promote flow reversals for $\Gamma = 2$.

Again we map out flow states in $\beta - Pr$ space for $Ra = 10^7$ and $\Gamma = 2$ in figure 6.8(a). We run at least $4\,000\,t_E$ to detect whether there exist flow reversals. As the position of the center of the main roll changes after reversals (this is especially evident for reversals with large β). The large eddy turnover time t_E is better defined

Figure 6.8: (a) Phase diagrams in the $\beta - Pr$ plane for $Ra = 10^7, \Gamma = 2$. (b) Phase diagrams in the $\beta - Ra$ plane for $Pr = 0.3, \Gamma = 2$. Red filled diamonds ($\blacklozenge$) correspond to detected flow reversals, blue filled circles ($\bullet$) denote double-roll state without flow reversals, green squares ($\blacksquare$) represent single-roll state without flow reversals.

by $t_E := (W + 2H)/u_{LSC}$, where u_{LSC} is the typical velocity of the LSC measured at $\sim H/10$ from the side walls and plates [255, 264]. Three different flow states are identified, which are the double-roll state without flow reversals, the double-roll state with flow reversals, and the single-roll state without flow reversals, respectively. One sees that for $Pr \leq 0.5$ and small β , the flow stays in the stable double-roll state without flow reversals. When β is beyond a critical value, the double-roll state is not stable anymore and flow reversals occur at intermediate range of β . However, when β increases further, the flow becomes stable one-roll state and no reversals can be observed anymore. For $0.71 \leq Pr \leq 4.38$, the flow transits directly from reversal state to one-roll state. It is interesting that for relatively small $Pr \leq 0.5$, flow reversals can only occur in tilted cells. It is worth mentioning that the reversals even occur at very small Pr and large β . For instance, there exist flow reversals for $Pr = 0.05$ and $\beta = 70^\circ$, as seen in figure 6.8(a). This is in stark contrast to that in a $\Gamma = 1$ cell where no reversals exist for $Pr \leq 0.71$ [114]. As flow reversals in tilted cells with $\Gamma = 2$ tend to occur for small Pr , thus the Ra dependence of flow reversals is studied for a representative small $Pr = 0.3$. Figure 6.8(b) maps out flow states over the $\beta - Ra$ parameter space for fixed $Pr = 0.3$. It is obvious that flow reversals can only be found in tilted cells, which even exist at $Ra = 10^8$. Based on the above findings, we show that the symmetry breaking of the system induced by tilt has considerably broadened the original $Ra - Pr$ parameter space of flow reversals for $\Gamma = 2$. However, the crucial question left in this section is that why reversals only occur in a limited range of β for $\Gamma = 2$.

6.3.2 Mode decomposition analysis

To find the answer to the question left in §6.3.1, a mode decomposition method is employed. We project the velocity components v and w onto the following Fourier basis:

$$\hat{v}^{m,n} = \sqrt{2} \sin(m\pi y/2) \cos(n\pi z), \quad (6.6)$$

$$\hat{w}^{m,n} = -\sqrt{2} \cos(m\pi y/2) \sin(n\pi z). \quad (6.7)$$

The velocity can be recovered by:

$$v(y, z, t) = \sum_{m,n} A_y^{m,n}(t) \hat{v}^{m,n}, \quad (6.8)$$

$$w(y, z, t) = \sum_{m,n} A_z^{m,n}(t) \hat{w}^{m,n}. \quad (6.9)$$

The projection is done component-wisely by a scalar product in the L_2 -space. The corresponding time series $A_y^{m,n}(t)$ and $A_z^{m,n}(t)$ can be obtained by:

$$A_y^{m,n}(t) = \int_0^{z_{max}} \int_0^{y_{max}} v(y, z, t) \hat{v}^{m,n} dy dz, \quad (6.10)$$

$$A_z^{m,n}(t) = \int_0^{z_{max}} \int_0^{y_{max}} w(y, z, t) \hat{w}^{m,n} dy dz. \quad (6.11)$$

A value $M^{m,n}(t)$ representing the contribution of different modes can be evaluated by $M^{m,n}(t) = \sqrt{(A_y^{m,n}(t))^2 + (A_z^{m,n}(t))^2}$. The mode (m, n) corresponds to a flow structure with m rolls in the y direction and n rolls in the z direction. Although Fourier basis functions do not satisfy the no-slip boundary conditions, it has been shown that they capture the convective flow profiles well [144, 220, 249, 251]. The Fourier mode decomposition method has been employed extensively to study 2D reversals [144, 220, 249, 251] and heat transfer in quasi 2D cells [118, 147].

We quantitatively analyze the flow mode competition for a representative case with $Ra = 10^7$, $Pr = 0.5$ and $\Gamma = 2$ where flow reversals only happen in a limited β range, i.e., $7^\circ \leq \beta \leq 13^\circ$. Figure 6.9 shows time evolution of the amplitudes of dominated modes (2,1) and (1,1), and the vertical velocity at the center of the cell w_c for $\beta = 8^\circ$. A sign change of w_c can nicely indicate a reversal. The flow near the times $t = 890$ and 2110 which are denoted by vertical dash dotted lines can be viewed as cessations in a $\Gamma = 2$ cell. The growth of corner rolls does not trigger successful reversals near these two moments. In figure 6.9, it is found that the transient $M^{2,1}(t)$ approaches zero during reversals, while $M^{1,1}(t)$ shows a spike. The vanishing of (2,1) mode and the appearance of (1,1) mode correspond to the appearance of the counterclockwise single roll structures during the reversal. So we conclude that the appearance of single-roll mode (1,1) plays a critical role in triggering a reversal, similar to that in a $\Gamma = 1$ cell where the (2,2) mode plays the dominant role.

Figure 6.9: For $Ra = 10^7$, $Pr = 0.5$ and $\beta = 8^\circ$, time evolution of (a) Flow mode amplitude of $M^{2,1}(t)$ (red line) and $M^{1,1}(t)$ (black line). (b) Vertical velocity component w_c at the center location $(1, 0.5)$.

Figure 6.10: Time averaged flow mode amplitudes $M^{1,1}$ and $M^{2,1}$ as a function of β for $Ra = 10^7$, $Pr = 0.5$ and $\Gamma = 2$, the grey area denotes the angle at which flow reversals appear.

To understand why the flow reversals occur in a limited range of β . The bifurcation diagram is further shown in figure 6.10. One sees that the flow is dominated by (2,1) mode for $0^\circ \leq \beta \leq 6^\circ$ and the amplitude of (1,1) mode approaches 0, which is referred to as regime I. The $M^{1,1}$ starts to grow rapidly beyond $\beta = 7^\circ$, then flow reversals occur for $7^\circ \leq \beta \leq 13^\circ$, and we call this regime II. In this regime, the flow is still dominated by (2,1) mode, but the $M^{2,1}$ and $M^{1,1}$ are comparable. For $15^\circ \leq \beta \leq 90^\circ$, the flow is dominated by (1,1) mode and the amplitude of (2,1) mode is close to 0, which is called regime III. It can be seen that the $M^{1,1}$ decreases monotonically with increasing β in this regime, this is because the flow strength which is denoted by the Reynolds number Re decreases monotonically with increasing β in this regime [59, 270]. It is clear that the reversals tend to happen in the intermediate β range where the (2,1) mode and (1,1) mode have the comparative strength. Based on the mode decomposition analysis, we conclude that the flow reversals in tilted cells for $\Gamma = 2$ can be viewed as a mode competition process between (2,1) mode and (1,1) mode.

6.3.3 Heat transfer during flow reversals

Finally, we discuss the heat transfer properties during a flow reversal. It is well recognized that the heat transfer is closely connected to the mean flow structures both in two dimensions [159] and three dimensions [115, 116, 158, 261, 272–274]. As shown above, during a flow reversal, the flow structures change greatly, so the significant change of heat transfer is expected. The Nusselt number (Nu), used to measure heat

Figure 6.11: Time evolution of (a) vertical velocity at the cell center w_c . (b) Nusselt number at the hot wall Nu_h . (c) Nusselt number at the cold wall Nu_c . for $Ra = 10^7$, $Pr = 0.71$ and $\beta = 1^\circ$.

transport, is defined as

$$Nu(t) = \frac{\hat{Q}}{\hat{k}\Delta\hat{T}/\hat{H}} = \frac{\left(\langle\hat{\rho}\hat{c}_p\hat{w}\hat{T}\rangle_A + \langle\hat{k}\partial_z\hat{T}\rangle_A\right)}{\hat{k}\Delta\hat{T}/\hat{H}} \quad (6.12)$$

where $\hat{Q}$ is heat flux across any horizontal plane (3D)/line (2D). $\langle\cdot\rangle$ denotes averaging over any horizontal plane (3D)/line (2D). Here, the Nu averaged over the hot and cold plate are denoted as Nu_h and Nu_c , respectively.

Figure 6.11 shows time evolution of the vertical velocity component at the cell center w_c , Nu_h and Nu_c for $Ra = 10^7$, $Pr = 0.71$, $\Gamma = 2$ and $\beta = 1^\circ$. We distinguish the two kinds of reversals by the sign change of w_c , referred to as REV1 and REV2. For REV1, the Nu_h experiences a minor decrease, and then a overshoot, after relatively a long time, it returns to its normal value. While the Nu_c has a drastic decrease during the reversal, after a overshoot, it return to its normal value quickly. For REV2, the behaviour of Nu_h is similar to that of Nu_c in REV1 and the Nu_c is similar to that of Nu_h in REV1. These phenomena are similar to that in a level case. The most significant feature for $\Gamma = 2$ is that Nu_h and Nu_c behave differently during a reversal.

To better illustrate this point, the time evolution of ensemble-averaged Nu_h and Nu_c are shown in figure 6.12. For $\Gamma = 2$, we first locate the data point where w_c becomes zero corresponding to the appearance of single-roll mode during reversals. Then starting from this data point we go forward and backward for 500 free-fall time

Figure 6.12: Time evolution of ensemble-averaged Nu_c and Nu_h during flow reversals for (a,b) $Ra = 10^7$, $Pr = 0.71$, $\beta = 1^\circ$, and $\Gamma = 2$ and (c,d) $Ra = 5 \times 10^7$, $Pr = 2$, $\beta = 0^\circ$, and $\Gamma = 1$. (a) REV1. (b) REV2. (c) The reversal from the clockwise single roll mode to the counterclockwise single roll mode. (d) The inverse process of (c).

units. We averaged for 59/62 samples for REV1/REV2, and the results are plotted in figure 6.12(a) and 6.12(b). Similar procedures were also used before [243, 252] and in the study of statistical properties of geomagnetic field reversals [275]. One can see from figure 6.12(a) that both Nu_h and Nu_c drop to their minima at $t \approx 0$ corresponding to the appearance of single roll mode during REV1. The Nu_c has a more drastic decrease than Nu_h , due to apparent top-down asymmetric flow structures as shown in figure 6.5(c). There is also a small overshoot for Nu_h before reaching its local minimum. After reaching their local minima, both Nu_h and Nu_c increase rapidly, the Nu_c experience some fluctuations around its local maximum, and then returns to its normal value. While for Nu_h , two local maxima are found, and it takes more time for Nu_h to return to its normal value. For REV2, the behaviors of Nu_h and Nu_c are reversed compared to REV1, as shown in figure 6.12(b). For comparison, Nu_h and Nu_c for $\Gamma = 1$ during two kinds of flow reversals are also plotted in figure 6.12(c) and 6.12(d). The time $t = 0$ is the time when the global angular momentum L becomes 0, corresponding to the appearance of (2,2) mode during the reversal, which is top-down symmetric. It is clear that Nu_h and Nu_c always have the same trend. It is interesting that the overshoot phenomenon is observed which is similar to that in a cylindrical cell [243]. In short, the heat transfer is directly connected with flow structures, the different trend of Nu_h and Nu_c during a reversal in a $\Gamma = 2$ cell is due to preference of single corner roll growth during a reversal. While for $\Gamma = 1$, two corner rolls grow simultaneously, and flow structure is symmetrical about the horizontal centerline, thus the Nu_c and Nu_h behave the same.

6.4 Conclusions

In this work, we investigate the effect of cell tilt on flow reversals in $\Gamma = 1$ and 2 cells by performing 2D DNS. It is found that flow reversals are suppressed by tilt for $\Gamma = 1$, however, flow reversals characterized by two main rolls are greatly promoted by tilt for $\Gamma = 2$. For $\Gamma = 1$, flow reversals in tilted cells are led by the simultaneous growth of two corner rolls, which is qualitatively the same as those in level cases. For $\Gamma = 2$, due to the introduction of horizontal buoyancy force by tilt, flow reversals in a tilted cell are always led by the growth of the bottom-right or the top-left corner roll, which is different from those in a level case where the four corner rolls all have opportunities to grow and trigger a reversal. Tilt is favorable for the thermal plumes to feed the bottom-right or the top-left corner roll, thus breaking the balance between the two main rolls and leads to a reversal. As a result, the reversals are promoted by tilt and are much more frequent than those in level cells. Moreover, flow reversals in tilted cells with $\Gamma = 2$ tend to occur in small Pr . Tilt broadens Ra - Pr parameter space for flow reversals to happen in a $\Gamma = 2$ cell. Our study expands the flow reversals in a canonical system to small Pr which is relevant to many geophysical and astrophysical flows [276, 277]. By exploiting the large-scale mode decomposition analysis, we demonstrate that flow reversals in a tilted cell with $\Gamma = 2$ can be viewed as a mode competition process between the single-roll mode and

horizontally adjacent double roll mode. Reversals can only occur in an intermediate range of β where $M^{1,1}$ and $M^{2,1}$ are comparable. Furthermore, we find that Nu_h and Nu_c behave the same during a reversal for $\Gamma = 1$, however, they behave differently for $\Gamma = 2$ due to preference of single corner roll growth during a reversal.

The present finding of more frequent flow reversals in a tilted $\Gamma = 2$ cell demonstrates that the effect of symmetry breaking on flow reversals are system-dependent, which should still be explored in asymmetrical systems to gain more physical insights into the various reversal phenomena in nature.

Chapter 7

Flow organization and heat transfer in two-dimensional tilted convection with aspect ratio 0.5°

Thermal convection in a two-dimensional tilted cell with aspect ratio (Γ =width/high) $1/2$ is studied using direct numerical simulations (DNS). The considered tilt angle β ranges from 0° to 90° . The Prandtl number Pr dependence is first studied in the range of $0.01 \leq Pr \leq 100$ for a fixed Rayleigh number $Ra = 10^7$. The Ra dependence is also investigated in the range of $10^6 \leq Ra \leq 10^9$ for a fixed $Pr = 0.71$. Different flow states are identified over the β - Pr parameter space. It is found that the flow tends to organize itself in a stable vertically-stacked double-roll state (DRS) for small Pr and small β , while this DRS becomes unstable and flow reversals happen with the increase of β . This finding complements our previous study of flow reversals in tilted cells with $\Gamma = 1$ and 2 (Wang *et al.*, *J. Fluid Mech.*, vol. 849, 2018, 355-372). For relatively larger Pr , the flow gives way to stable triple-roll state or unstable triple-roll state for small β . Moreover, multiple states in the turbulent regime are found for $Ra \geq 10^8$, between which the flow can or can not switch. In the latter case the Nu are different for the two states with the same number of convection rolls but different orientations. It is found that the $Nu(\beta)/Nu(0)$ and $Re(\beta)/Re(0)$ dependence is strongly influenced by a combination of Ra and Pr . In the present system, we interestingly find that the earlier conclusion that Nu decreases with increasing β close to $\beta = 90^\circ$ for $\Gamma = 1$ does not hold for the present $\Gamma = 1/2$ case with small Pr .

^oBased on: **Qi Wang**, Zhen-Hua Wan, Rui Yan, De-Jun Sun, *Flow organization and heat transfer in two-dimensional tilted convection with aspect ratio 0.5*, *Phys. Fluids*. **31**, 025102 (2019).

7.1 Introduction

Fluid motion driven by a temperature gradient is ubiquitous in nature and is also important in many industrial processes. Two classical models are frequently used to study thermal convection: one is Rayleigh-Bénard convection (RBC) [53–56] where a fluid layer in a cavity is heated from the below and cooled from the above, the other is vertical convection (VC) [57, 58, 109, 149, 278] where the cavity is heated and cooled from the sides. In practice, the angle β between the gravity acceleration $\hat{g}$ and the inverse temperature gradient $-\nabla\hat{T}$ often varies in the range of $0^\circ \leq \beta \leq 90^\circ$ ($\beta = 0^\circ$ corresponds to RBC while $\beta = 90^\circ$ denotes VC). A relevant example is the process of crystal growth from melts in tilted ampoules with axial heating, larger mass transport rates are typically obtained by slightly tilting the horizontal Bridgman configuration [66]. In the same way, the heat transfer in heat exchangers and thermosiphons can be enhanced by selecting an optimal tilt angle [66]. This has been confirmed in recent experiments [279–282] and simulations [283] with low-Prandtl-number fluids, which show that any tilt angle β would increase the Nusselt number Nu , compared to that of RBC.

The influence of minor tilting on RBC has been investigated extensively over the past decades [155, 160, 261, 284–288]. For a convection cell with aspect ratio $\Gamma = 1$, Ref. [284] experimentally found that the Nu did not change for $Ra = 5 \times 10^{10}$, $Pr = 3$ with $\beta = 10^\circ$. A similar conclusion was drawn by Ref. [285] for mercury with $Pr \approx 0.025$, where they found that the heat transfer is unchanged (by no more than 1% even for a angle of 4°). Ref. [287] found a very small reduction of Nu by about 0.5% for $\beta = 10^\circ$, although the changes of amplitudes and dynamics of the large scale circulation (LSC) were found to be substantial, and they discovered a much stronger Reynolds number Re dependence on the tilt angle β , namely $Re(\beta)/Re(0) \approx 1 + (1.85 \pm 0.21)|\beta| - (5.9 \pm 1.7)\beta^2$. These results are all performed with relatively small β . In recent years, thermal convection in tilted cells with a wide range of β ($0^\circ \leq \beta \leq 90^\circ$) has also been investigated [59, 104, 106, 107, 270, 271, 282, 283, 289–291]. Ref. [270] experimentally studied thermal convection in a tilted cavity for $Ra \approx 4.3 \times 10^9$ and $Pr \approx 6.7$, they found that the LSC is sensitive to the symmetry of the system and the LSC changes gradually from oblique ellipse-like to square-like, they also found that the Nu monotonically decreases with increasing β for $0^\circ \leq \beta \leq 90^\circ$. Ref. [59] performed detailed direct numerical simulations (DNS) for $10^6 \leq Ra \leq 10^8$ and $0.1 \leq Pr \leq 100$ in a cylindrical cell with $\Gamma = 1$, they showed that $Nu(\beta)/Nu(0)$ dependence is not universal and is strongly influenced by a combination of Ra and Pr , and the $Nu(\beta)/Nu(0)$ dependence is a complicated, non-monotonic function of β . Ref. [271] performed experimental study of the evolution and statistics of thermal plumes in tilted convection using the shadowgraph technique. A recent experimental work of Ref. [282] using liquid sodium with Prandtl number 0.0094 found that the Nu is minimal at $\beta = 0^\circ$ and demonstrates a smooth increase with increasing β , reaching a maximum at $\beta = 70^\circ$ and decreasing with a further increase of β . The maximal value of $Nu(\beta)/Nu(0)$ was 1.21. Ref. [107] studied the effect of cell tilting

on the flow reversals of the large-scale circulation (LSC). They found interestingly that tilting will promote flow reversals of the horizontal-adjacent double-roll state for $\Gamma = 2$, which is different from $\Gamma = 1$ cases where flow reversals are greatly suppressed by tilting. Ref. [291] studied tilted convection with large Prandtl number $Pr \approx 480$ using a combination of experiments and 2D DNS. The general trends of $Nu(\beta)/Nu(0)$ as a function of β were found to agree with each other for quasi-2D experiments and 2D DNS. The global heat transfer was found to be insensitive to the drastic reduction of turbulent intensity. The maximum enhancement of Nu was about 20% for $Ra = 10^8$ and 10% for 5×10^8 . More recently, Ref. [106] reported multiple states for tilted convection with large aspect ratios, they also found that Nu generally decreases monotonically with increasing β for $\Gamma \geq 8$.

Apart from $\Gamma = 1$, another well used aspect ratio in the studies of RBC is $\Gamma = 1/2$. In fact, this geometry has been used in many experiments [292, 293] and simulations [227, 294–296] to achieve higher Ra . However, compared with unit aspect ratio, there are less works devoted to tilted convection with $\Gamma = 1/2$. An experimental study by Ref. [286] that uses water with $Pr \approx 2$ in a $\Gamma = 1/2$ cylindrical cell found a decrease of Nu up to 5% for tilt angles as small as 1.5° , another measurements in a cylindrical cell by Ref. [261] ($\Gamma = 1/2, Pr = 5.3$) also found a decrease of Nu by about 2% for $\beta = 2^\circ$. Ref. [297] also found that for $\beta = 3.6^\circ$ the Nu was reduced by about 1.5% in comparison with that for $\beta = 1.3^\circ$. However, Ref. [160] did not confirm these results and instead they found a very small increase of the heat transport, they explained this increase by a stabilization of the single-roll state of the LSC and a destabilization of the double-roll state. It is still ambiguous how the Nu is modified by tilting the cell for relatively high Ra with $\Gamma = 1/2$. A recent numerical work [104] indeed observed an increase of Nu as the cell is minor tilted for $Ra = 10^7$, however, the Ra is much lower than that in the previous experiments. Most of the available studies on tilted convection in a $\Gamma = 0.5$ cell are performed for relatively small β close to 0° , and are restricted to a fluid with fixed Pr . Thus a systematic study of the Pr dependence is still lacking, which motivates us to investigate tilted convection with $\Gamma = 1/2$ over a wide range of both β - Pr and β - Ra parameter space. Another motivation of this study is the flow reversals in a $\Gamma = 1/2$ cell. Flow reversals in a two-dimensional(2D)/quasi 2D cell with $\Gamma = 1$ [79, 114, 144, 220, 237, 251, 252, 255, 256] and $\Gamma = 2$ [107, 220, 251] have been reported in previous studies. However, flow reversals in a 2D/quasi-2D cell with $\Gamma = 1/2$ is largely unexplored and many questions remain open. For example, do flow reversals exist in a 2D/quasi-2D cell with $\Gamma = 1/2$? Can the conclusion that tilting promotes flow reversals in a $\Gamma = 2$ cell [107] be applied to a $\Gamma = 1/2$ cell? These questions are still open but are important for building a unified view on flow reversals in convection cells with different aspect ratios. We will also try to answer these questions in the present work.

In this chapter, we aim to investigate tilted convection in a $\Gamma = 1/2$ cell by means of 2D DNS. Two considerations promote us to use 2D geometry: (i) the numerical cost is cheaper so that we can study tilted convection over wide range of both β - Pr and β - Ra parameter space; (ii) the flow visualization is much easier and we can

Figure 7.1: (a) Sketch of two-dimensional tilted cavity with $\Gamma = 1/2$. (b) A example of non-uniform grid used in our simulations for $N_y \times N_z = 128 \times 256$. (c) A enlargement of (b) for $0 \leq y \leq 0.1$ and $0.1 \leq z \leq 0.2$

obtain a detailed scenario for flow reversals in a $\Gamma = 1/2$ cell. Due to the flexibility of 2D DNS, it has been extensively used to study turbulent thermal convection in recent years [122, 148, 183, 204, 267–269, 298]. We show for the first time the flow reversal scenario in a tilted cell with $\Gamma = 1/2$, and demonstrate that the finding that tilting promotes flow reversals in a $\Gamma = 2$ cell [107] can also be applied to a $\Gamma = 1/2$ case. Moreover, we report multiple states in the turbulent regime for small β , between which the flow can or cannot switch. Another interesting finding is that the conclusion that Nu decreases with increasing β close to $\beta = 90^\circ$ for a $\Gamma = 1$ cell [59, 270, 282, 283] does not hold for the present $\Gamma = 1/2$ configuration for small- Pr cases.

The remainder of this chapter is organized as follows. In §7.2., we introduce the governing equations and numerical methods. The main results are presented and discussed in §7.3. Finally, we summarize our findings in §7.4.

7.2 Numerical settings

The considered problem is sketched in figure 7.1(a). The bottom plate has a high temperature $\hat{T}_H$ (quantities marked with a circumflex are dimensional) while the top plate has a low temperature $\hat{T}_C$. The two side walls are insulated. The configuration is counterclockwise tilted with an angle β relative to the RBC configuration. The cavity has a width $\hat{W}$ and height $\hat{H}$, and the aspect ratio $\Gamma = \hat{W}/\hat{H}$ is fixed to $1/2$. The dimensionless incompressible Navier-Stokes equations in Cartesian coordinates

within the Oberbeck-Boussinesq approximation read

$$\nabla \cdot \mathbf{u} = 0, \quad (7.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \sqrt{\frac{Pr}{Ra}} \nabla^2 \mathbf{u} + T \vec{e}, \quad (7.2)$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{1}{\sqrt{RaPr}} \nabla^2 T, \quad (7.3)$$

where $\mathbf{u} = (v, w)$ are the velocity components in the horizontal and vertical directions, t is the time, p is the pressure, T is the temperature, $\vec{e}$ is a unit vector defined by $\vec{e} = (\sin \beta, \cos \beta)$, and β is the tilt angle. The dimensionless control parameters are the Rayleigh number $Ra = \hat{g} \hat{\alpha} \Delta \hat{T} \hat{H}^3 / \hat{\nu} \hat{\kappa}$ and Prandtl number $Pr = \hat{\nu} / \hat{\kappa}$, where $\hat{\alpha}$, $\hat{\nu}$ and $\hat{\kappa}$ denote the isobaric thermal expansion coefficient, kinematic viscosity and thermal diffusivity of the fluid, $\hat{g}$ is the gravitational acceleration, $\hat{\alpha}$ is the isobaric thermal expansion coefficient, $\Delta \hat{T} = \hat{T}_H - \hat{T}_C$ is the temperature difference. For non-dimensionalization, we choose the free-fall velocity $\hat{U} = \sqrt{\hat{g} \hat{\alpha} \Delta \hat{T} \hat{H}}$ as reference velocity and the height of the container $\hat{H}$ as reference length, the reference time is free-fall time $\hat{t}_f = \hat{H} / \hat{U}$, the temperature is non-dimensionalized by $T = (\hat{T} - \hat{T}_C) / \Delta \hat{T}$. The dimensionless pressure is defined as $p = \hat{p} / \hat{\rho}_0 \hat{U}^2$, where $\hat{\rho}_0$ is the density at the reference temperature $\hat{T}_0 = \hat{T}_C$. No-slip and impermeability velocity conditions are imposed at all solid walls. Two important response parameters are the Nusselt number Nu and Reynolds number Re which are defined as:

$$Nu = \frac{\hat{Q}}{\hat{\kappa} \Delta \hat{T} / \hat{H}}, \quad Re = \frac{\sqrt{\langle \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \rangle_{V,t}} \hat{H}}{\hat{\nu}}, \quad (7.4)$$

where $\hat{Q}$ is the time-averaged heat flux across the fluid layer subjected to an imposed temperature difference $\Delta \hat{T}$. The Nu used in this study is averaged over the bottom plate with $z = 0$. $\hat{\mathbf{u}}$ is the velocity vector. $\langle \rangle_{V,t}$ denotes average over the whole area and time. The Nu and Re are averaged for at least 1000 free-fall time units for the chaotic cases.

The governing equations together with proper boundary conditions are solved by an in-house code lMn2d [79]. All spatial terms are discretized using the central second-order finite difference method. Time integration is accomplished using the Adams-Bashforth scheme for the nonlinear terms and Crank-Nicolson scheme for the viscous and diffusion terms. A non-uniform grid with clustered points near walls is adopted to ensure grid resolution in boundary layers. Specifically, the relation between the physical grid y and the computation grid η is expressed as:

$$y = \left\{ \frac{\tanh[(\eta / Ny - 0.5) * Ag]}{\tanh[0.5 * Ag]} + 1 \right\} * 0.5 * Ly, \quad (7.5)$$

where Ag is an adjustable parameter describing the grid stretching near walls, and Ag is fixed to 2.5 in this study. Ly is dimensionless horizontal length which is 0.5 in this study. Similar grid strategy is also used to z direction. Figure 7.1(b) shows an example of the non-uniform grid with $N_y \times N_z = 128 \times 256$, and figure 7.1(c) is a zoom of figure 7.1(b). The Poisson equation is solved using a multi-grid strategy to accelerate convergence. The grid resolution has been chosen according to the requirement for DNS [110]. The simulation parameters and grid resolutions of this chapter are listed in Table 7.1.

Ra	Pr	β	$\frac{Nu(\beta^*)}{Nu(0)}$	$N_y \times N_z$
10^6	0.71	$[0^\circ, 90^\circ]$	1.51	128×256
10^7	[0.01,100]	$[0^\circ, 90^\circ]$	2.37	128×256
10^8		$[0^\circ, 90^\circ]$	1.68	128×256
10^9	0.71	$[0^\circ, 90^\circ]$	1.73	256×512

Table 7.1: Simulation parameters and grid resolutions. The columns from left to right indicate: Rayleigh number Ra , Prandtl number Pr , tilt angle β , the maximal normalized Nusselt number $\frac{Nu(\beta^*)}{Nu(0)}$ (β^* is the angle at which the Nu is maximal) and grid resolutions $N_y \times N_z$.

7.3 Results and discussion

7.3.1 Prandtl number dependence

We first explore different flow organizations over the β - Pr parameter space for fixed $Ra = 10^7$. Figure 7.2(a) and 7.2(b) show time evolution of vertical velocity at the location (0.06, 0.5) for $Ra = 10^7$, $Pr = 0.71$ with $\beta = 0^\circ$ and $\beta = 9^\circ$, respectively. The instantaneous flow fields at different times denoted by vertical dashed lines in figure 7.2(a) are shown in figure 7.2(c-f). It is seen that the flow constantly changes between two triple-roll states (TRS) with different orientations of the rolls. The top and bottom rolls in figure 7.2(c) grow until they collide with each other as shown in figure 7.2(d), this will result in enhanced mixing and heat transport. This kind of flow is referred to as unstable triple-roll state (UTRS). Figure 7.2(g) shows instantaneous flow field at the time $t = 100$ for $\beta = 9^\circ$. The flow for $\beta = 9^\circ$ always stays in the stable triple-roll state (STRS) as shown in figure 7.2(g), and the corresponding Nu is lower than that of UTRS. Similar UTRS and STRS have also been reported in prior 2D DNS where the flow stays in STRS for $\Gamma = 0.45$ and UTRS for $\Gamma = 0.55$ with $Ra = 10^8$, $Pr = 0.71$ [159].

It is well known that large-scale circulation (LSC) may spontaneously reverse its orientation for certain Ra and Pr . The reversals of LSC are led by the simultaneous growth of the two corner rolls for $\Gamma = 1$ [114, 144, 220], while for $\Gamma = 2$, only single corner roll growth leads to a reversal [107, 251]. For a cylindrical cell with $\Gamma = 1/2$, it

Figure 7.2: Time evolution of vertical velocity w at the location $(0.06, 0.5)$ marked with the black circle for $Ra = 10^7, Pr = 0.71$ with (a) $\beta = 0^\circ$. (b) $\beta = 9^\circ$. (c-f) Instantaneous temperature fields superimposed with velocity vectors at different times denoted by vertical dashed lines in (a). (g) Instantaneous temperature field superimposed with velocity vectors for $\beta = 9^\circ$ at $t = 100$. The black arrow indicates the direction of gravity.

Figure 7.3: (a) Time evolution of horizontal velocity component v at the center point $(y, z) = (0.25, 0.5)$ marked with the black circle for $Ra = 10^7$, $Pr = 0.021$, $\beta = 30^\circ$. (b) Time evolution of three dominating modes $M^{1,1}$, $M^{1,2}$ and $M^{1,3}$. (c-h) Six snapshots of the temperature fields superimposed with velocity vectors during the flow reversal shown in (a) at (c) $t = 42.6$. (d) $t = 57.9$. (e) $t = 65.1$. (f) $t = 68.3$. (g) $t = 88.6$. (h) $t = 104.5$. The black arrow indicates the direction of gravity. Colour scales as in figure 7.2

was reported that flow mode transitions between single-roll state (SRS) and double-roll state (DRS) exist [158]. For the present 2D configuration, we found that the flow tends to organize in vertically-stacked DRS for low Pr with small β . When β increases beyond a critical value, this DRS will no longer be stable and flow reversals may occur. Figure 7.3(a) shows time evolution of horizontal velocity v at the center point (0.25,0.5) for $Pr = 0.021$ with $\beta = 30^\circ$. A sign change of v at this point can nicely reveal a reversal. Figure 7.3 (c-h) show six instantaneous flow fields at different times denoted by vertical dashed lines in figure 7.3(a). It can be seen from figure 7.3(c) that there are two main rolls that almost occupy the whole area of the convection cell, and two small corner rolls appear in the bottom-right and top-right of the cell. As tilting introduces a horizontal ‘gravity’ which points to the left, the hot fluid near the bottom plate tends to move to the right and enters the bottom-right corner roll. Thus the bottom-right corner roll grows and squeezes the lower main roll as shown in figure 7.3(d). The TRS shown in figure 7.3(d) is not stable, the further growth of the bottom roll in figure 7.3(d) will lead to collision with the top roll as shown in figure 7.3(e), which will cause mixing and enhanced heat transport as mentioned before. This UTRS will continue for several rounds as shown in figure 7.3(f) and 7.3(g), and then the flow gradually returns to DRS as displayed in figure 7.3(h) which has different orientations of the the main rolls compared to figure 7.3(a). A reversal process thus has been completed. For the next reversal cycle, the top-left corner roll shown in figure 7.3(h) will grow due to the horizontal ‘gravity’ introduced by tilting the cell. It is clear that the growth of the bottom-right corner roll shown in figure 7.3(a) or top-left corner roll shown in figure 7.3(h) plays a critical role in triggering a reversal. This is very similar to flow reversals in a tilted $\Gamma = 2$ cell [107]. The introduction of tilting makes one of the corner rolls have more opportunities to grow, thus breaking the balance between the main rolls and the corner rolls. As a result, flow reversals are more easily to happen in tilted cells. Thus the conclusion that tilting promotes flow reversals in a 2D cell with $\Gamma = 2$ [107] can also be generalized to a slimmer cell with $\Gamma = 1/2$. We also used Fourier mode decomposition method [118, 144, 147, 220] to quantitatively describe the behaviours of different modes during a reversal. The velocity components v and w are projected onto the following Fourier basis:

$$\hat{v}^{m,n} = 2\sqrt{2} \sin(2m\pi y) \cos(n\pi z), \quad (7.6)$$

$$\hat{w}^{m,n} = -2\sqrt{2} \cos(2m\pi y) \sin(n\pi z). \quad (7.7)$$

The velocity can be recovered by:

$$v(y, z, t) = \sum_{m,n} A_y^{m,n}(t) \hat{v}^{m,n}, \quad (7.8)$$

$$w(y, z, t) = \sum_{m,n} A_z^{m,n}(t) \hat{w}^{m,n}. \quad (7.9)$$

The projection is done component-wisely by a scalar product in the L_2 -space. The

Figure 7.4: (a) Time evolution of horizontal velocity component v at the center point $(y, z) = (0.25, 0.5)$ marked with the black circle for $Ra = 10^7$, $Pr = 0.71$, $\beta = 30^\circ$. (b-g) temperature fields superimposed with velocity vectors at different times shown in (a). (b) $t = 96$. (c) $t = 229$. (d) $t = 405$. (e) $t = 592$. (f) $t = 717$. (g) $t = 771$. The black arrow indicates the direction of gravity. Colour scales as in figure 7.2

corresponding time series $A_y^{m,n}(t)$ and $A_z^{m,n}(t)$ can be obtained by:

$$A_y^{m,n}(t) = \int_0^1 \int_0^{0.5} v(y, z, t) \hat{v}^{m,n} dy dz, \quad (7.10)$$

$$A_z^{m,n}(t) = \int_0^1 \int_0^{0.5} w(y, z, t) \hat{w}^{m,n} dy dz. \quad (7.11)$$

A value $M^{m,n}(t)$ representing the contribution of different modes can be evaluated by $M^{m,n}(t) = \sqrt{(A_y^{m,n}(t)^2 + A_z^{m,n}(t)^2)}$. The mode (m, n) corresponds to a flow structure with m rolls in the y direction and n rolls in the z direction. Figure 7.3(b) shows time evolution of the three dominating modes $M^{1,1}$, $M^{1,2}$, $M^{1,3}$. One can see that the $(1,2)$ mode decreases in strength during a reversal, while the $(1,3)$ and $(1,1)$ modes play important roles during the reversal, corresponding to the appearance of UTRS during the reversal.

For certain combinations of Pr and β , there also exists another different organization which we define as mixed state. Figure 7.4(a) shows time evolution of horizontal velocity v at the center point $(0.25, 0.5)$ for $Pr = 0.71$, $\beta = 30^\circ$ where the flow

Figure 7.5: Phase diagrams in the β - Pr plane for $Ra = 10^7$. Blue hollow diamonds ($\diamond$) denote stable vertically-stacked double-roll state without flow reversals, red filled diamonds ($\blacklozenge$) correspond to double-roll state with flow reversals, green hollow triangles ($\triangle$) correspond to stable triple-roll state, orange filled triangles ($\blacktriangle$) correspond to unstable triple-roll state, blue filled left triangles ($\blacktriangleleft$) denote mixed states, green filled circles ($\bullet$) represent single-roll state without flow reversals.

stays in a mixed state. Figure 7.4(b-g) display instantaneous flow fields at different times shown in figure 7.4(a). One sees that the flow is very complicated, it constantly changes among different flow states, the SRS, DRS, TRS and other more complicated flow structures all can be identified at different times.

These findings led us to map out flow states over a considerable fraction of the β - Pr parameter space in figure 7.5. For $\Gamma = 1$, there is only a main roll that spans the whole size of the convection cell in the range of $0^\circ \leq \beta \leq 90^\circ$ for not too high Pr [59, 270], while it is seen that the phase diagram in figure 7.5 is much more complicated than the $\Gamma = 1$ case. The flow tends to stay in stable DRS without flow reversals for small Pr and relatively small β . For larger β , the DRS is not stable anymore and flow reversals exist. We perform long-time simulations for the cases near the reversing/non-reversing boundaries. Figure 7.6 shows time evolution of horizontal velocity at the center of the cell for $\beta = 15^\circ$ and 20° with $Ra = 10^7, Pr = 0.021$. One can see that the DRS is stable for very long time up to 70 000 free-fall time units for 15° , while many reversals are observed when β increases to 20° . We still do not know whether there are also flow reversals for $\beta = 15^\circ$ if we perform simulations for much longer time. From figure 7.6(b), we can also find that the two DRSs seem to remain equally probable, which is also quite different from $\Gamma = 1$ case where the counterclockwise circulation is much more probable than the clockwise one when the cell is tilted [107]. Since tilting favors counterclockwise circulation, and there is always a counterclockwise and a clockwise roll for these two DRSs, they have similar stability

Figure 7.6: Time evolution of horizontal velocity at the center of the cell for $Ra = 10^7, Pr = 0.021$. (a) $\beta = 15^\circ$, (b) $\beta = 20^\circ$.

properties and happen equally probable. We also find that the conditionally-averaged Nu for the two DRSs seem to have no difference. For larger Pr , the flow organizes in STRS or UTRS for small β .

The phase diagram in figure 7.5 has shown abundant flow organizations for a tilted cell with $\Gamma = 1/2$. Figure 7.7 further shows instantaneous flow fields for different combinations of Pr and β . We can see that the flow organizations sensitively depend on the combination of Pr and β . For small Pr , the flow organizes in DRS within a large range of β , and the thermal boundary layers are thicker, leading to inefficient heat transport [299]. For $Pr = 4.38, \beta = 10^\circ$, the top and bottom rolls seem more like ‘corner rolls’, so we regarded this kind flow as single single-roll state.

We now turn to discussing the influence of cell tilting on the Nu and Re . The Nu has been found to be closely connected to the mean flow structures both in two [159] and three dimensions [115–117, 119, 158, 161]. The abundant flow structures shown in figure 7.5 imply that Nu may vary drastically in a tilted cell with $\Gamma = 1/2$ due to the various flow organizations. Figure 7.8 shows absolute and normalized Nu and Re as a function of β . From figure 7.8(a) one sees that the Nu is smaller with small Pr , due to the enhanced thermal diffusion with thicker thermal boundary layers as seen in figure 7.7. The lower Nu for small Pr has been reported in many previous works [53, 181, 227, 277] and is also predicted by the unify theory proposed by Grossmann and Lohse [222, 223]. Figure 7.8(b) shows that the Nu does not change much for small tilt angle $\beta \leq 5^\circ$ for the six representative Pr . However, the curves of $Nu(\beta)/Nu(0)$ are quite different for larger β with different Pr . For large Pr ($Pr = 4.38, 100$), the Nu first increases with increasing β until $\beta \approx 80^\circ$, where it reaches its global maximum, and after that it decreases monotonically with increasing β . For $Pr = 0.71$, the Nu experiences a sudden drop near $\beta = 7^\circ$, due to the fact that the flow changes

Figure 7.7: Instantaneous temperature fields superimposed with velocity vectors for different Pr and β . The black arrow indicates the direction of gravity. Colour scales as in figure 7.2

Figure 7.8: Absolute and normalized Nusselt and Reynolds numbers as a function of tilt angle β for fixed $Ra = 10^7$ and different Pr . (a) Absolute Nusselt numbers Nu . (b) Normalized Nusselt numbers $Nu(\beta)/Nu(0)$. (c) Absolute Reynolds numbers Re . (d) Normalized Reynolds numbers $Re(\beta)/Re(0)$.

Figure 7.9: Phase diagram in the $Pr - \beta$ plane for fixed $Ra = 10^7$ for (a) $Nu(\beta)/Nu(0)$ and (b) $Re(\beta)/Re(0)$.

from UTRS to STRS, and UTRS transports heat more efficiently than STRS. After reaching its local minimum at $\beta \approx 13^\circ$, it increases monotonically with increasing β until reaching its maximum at $\beta \approx 80^\circ$, it then decreases with increasing β . For $Pr = 0.1$, the Nu first decreases with increasing β until $\beta \approx 17^\circ$ and then sharply increases after $\beta > 20^\circ$. This corresponds to the flow changing from stale DRS ($\beta = 20^\circ$) to UTRS ($\beta = 23^\circ$) as shown in figure 7.5, and the Nu for UTRS is larger. After reaching its local maximum at around $\beta = 35^\circ$, it then experiences a sudden decrease after $\beta = 35^\circ$. This corresponds to the flow changing from UTRS ($\beta = 35^\circ$) to reversal state of DRS ($\beta = 40^\circ$). The Nu continues to increase after $\beta = 40^\circ$ until 80° , and then decreases with increasing β when $\beta > 80^\circ$. For small Pr , the behaviour of Nu is quite different. It decreases monotonically with increasing β until reaching its local minimum at $\beta \approx 75^\circ$ ($Pr = 0.021$) or $\beta \approx 80^\circ$ ($Pr \approx 80^\circ$), which corresponds to the maximal β for the existence of DRS as shown in figure 7.5. With further increasing β , the flow becomes SRS and the Nu increases monotonically until $\beta = 90^\circ$. In a word, the Nu is directly connected with the complicated flow organization shown in figure 7.5. It is clear that the Nu has a different trend near $\beta = 90^\circ$ for small Pr . Previous works all showed that the Nu drops monotonically near $\beta = 90^\circ$ for a $\Gamma = 1$ cell [59, 270, 282, 283]. In the present system, we find that this conclusion does not hold for a $\Gamma = 1/2$ cell with small Pr .

The absolute and normalized Reynolds number are presented in figure 7.8(c) and 7.8(d). It is clear that the Re is an increase function of Pr . As seen from figure 7.8(d), the Re shows a similar trend for all Pr for $\beta > 30^\circ$, it decreases monotonically with increasing β when $\beta > 30^\circ$ for all considered Pr , the similar trend was also found in a $\Gamma = 1$ cell [59, 270]. However, the Re behaves quite differently for small β . For large Pr ($Pr = 4.38, 100$), the Re first increases with increasing β , until reaching its local maximum, then it decreases with increasing β . For $Pr = 0.71$ and 0.1 , the Re drops, increases, and then drops again. The local minimum corresponds to the STRS ($Pr = 0.71$) or DRS ($Pr = 0.1$) which coincides with the local minimum for Nu as shown in figure 7.8(b). While for small Pr ($Pr = 0.021, 0.01$), the Re drops monotonically with increasing β . These complicated relationships demonstrate that the normalized Re is also strongly influenced by Pr .

We further show contours of the $Nu/(\beta)/Nu(0)$ and $Re(\beta)/Re(0)$ over $Pr - \beta$ space in figure 7.9. From figure 7.9(a), we can see that the maximal enhancement of Nu happens near $Pr = 0.3$ for relatively large β close to $\beta = 80^\circ$. The maximal value of $Nu(\beta)/Nu(0)$ is 2.37 corresponding to $Pr = 0.3, \beta = 80^\circ$. In figure 7.9(b), it is readily seen that the enhancement of Re happens at large Pr and small β .

7.3.2 Rayleigh number dependence

In this section, we discuss the Ra dependence with a fixed $Pr = 0.71$ for $10^6 \leq Ra \leq 10^9$. We first report multiple states in the turbulent regime found in the present system. Previous studies have reported multiple states in transitional as well as fully turbulent flow, e.g., in cylindrical RBC with $\Gamma = 1/2$ [158], in 2D RBC [159], in von

Figure 7.10: (a,b) Time evolution of vertical velocity w at the location $(0.06, 0.5)$ and (c,d) the Nusselt number Nu for $Ra = 10^8, Pr = 0.71$ with different β . (a,c) $\beta = 4^\circ$. (b,d) $\beta = 8^\circ$.

Kármán swirling flow [167, 169], in high-Reynolds number Taylor-Couette flow [163], in spherical Couette flow [171], in turbulent plane Couette flow with spanwise rotation [172]. Ref. [158] reported flow mode transitions between SRS and DRS. Recently, Ref. [172] reported multiple states in turbulent plane Couette flow with spanwise rotation, they found one state corresponds to two pairs of roll cells, and the other shows three pairs. More recently, Ref. [161] reported an experimental observation of a flow topology transition in turbulent RBC using an annulus convection cell, they found that the flow jumps between the two states with four or two large-scale rolls [163]. For $Ra = 10^8$ with small β , we also find flow mode transitions between different turbulent states. Figure 7.10(a) shows time evolution of vertical velocity w at the location $(0.06, 0.5)$ for $Ra = 10^8$ and $\beta = 4^\circ$. Figure 7.10(c) shows the time evolution of the Nu for the same case. It is clearly seen that the flow constantly transits between two states. For the interval marked with green, the flow is UTRS with large fluctuations of w , and the corresponding Nu is also evidently larger, for the time interval marked with blue, the flow stays in STRS, the fluctuations of w are smaller and the Nu is also smaller. The conditionally averaged Nu of the UTRS ($Nu = 24.19$) is larger than that of STRS ($Nu = 15.93$) by 52%. Further increasing β will lead the flow to become STRS as shown in figure 7.10(b) and 7.10(d) for $\beta = 8^\circ$. We perform a long-time simulation up to 40,000 free-fall time units for this case, and no flow mode transitions between UTRS and STRS are observed.

For $Ra = 10^9$ with small β , we also observe different turbulent states with the same control parameters. When we use the conduction state as initial conditions, the fully developed flow for $\beta = 10^\circ$ is STRS with only one counterclockwise roll (denoted as STRS_1) as shown in figure 7.11(a). While for $\beta = 20^\circ$, the fully developed flow

Figure 7.11: Temperature field superimposed with velocity vectors for two different states obtained using different initial conditions for $Ra = 10^9, \beta = 10^\circ$ (a) Stable triple-roll state with one counterclockwise roll(STRS_1) (b) Stable triple-roll state with two counterclockwise rolls(STRS_2). (c) Time evolution of the Nusselt number Nu for STRS_2 and STRS_1 with $Ra = 10^9, \beta = 10^\circ$. The black arrow indicates the direction of gravity. Colour scales as in figure 7.2

is STRS with two counterclockwise rolls (STRS_2). Using this state as the initial condition for $\beta = 10^\circ$, it is found that STRS_2 can also be stable state for $\beta = 10^\circ$ as shown in figure 7.11(b). Figure 7.11(c) shows time evolution of the Nu for these two states for $\beta = 10^\circ$. It is seen that the two states can be stable for a long time. The time-averaged Nu for STRS_1 is 40.43, larger than that of STRS_2 (36.00) by 12%. The recent work showed that the Nu are different for flow states with different numbers of large-scale convection rolls. [161] Here, it is found that the Nu can still be different for different turbulent states with the same number of convection rolls but with different orientations when the cell is tilted. As the two rolls near the bottom and top plates are clockwise for STRS_1, the flow velocity near the bottom and top plates are always in opposite direction to buoyancy direction, thus this state is less stable compared with STRS_2, and we find that STRS_1 can be stable only for $0^\circ \leq \beta \leq 15^\circ$, when β increases to $\beta = 16^\circ$, this STRS_1 will transit to STRS_2.

Figure 7.12 shows instantaneous flow fields for different combinations of Ra and β with $Pr = 0.71$. For $Ra = 10^6$, the flow is laminar with small β . For $\beta = 0^\circ$, the flow is UTRS, when β increases to 9° , the flow becomes stable DRS. With even larger $\beta \geq 20^\circ$, the flow becomes SRS. For $Ra = 10^8$, flow mode transitions between UTRS and STRS are found for $\beta < 8^\circ$. At $\beta = 8^\circ$ and 10° , the flow is STRS. Further increasing β will lead the flow to become mixed state as seen from the flow fields with $\beta = 20^\circ$ and 30° . At larger β , the flow becomes SRS. For $Ra = 10^9$, both STRS_1 and STRS_2 can be stable state for $0^\circ \leq \beta \leq 15^\circ$, while only the STRS_2 exist for $16^\circ \leq \beta \leq 20^\circ$. Further increasing β to 30° leads the flow to become mixed state. At larger β , the flow gives way to SRS.

Figure 7.12: Instantaneous temperature field superposed with velocity vectors for different different Ra and β . The black arrow indicates the direction of gravity. Colour scales as in figure 7.2

Figure 7.13: Absolute and normalized Nusselt and Reynolds numbers as a function of tilt angle β for fixed $Pr = 0.71$ and different Ra . (a) Absolute Nusselt numbers Nu . (b) Normalized Nusselt numbers $Nu(\beta)/Nu(0)$. (c) Absolute Reynolds numbers Re . (d) Normalized Reynolds numbers $Re(\beta)/Re(0)$.

We further show absolute and normalized Nu and Re in figure 7.13. For $10^6 \leq Ra \leq 10^8$, it is seen that the Nu generally decreases with increasing β until reaching its local minimum, it then increases monotonically with increasing β , after reaching its local maximum at around $\beta = 80^\circ$ it then decreases with increasing β . The flow states correspond to the minimal Nu are DRS for $Ra = 10^6$ and STRS for $Ra = 10^7$ and 10^8 . For $Ra = 10^9$, there are two branches of Nu at small β , corresponding to STRS_1 and STRS_2. Both the Nu and Re are larger for less stable state STRS_1. The Nu generally increases monotonically with increasing β until reaching its maximum at around $\beta = 80^\circ$, it then decreases with increasing β . These results show the Nu is directly connected to the flow organization, and $Nu(\beta)/Nu(0)$ is also a complicated function of Ra . The Nu can either increase or decrease for small β when Ra is varied.

The absolute and normalized Re are shown in figure 7.13(c) and 7.13(d). It can be seen that the Re all decrease with increasing β at large $\beta \geq 30^\circ$ for different Ra , while they show some variations for small β for different Ra . For $10^6 \leq Ra \leq 10^8$, similar to Nu , the Re also decreases with increasing β initially until reaching its local minimum, after which it increases until its local maximum. For $Ra = 10^9$, the Re of STRS_1 and STRS_2 behave differently, the Re of STRS_1 increases with increasing β while STRS_2 decreases with increasing β . The complicated behaviours of Re at small β are also directly connected with the flow organizations, and the $Re(\beta)/Re(0)$ dependence is also a complicated function of Ra at small β .

7.4 Conclusions

In summary, we have studied flow organizations and heat transfer in a two-dimensional tilted cell with aspect ratio $1/2$. Abundant flow structures are identified. It is found that vertically-stacked double-roll state exists for small Pr with small β . When β increases beyond a critical value, the double-roll state is not stable anymore and flow reversals happen. The growth of the bottom-right or top-left corner rolls due to horizontal buoyancy force introduced by tilting plays a key role in triggering a reversal. This finding complements our previous study of flow reversals in tilted cells with $\Gamma = 1$ and 2 [107], demonstrating the conclusion that tilting promotes flow reversals in a $\Gamma = 2$ cell [107] can also be applied to a $\Gamma = 1/2$ case. Multiple states in the turbulent regime are also reported, and the different states can ($Ra = 10^8$) or cannot ($Ra = 10^9$) switch between each other. In the tilted cases, the Nu are different for the two states with the same number of convection rolls but with different orientations. These results enrich the studies on multiple states in turbulence. We further show that $Nu(\beta)/Nu(0)$ and $Re(\beta)/Re(0)$ are strongly influenced by a combination of Ra and Pr , and are closely related to different flow organizations. In particular, the earlier conclusion that the Nu decreases with increasing β close to $\beta = 90^\circ$ for a cell with $\Gamma = 1$ [59, 270, 282, 283] does not hold for the present $\Gamma = 1/2$ case with small Pr .

Chapter 8

Multiple states and heat transfer in two-dimensional tilted convection with large aspect ratios.[◦]

The coexistence of multiple turbulent states were reported in several recent studies in different flows. We present in this work that multiple turbulent states also exist for thermal convection in two-dimensional tilted cells with large aspect ratios (Γ =width/height) through direct numerical simulations for the Rayleigh number $Ra = 10^7$ and the Prandtl number $Pr = 0.71$. The considered Γ ranges from 1 to 16. The tilt angle β varies from 0° to 180° . Multiple states are identified for small β with $\Gamma \geq 2$, where the different flow states are reflected in different numbers of convection rolls. The corresponding Nusselt number Nu is generally higher for the flow state with more convection rolls. Moreover, flow mode transitions between different roll states are observed for large $\Gamma \geq 8$ when β is larger than a critical value. The effect of cell tilting on the Nusselt number Nu and Reynolds number Re are also investigated. It is found that for $\Gamma \leq 4$, Nu first increases with increasing β , and then declines after reaching its local maximum. However, the Nu decreases monotonically with increasing β for $\Gamma = 8, 12$, and 16 . This indicates that the idea to enhance heat transfer by tilting the cell can be realized only for relatively small Γ for the present system. It is also found that the previous finding that Re decreases monotonically with increasing β for large β with $\Gamma = 1$ does not hold for large- Γ cases.

[◦]Based on: **Qi Wang**, Zhen-Hua Wan, Rui Yan, De-Jun Sun, *Multiple states and heat transfer in two-dimensional tilted convection with large aspect ratios*, Phys. Rev. Fluids **3**, 113503 (2018).

8.1 Introduction

Turbulence is ubiquitous in flows in nature and engineering applications. For a flow in turbulent state, to extrapolate laboratory data to a much higher Reynolds number flow in nature, one needs to know whether there are transitions and whether multiple states can coexist in the high-Reynolds number flow. Over the past few years, the community became aware of the possibility of the coexistence of different turbulent states. Several studies have reported multiple states in transitional as well as fully turbulent flow. Examples include different turbulent states in von Kármán swirling flow [167–170, 300], in high-Reynolds number Taylor-Couette flow [163, 164], in spherical Couette flow [171], in Rayleigh-Bénard convection [158, 159, 161], in rotating Rayleigh-Bénard convection [301–303], and in turbulent plane Couette flow with spanwise rotation [172]. Ref. [158] experimentally found flow mode transitions between single-roll state and vertically-stacked double-roll state in Rayleigh-Bénard convection in a $\Gamma = 1/2$ cylindrical cell. Ref. [163] experimentally found evidence for different turbulent states in Taylor-Couette flow, the different turbulent states are reflected in different numbers of Taylor rolls (three or four rolls in top half of the system), the Nusselt number Nu_ω is larger for the state with more rolls. A recent numerical work [172] reported multiple states in turbulent plane Couette flow with spanwise rotation, with two different initial flow fields, they found one state corresponds to two pairs of roll cells, and the other shows three pairs, however, it is still unknown whether one state will change to the other at longer time duration. More recently, Ref. [161] reported an experimental observation of a flow topology transition in turbulent Rayleigh-Bénard convection using an annulus convection cell, they found that the flow jumps between the two states with four or two large-scale rolls and the Nusselt number Nu is higher for the four-roll state, this is similar to that in turbulent Taylor-Couette flow [163].

Rayleigh-Bénard convection (RBC) where a fluid in a cavity heated from below and cooled from above is an ideal model problem for the study of turbulent convection, and it has been extensively investigated in the past years [53–56, 236]. In industrial processes, the heat transfer in heat exchangers can be enhanced by selecting an optimum tilt angle β . The convection in a fluid layer tilted with respect to gravity is of particular meteorological and oceanographic interest. This system is well suited for the study of buoyancy- and shear flow-driven instabilities in the Earth's atmosphere and hydrosphere [66]. Due to its importance, thermal convection in tilted containers has been studied extensively in recent years both with small tilt angles [155, 160, 261, 286–288] and with relatively wide range of tilt angles ($0^\circ < \beta < 90^\circ$) [59, 104, 107, 270, 271, 279–283, 289–291]. For a convection cell with aspect ratio $\Gamma = 1$, Ref. [270] experimentally studied convection in a tilted cavity for the Rayleigh number $Ra \approx 4.3 \times 10^9$ and the Prandtl number $Pr \approx 6.7$, they found that the large-scale circulation (LSC) is sensitive to the symmetry of the system and the LSC changes gradually from oblique ellipse-like to square-like, they also found that the Nu monotonically decreases with increasing β for $0^\circ \leq \beta \leq 90^\circ$. Ref. [59] performed

detailed direct numerical simulations (DNS) for $10^6 \leq Ra \leq 10^8$ and $0.1 \leq Pr \leq 100$ in a cylindrical cell of unit aspect ratio, they showed that $Nu(\beta)/Nu(0)$ dependence is not universal and is strongly influenced by a combination of Ra and Pr , and the $Nu(\beta)/Nu(0)$ dependence is a complicated, non-monotonic function of β . A recent experimental work [282] using liquid sodium with $Ra \approx 1.47 \times 10^7$ and $Pr = 0.0094$ found that the Nusselt number is minimal at $\beta = 0^\circ$ and demonstrates a smooth growth with the increase of the tilt angle, reaching a maximum at angle $\beta = 70^\circ$ and decreasing with a further increase of β . The maximal value of the normalized Nusselt number $Nu(\beta)/Nu(0)$ was 1.21. Several studies performed in convection cells with $\Gamma < 1$ also reported that tilt can induce heat transport enhancement [279–281, 283]. Recently, Ref. [290] studied thermal convection in a tilted cylindrical cell with $\Gamma = 1/5$ for $Pr = 0.1$ and 1.0 using DNS. They determined the optimal tilt angle, which provides the maximal global heat transport. They found that the formation of two system-sized plume columns are related to a strong increase in the heat transport.

Most of the previous results were performed in a cell with $\Gamma \leq 1$. In reality, the aspect ratio of the convection problem can be varied greatly [55] and it is extremely large in many natural instances of convection [215, 216]. It is already known that Γ has a great influence on the heat transport and flow organization for RBC both in two-dimensions (2D) [145, 159] and three-dimensions (3D) [115–117, 119, 147, 272]. However, there are less works devoted to the influence of Γ on tilted convection especially those with large Γ which is of great interest in geophysical settings. In this paper, we try to fill this gap by investigating thermal convection in tilted cells with widely varying aspect ratios $1 \leq \Gamma \leq 16$ using two-dimensional (2D) DNS. We report multiple stable states found for large Γ ($\Gamma \geq 2$) with small β . Moreover, flow mode transitions between different roll states are observed when β is large than a critical value for $\Gamma \geq 8$. Another interesting finding is that the Nu decreases monotonically with increasing β for large $\Gamma \geq 8$ which is quite different from those of small- Γ cases where the Nu reaches its local maximum at an intermediate angle. The reason why we choose 2D DNS is that it is computationally cheaper so that we can study tilted convection over broad $\Gamma - \beta$ parameter space. Moreover, we can perform long-time simulations to detect whether there are flow mode transitions. Furthermore, the flow visualizations are much easier so that a direct connection between the flow organization and the heat-transfer properties is possible. The 2D simulations are also found to reflect many physics in the true three-dimensional (3D) system [181]. Due to its advantages, the 2D simulation has been utilized widely to study RBC in recent years [79, 121, 122, 148, 183, 204, 262, 267–269].

The remainder of this chapter is organized as follows. In §8.2, we describe the governing equations and numerical methods. In §8.3, we first report the coexistence of multiple states in tilted convection with large Γ . Furthermore, we discuss the influence of cell tilting on the Nusselt number Nu and the Reynolds number Re . Finally, we summarize our findings in §8.4.

8.2 Numerical settings

The problem is described in figure 8.1. The rectangular cell has width $\hat{W}$ (hatted quantities are dimensional) and height $\hat{H}$. The bottom wall has constant high temperature $\hat{T}_H$ while the top wall has constant low temperature $\hat{T}_C$. The sidewalls are insulated. The dimensionless governing equations with Oberbeck-Boussinesq (OB) approximation read

$$\nabla \cdot \mathbf{u} = 0, \quad (8.1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \sqrt{\frac{Pr}{Ra}} \nabla^2 \mathbf{u} + T \vec{e}, \quad (8.2)$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{1}{\sqrt{RaPr}} \nabla^2 T, \quad (8.3)$$

where $\mathbf{u} = (v, w)$ are the velocity components in the horizontal and vertical directions, t is the time, p is the reduced kinetic pressure, T is the temperature, $\vec{e}$ is a unit vector defined by $\vec{e} = (\sin \beta, \cos \beta)$. β is the tilt angle which is positive when the cell rotates counterclockwisely. The dimensionless control parameters are the Rayleigh number $Ra = \hat{g} \hat{\alpha} \Delta \hat{T} \hat{H}^3 / \hat{\nu} \hat{\kappa}$ and Prandtl number $Pr = \hat{\nu} / \hat{\kappa}$, where $\hat{g}$ is the gravitational acceleration, $\hat{\alpha}$ is the isobaric thermal expansion coefficient, $\Delta \hat{T} = \hat{T}_H - \hat{T}_C$ is the temperature difference. $\hat{\kappa}$ is the thermal diffusivity, and $\hat{\nu}$ is the kinematic viscosity.

The free fall velocity $\hat{U} = \sqrt{\hat{g} \hat{\alpha} \Delta \hat{T} \hat{H}}$ and the height of the container $\hat{H}$ are used for non-dimensionalization, the reference time is free-fall time $\hat{t}_f = \hat{H} / \hat{U}$, the temperature is made dimensionless by $T = (\hat{T} - \hat{T}_C) / \Delta \hat{T}$. No-slip and impermeability conditions are imposed at all walls. Two important response parameters are the Nusselt number Nu and the Reynolds number Re which are defined as:

$$Nu = \frac{\hat{Q}}{\hat{\kappa} \Delta \hat{T} / \hat{H}}, \quad Re = \frac{\sqrt{\langle \hat{\mathbf{u}} \cdot \hat{\mathbf{u}} \rangle_{v,t}} \hat{H}}{\hat{\nu}}, \quad (8.4)$$

where $\hat{Q}$ is the time-averaged heat flux input into the system, The Nusselt number used in this study is averaged over the hot plate ($z = 0$). $\hat{\mathbf{u}}$ is the velocity vector. $\langle \rangle_{v,t}$ denotes average over the whole area and time.

The governing equations along with proper boundary conditions are solved by an in-house code lMn2d [79]. All spatial terms are discretized using central second-order finite difference method. Time integration is accomplished using an Adams-Bashforth scheme for the nonlinear terms and Crank-Nicolson scheme for the viscous and diffusion terms. A non-uniform grid with clustered points near walls is adopted to better resolve the thin boundary layers. The Poisson equation is solved using a multi-grid strategy to accelerate the iteration process. The code has been validated and described in our previous works [79, 104, 107, 109].

The considered Ra is fixed to 10^7 and the Pr is 0.71. According to previous studies [114, 159, 220], the flow is already turbulent for the considered Ra and Pr in

Figure 8.1: Sketch of the two-dimensional tilted cavity

this study. The aspect ratios $\Gamma = \hat{W}/\hat{H}$ ranges from 1 to 16. The grid resolution has been chosen according to the requirement for DNS [110]. We set 128 grid points in the vertical direction and the largest grid is 2048×128 for $\Gamma = 16$. This guarantees that there are at least 9 grid points to resolve the thermal boundary layer (BL), which fulfills the requirement that 3.9 or more grid points are needed to resolve the BLs [110]. Different initial flow fields were used in our simulations. We first use the conduction state with small perturbations added to the temperature as initial fields for all β . After the flow is fully developed, we gradually decreases β using the single-roll state obtained for large β as initial conditions. In this way, we can obtain different flow states with different numbers of convection rolls for small β . We then use these different roll states as initial conditions and change β so that we can obtain the β range where a specific roll state can stably exist. It is known that time scales at which LSC structures change can become very long [304–306]. Thus long-time simulations are needed to conclude whether one state can stably exist or flow mode transitions exist. We perform long-time simulations up to 20 000 $\hat{t}_f$ for $\Gamma = 12$ and 16 cases, while for $\Gamma \leq 8$, we generally perform at least 40 000 $\hat{t}_f$.

8.3 Results and discussion

8.3.1 Multiple states in tilted convection with large Γ

We start by showing the influence of cell tilting on the flow organization. We particularly focus our attention on $\Gamma \geq 2$ cases in this study. For $\Gamma = 2$, previous studies showed that the flow organizes in horizontally-adjacent double-roll state (DRS) for $\beta = 0^\circ$ as shown in figure 8.2(a), this DRS is not stable, it experiences spontaneous flow reversals of the two main rolls [107, 220, 251]. Interestingly, it was recently found that the reversals are much more frequent when the cell is tilted which is different

Figure 8.2: Instantaneous temperature field superposed with velocity vectors for different β and $\Gamma = 2$ (a) Double-roll state for $\beta = 0^\circ$. (b) Single-roll state for $\beta = 0^\circ$. (c) $\beta = 15^\circ$. (d) $\beta = 30^\circ$. (e) $\beta = 80^\circ$. (f) $\beta = 90^\circ$. The black arrows indicate the direction of gravity.

Figure 8.3: Instantaneous temperature field superposed with velocity vectors for different β with $\Gamma = 4$. (a) Four-roll state for $\beta = 0^\circ$. (b) Three-roll state for $\beta = 0^\circ$. (c) Single-roll state for $\beta = 0^\circ$. (d) Single-roll state for $\beta = 19^\circ$. The black arrows indicate the direction of gravity. Colour scale as in figure 8.2

from that in a $\Gamma = 1$ cell where flow reversals are suppressed by tilting the cell [107]. As β is further increased to 9° , this DRS will transit to a stable single-roll state (SRS), and the flow reversals are absent for this SRS [107]. Using this SRS as initial conditions for $\beta = 0^\circ$, we found that the SRS can also be stable state for $\beta = 0^\circ$ as shown in figure 8.2(b), while this SRS at $\beta = 0^\circ$ was not reported in previous studies [107, 220, 251]. The Nu of DRS is larger than that of SRS by 13% for $\beta = 0^\circ$. Figure 8.2(c-f) show flow fields for $\beta = 15^\circ, 30^\circ, 80^\circ$ and 90° , respectively. For clarity, the cells are shown in a level manner. We can see that the upper-left and lower-right corner rolls shown in figure 8.2(b) are squeezed when the cell is tilted as illustrated in figure 8.2(c), and the LSC changes from ellipse-like to rectangular-like which is similar to $\Gamma = 1$ cases [270]. As the cell is further tilted to 30° , the high-velocity band concentrates mainly near the walls, the velocity of the fluid in the bulk of the cell is relatively small as illustrated in figure 8.2(d). For large β cases with $\beta = 80^\circ$ and 90° as shown in figure 8.2(e) and 8.2(f), the fluid motion concentrates mainly near the two isothermal plates. The flow is stably-stratified in the bulk, and the fluid there is almost motionless. This is the reason why we can hardly see the velocity field in the bulk region.

For $\Gamma = 4$ we also observe multiple states for small β . Figure 8.3(a)-(c) display three different flow states for $\beta = 0^\circ$. The four-roll state is stable for $\beta \leq 10^\circ$, while the three-roll state is stable up to $\beta = 17^\circ$. At larger $\beta = 19^\circ$, only SRS shown in figure 8.3(d) can stably exist. This SRS is also more rectangular-like than that of $\beta = 0^\circ$ shown in figure 8.3(c).

Multiple states are also observed for larger aspect ratios $\Gamma = 8, 12$ and 16 . Figure 8.4 shows the temperature fields for $\Gamma = 8$ with different β . The initial fields used in this figure all are conduction state superposed with a small perturbation added to the temperature. It is also seen that the flow develops in multiple-roll states (MRS) for small β . While the number of rolls decreases as β increases. There are 8, 7, 5 rolls for $\beta = 0^\circ, 10^\circ, 20^\circ$, respectively. These three different roll states all can be

Figure 8.4: Instantaneous temperature field for $Ra = 10^7$, $Pr = 0.71$, $\Gamma = 8$ for different β . Colour scale as in figure 8.2

Figure 8.5: Instantaneous flow fields for different flow states obtained using different initial conditions for $Ra = 10^7$, $\Gamma = 8$, $\beta = 5^\circ$. (a) Eight-roll state. (b) Seven-roll state. (c) Five-roll state. (d) Three-roll state. (e) Time evolution of the Nu for the four different roll states shown in (a-d), for clarity the Nu for the seven-roll and eight-roll states are shifted upwards by a value of 3 and 7 respectively. The roll numbers decrease from top to bottom. The black arrows indicate the direction of gravity. Colour scale as in figure 8.2

Figure 8.6: Time evolution of the Nusselt number Nu for $\Gamma = 8$ and (a) $\beta = 23^\circ$, (b) $\beta = 25^\circ$. (c) Instantaneous temperature fields at different times in (b). The black arrows indicate the direction of gravity. Colour scale as in figure 8.2

Figure 8.7: The phase diagram in the $\beta - \Gamma$ plane. Red circles (●) correspond to coexistence of multiple states, blue diamonds (◆) denote single-roll state, green squares (■) stand for the state with flow mode transitions.

stable in a certain range of β . For larger $\beta = 30^\circ$, the MRS disappears and SRS develops. For $\beta \geq 30^\circ$, the hot/cold fluid occupies more area near the right/left adiabatic wall with increasing β . As β increases beyond 90° , there exists a stabilizing temperature gradient along z direction, thus the fluid flow gradually becomes laminar. At $\beta = 180^\circ$, we obtain conduction state. When we decrease β using SRS for $\beta = 30^\circ$ as initial conditions, a three-roll state can also be identified. Figure 8.5(a-d) show instantaneous flow fields for four different states with $\beta = 5^\circ$. The time evolution of the Nu for these four states are shown in figure 8.5(e), it is seen that these four states can be stable for very long time up to 60 000 free-fall time units. Moreover, it is clearly found that the Nu is larger for the state with more rolls. This is very similar to RBC in an annulus cell where the Nu for the four-roll state is larger than that of the two-roll state [161], and is also similar to Taylor-Couette flow where Nu_ω is larger for the state with more Taylor rolls [163]. Actually, this point has been mentioned and studied in a previous work in a 2D cell with large Γ [145]. When one state jumps to another state with more convection rolls with increasing Γ , there is a sharp increase of Nu (see figure 4 in Ref. [145]), which is just the same as present conclusion. The reason is that, for the state with more rolls, there are more plume-ejected regions and these regions are favourable for heat transport, leading to larger Nu [145]. However, they did not report multiple states for a fixed large Γ . When considering possible multiple states at fixed Γ , the situation in figure 4 of Ref. [145] will become more complicated. We note that Ref. [272] also studied aspect ratio dependence of Nu , their configurations are 3D cylindrical cells. They found that a minimum of $Nu(\Gamma)$ is found at $\Gamma \approx 2.5$ and $\Gamma \approx 2.25$ for $Ra = 10^7$ and $Ra = 10^8$, respectively. This is the point where the LSC undergoes a transition from a single-roll to a double-roll

pattern. It can be seen the conclusion in 3D cylindrical cells is opposite to that of the 2D ones [145]. However, the reasons are still unknown, and further work is needed to clarify this.

We also find that flow mode transitions between different roll states exist for $\Gamma \geq 8$ when β increases beyond a critical value. Figure 8.6(a) and 8.6(b) show time evolution of Nu for $\Gamma = 8$ with $\beta = 23^\circ$ and 25° respectively. As mentioned before, the Nu are different for different states, thus the Nu changes sharply when flow mode transitions happen. It is easily seen that flow mode transitions happen more frequently when β increases from 23° to 25° . Figure 8.6(c) show instantaneous temperature fields at different times in figure 8.6(b). At $t = 1000$, the flow stays in three-roll state with relatively low Nu . At $t = 1310$, flow mode transition happens and the flow reorganizes itself until five-roll state appears at around $t = 1470$. Before the appearance of this five-roll state, the six-roll state as shown at $t = 1380$ also appears as intermediate flow state. The five-roll state last for a while and then transits to three-roll state as illustrated at $t = 1930$. Before the appearance of three-roll state, there also exist four-roll state as shown at $t = 1720$. At most of the time, the flow stays in three-roll state. It is also found that the three-roll state does not always transits to a five-roll state. For example, at $t = 5660$, there is a decrease of the Nu , and the three-roll state transits to a state with a large-scale roll that occupies almost the whole size of the convection cell as illustrated in temperature fields at $t = 5663$. It was found that flow reversals happen when β is larger than a critical value for small Pr (see figure 8 in Ref. [107]), this is similar to the present finding that flow mode transitions happen when β is larger than a critical value for $\Gamma \geq 8$.

These findings lead us to map out flow states over the $\beta - \Gamma$ parameter space, which is shown in figure 8.7. It is seen that multiple stable states can coexist for small β for $\Gamma \geq 2$. For relatively large $\Gamma = 8, 12$ and 16 , flow mode transition happens when β increase beyond a critical value. At large $\beta = 30^\circ$, the flow constantly jumps between different roll states for $\Gamma = 12, 16$. With further increasing β . Multiple-roll state disappears and we generally obtain single-roll state. As multiple-roll states are stable for small β , while the flow develops in single-roll state for large β , thus the flow mode transitions between different roll states at intermediate β can be viewed as mode competition between these two different states.

8.3.2 Nusselt number and Reynolds number

Γ	$\beta(deg.)$	Nu/n	Nu/n	Nu/n	Nu/n	Nu/n
2	0	12.13/2	10.82/1			
2	3	12.41/2	11.22/1			
2	5	12.49/2	11.41/1			
2	8	12.57/2	11.68/1			
4	0	12.89/4	12.92/3	9.07/1		
4	3	13.06/4	13.15/3	9.44/1		
4	5	13.42/4	13.21/3	9.72/1		

Γ	$\beta(deg.)$	Nu/n	Nu/n	Nu/n	Nu/n	Nu/n
4	10	13.42/4	12.77/3	10.26/1		
4	15		13.08/3	10.75/1		
4	17		12.52/3	11.04/1		
8	0	14.02/8	14.01/7	12.60/5		
8	3	13.94/8	13.90/7	12.60/5		
8	5	13.91/8	13.68/7	12.78/5	10.69/3	
8	10		13.66/7	13.08/5	11.34/3	
8	15		13.72/7	12.57/5	11.67/3	
8	20		13.93/7	12.87/5	11.79/3	
12	0	14.11/12	14.25/11	13.13/9	12.44/7	
12	3	14.03/12	14.15/11	13.34/9	12.32/7	
12	5	13.97/12	13.92/11	13.55/9	12.71/7	
12	10		13.77/11	13.05/9	12.23/7	
12	15		13.97/11	13.30/9	12.53/7	
12	20			13.19/9	12.70/7	
16	0	14.32/16	14.34/15	12.94/11	12.93/10	
16	3	14.14/16	14.25/15	12.97/11	12.67/10	12.21/9
16	5	14.10/16	14.05/15	13.11/11	12.49/10	12.12/9
16	10		13.88/15	12.81/11	12.54/10	12.13/9
16	15		14.03/15	12.96/11	12.64/10	12.28/9
16	20		14.13/15	13.03/11		12.59/9

Table 8.1: The Nusselt number and corresponding roll number (n) for different Γ and β where multiple stable states exist.

We then investigate the effect of cell tilting on the Nu and the Re . Table 8.1 lists the Nu for different states with different numbers of convection rolls. It is even found that five flow states are able to coexist with the same parameters for the largest aspect ratio $\Gamma = 16$. The Nu is generally larger for the state with more convection rolls. Figure 8.8(a) shows Nu as a function of β for different Γ . The Nu for different states for $\Gamma = 4, 8, 12$ and 16 with small β has been listed in table 8.1, so for the MRS, we only show Nu for the state with most convection rolls for a specific combination of Γ and β . For $\Gamma = 2$ and 4 , the two branches of Nu for small β correspond to SRS (lower branch) and MRS (upper branch), respectively. It is also seen that the MRS transport heat more efficiently than SRS. When MRS transits to SRS, the Nu has a sudden drop. When $\beta > 90^\circ$, the Nu decreases monotonically with increasing β for all considered Γ due to the fact that the stabilized temperature gradient along z direction exists when $\beta > 90^\circ$. However, the trend of Nu in the range of $0^\circ < \beta < 90^\circ$ differs greatly for different Γ . For relatively small aspect ratios $\Gamma \leq 4$, the Nu first increases with increasing β , and after reaching their maxima which are approximately located at $\beta = 80^\circ$ for $\Gamma = 1$ and $\beta = 70^\circ$ for $\Gamma = 2$ and 4 , they decrease monotonically with increasing β . In prior results [59, 282], the maximal heat transport was also achieved

Figure 8.8: Absolute and normalized Nusselt numbers Nu and Reynolds numbers Re as a function of β for fixed $Ra = 10^7$, $Pr = 0.71$ with different aspect ratios. (a) Absolute Nusselt numbers Nu . (b) Normalized Nusselt numbers $Nu(\beta)/Nu(0)$. (c) Absolute Reynolds numbers Re . (d) Normalized Reynolds numbers $Re(\beta)/Re(0)$. The normalization for $\Gamma = 2$ and 4 uses the corresponding values for single-roll state for RBC ($\beta = 0^\circ$).

at an intermediate β in a cylindrical cell of unit aspect ratio, and neither vertical convection ($\beta = 90^\circ$) nor RBC seems to be optimal state in terms of heat transport. However, the behaviour of Nu as a function of β for large Γ is qualitatively different, it is seen that the Nu generally decreases monotonically with increasing β for large $\Gamma = 8, 12$ and 16 , which is more evidently shown in the normalized Nusselt number $Nu(\beta)/Nu(0)$ in figure 8.8(b). When flow mode transitions happen with increasing β , the Nu experiences a sudden drop around $20^\circ < \beta < 30^\circ$, it seems that the RBC configuration transports heat most efficiently for large aspect ratios. Therefore, in the present system, the idea to enhance heat transport by tilting the cell can be realized only for relatively small Γ . Another feature is that the curves of Nu for the largest Γ ($\Gamma = 12, 16$) collapse well with each other apart from the β range where flow mode transitions exist. This imply that the influence of Γ on the heat transport decreases with increasing Γ for all the considered β .

Figure 8.8(c) shows the Re as a function of β for different Γ . We can see that the Re of MRS is also higher than that of SRS. An apparent feature of Re is that it does not change much for small β for all considered Γ . For $\Gamma \leq 4$, the Re all decrease with increasing β for large β ($\beta \geq 25^\circ$). This trend was also found in a 3D cell with unit aspect ratio [59, 270]. Interestingly, for larger aspect ratios $\Gamma \geq 8$, the trend is qualitatively different. When the Re reach their local minimum at around $\beta \sim 25^\circ$, they increase with increasing β until reaching their local maxima, after which they again decrease with increasing β until $\beta = 90^\circ$.

8.4 Conclusions

In summary, we have studied the effect of cell tilting on thermal convection in two-dimensional rectangular cells with widely varying aspect ratios $1 \leq \Gamma \leq 16$ using DNS. We found that multiple flow states can coexist for small β with large aspect ratios ($\Gamma \geq 2$). In some cases with $\Gamma = 16$, up to five different states are even found with the same parameters. For $\Gamma = 2$, the stable single-roll state is identified for small β , which was not reported in 2D RBC in previous works [107, 220, 251]. The multiple states at small β are stable without flow mode transitions although we perform very long-time simulations. However, when β increases beyond a critical value for $\Gamma \geq 8$, flow mode transitions between different roll states are observed. It is also found that Nu is higher for the flow state with more convection rolls. The Nu as a function of β is found to be highly dependent on the aspect ratio. For relatively small aspect ratios ($\Gamma \leq 4$), the Nu first increases with increasing β , and after reaching its local maximum, it decreases with increasing β . However, the Nu generally decreases monotonically with increasing β for larger aspect ratios ($\Gamma \geq 8$). This finding demonstrates that the idea to enhance heat transfer by tilting the cell can only be realized for relatively small Γ in the present system.

The present results enrich the studies on multiple states in turbulence, and demonstrate that special attention should further be paid when studying tilted convection with large aspect ratios. The question of whether multiple states can coexist at much higher Ra in tilted cells with large Γ is still an open question and demand further investigations. Obviously, the Prandtl number Pr is another important control parameter, and the effect of Pr on the multiple states should also be explored in the future.

Chapter 9

Thermal convection in a tilted rectangular cell with aspect ratio 0.5.[◦]

Thermal convection in a three-dimensional tilted rectangular cell with aspect ratio 1/2 is studied using direct numerical simulations (DNS) both within Oberbeck-Boussinesq (OB) approximation and with strong non-Oberbeck-Boussinesq (NOB) effects. The considered Rayleigh numbers Ra range from 10^5 to 10^7 , the working fluid is air at 300K and the corresponding Prandtl number Pr is 0.71. Within OB approximation, it is found there exist multiple states for $Ra = 10^5$ and hysteresis for $Ra = 10^6$. For a relatively small tilt angle β , the large-scale-circulation (LSC) can either orient along one of the vertical diagonal planes (denoted by M_d mode) or orient parallel to the front wall (denoted by M_p mode). Which of the two modes transports heat more efficiently is not definitive, and it depends on Rayleigh number Ra . For $Ra = 10^7, \beta = 0^\circ$, the time-averaged flow field contains four rolls in the upper half and lower half of the cell, respectively, the M_d and M_p modes only developing in tilted cells. By investigating NOB effects in tilted convection for fixed $Ra = 10^6$, it is found the influence of NOB effects on Nusselt number Nu , Reynolds number Re , and central temperature T_c is different for different β range. NOB effects can either increase or decrease Nu , Re , and T_c when β is varied.

[◦]Based on: **Qi Wang**, Bo-Lun Xu, Shu-Ning Xia, Zhen-Hua Wan, De-Jun Sun, *Thermal convection in a tilted rectangular cell with aspect ratio 0.5*, Chin. Phys. Lett. **34**, 104401 (2017).

9.1 Introduction

Thermal convection is ubiquitous in nature and is important in many industrial processes. An important model to study thermal convection is Rayleigh-Bénard convection (RBC) where a fluid layer in a box is heated from below and cooled from above [53, 54, 56, 236]. In industrial process, the heat transfer in heat exchangers can be enhanced by tilting the cell. The relation between the tilt angle and the Nusselt number Nu has a direct interest for reducing the loss of energy in honeycomb solar collector plates. In recent years, the effect of cell tilting on RBC has been studied both numerically [59] and experimentally [270, 271]. These results are all performed within Oberbeck-Boussinesq (OB) approximation, and they are all restricted to a cell of unit aspect ratio. Another typical aspect ratio in the studies of RBC is 1/2 [296, 307]. However, there are less work devoted to tilted convection in a cell with aspect ratio 1/2. In this work, we try to fill this gap by investigating thermal convection in a tilted cell with aspect ratio 1/2 within OB approximation.

The OB approximation, being more mathematical than physical, assumes the thermal properties of the working fluid to be constant and the density is constant everywhere except within the buoyancy term. Obviously, this approximation is valid only for small temperature differences. In many practical applications, such as thermal insulation systems in nuclear reactors, the temperature differences can be as large as several hundred degrees Kelvin. Thus, non-Oberbeck-Boussinesq (NOB) effects should be taken into consideration and are worthy of in-depth study. The influence of NOB effects on RBC has been studied extensively in recent years [79, 95, 96, 100]. However, there are less work devoted to NOB effects in tilted convection. As tilted convection in the industrial process may be subjected to large temperature differences. In this letter, we also study NOB effects in tilted convection.

9.2 Numerical settings

The problem is described in figure 9.1. The tilt angle β is such that $\beta = 0$ corresponds to RBC. When the cell rotates counterclockwise about x axis, the tilt angle increases. The considered Rayleigh numbers Ra range from 10^5 to 10^7 . The working fluid is air with reference temperature 300K and the corresponding Pr is 0.71. Both the length-to-height aspect ratio and the width-to-height aspect ratio are fixed to 1/2. Two important response parameters of the system are Nusselt number Nu and Reynolds number Re which are defined by:

$$Nu = \frac{Q}{k\Delta T/H}, Re = \frac{\sqrt{\langle \mathbf{u} \cdot \mathbf{u} \rangle} H}{\nu} \quad (9.1)$$

where Q is the heat flux across any horizontal plane, k the thermal conductivity of the fluid, $\Delta T = T_H - T_C$ the temperature difference between the hot and cold plate, H the height of the cell, ν the kinematic viscosity of the fluid, $\mathbf{u}$ the velocity vector.

Figure 9.1: Sketch of a tilted rectangular cell with length L , width W and height H .

The low-Mach-number equations [132] are adopted in order to study NOB effects. The governing equations are solved by our in-house code lMn3d. All spatial terms are discretized using second-order central difference scheme. Time integration is accomplished using an Adams-Bashforth scheme for the nonlinear terms and Crank-Nicolson scheme for the viscous and diffusion terms. The two-dimensional version of the code lMn2d which is directly developed from lMn3d has already been validated and described in our previous work [79]. Computation grids of $48 \times 48 \times 96$, $64 \times 64 \times 128$, $96 \times 96 \times 192$ are used for $Ra = 10^5$, $Ra = 10^6$ and $Ra = 10^7$, respectively. The according time steps for these three grids are 0.01, 0.008, 0.003.

9.3 Results

We first discuss the influence of cell tilting on the flow organization. Figure 9.2 shows time-averaged velocity streamlines for different combinations of Ra and β , the colour contour denotes the magnitude of the time-averaged vertical velocity which is normalized by free-fall velocity w/U_f . Note the free-fall velocity is defined by: $U_f = \sqrt{g\alpha\Delta TH}$, where α is the thermal expansion coefficient, g is acceleration due to gravity. For $Ra = 10^5$, we observe two different flow states by using different initial conditions. One contains a large-scale-circulation(LSC) oriented along one of the

Figure 9.2: Time-averaged velocity streamlines for (a, d) $Ra = 10^5$. (b, e) $Ra = 10^6$. (c, f) $Ra = 10^7$. (a) M_d mode for $\beta = 0^\circ$. (b) M_d mode for $\beta = 3^\circ$. (c) $\beta = 0^\circ$. (d) M_p mode for $\beta = 0^\circ$. (e) M_p mode for $\beta = 3^\circ$. (f) M_d mode for $\beta = 5^\circ$. The colour coding denotes the magnitude of time-averaged vertical velocity normalized by free-fall velocity w/U_f

diagonal planes which is denoted by M_d mode as shown in figure 9.2(a), this kind of flow structure also prevails in a cubic cell even at much larger Ra . The other contains a LSC oriented parallel to the front wall which is denoted by M_p mode as shown in figure 9.2(d). For small β , both of these two modes can be stable, while only M_p mode exists for $\beta > 8^\circ$. The Nu of M_d mode is larger than that of M_p mode which may be related to larger flow intensity denoted by Reynolds number Re as will be shown later. For $Ra = 10^6$, these two modes can also both be stable at small β as shown for $\beta = 3^\circ$ in figure 9.2(b) and 9.2(e). However, different from $Ra = 10^5$, the Nu of M_p mode is larger than that of M_d mode for $Ra = 10^6$. Therefore, which of the two modes transports heat more efficiently is not definitive, and it depends on Ra . For $Ra = 10^7$, the flow organization is qualitatively different for $\beta = 0^\circ$ as shown in figure 9.2(c). Four rolls develop in the upper half and lower half of the cell respectively, and there is no apparent LSC. The M_d mode only exists when the cell is tilted as shown in figure 9.2(f) for $\beta = 5^\circ$. For even larger β , the flow will organize in M_p mode.

We then investigate the influence of cell tilting on Nu and Re . Figure 9.3(a) and 9.3(b) show absolute and normalized Nu as functions of β for different Ra . It should be mentioned that for $Ra = 10^5$, the Nu of M_p mode for RBC is used for normalization, while for $Ra = 10^6$ only M_d mode exists for RBC, and we use the corresponding Nu for normalization. Multiple states are found for $Ra = 10^5$ and hysteresis is found for $Ra = 10^6$ for small β when β increases or decreases. The hysteresis of $Ra = 10^6$ is more clearly shown in figure 9.5(a) and 9.5(b). From figure 9.3(b), it is seen at first sight that the $Nu(\beta)/Nu(0)$ is strongly influenced by Ra while no apparent regularity-dependence on Ra can be found. For $Ra = 10^5$, the Nu of M_p mode first increases as β increases until it reaches maximum value at about 50° , it then decreases with increasing β . While for M_d mode, the Nu decreases monotonically with increasing β . For $Ra = 10^6$, the behaviour of $Nu(\beta)/Nu(0)$ is much more complicated, there at least exist three maxima and two minima. For $Ra = 10^7$, the Nu experiences an increase when the cell is minor tilted, which is different from that in a cylindrical cell with aspect ratio $1/2$ where Nu decreases in a tilted cell at higher Ra [261]. This demonstrates that the Rayleigh number plays an important role in the influence of tilting on Nu , which is also found in an aspect ratio 1 cell [59, 270]. The Nu has two maxima and one minimum, the first and second maximum locate near $\beta = 30^\circ$ and $\beta = 80^\circ$ respectively.

Compared with Nusselt number Nu , the Reynolds number Re , however, shows a much more regular dependence on β as shown in figure 9.3(c) and 9.3(d). The normalization in figure 9.3(d) is the same as that of Nu . Normally, the Re first increases with β , after reaching its maximum, it decreases monotonously with increasing β , which is similar to the cases in a cylindrical cell of unity aspect ratio [59]. It is also seen the Re of M_d mode is higher than that of M_p mode for $Ra = 10^5$ which is in contrary to that of $Ra = 10^6$. The same conclusion is also drawn for Nu as shown before, this implies that the higher Nu may be related with faster fluid motion.

The above results are performed within strict OB approximation. We now investigate strong NOB effects on the flow organization, Nusselt number Nu , Reynolds

Figure 9.3: Absolute and normalized Nusselt and Reynolds numbers in tilted convection for $Ra = 10^5$ (squares), $Ra = 10^6$ (circles) and $Ra = 10^7$ (diamonds), as functions of tilt angle β . (a) Absolute Nusselt numbers Nu . (b) Normalized Nusselt numbers $Nu(\beta)/Nu(0)$. (c) Absolute Reynolds numbers Re . (d) Normalized Reynolds numbers $Re(\beta)/Re(0)$. The corresponding values for RBC ($\beta = 0^\circ$) are used for normalization. For $Ra = 10^5$ the Nu and Re of M_p mode for RBC are used for normalization, while for $Ra = 10^6$, we use the Nu and Re of M_d mode.

Figure 9.4: Time evolution of Nusselt number Nu for $Ra = 10^6$ with NOB effects where temperature difference $\Delta T = 360\text{K}$ for (a) $\beta = 1^\circ$. (b) $\beta = 2^\circ$. (c,d,e) Instantaneous streamlines at different times in (a). (c) $t = 960t_f$. (d) $t = 5760t_f$. (e) $t = 10560t_f$.

Figure 9.5: (a) Nusselt number Nu as a function of tilt angle β within OB approximation (green circles) and with NOB effects (red squares). (b) Reynolds number Re as a function of tilt angle β within OB approximation (green circles) and with NOB effects (red squares). (c) Central temperature deviation $T_c - T_m$ as a function of tilt angle for NOB cases.

number Re , and central temperature T_c in tilted convection for fixed $Ra = 10^6$. The temperature difference ΔT is 360K for the considered NOB effects, corresponding to $T_H = 480\text{K}$ and $T_C = 120\text{K}$. For $\beta = 0^\circ$ and $\beta = 1^\circ$, there exists flow mode transition between single-roll mode(SRM) and double-roll mode(DRM) for the NOB cases. Figure 9.4(a) shows time evolution of Nu for $\beta = 1^\circ$ and figure 9.4(c-e) show the instantaneous streamlines at the time $t = 960t_f$, $t = 5760t_f$ and $t = 10560t_f$ respectively. It is seen the flow organizes in SRM at time $t = 960t_f$ and $t = 5760t_f$ with relatively larger Nu , while the orientation of the LSC is different. At time $t = 10560t_f$, the flow organizes in DRM with smaller Nu . It is already experimentally found that the Nu of SRM is larger than that of DRM at higher Ra within OB approximation [274]. The flow mode transition is also observed in experiment within OB approximation [158]. Thus our numerical results confirm the phenomenon and show it also exists when strong NOB effects exist. As the β increases to 2° , the flow organizes in stable M_d mode similar to figure 9.4(d), the reorientation of LSC and flow mode transition is suppressed. Figure 9.4(b) shows time evolution of Nu for $\beta = 2^\circ$, it is evident the Nu of 2° is larger than that of 1° .

Figure 9.5(a) and 9.5(b) show Nu and Re as functions of β for OB and NOB cases. It is evident that the influence of NOB effects is different for different β range. For $\beta = 0^\circ$ and 1° , flow mode transition exists for NOB cases and both Nu and Re are decreased compared to OB cases. For $2^\circ \leq \beta \leq 23^\circ$, the Nu and Re of both M_d and M_p modes are increased when considering NOB effects. While for larger $\beta > 23^\circ$, the Nu and Re decrease compared to OB cases, and the influence of NOB effects on Re at $\beta \geq 60^\circ$ is very small. Quantitatively, the maximum reduction of Nu is 11% at $\beta = 0^\circ$ which is due to changed flow modes, and the maximum enhancement of Nu is 7% at $\beta = 5^\circ$ in M_p mode. For Re , the maximum reduction is 4% at $\beta = 0^\circ$ and 30° while the maximum enhancement is 7% at $\beta = 3^\circ$ in M_d mode. The present finding demonstrates complicated influence of NOB effects on tilted convection at relatively low $Ra = 10^6$. For higher Ra where the flow is in the fully turbulent state, we expect this influence will be small as already found in turbulent RBC [95]. The influence of NOB effects on tilted convection at higher Ra will be the goal of our future research.

A major influence of NOB effects on RBC is the breaking of top-down-symmetry, this will cause the deviation of central temperature T_c from the arithmetic mean temperature of hot and cold plates $T_m = (T_H + T_C)/2$. We show in figure 9.5(c) this temperature deviation $T_c - T_m$ as a function of β . It is seen at small β the central temperature decreases compared to T_m , while at larger β the T_c is normally increased except for $\beta = 40^\circ$ where T_c is minorly decreased. This demonstrates that for different β , NOB effects have a varying influence on T_c .

9.4 Conclusions

In summary, the influence of cell tilting on thermal convection in a three-dimensional rectangular cell with aspect ratio 1/2 is numerically studied both within OB approximation and with strong NOB effects caused by large temperature differences. It is

found there exist multiple states for $Ra = 10^5$ and hysteresis for $Ra = 10^6$. Both M_d and M_p modes can be stable at a specific β range. Which of the two modes transports heat more efficiently is not definitive, and it depends on Ra number. For $Ra = 10^7, \beta = 0^\circ$, the time-averaged flow field contains four rolls in the upper half and lower half of the cell respectively. The M_d and M_p modes only develop in tilted cells. By investigating the influence of NOB effects on tilted convection for fixed $Ra = 10^6$, it is found there exists flow mode transition between SRM and DRM for $\beta = 0^\circ, 1^\circ$. The SRM is shown to transport heat more efficiently. In different β range, the influence of NOB effects on Nu , Re and T_c is different for considered $Ra = 10^6$. NOB effects can either increase or decrease Nu , Re , T_c when β is varied.

Part III

Vertical convection

Chapter 10

Non-OberbeckBoussinesq effects due to large temperature differences in a differentially heated square cavity filled with air^o

We numerically investigate non-Oberbeck-Boussinesq (NOB) effects due to large temperature differences in a two-dimensional differentially heated square cavity using low-Mach-number equations. The working fluid is air and the Prandtl number Pr is 0.71 for the reference state. The considered Rayleigh numbers Ra range from 10^5 to 10^9 . Various temperature differences $\Delta\hat{T}$ between the hot and cold plates are considered and the maximum value is up to 360K. The critical Rayleigh number for the onset of unsteadiness decrease with increasing temperature differences. The NOB effects on the temperature and velocity fields are investigated. It is found that both the thermal and velocity boundary layers become thicker near the hot plate while they get thinner near the cold plates under NOB conditions. The central temperature is increased compared to Oberbeck-Boussinesq (OB) cases considering NOB effects. The normalized center temperature θ_c roughly increases linearly with increasing ϵ . The horizontal velocity near the top plate is normally enhanced by NOB effects while it is generally decreased near the bottom plate under NOB conditions. Despite these marked qualitative differences in the NOB flow relative to OB flow, the overall integral quantities like the Nusselt number Nu and the Reynolds number

^oBased on: **Qi Wang**, Shu-Ning Xia, Rui Yan, De-Jun Sun, Zhen-Hua Wan, *Non-OberbeckBoussinesq effects due to large temperature differences in a differentially heated square cavity filled with air*, Int. J. Heat Mass Trans. **128**, 479-491 (2018).

Re are insensitive to NOB effects and retain their Ra -scaling exponents. The Nu has a minor decrease under NOB conditions. The maximum decrease is only about 3% for the temperature difference as large as 360K. Within OB approximation, the Nusselt number Nu is found to scale as $\sim Ra^{0.27}$, and the inverse thermal boundary layer thickness λ_θ^{-1} also scale as $\sim Ra^{0.27}$. The scaling exponent does not change under NOB conditions. The Reynolds number based on the root mean square (r.m.s) velocity Re^{rms} increases slightly for NOB cases. We also find $Re^{rms} \sim Ra^{0.37}$, $Re^w \sim Ra^{0.50}$, and $Re^v \sim Ra^{0.45}$ for OB cases, where Re^w/Re^v are Reynolds number based on the maximum magnitude of vertical/horizontal velocity over the whole cell. The NOB effects almost have no influence on these scaling exponents.

10.1 Introduction

Thermal convection driven by temperature differences is important in many industrial processes and is also ubiquitous in nature environments. Two important model problems for the study of thermal convection are Rayleigh-Bénard convection (RBC) [53–56] where a cavity is heated from the below and cooled from the above, and convection in a differentially heated cavity (DHC) [308–310] where the cavity is heated/cooled from the sides. The latter one is also known as vertical convection [57–59, 149, 278, 311]. The present work focuses on the DHC configuration. Examples of typical applications for DHC include thermal insulation using double-pane windows or double walls, ventilation of rooms, improvement of the efficiency of solar collectors, nuclear reactors, fire within buildings and cooling of electronic devices. Apart from its various applications in engineer processes, it is also served as a benchmark case for the development of numerical algorithms and validation of codes [312–316].

Since the pioneering work of Batchelor [317], this problem has been investigated extensively over the past years. For relatively low Ra (e.g. $Ra < 10^3$), the flow is weak and heat is transferred mainly by thermal conduction. As Ra increases, the flow in the cell is strengthened and stratified flow structures appear in the center [312, 318, 319], however, the flow is still steady. As Ra increases further, the flow becomes unsteady with periodical or quasi-periodical motion [310, 320] or chaotic [308]. Eventually, the flow becomes turbulent at higher Ra [149, 278, 321–324]. Of particular research interest in DHC is the relationship between the Nusselt number Nu and the Rayleigh number Ra . Previous studies have shown a preference for the power-law form for fixed Pr , i.e. $Nu \sim \alpha Ra^\beta$. But the scaling exponent β is found to range anywhere between $1/3$ and $1/4$ [57, 58, 311, 325–327]. More efficient heat transfer is often desirable in industrial applications, for instance, this can be achieved by mounting a fin on the side wall [328]. The transient behaviour of DHC with a fin on the side wall has also been investigated in recent years [329–332].

Thermal convection is usually investigated under the Oberbeck-Boussinesq (OB) approximation. This approximation has four main assumptions. (i): The density is constant everywhere except within the buoyancy term. (ii): In buoyancy term, the density ρ is a linear function of temperature T . (iii): All material properties such as thermal diffusivity κ , viscosity ν are assumed to be constant. (iv): Energy dissipation due to viscosity is neglected in the temperature equation. Any deviations of these four assumptions are commonly referred to as non-Oberbeck-Boussinesq (NOB) effects. For most fluids under small temperature differences, OB approximation is found to be sufficiently accurate. However, when the temperature difference is relatively large, the material properties may depend strongly on the temperature and the density is not a linear function of the temperature anymore, therefore the NOB effects should be taken account. It is shown by Ref. [123] that the maximum temperature differences for air/water for which OB approximation is applicable are 28.6K/1.25K. However, large temperature differences are ubiquitous in industrial processes, examples include thermal insulation systems in nuclear reactors where the typical temperature differ-

ences are of the order of several hundred degrees Kelvin. This situation also takes place in foundry processes and astrophysical flows. Therefore it is important to study the influence of NOB effects on thermal convection.

The NOB effects in RBC have already been studied extensively [53,55], which can serve as reference to understand NOB effects in DHC. Two paths can lead to NOB effects [55]. One is compressibility, and this usually happens in gases especially those in the vicinity of critical points [96]. The other is sensitive temperature dependence of material properties which is usually found in liquids [95]. The NOB effects in RBC for various fluids such as gaseous helium [97,124], gaseous ethane [96,127], SF₆ [129], water [95,99,101], glycerol [100,125,126] and air [79,105] have been investigated extensively. Ref. [94] confirmed experimentally that NOB effects will lead to a top-down symmetry breaking, the thickness of top and bottom boundary will be different and this will cause a shift of central temperature, the temperature in the center of the cell is not the arithmetic average of the temperature at the top and bottom plate i.e. $\hat{T}_c \neq (\hat{T}_b + \hat{T}_t)/2$. For NOB effects in water, Ref. [95] experimentally found that the Nusselt number Nu and Reynolds number Re are rather insensitive against NOB effects. The Nu was found to be only slightly smaller (at most 1.4%) under NOB conditions and the Re in the OB and NOB cases agree with each other within the experimental resolution of about 2%. The central temperature is increased due to NOB effects. While for gaseous ethane, Ref. [96] found a decrease of the central temperature and an increase of the Nu which are of opposite and greater magnitude than those for NOB effects in water. More recently, Ref. [131] performed experimental study of the whole velocity field under NOB conditions. The accumulated knowledge shows that the NOB effects depend very sensitively on the particular working fluid, particular material parameters can dominate in one fluid, but are subdominant in another one [98], and we are still far away from a complete understanding of this variety of NOB effects [55]. The NOB effects are especially relevant to high- Ra convection where the temperature difference is usually large, however, the NOB effects in various fluids at high Ra is still poorly understood.

Compared with NOB effects in RBC, there are less works devoted to NOB effects in DHC. Earlier results regarding NOB effects in DHC mainly focused on stability analysis [320,333–337]. These results all show that for any finite temperature difference between the walls the artificial symmetry of the flow obtained with OB approximation is broken because of material property variations. Ref. [320] found that temperature and velocity distributions are very sensitive to property variations and the critical Ra for stationary and oscillatory instabilities are reduced with increasing temperature differences. Ref. [335] examined linear stability of DHC in a tall enclosure considering NOB effects, they investigated the dependence of the critical Ra on the temperature difference, and showed that two different modes of instability are possible. Later, Ref. [337] performed nonlinear analysis of convection flow in a tall vertical enclosure considering NOB effects, they found that for small temperature differences the basic flow bifurcates supercritically due to shear-driven disturbances, while this bifurcation becomes subcritical for larger temperature differences. Recently, Ref. [315] proposed

a new variation of smoothed particle hydrodynamics (SPH) method, and obtained steady laminar natural solutions for a square cavity of $Ra = 10^5$. Ref. [338] accessed the limits of OB approximation in a tall differentially heated cavity filled with water, they found NOB effects are relevant especially beyond the temperature difference of 30°C . More recently, Ref. [339] investigated effects of variable air properties on transient natural convection, showing that the effects of variable properties are especially relevant along the flow development period, during which the average Nu can differ up to roughly 10% with respect to the constant properties case.

In this paper, we aim at studying NOB effects in DHC due to large temperature differences with the air as the working fluid by means of two dimensional (2D) direct numerical simulations (DNS). The 2D DNS is numerically cheaper so that a good resolution can be achieved, also the flow visualization is much easier. It is concluded by Ref. [340] that for $Pr \geq 1$ various properties observed in three dimensional (3D) DNS are well reflected in the 2D DNS. Ref. [181] also found for higher Pr , the $Nu \sim Ra$ scalings are very similar for 2D and 3D, differing only by a constant factor up to $Ra = 10^{10}$. Ref. [99] also performed 2D DNS for RBC considering NOB effects, showing that 2D approach reflects well the dynamics of the boundary and the behaviour of central temperature. Ref. [322] performed 2D and 3D simulations of DHC for Ra up to 10^{10} , and they found that 2D computations were in good agreement with 3D ones in terms of time-averaged structures and overall Nusselt numbers. In view of the advantages of the 2D simulations, they have been used widely to study thermal convection in recent years [122, 148, 183, 204, 267–269]. The main purpose of this paper is to illustrate the differences between OB and NOB cases in detail. The 2D results presented here can also shed some light on NOB effects in true 3D DHC.

The remainder of this chapter is organized as follows. In §10.2 the problem is briefly described, the governing equations and numerical methods are introduced. The results are presented and analysed in §10.3, which is divided into three parts. Section 10.3.1 studies NOB effects on the critical Rayleigh number for the onset of unsteadiness, the temperature and velocity distributions. In §10.3.2, we investigate NOB effects on the Nu . Section 10.3.3 discusses NOB effects on the Re . We finally summarize our findings in §10.4.

10.2 Numerical settings

10.2.1 Governing equations

Consider a 2D cavity filled with air with width $\hat{W}$ (hatted quantities are dimensional) and height $\hat{H}$. The left and right walls are kept at temperatures of $\hat{T}_H$ and $\hat{T}_C$, respectively, where $\hat{T}_H > \hat{T}_C$. The top and bottom walls are insulated. All walls are assumed to be no-slip and the velocities are set to zero. The above problem is illustrated in Figure 10.1. The low-Mach-number equations [132] capable of treating NOB effects in air [320] are adopted. The dimensionless low-Mach-number equations read

Figure 10.1: Sketch of a differentially heated square cavity.

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} = 0, \quad (10.1)$$

$$\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} = -\frac{\partial \pi}{\partial x_j} + \sqrt{\frac{Pr}{Ra}} \frac{\partial \tau_{ij}}{\partial x_j} + \frac{1}{2\epsilon} (\rho - 1) n_i, \quad (10.2)$$

$$\rho c_p \left(\frac{\partial T}{\partial t} + \frac{\partial u_j T}{\partial x_j} \right) = \frac{1}{\sqrt{Ra Pr}} \frac{\partial}{\partial x_j} k \frac{\partial T}{\partial x_j} + \Gamma \frac{dp}{dt}, \quad (10.3)$$

$$p = \rho T, \quad (10.4)$$

where $u_i = (v, w)$ and $x_i = (y, z)$ are the velocity components and coordinates in the horizontal and vertical directions. T is the temperature, ρ is the density, p is the thermodynamic pressure which is uniform in space and is only a function of time. π is the reduced pressure which accounts for the hydrostatic and dynamic effects. $n_i = (0, -1)$ is the unit vector in the direction of gravity. τ_{ij} is the viscous stress tensor given by

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \mu \frac{\partial u_k}{\partial x_k}, \quad (10.5)$$

where δ_{ij} is the Kronecker delta function. $\gamma = 1.4$ is the ratio of specific heats and $\Gamma = (\gamma - 1)/\gamma$ is a measure of the resilience of the fluid. $c_p = 1$ is the isobaric specific heat and k is the thermal conductivity. ϵ is the temperature differential which will be defined later. The low-Mach-number equations are non-dimensionalized by:

$$x_i = \hat{x}/\hat{W}, u = \hat{u}/\hat{u}_0, t = \hat{t}/t_0, \quad (10.6)$$

$$\rho = \hat{\rho}/\rho_0, p = \hat{p}/p_0, \pi = \hat{\pi}/(\hat{\rho}_0 \hat{u}_0^2), T = \hat{T}/\hat{T}_0, \quad (10.7)$$

$$\mu = \hat{\mu}/\mu_0, k = \hat{k}/k_0, c_p = \hat{c}_p/\hat{c}_{p0}. \quad (10.8)$$

The reference temperature $\hat{T}_0 = (\hat{T}_H + \hat{T}_C)/2$ is 300K, $\hat{\mu}_0, \hat{k}_0, \hat{c}_{p0}$ are also measured at the reference temperature. These reference quantities are not always independent, e.g. $\hat{p}_0 = \hat{\rho}_0 \hat{R} \hat{T}_0$, with $\hat{R}$ the gas constant $8.314 \text{ JK}^{-1} \text{ mol}^{-1}$. We choose $\hat{u}_0 = (2\epsilon \hat{g} \hat{W})^{1/2}$ as the reference velocity with ϵ defined later, and the reference time $\hat{t}_0 = \hat{W}/\hat{u}_0$. The four dimensionless parameters controlling this problem are the temperature differential ϵ , Rayleigh number Ra , Prandtl number Pr and aspect ratio A defined by

$$\epsilon = \frac{\Delta \hat{T}}{2\hat{T}_0}, Ra = \frac{2\epsilon \hat{c}_{p0} \hat{\rho}_0^2 \hat{g} \hat{W}^3}{\hat{\mu}_0 \hat{k}_0}, Pr = \frac{\hat{c}_{p0} \hat{\mu}_0}{\hat{k}_0}, A = \frac{\hat{H}}{\hat{W}}, \quad (10.9)$$

where $\Delta \hat{T} = \hat{T}_H - \hat{T}_C$ is the temperature difference between the hot and cold walls. Temperature differential ϵ is a measure of NOB effects, The larger ϵ is, the stronger the NOB effects are. The reference Pr at 300K is 0.71 and the considered A is fixed to 1 in the present study. The dimensionless thermal conductivity and dynamic viscosity are obtained by the Sutherland laws

$$\begin{aligned} k &= T^{1.5} \frac{1 + S_k}{T + S_k}, \\ \mu &= T^{1.5} \frac{1 + S_\mu}{T + S_\mu}, \end{aligned} \quad (10.10)$$

where $S_k = 0.648$ and $S_\mu = 0.368$ for $\hat{T}_0 = 300\text{K}$. Since $S_k \neq S_\mu$, Pr is not a global constant. Since Eq. (10.10) for air degrades rapidly above $\epsilon = 0.6$, we set $0.005 \leq \epsilon \leq 0.6$. The dimensionless temperature at the left wall and right wall is $1 + \epsilon$ and $1 - \epsilon$ respectively. The dimensional temperature difference is $\Delta \hat{T} = 2\epsilon \hat{T}_0$, note for $\epsilon = 0.6$, $\hat{T}_H = 480\text{K}$, $\hat{T}_C = 120\text{K}$, $\Delta \hat{T} = 360\text{K}$. In the rest of the paper, we also use normalized temperature which is defined by $\theta = (T - 1)/(2\epsilon)$, and $-0.5 \leq \theta \leq 0.5$.

For a closed system with zero velocities at the boundary, we can obtain divergency of velocity by combining Eqs.(10.1), (10.3) and (10.4)

$$\frac{\partial u_j}{\partial x_j} = \frac{1}{p} \left[(\Gamma - 1) \frac{dp}{dt} + \frac{1}{\sqrt{RaPr}} \frac{\partial}{\partial x_j} k \frac{\partial T}{\partial x_j} \right]. \quad (10.11)$$

Integrating over the flow domain V leads to the following relationship for the thermodynamic pressure in a closed system

$$\frac{dp}{dt} = \frac{1}{(1 - \Gamma)V} \frac{1}{\sqrt{RaPr}} \oint_S k \frac{\partial T}{\partial x_j} n_j dS, \quad (10.12)$$

where S is the surface surrounding the flow domain. The total mass M in a closed cavity is a constant over time. By integrating the equation of state over the whole domain one obtains the following expression for the thermodynamic pressure

$$p = M \left(\int_V \frac{dV}{T} \right)^{-1}. \quad (10.13)$$

The solution (ρ, u_i, p, π, T) is completely described by Eqs(10.1)-(10.4) and (10.12) [341]. In order to compare with OB cases, we set $\epsilon = 0.005$, $\mu = k = p = 1$, $u_{j,j} = 0$ and $\rho = 1$ except in the buoyancy term $\rho = \frac{1}{T} = \frac{1}{1+\delta T} \approx 1 - \delta T = 2 - T$, thus pure OB approximation can also be realized in our code.

It is interesting to note that the results obtained here can be generalized to gases other than air, although the temperature range may be somewhat different. For example, hydrogen has a Pr of 0.7 for the reference temperature 260K, and $S_k = 0.641$, $S_k = 0.372$, so the results should be approximately the same as those for air.

10.2.2 Numerical methods

The governing equations are solved numerically by a finite-difference method with the in-house code lMn2d. All spatial terms are discretised using a second-order central difference scheme which intrinsically conserves energy. There are also several other codes specialized for thermal convection using second-order finite-difference scheme [80]. A recent work compared dedicated second-order finite-difference code and a specialized fourth-order finite-volume code with a general purpose finite-volume approach and a general purpose spectral-element code for DNS of RBC [182]. They found that dedicated special purpose codes with second-order finite-difference schemes or higher-order finite-volume schemes excel in efficiency. A non-uniform grid with clustered points near walls is used. Scalar and vector variables are staggered in space in order to avoid pressure-velocity decoupling. We used a fractional-step method to solve the momentum equations [141], the pressure Poisson equation is solved using a multi-grid strategy [142]. Time integration is accomplished using an Adams-Bashforth scheme for the nonlinear terms and Crank-Nicolson scheme for the viscous and diffusion terms. The descriptions and validations of the code can be found in our previous works [79, 104].

Ra	$N_y \times N_z$	Nu				Re			
		OB	$\epsilon = 0.2$	$\epsilon = 0.4$	$\epsilon = 0.6$	OB	$\epsilon = 0.2$	$\epsilon = 0.4$	$\epsilon = 0.6$
10^5	128×128	4.524	4.515	4.486	4.429	41.44	41.52	41.72	41.95
3×10^5	128×128	6.265	6.249	6.198	6.106	63.32	63.37	63.47	63.74
5×10^5	128×128	7.260	7.240	7.157	7.070	76.72	76.74	76.91	77.30
8×10^5	128×128	8.294	8.272	8.201	8.079	91.31	91.36	91.61	92.23
10^6	128×128	8.830	8.806	8.732	8.602	99.11	99.19	99.53	100.26
3×10^6	128×128	11.949	11.918	11.817	11.628	148.10	148.49	149.53	151.14

Ra	$N_y \times N_z$	Nu				Re			
5×10^6	128×128	13.723	13.688	13.571	13.354	178.35	178.90	180.34	182.61
8×10^6	128×128	15.571	15.530	15.394	15.152	211.70	212.34	214.18	217.19
10^7	256×256	16.528	16.483	16.339	16.083	229.70	230.36	232.38	235.79
3×10^7	256×256	22.101	22.040	21.842	21.486	343.63	344.49	347.38	352.85
5×10^7	256×256	25.260	25.193	24.966	24.564	414.44	415.63	419.10	425.72
8×10^7	256×256	28.543	28.469	28.197	27.782	492.32	493.98	498.44	506.37
10^8	256×256	30.242	30.164	29.884	29.445	534.23	536.97	541.98	550.7
3×10^8	512×512	40.097	39.982	39.636	38.980	804.52	808.59	816.10	829.20
5×10^8	512×512	45.717	45.584	45.162	44.416	980.08	982.66	991.67	1005.68
8×10^8	512×512	51.610	51.457	50.966	50.102	1173.88	1178.28	1186.65	1206.67
10^9	512×512	54.691	54.514	53.979	53.069	1284.29	1287.96	1294.25	1314.68

Table 10.1: List of simulation parameters, grid resolutions, Nusselt numbers Nu and Reynolds numbers Re . The Prandtl number is fixed to $Pr = 0.71$, the Re is based on root mean square (r.m.s) velocity

The simulation parameters and grid resolutions are listed in Table 10.1. The resolution is chosen to resolve the smallest scales of the problem, i.e. the Kolmogorov scale η_K and the Batchelor scale η_B [110]. For all runs the maximum grid spacing $\Delta_{max} < 0.81\eta_K$ and $\Delta_{max} < 0.69\eta_B$. The Nu of $Ra = 10^8$ with OB approximation is 30.24, in agreement with the benchmark results 30.31 [342] to a precision about 0.2%. For chaotic cases, we waited 400 time units before collecting data to make sure all transits have been dissipated, both Nu and Re are averaged for more than 800 free fall time units.

10.3 Results and discussion

10.3.1 Flow organization

Onset of unsteady motion

If a critical Rayleigh number Ra_c is reached, DHC becomes unsteady and for larger Ra even chaotic or turbulent. The critical value for the onset of unsteadiness with OB approximation has been studied widely. For a two dimensional square cavity with unit aspect ratio, Paolucci & Chenoweth determined the critical number as $Ra_c = 1.93 \times 10^8$ [308], Henkes identified the critical number as $Ra_c = 1.9 \times 10^8$ [343]. The Ra_c obtained in our study is 1.92×10^8 , which is in excellent agreement with these previous studies. We also studied the influence of NOB effects on the Ra_c , Table 10.2 lists Ra_c for the OB case and NOB cases with different ϵ . We can clearly see that Ra_c monotonically decreases with increasing ϵ , suggesting that NOB effects play a destabilizing effect on the flow.

	OB	$\epsilon = 0.2$	$\epsilon = 0.4$	$\epsilon = 0.6$
Ra_c	$(1.92 \pm 0.02) \times 10^8$	$(1.02 \pm 0.01) \times 10^8$	$(5.45 \pm 0.05) \times 10^7$	$(2.35 \pm 0.05) \times 10^7$

Table 10.2: The critical Rayleigh number Ra_c for the onset of unsteadiness for the OB case and NOB cases with different ϵ

Mean temperature profiles and central temperature

We then investigate NOB effects on the temperature field. Figure 10.2(a) and 10.2(d) show the instantaneous temperature field of DHC for the OB cases with $Ra = 10^5$ and $Ra = 10^9$. For $Ra = 10^5$, the flow is steady with stratified temperature in the bulk region. For $Ra = 10^9$, next to the heated wall, the boundary layer flow has a laminar character in the lower part of cavity. This region is followed by a transition and finally by a turbulent boundary in the downstream. The central region is stably stratified with low-frequency, low-velocity oscillation, it is stirred by random buoyant elements of finite energy which are discharged into it. Figure 10.2(b) and 10.2(e) show the temperature field under strong NOB effects with $\epsilon = 0.6$. It is seen that the flow organization is qualitatively the same as that for OB cases. According to the Sutherland laws, the thermal conductivity and dynamic viscosity of air increase with increasing temperature, thus both thermal boundary layer and velocity boundary layer are thicker near the hot plate. To quantitatively reflect how the local temperature is influenced by strong NOB effects, we show the temperature difference $\theta_{\epsilon=0.6} - \theta_{OB}$ in Figure 10.2(c) and 10.2(f). It is seen that the temperature near the two isothermal plates is enhanced, while the temperature near the top and bottom plates is slightly reduced under NOB conditions. The temperature in the central region is also enhanced for the NOB cases.

Figure 10.3(a) and 10.3(b) show temperature profiles at the line with $z = 0.5$ for $Ra = 10^5$ and 10^9 for OB cases and NOB cases with different ϵ . We can see the profiles of $\epsilon = 0.005$ using low-Mach-number equations collapse well with those of OB cases, indicating that the low-Mach-number equations will recover to OB cases for small ϵ . It is evident that the thermal boundary layer near the hot plate is thicker than that near the cold plate, which will cause an increase of local temperature in the thermal boundary layers when considering NOB effects. For OB cases, the temperature at the center of the cell T_c equals to the arithmetic mean temperature of the hot and cold plate T_m , thus $\theta_c = 0$. As the symmetry is broken by NOB effects, T_c does not equal to T_m anymore. From figure 10.3(a) and 10.3(b), we can see that the central temperature is enhanced under NOB conditions. Figure 10.3(c) and 10.3(d) show temperature profiles at the line with $y = 0.5$ for $Ra = 10^5$ and 10^9 . It is clearly seen that the temperature is enhanced in the central region for NOB cases, and the central region where the temperature is enhanced expands for large $Ra = 10^9$.

To quantitatively investigate the central temperature change due to NOB effects, we plot normalized temperature at the center of the cell (θ_c) as a function of Ra for various ϵ in Figure 10.4(a) and as a function of ϵ for various Ra in Figure 10.4(b).

Figure 10.2: Instantaneous normalized temperature θ contours for (a,d) OB cases and (b,e) strong NOB cases with $\epsilon = 0.6$. (c,f) Contours of time-averaged normalized temperature difference $\theta_{\epsilon=0.6} - \theta_{OB}$. The top row for $Ra = 10^5$, the bottom row for $Ra = 10^9$.

Figure 10.3: Mean normalized temperature profiles at $z = 0.5$ (top row) and $y = 0.5$ (bottom row) (a, c) $Ra = 10^5$. (b, d) $Ra = 10^9$.

Figure 10.4: (a) Reduced central temperature θ_c as a function of Ra ; (b) Reduced central temperature θ_c as function of ϵ .

We can see in Figure 10.4(a) that θ_c increases with increasing Ra , and it increases relatively more rapidly in the range $10^5 \leq Ra \leq 10^7$. For fixed Ra , it is found that the larger the ϵ , the larger the θ_c . From Figure 10.4(b), it is shown that θ_c roughly increases linearly with ϵ for $Ra \geq 10^6$. While the curve for $Ra = 10^5$ is lower than those for higher Ra .

Velocity field

Apart from the temperature distribution and central temperature, NOB effects also have a great influence on the velocity field which has already been investigated in detail in RBC with the working fluid of water [99]. In this section, we are devoted to investigating NOB effects on the velocity distribution and quantify the asymmetrical velocity distribution due to NOB effects.

We first show the horizontal velocity v distribution for OB cases and strong NOB cases with $\epsilon = 0.6$ for $Ra = 10^5$ and $Ra = 10^9$ in Figure 10.5. For $Ra = 10^5$ where the flow is steady, the magnitude of v is large near the top and bottom plates as shown in Figure 10.5(a) for OB case and Figure 10.5(b) for NOB case. Figure 10.5(c) shows the distribution of the change of the magnitude of v due to NOB effects, namely $|v_{\epsilon=0.6}| - |v_{OB}|$. It is seen for a large portion near the top plate, the magnitude of v increases due to NOB effects, while for a larger portion near the bottom plate, the magnitude of v decreases due to NOB effects. There is also a small portion in the upper-left of the cell where $|v|$ decreases and a small portion in the lower-right of the cell where $|v|$ increases due to NOB effects. For $Ra = 10^9$, the area where large magnitude of v is found is located in the upper-left and lower-right of the cell as shown in Figure 10.5(d) for the OB case and Figure 10.5(e) for the NOB case. For NOB case, the largest v is about 0.1 in the upper-left which is very close to the OB case, while the smallest v is about -0.13 in the lower-right, this may be caused by smaller viscosity of the fluid in the lower-right of the cell. Figure 10.5(f) shows the

distribution of $|v_{\epsilon=0.6}| - |v_{OB}|$ for $Ra = 10^9$, we can see that there is a region in the upper-left of the cell where $|v|$ increases due to NOB effects, while in the lower-right of the cell there is a region where $|v|$ decreases due to NOB effects.

Figure 10.5: Time-averaged horizontal velocity v contours for (a, d)OB cases and (b, e)strong NOB cases with $\epsilon = 0.6$. (c, f) Contours of time-averaged horizontal velocity difference $|v_{\epsilon=0.6}| - |v_{OB}|$. The top row for $Ra = 10^5$, the bottom row for $Ra = 10^9$.

Figure 10.6 shows the vertical velocity w distribution for OB cases and strong NOB cases with $\epsilon = 0.6$ for $Ra = 10^5$ and $Ra = 10^9$. For $Ra = 10^5$, the vertical motion of the fluid mainly concentrates near the two isothermal plate as shown in Figure 10.6 (a) for the OB case and Figure 10.6 (b) for the NOB case. Figure 10.6 (c) shows the distribution of the change of the magnitude of w due to NOB effects, namely $|w_{\epsilon=0.6}| - |w_{OB}|$. There is a region where $|w|$ decreases due to NOB effects near the hot plate, further away from the hot plate, there is a larger region where w increases due to NOB effects. For $Ra = 10^9$, the vertical motion is restricted to a much narrower region near the two isothermal plates as shown in Figure 10.6 (d) and 10.6 (e). In Figure 10.6 (f), there is a small region near the hot plate where $|w|$ decreases due to NOB effects, further away from the hot plate is a region where $|w|$ increases due to NOB effects.

We now quantitatively demonstrate the NOB effects on the horizontal velocity v profile at the location of $y = 0.5$ and vertical velocity w profile at the location $z = 0.5$ in Figure 10.7. Due to the inherent symmetry of the system, the maximum value of velocity and the minimum value of the velocity are the same in magnitude for OB

Figure 10.6: Time-averaged vertical velocity w contours for (a, d) OB approximation and (b, e) strong NOB cases with $\epsilon = 0.6$. (c, f) Contours of time-averaged vertical velocity difference $|w_{\epsilon=0.6}| - |w_{OB}|$. The top row for $Ra = 10^5$, the bottom row for $Ra = 10^9$.

Figure 10.7: Mean velocity profiles of v at $y = 0.5$ (top row) and w at $z = 0.5$ (bottom row) for (a, c) $Ra = 10^5$. (b, d) $Ra = 10^9$.

cases. It is seen that the curves for OB cases collapse well with those of low-Mach-number cases with $\epsilon = 0.005$, further suggesting that low-Mach-number model will recover to OB approximation for small ϵ . We can see from Figure 10.7(a) and 10.7(b) that the horizontal velocity near the top plate is larger than that near the bottom plate for NOB cases. From Figure 10.7(c) and 10.7(d), it is found that the vertical velocity peak moves to the right, indicating that the viscous boundary layer is thicker near the hot plate, the vertical motion of the fluid mainly concentrates in the area near the two isothermal plates, and the vertical velocity is almost zero in the central region.

To quantify the NOB effects on velocity asymmetry, we also investigate NOB effects on the absolute value of the ratio of horizontal maximum velocity v_{max} and horizontal minimum velocity v_{min} , namely $|v_{max}/v_{min}|$, the influence of NOB effects on $|w_{max}/w_{min}|$ is also investigated. Figure 10.8(a) shows $|v_{max}/v_{min}|$ as a function of Ra for different ϵ . Within OB approximation, due to inherent symmetry of the system,

Figure 10.8: (a) $|v_{max}/v_{min}|$ versus Ra for NOB cases with different ϵ . (b) $|w_{max}/w_{min}|$ versus Ra for NOB cases with different ϵ . These two quantities equal 1 for OB cases while deviate from 1 due to symmetry breaking by NOB effects.

this value equals to 1, while it deviates from 1 considering NOB effects. It is shown that $|v_{max}/v_{min}|$ is a non-monotonic function of Ra , for most value of Ra , $|v_{max}/v_{min}|$ is smaller than one, suggesting that the maximum magnitude of horizontal velocity in the lower-right of the cell is larger than that in the upper-left of the cell. Generally, the larger the ϵ , the smaller the $|v_{max}/v_{min}|$. Figure 10.8 (b) shows $|w_{max}/w_{min}|$ as a function of Ra for different ϵ . We can see for different ϵ , $|w_{max}/w_{min}|$ all increase with increasing Ra for $Ra \leq 10^8$, and it seems to be constant for $10^8 \leq Ra \leq 10^9$. This demonstrates that the vertical fluid motion near the hot plate is greater than that near the cold plate.

10.3.2 Nusselt number

We now discuss the NOB effects on the heat transfer which is characterized by Nusselt number Nu . The Nu in DHC is defined as

$$Nu = \frac{\hat{Q}}{\hat{k}_0 \Delta \hat{T} / \hat{W}}, \quad (10.14)$$

where $\hat{Q}$ is the heat flux across any vertical plane. Since the Nu averaged over any vertical line should be the same, here the Nu is averaged over the hot plate. As stated before, for fixed Pr , the relationship of Nu and Ra is usually expressed by power-law scaling, i.e. $Nu \sim \alpha Ra^\beta$. Figure 10.9(a) shows Nu as a function of Ra for the OB case and NOB cases. For clarity, each data set for the NOB cases has been shifted downwards from its upper neighbour by a factor of 0.6. The scaling exponents for both the OB case and NOB cases are found to be 0.27, which lie between $1/4$ [58, 327] and $1/3$ [57, 311, 325] scalings. Figure 10.9(b) shows compensated $Nu/Ra^{1/4}$ as a

Figure 10.9: (a) Log-Log plot of Nu as a function of Ra for the OB case the NOB cases with different ϵ . For clarity, each data set for the NOB cases has been shifted downwards from its upper neighbour by a factor of 0.6. (b) Log-Log plot of $Nu/Ra^{1/4}$ as a function of Ra . (c) Log-Log plot of $Nu/Ra^{1/3}$ as a function of Ra .

function of Ra and Figure 10.9(c) shows compensated $Nu/Ra^{1/3}$ as a function of Ra . It can be seen that the Nu grows slightly faster for $Ra \leq 10^7$. We list the obtained scaling relations in different Ra range for OB case and NOB cases in Table 10.3. It is seen that the NOB effects almost have no influence on the scaling exponents, it is also shown that for $10^5 \leq Ra \leq 10^7$ the scaling exponents are close to 0.28, while the scaling exponents of 0.259 in the $10^7 \leq Ra \leq 10^9$ range is smaller, and is in good agreement with the theoretical predication of 0.25 [58]. A similar increase of the scaling exponent for small Ra was also found in several previous works [57, 58, 325].

Ra range	OB	$\epsilon = 0.2$	$\epsilon = 0.4$	$\epsilon = 0.6$
$[10^5, 10^9]$	$0.2162Ra^{0.268}$	$0.2158Ra^{0.268}$	$0.2144Ra^{0.268}$	$0.2124Ra^{0.267}$
$[10^5, 10^7]$	$0.1836Ra^{0.280}$	$0.1834Ra^{0.280}$	$0.1823Ra^{0.279}$	$0.1814Ra^{0.279}$
$[10^7, 10^9]$	$0.2551Ra^{0.259}$	$0.2549Ra^{0.259}$	$0.2532Ra^{0.259}$	$0.2503Ra^{0.259}$

Table 10.3: Power law scalings of Nu and Ra for OB case and NOB cases with different ϵ for different Ra range.

We now quantitatively investigate the NOB effects on the Nu . Figure 10.10(a) shows the Nusselt number ratio Nu_{NOB}/Nu_{OB} as a function of Ra for different ϵ . It is shown that the Nu is slightly reduced when considering NOB effects. The stronger the NOB effects, the smaller the Nu_{NOB}/Nu_{OB} . Figure 10.10(b) shows Nu_{NOB}/Nu_{OB} as a function of ϵ for different Ra . For different Ra , the Nu_{NOB}/Nu_{OB} decreases monotonically with increasing ϵ . However, it can be seen that the maximum reduction of Nu due to NOB effects is only about 3%, demonstrating that the Nu is insensitive to NOB effects in DHC. For RBC, it is also found that the Nu is insensitive to NOB effects for the working fluid of water [95, 99] and glycerol [100].

We then discuss how the NOB effects influence the distribution of local Nusselt number $Nu(z)$ at the hot and cold plates. The $Nu(z)$ is defined as:

$$Nu(z) = \left| k \frac{\partial \theta}{\partial y} \right|_{y=0,1}. \quad (10.15)$$

The Nusselt number at the hot wall is obtained when $y = 0$ and $y = 1$ denotes Nusselt number at the cold wall. Figure 10.11(a) and 10.11(b) show the distribution of Nu at the hot plate and cold plate for OB case and NOB cases for $Ra = 10^5$ and $Ra = 10^9$ respectively. It is seen that the Nu in the lower part of the hot plate is enhanced while the Nu in the upper part of the hot plate is reduced under NOB conditions. For the cold plate, the trend is the same, namely the Nu in the upper part of the cold plate is reduced while the Nu in the lower part of the cold plate is enhanced. From Figure 10.11(b), we can see that the peak value of Nu at the hot plate is achieved quite close to the bottom plate. The Nu also increases in the lower part of the plate while it decreases in the upper part of the plate for NOB cases.

We now investigate NOB effects on thermal boundary layer thickness, which is directly related to Nu . As mentioned before, the thermal conductivity increases as temperature increases, thus the thermal boundary layer near the hot plate will become thicker with NOB effects. The thickness of thermal boundary layer near the two isothermal plates are defined as [325]:

$$\lambda_{\theta}^{hot}(z) = (\theta_{(0.5,z)} - \theta_h) / \left(\frac{\partial \theta}{\partial y} \right)_{(0,z)}, \lambda_{\theta}^{cold}(z) = (\theta_c - \theta_{0.5,z}) / \left(\frac{\partial \theta}{\partial y} \right)_{(1,z)}. \quad (10.16)$$

Figure 10.12 shows the mean inverse thermal boundary thickness computed as $\lambda_{\theta}^{-1} = \int_0^1 1/\lambda_{\theta}(z) dz$ as a function of Ra . As the heat transfer near the side boundaries is dominated by conduction, the growth of Nu should be roughly proportional to λ_{θ}^{-1} . The detailed power law scalings of λ_{θ}^{-1} and Ra are listed in Table 10.4. We can see $\lambda_{\theta}^{-1} \sim aRa^{0.27}$ (a is a constant) for almost all present cases. The scaling exponent

Figure 10.10: (a) Nu_{NOB}/Nu_{OB} as a function of Ra for different ϵ . (b) Nu_{NOB}/Nu_{OB} as a function of ϵ for different Ra . For all the data, the error bar is smaller than the symbol size.

Figure 10.11: Local Nusselt number distribution at the hot (solid lines) and cold walls (dashed lines), red lines for OB approximation, green lines for $\epsilon = 0.2$, blue lines for $\epsilon = 0.4$, grey lines for $\epsilon = 0.6$. (a) $Ra = 10^5$. (b) $Ra = 10^9$.

Figure 10.12: Log-log plot of λ_θ^{-1} as a function of Ra for hot plate (solid lines) and cold plate (dashed lines).

0.27 is the same as the scaling exponent of Nu and Ra . Thus the NOB effects only modify the perfectors. The local Nu can be computed as $Nu = k\delta\theta/\lambda_\theta$, where $\delta\theta$ is the temperature change in the thermal boundary layer, for the Nu on the hot wall, $\delta\theta = \theta_h - \theta_{(0.5,z)}$ while for the cold wall $\delta\theta = \theta_{(0.5,z)} - \theta_c$. As NOB effects only have a minor influence on the Nu and the temperature change in the thermal boundary layer, we expect the product of k and a should be approximately the same for the OB and NOB cases. Table 10.5 confirms this argument where we can see the products of a and k are all around 0.4.

	OB	$\epsilon = 0.2$	$\epsilon = 0.4$	$\epsilon = 0.6$
hot	$0.4304Ra^{0.273}$	$0.3511Ra^{0.262}$	$0.3048Ra^{0.269}$	$0.2693Ra^{0.269}$
cold	$0.4304Ra^{0.273}$	$0.5099Ra^{0.268}$	$0.6736Ra^{0.267}$	$1.0189Ra^{0.266}$

Table 10.4: Power law scalings of $\lambda_\theta^{-1} \sim aRa^b$ for OB case and NOB cases with different ϵ

	OB	$\epsilon = 0.2$	$\epsilon = 0.4$	$\epsilon = 0.6$
hot	0.4304	0.4116	0.4062	0.3996
cold	0.4304	0.4152	0.4134	0.4053

Table 10.5: The product of thermal conductivity k and the perfectors a of $\lambda_\theta^{-1} \sim aRa^b$

Figure 10.13: Log-Log plots of Re as function of Ra for OB approximation. Re^{rms} is Reynolds number based on r.m.s velocity, while Re^w/Re^v are based on maximum magnitude of vertical/horizontal velocity over the whole cell.

10.3.3 Reynolds number

Apart from Nu , the other important response parameter of DHC is Reynolds number Re . The Reynolds number is defined by:

$$Re = \frac{\hat{U}\hat{W}}{\hat{\nu}_0} = \sqrt{Ra/Pr}U, \quad (10.17)$$

where $\hat{U}/U$ are the dimensional/dimensionless characteristic velocity which can be chosen in several ways. For fixed Pr , the relationship of Re with Ra is also expressed as the power law scaling, i.e. $Re \sim \alpha' Ra^{\beta'}$, this is also true for RBC. The Re in RBC has been investigated extensively [53, 99, 122]. For 3D RBC, the scaling exponent is found to be $0.42 \sim 0.5$ [53] while for 2D RBC, the scaling exponent is found to be 0.6 [99, 122]. Compared with RBC, there are less studies devoted to the scaling of Re in DHC. If the characteristic velocity is chosen to be the maximum vertical velocity amplitude in the flow field, the scaling exponent is found to be 0.54 [325]. If the characteristic velocity is chosen to be the maximal mean velocity along the heated plate, the scaling exponent is found to be 0.5 [58].

In this study, we choose three different characteristic velocities, the first one is root mean square (r.m.s) velocity of the flow field which is defined as $U^{rms} = \sqrt{\langle \mathbf{u} \cdot \mathbf{u} \rangle_{V,t}}$ where $\mathbf{u}$ is the velocity vector, $\langle \rangle_{V,t}$ denotes average over space and time. The other two are the time-averaged maximum vertical/horizontal velocity magnitude over the whole cell which are defined as $U^w = \max|w|/U^v = \max|v|$. The Reynolds numbers based on these three characteristic velocities are denoted as Re^{rms}, Re^w and Re^v , respectively. The various Re as a function of Ra are displayed in Figure 10.13. It is

Figure 10.14: Re_{NOB}/Re_{OB} versus Ra (a, c, e) or ϵ (b, d, f). (a,b) Re based on rms velocity. (c, d) Re based on maximum vertical velocity (in magnitude). (e, f) Re based on maximum horizontal velocity (in magnitude).

found that Re^w is always larger than Re^v , implying that the vertical motion is greater than horizontal motion in DHC. We note the scaling exponent for Re^w obtained here is 0.5, which is in excellent agreement with the theoretical predication of 0.5 [58], further demonstrating the robustness of this scaling theory for DHC, it is also quite close to previous simulations of 0.54 [325]. The scaling exponent for Re^v is 0.45 which is a little bit smaller than that of Re^w . The scaling exponent of Re^{rms} is only 0.37, which is evidently smaller than 0.6 for two-dimensional RBC [99, 122]. Table 10.6 compares the scalings for different Reynolds number for OB cases with those of NOB cases. Again we can find that NOB effects have a negligible influence on the scaling exponents.

	OB	$\epsilon = 0.2$	$\epsilon = 0.4$	$\epsilon = 0.6$
Re^{rms}	$0.5946Ra^{0.3697}$	$0.5927Ra^{0.3701}$	$0.5906Ra^{0.3707}$	$0.5843Ra^{0.3721}$
Re^w	$0.2995Ra^{0.5025}$	$0.3004Ra^{0.5032}$	$0.3019Ra^{0.5038}$	$0.3025Ra^{0.5046}$
Re^v	$0.3451Ra^{0.4544}$	$0.3353Ra^{0.4571}$	$0.3127Ra^{0.4626}$	$0.2808Ra^{0.4705}$

Table 10.6: Power law scalings for the different Re with Ra for OB case and NOB cases with different ϵ

To quantitatively investigate how the three Reynolds numbers are influenced by NOB effects, we show the Reynold number ratio Re_{NOB}/Re_{OB} as functions of Ra and ϵ in Figure 10.14. Figure 10.14(a) shows that $Re_{NOB}^{rms}/Re_{OB}^{rms}$ is a non-monotonic function of Ra for different ϵ , the Re^{rms} increases with NOB effects. For fixed Ra , the $Re_{NOB}^{rms}/Re_{OB}^{rms}$ increases with increasing ϵ , as is shown in Figure 10.14 (b). The NOB effects on Re^{rms} is small which is also true for NOB effects in RBC with the working fluid of water [99], and the maximum increase due to NOB effects is only about 3%. From Figure 10.14(c), it can be seen that the Re^w is also slightly increased considering NOB effects, the Re_{NOB}^w/Re_{OB}^w is also a non-monotonic function of Ra for different ϵ . Based on the present data, it increases with increasing Ra for $Ra \leq 10^8$, while it experiences a sudden decrease for $Ra \geq 10^8$. The Re_{NOB}^w/Re_{OB}^w is monotonic increasing function of ϵ as can be seen in Figure 10.14(d). In Figure 10.14(e), we can see that the Re_{NOB}^v/Re_{OB}^v is also a non-monotonic function of Ra for different ϵ . It firstly increases with increasing Ra , and then decreases with increasing Ra , for $Ra \geq 3 \times 10^7$, it increases with increasing Ra again. For the largest Ra considered in this work, we can see that the maximum enhancement of Re^v is up to 30% for $\epsilon = 0.6$, as mentioned before, the maximum horizontal velocity magnitude is 0.13 which locates in the lower-right of the cell, and it is larger than the OB case of 0.1, this may be due to smaller viscosity of the fluid in the lower-right of the cell.

10.4 Conclusions

In conclusion, we have studied NOB effects in a differentially-heated cavity with unit aspect ratio by means of two-dimensional DNS using low-Mach-number equations over

the Rayleigh number range $10^5 \leq Ra \leq 10^9$. Extremely large temperature differences up to 360K between the two isothermal plate are considered. As Sutherland law is used, the results obtained in this work can be applied to gases other than air. The main conclusions are drawn as follows:

- (i) The critical Rayleigh number for the onset of unsteadiness decrease with increasing temperature differences. Both the thermal and velocity boundary layers become thicker near the hot plate while they get thinner near the cold plates under NOB conditions. The central temperature is enhanced with NOB effects. The normalized center temperature θ_c roughly increases linearly with increasing ϵ . The horizontal velocity near the top plate is normally enhanced by NOB effects while it is normally decreased near the bottom plate.
- (ii) In spite of the marked qualitative differences in the NOB flow relative to OB flow, the overall integral quantities like the Nu and Re are insensitive to NOB effects and retain their Ra -scaling exponents. The Nu has a minor decrease under NOB conditions. The maximum decrease is only about 3% for temperature difference as large as 360K. The $Nu \sim Ra^{0.27}$ and $\lambda_\theta^{-1} \sim Ra^{0.27}$ scalings are obtained for OB approximation. These scaling exponents does not change under NOB conditions.
- (iii) The Re based on three characteristic velocities all normally increase with NOB effects. The influence of NOB effects on Re^{rms} and Re^w is small, while the Re^v can be enhanced up to 30% for $Ra = 10^9, \epsilon = 0.6$. Different scalings for the Re based on different characteristic velocities are identified: $Re^{rms} \sim Ra^{0.37}$, $Re^w \sim Ra^{0.5}$, $Re^v \sim Ra^{0.45}$. These scaling exponents are also quite robust to NOB effects. The 0.37 scaling exponent for Re^{rms} is evidently smaller than that of 0.6 in RBC. The 0.5 scaling exponent for Re^w is in perfect agreement with recent theoretical predictions [58].

The present results can serve as reference to understand NOB effects in DHC for higher Ra , which is relevant in industrial processes and many high- Ra experiments.

Part IV

Conclusions and outlook

Chapter 11

Conclusions and outlook

In this thesis we have studied turbulent thermal convection by direct numerical simulations. We focused on three different model problems, i.e., (I) Rayleigh-Bénard convection (RBC), (II) tilted convection, and (III) vertical convection (VC). Here, tilted convection serves as a bridge to connect RBC where the tilt angle $\beta = 0^\circ$ and VC where the tilt angle $\beta = 90^\circ$.

Below we provide the main conclusions for each part of this thesis. A outlook to the future research on turbulent thermal convection is given at the end of this chapter.

11.1 Part I: Rayleigh-Bénard convection

It is known that Oberbeck-Boussinesq (OB) approximation works well for thermal convection with small temperature difference. However, non-Oberbeck-Boussinesq (NOB) effects can naturally arise whenever temperature differences are large or the density has a nonlinear dependence on the temperature. It is thus of importance to study the influence of NOB effects on thermal convection. We have studied NOB effects either due to the density maximum of cold water near 4°C (§2) or due to large temperature differences (§3) in turbulent Rayleigh-Bénard convection.

In §2, we studied penetrative turbulent RBC which depends on the density maximum of water near 4°C using 2D and 3D DNS. The system is controlled by an important dimensionless parameter, namely the density inversion parameter θ_m . Several universal properties were revealed in this system. We found that the ratio of the top and bottom thermal boundary-layer thickness ($F_\lambda = \lambda_t^\theta / \lambda_b^\theta$) increases with increasing θ_m , and the relationship between F_λ and θ_m seems to be independent of Ra . The center temperature θ_c is enhanced compared to that of the OB cases, as θ_c is related to F_λ with $1/\theta_c = 1/F_\lambda + 1$, θ_c is also found to have a universal relationship with θ_m , which is independent of Ra . Both the Nusselt number Nu and the Reynolds number Re decrease with increasing θ_m , the normalized Nusselt number $Nu(\theta_m)/Nu(0)$ and Reynolds number $Re(\theta_m)/Re(0)$ also have universal relationships with θ_m , which

seem to be independent of both Ra and the aspect ratio Γ . The effective scaling exponents of $Nu \sim Ra^\alpha$ and $Re \sim Ra^\beta$ are found to be insensitive to θ_m despite the remarkable change of the flow organizations.

In §3, we considered NOB effects due to large temperature differences. Perfect air was chosen as the operating fluid. We found that due to top-down symmetry breaking under NOB conditions, the center temperature T_c is enhanced compared to the arithmetic mean temperature T_m of the top and bottom plates, and the shifts of T_c are strongly dependent on Ra and temperature differential ϵ . The NOB effects on the Nusselt number Nu are quite small ($\lesssim 2\%$). The power-law scalings of Nu versus Ra are robust against NOB effects, even for the extremely large temperature difference 240K, which has never been reached in previous experiments. The Reynolds numbers Re , as well as the scalings of Re versus Ra , are also insensitive to NOB effects. It is noteworthy that the influence of NOB effects on Nu and Re in 3D RB flow are weaker than its 2D counterpart. Furthermore, the extended laminar boundary layer (BL) equations were developed based on the low-Mach-number Navier-Stokes equations, which qualitatively predict the NOB effects on velocity profiles. DNS results indicate that the top and bottom thermal BLs can compensate each other much better than the velocity BLs under NOB conditions, which contribute to the robustness of Nu .

Multistability and metastability are fascinating phenomena in turbulent flows. Taking large-aspect ratio 2D turbulent RBC as the model problem, we shifted our attention to these phenomena in §4 and §5.

In §4, we analyzed the multiple states in large-aspect ratio 2D RBC with no-slip plates and horizontally periodic boundary conditions. We determined the number n of convection rolls, their mean aspect ratios $\Gamma_r = \Gamma/n$, and the corresponding transport properties of the flow (i.e., the Nusselt number Nu), as function of the control parameters Ra and Pr . The effective scaling exponent β in $Nu \sim Ra^\beta$ is found to depend on the realized state and thus Γ_r , with a larger value for the smaller Γ_r . By making use of a generalized Friedrichs inequality, we showed that the elliptical shape of the rolls and viscous damping determine the Γ_r -window for the realizable turbulent states. The theoretical results are in excellent agreement with our numerical finding $2/3 \leq \Gamma_r \leq 4/3$, where the lower threshold is approached for the larger Ra . Finally, we showed that the theoretical approach to frame Γ_r also works for free-slip boundary conditions.

In §5, we investigated RBC with free-slip plates and horizontally periodic boundary conditions. Two configurations were considered, one is 2D RBC and the other one 3D RBC with a rotating axis parallel to the plate, which for strong rotation mimics 2D RBC. We showed that zonal flow, which was observed, for example, by Ref. [81] for $\Gamma = 2$, is only stable when Γ is smaller than a critical value, which depends on Ra and Pr . The regime in which only zonal flow can exist is called the first regime in this thesis. With increasing Γ , we find a second regime in which both zonal flow and different convection roll states can be statistically stable. For even larger Γ , in a third regime, only convection roll states are statistically stable and zonal flow is not sustained. How many convection rolls form (or in other words, what the mean aspect

ratio of an individual roll is), depends on the initial conditions and on Ra and Pr . For instance, for $Ra = 10^8$ and $Pr = 10$, the aspect ratio Γ_r of an individual, statistically stable convection roll can vary in a large range between 16/11 and 64. A convection roll with an as large aspect ratio of $\Gamma_r = 64$ or more generally already with $\Gamma_r \gg 10$ can be seen as “localized” zonal flow, and indeed carries over various properties of the global zonal flow. For the 3D simulations, we fixed $Ra = 10^7$ and $Pr = 0.71$, and compared the flow for $\Gamma = 8$ and $\Gamma = 16$. We first showed that with increasing rotation rate both the flow structures and global quantities like the Nusselt number Nu and the Reynolds number Re increasingly behave like in the 2D case. We then demonstrated that with increasing aspect ratio Γ , zonal flow, which was observed for small $\Gamma = 2\pi$ by Ref. [82], completely disappears for $\Gamma = 16$. For such large Γ only convection roll states are statistically stable. In between, here for medium aspect ratio $\Gamma = 8$, the convection roll state and the zonal flow state are both statistically stable. What state is taken depends on the initial conditions, similarly as we found for the 2D case.

11.2 Part II: Tilted convection

In this part, we have investigated the properties of large-scale convection roll structures and heat and momentum transport in tilted convection. We mainly focused on 2D geometries with different aspect ratios (§6-§8), but later we also examined the 3D configuration with $\Gamma = 1/2$ (§9).

In §6, we focused on the influence of tilt on flow reversals in 2D thermal convection for both $\Gamma = 1$ and $\Gamma = 2$. For $\Gamma = 1$, tilt tends to suppress flow reversals. However, it is found that flow reversals characterized by two main rolls are promoted by tilt for $\Gamma = 2$, which are even observed for some small Pr and large tilt angle (β) cases. Different from level cases where the four corner rolls all have opportunities to grow and trigger a flow reversal, the reversals in a counterclockwise tilted cell with $\Gamma = 2$ are always led by the growth of the bottom-right or the top-left corner roll. Tilt is favorable for the growth of the bottom-right or the top-left corner roll and thus breaking the balance between the two main rolls and trigger a flow reversal. The mode decomposition analysis shows that the appearance of the intermediate single-roll mode is crucial for reversals, and flow reversals in a tilted cell with $\Gamma = 2$ can be viewed as a mode competition process between single-roll mode and horizontally adjacent double-roll mode. They can only occur in a limited range of β where the two modes have the comparative strength. Furthermore, the Nusselt numbers at the hot plate Nu_h and at the cold plate Nu_c behave differently during a flow reversal for $\Gamma = 2$ due to the preference of single corner roll growth.

One may wonder whether the finding in §6 that tilt promotes flow reversals in a convection cell with $\Gamma = 2$ can also be generalized to slimmer cells. To address this question, in §7, we tested a slender 2D tilted cell with $\Gamma = 0.5$. Different flow states were identified over the β - Pr parameter space. We found that the flow tends to organize itself in a stable vertically-stacked double-roll state (DRS) for small Pr

and small β , while this DRS becomes unstable and flow reversals happen with the increase of β . This finding thus complements our previous study of flow reversals in tilted cells with $\Gamma = 2$ (§6). For relatively larger Pr , the flow gives way to stable triple-roll state or unstable triple-roll state for small β . Moreover, multiple states in the turbulent regime are found for $Ra \geq 10^8$, between which the flow can or can not switch. In the latter case the Nu are different for the two states with the same number of convection rolls but different orientations. It is found that the $Nu(\beta)/Nu(0)$ and $Re(\beta)/Re(0)$ dependence is strongly influenced by a combination of Ra and Pr . In the present system, we interestingly found that the earlier conclusion that Nu decreases with increasing β close to $\beta = 90^\circ$ for $\Gamma = 1$ does not hold for the present $\Gamma = 0.5$ case with small Pr .

In §8, we further extended the parameter space to much larger aspect ratios up to $\Gamma = 16$ with fixed $Ra = 10^7$ and $Pr = 0.71$. Multiple states were identified for small β with $\Gamma \geq 2$, where the different flow states are reflected in different numbers of convection rolls. Moreover, flow mode transitions between different roll states were observed for large $\Gamma \geq 8$ when β is larger than a critical value. The effect of cell tilting on Nu and Re were also investigated. It is found that for $\Gamma \leq 4$, Nu first increases with increasing β , and then declines after reaching its local maximum. However, the Nu decreases monotonically with increasing β for $\Gamma = 8, 12$, and 16 . This indicates that the idea to enhance heat transport by tilting the cell can be realized only for relatively small Γ for the present system. It is also found that the previous finding that Re decreases monotonically with increasing β for large β with $\Gamma = 1$ does not hold for large- Γ cases.

In §9, we further explored 3D tilted convection with aspect ratio $\Gamma = 0.5$ both for OB cases and considering NOB effects. Within OB approximation, it is found that there exist different states for $Ra = 10^5$ and hysteresis for $Ra = 10^6$. For a relatively small tilt angle β , the large-scale-circulation (LSC) can either orient along one of the vertical diagonal planes (denoted by M_d mode) or orient parallel to the front wall (denoted by M_p mode). Which of the two modes transports heat more efficiently depends on Ra . For $Ra = 10^7, \beta = 0^\circ$, the time-averaged flow field contains four rolls in the upper half and lower half of the cell, respectively, the M_d and M_p modes only develop in tilted cells. By investigating NOB effects in tilted convection for fixed $Ra = 10^6$, it is found that the influence of NOB effects on Nu , Re , and central temperature T_c are different for different β range. NOB effects can either increase or decrease Nu , Re , and T_c when β is varied.

11.3 Part III: Vertical convection

In §10, we focused on NOB effects in 2D VC. The working fluid is air and Pr is 0.71 for the reference state. We found that the critical Rayleigh number for the onset of unsteadiness decreases with increasing temperature differences. We also investigated NOB effects on the temperature and velocity fields. It is found that both the thermal and velocity boundary layers become thicker near the hot plate while they get thinner

near the cold plates under NOB conditions. The center temperature is increased when considering NOB effects compared to OB cases. The normalized center temperature θ_c roughly increases linearly with increasing ϵ . The horizontal velocity near the top plate is normally enhanced by NOB effects while it is generally decreased near the bottom plate under NOB conditions. Despite these marked qualitative differences in the NOB flow relative to OB flow, the overall integral quantities like the Nusselt number Nu and the Reynolds number Re are insensitive to NOB effects and retain their Ra -scaling exponents. The Nu has a minor decrease under NOB conditions. The maximum decrease is only about 3% for the temperature difference as large as 360K. Within OB approximation, Nu is found to scale as $\sim Ra^{0.27}$, and the inverse thermal boundary layer thickness λ_θ^{-1} also scales as $\sim Ra^{0.27}$. The scaling exponent does not change under NOB conditions. The Reynolds number based on the root mean square (r.m.s) velocity Re^{rms} increases slightly for NOB cases. We also find $Re^{rms} \sim Ra^{0.37}$, $Re^w \sim Ra^{0.50}$, and $Re^v \sim Ra^{0.45}$ for OB cases, where Re^w/Re^v are Reynolds number based on the maximum magnitude of vertical/horizontal velocity over the whole cell. The NOB effects almost have no influence on these scaling exponents.

In short, this thesis has addressed the importance of the direction of the temperature gradient in relative to the gravity, which is denoted by the tilt angle β of the convection cell, to the global flow organizations and transport properties in turbulent thermal convection. The two extreme cases of RBC ($\beta = 0^\circ$) and VC ($\beta = 90^\circ$) are particularly investigated. We have focused on several relevant important questions as reviewed in Ref. [53]. These include global heat and momentum transport, NOB effects, multistability and metastability phenomena in turbulent flows, flow reversals of the large-scale circulation, and so on.

11.4 Future research directions

This thesis has addressed a number of topics related to the three classical model problems for turbulent thermal convection, namely Rayleigh-Bénard convection, tilted convection and vertical convection. Many issues still remain unexplored and open for future research. Below, we list some directions which can be continued on this thesis.

11.4.1 Rayleigh-Bénard convection

In §2 we have considered penetrative convection in confined geometries with side-walls. However, penetrative convection in astrophysical settings usually happens in horizontally unbounded domains where the aspect ratio can be very large. Thus it may become more relevant to address penetrative RBC using horizontally periodic convection cells with large aspect ratios. In recent years, large-aspect ratio RBC has been extensively studied [213–218, 232, 344–347]. For penetrative RBC it is known that the top-bottom symmetry is broken, the flow structures near the bottom and

top boundaries will be different and may have different sizes, it is thus of interest to quantify the different structures near the top and bottom boundaries.

Furthermore, rotation also plays an important role in astrophysical penetrative convection. It is therefore of great interest to study the influence of rotation on the penetrative convection, a similar problem has recently been addressed in Ref. [348], where they have focused on the generation of large-scale vortices. As the top-bottom symmetry is broken, the Ekman vortices [231] near the top and bottom plates also have different behaviours. The density inversion parameter θ_m is sure to have significant influence on the behaviour of Ekman vortices, especially on those near the top plate, which may develop from a upper stably stratified layer instead unstable convective layer near the bottom plate. At high rotation rate, Taylor column structures arise and a recent study showed that the movement of these structures can be described on the framework of Brownian particles where they move ballistically for the time shorter than some critical timescales, and then move diffusively [203]. It is also interesting to look at the movement of Taylor columns in such top-bottom symmetry-breaking system.

In §5, we find surprisingly large-scale convection roll structures in 2D and spanwise rotating 3D convection cells with free-slip plates, i.e., the mean aspect ratio of the roll $\Gamma_r = 64$ in a 2D cell, and $\Gamma_r = 8$ in a 3D cell with spanwise rotation. Such large-scale convection roll structures have never been reported before. An open question is whether the convection roll size continues to grow or saturates to a finite value with further increasing the aspect ratio Γ of the convection cell. The theoretical argument in §4 has shown that the convection roll size is possible to have very large values. More simulations with larger Γ are needed to further clarify this problem and to further validate the theoretical argument.

11.4.2 Tilted convection

An important issue for tilted convection is to enhance heat transport by tilting the cell with a suitable angle. For examples, various studies have shown that Nu can reach a maximal value with an optimal tilt angle [59, 279, 282, 283, 290, 349]. Experiments by [279] even observed that Nu is enhanced by more than 10 times for $Ra = 2.4 \times 10^6$, $Pr = 0.0089$, $\Gamma = 1/20$. As accurate measurements in experiments are challenging for liquid metal with such small Pr , it is important to perform one-to-one simulations to compare with the experiments and further address the physics behind such significant heat transport enhancement.

Moreover, the tilted convection with large $Ra \geq 10^{10}$ has seldom been addressed. A few early experiments were only restricted to small tilt angles and they sometimes even gave contrary conclusions. For example, for a cylindrical cell with $10^9 \leq Ra \leq 7 \times 10^{10}$, $Pr = 5.3$, $\Gamma = 1/2$, Ref. [261] reported a decrease of Nu when the cell is tilted by 2° . However, under similar conditions ($Ra = 7.2 \times 10^8$, $Pr = 4.38$, $\Gamma = 1/2$), Ref. [160] observed a tiny increase of Nu when $\beta < 7^\circ$. Such discrepancy is still not well understood, more simulations and experiments are needed to resolve this.

11.4.3 Vertical convection

In comparison to RBC, there have been much less studies devoted to vertical convection. Most of the early studies mainly focused on a limited parameter space with $Ra \lesssim 10^{10}$ and $Pr \sim 1$. What will happen for much larger Ra and much smaller Pr is an interesting topic to explore in the future.

11.4.4 Other models for turbulent thermal convection

As already mentioned in §1.2. There are many other model problems to approximate various convection flows in geo- and astrophysical settings. Here, we particularly mention internally heated convection.

The thermal driving in different convection flows can be due to the temperature boundary conditions such as in Rayleigh–Bénard convection, tilted convection, vertical convection and many others. However, the thermal driving can also be thanks to internal heating, where the temperature field is driven by some forcing in the bulk. In many cases in nature, both ways of driving play a role at the same time. E.g., this holds for the Earth’s mantle due to the driving through the hot inner core of the Earth and an additional driving due to the decay of radioactive material inside the core, producing heat [350–357]. Thus, in the Earth’s mantle, about 10–20% of the heat is transferred from the core, while the rest occurs due to the internal heating [357]. The internal heating dominates also in the atmosphere of Venus [358, 359], which is heated up due to the absorption of sun light. One more example is the formation of Pluto’s polygonal terrain, which is caused not only by convection of Rayleigh–Bénard type [22, 360], but also by internally heated convection [44]. And, of course, internally heated convection is relevant in many engineering applications, e.g., liquid-metal batteries [361, 362].

Recently, we offered a unifying theory for turbulent purely internally heated convection, generalizing the unifying theories of Grossmann and Lohse [222, 223] for Rayleigh–Bénard turbulence and of Shishkina, Grossmann and Lohse [68] for turbulent horizontal convection [76]. We obtained the mean temperature of the system and the Reynolds number as function of the control parameters, namely the internal thermal driving strength (called, when nondimensionalized, the Rayleigh–Roberts number) and the Prandtl number. The results of the theory are consistent with our direct numerical simulations.

Many questions remain open even for purely internally heated convection. What will happen at very high internal thermal driving is poorly understood and much more effort is needed to further explore the parameter space.

As one can see, even when we restrict ourself to single-phase turbulent convection, there are lots of interesting open questions to explore. If we further consider more complex physics, like including particles [363–367], bubbles [368–370], or modulation [208, 210], the possibilities seem endless. One message we want to convey is that despite centuries of study, thermal convection continues to yield new aspects and it

will continue as a model of complexity for quite some time.

Bibliography

- [1] D. M. Hassler, I. E. Dammasch, P. Lemaire, P. Brekke, W. Curdt, H. E. Mason, J.-C. Vial, and K. Wilhelm, “Solar wind outflow and the chromospheric magnetic network”, *Science* **283**, 810–813 (1999).
- [2] R. T. Merrill and M. W. McElhinny, “The Earth’s magnetic field: Its history, origin and planetary perspective”, *International Geophysics Series* **32** (1983).
- [3] P. H. Roberts and E. M. King, “On the genesis of the Earth’s magnetism”, *Rep. Prog. Phys.* **76**, 096801 (2013).
- [4] G. A. Glatzmaier and P. H. Roberts, “A three-dimensional self-consistent computer simulation of a geomagnetic field reversal”, *Nature* **377**, 203–209 (1995).
- [5] G. A. Glatzmaier, R. S. Coe, L. Hongre, and P. H. Roberts, “The role of the earth’s mantle in controlling the frequency of geomagnetic reversals”, *Nature* **401**, 885–890 (1999).
- [6] J. Li, T. Sato, and A. Kageyama, “Repeated and sudden reversals of the dipole field generated by a spherical dynamo action”, *Science* **295**, 1887–1890 (2002).
- [7] F. Takahashi, M. Matsushima, and Y. Honkura, “Simulations of a Quasi-Taylor state geomagnetic field including polarity reversals on the Earth Simulator”, *Science* **309**, 459–461 (2005).
- [8] D. P. McKenzie, J. M. Roberts, and N. O. Weiss, “Convection in the Earth’s mantle: towards a numerical simulation”, *J. Fluid Mech.* **62**, 465–538 (1974).
- [9] D. Bercovici, G. Schubert, and G. A. Glatzmaier, “Three-dimensional spherical models of convection in the Earth’s mantle”, *Science* **244**, 950–955 (1989).
- [10] R. Trompert and U. Hansen, “Mantle convection simulations with rheologies that generate plate-like behaviour”, *Nature* **395**, 686–689 (1998).
- [11] M. Gurnis, “Large-scale mantle convection and the aggregation and dispersal of supercontinents”, *Nature* **332**, 695–699 (1988).
- [12] S. Zhong and M. Gurnis, “Mantle convection with plates and mobile, faulted plate margins”, *Science* **267**, 838–843 (1995).

- [13] P. J. Tackley, “Mantle convection and plate tectonics: Toward an integrated physical and chemical theory”, *Science* **288**, 2002–2007 (2000).
- [14] G. Houseman, “The dependence of convection planform on mode of heating”, *Nature* **332**, 346–349 (1988).
- [15] A. Davaille, “Simultaneous generation of hotspots and superswells by convection in a heterogeneous planetary mantle”, *Nature* **402**, 756–760 (1999).
- [16] G. Hadley, “Concerning the cause of the general trade-winds”, *Philos. Trans. R. Soc. London.* **39**, 58–62 (1735).
- [17] “Convection in the Earth’s outer core and mantle”, URL <https://storymaps.arcgis.com/stories/79b387dda0844e0f8ff38273f1f904e4>.
- [18] “Convection rolls in the atmosphere”, URL <https://wattsupwiththat.com/2013/12/28/climate-as-a-heat-engine>.
- [19] “Convection in the ocean”, URL <https://biophilesblog.wordpress.com/2015/07/29/fire-flood-and-climate-change>.
- [20] “Convection in the interior of the Sun”, URL <https://www.zmescience.com/science/news-science/sun-core-rotates-faster>.
- [21] Y. Kaspi, E. Galanti, W. B. Hubbard, D. J. Stevenson, S. J. Bolton, L. Iess, T. Guillot, J. Bloxham, J. E. P. Connerney, and H. Cao, “Jupiter’s atmospheric jet streams extend thousands of kilometres deep”, *Nature* **555**, 223 (2018).
- [22] A. J. Trowbridge, H. J. Melosh, J. K. Steckloff, and A. M. Freed, “Vigorous convection as the explanation for Pluto’s polygonal terrain”, *Nature* **534**, 79–81 (2016).
- [23] C. Wunsch, “What is the thermohaline circulation?”, *Science* **298**, 1179–1181 (2002).
- [24] S. Rahmstorf, “Thermohaline circulation: The current climate”, *Nature* **421**, 699–699 (2003).
- [25] W. S. Broecker, D. M. Peteet, and D. Rind, “Does the ocean–atmosphere system have more than one stable mode of operation?”, *Nature* **315**, 21 (1985).
- [26] S. Rahmstorf, “Ocean circulation and climate during the past 120,000 years”, *Nature* **419**, 207–214 (2002).
- [27] A. Ganopolski and S. Rahmstorf, “Abrupt glacial climate changes due to stochastic resonance”, *Phys. Rev. Lett.* **88**, 038501 (2002).
- [28] W. S. Broecker, “Thermohaline circulation, the Achilles heel of our climate system: Will man-made CO₂ upset the current balance?”, *Science* **278**, 1582–1588 (1997).
- [29] G. Veronis, “Penetrative convection.”, *Astrophys. J.* **137**, 641 (1963).

- [30] B. Dintrans, A. Brandenburg, Å. Nordlund, and R. F. Stein, “Spectrum and amplitudes of internal gravity waves excited by penetrative convection in solar-type stars”, *Astron. Astrophys* **438**, 365–376 (2005).
- [31] M. Heimpel, J. Aurnou, and J. Wicht, “Simulation of equatorial and high-latitude jets on Jupiter in a deep convection model”, *Nature* **438**, 193–196 (2005).
- [32] D. Kong, K. Zhang, G. Schubert, and J. D. Anderson, “Origin of Jupiter’s cloud-level zonal winds remains a puzzle even after Juno”, *Proc. N. Acad. Sci.* **115**, 8499–8504 (2018).
- [33] A. P. Ingersoll, “Atmospheric dynamics of the outer planets”, *Science* **248**, 308–315 (1990).
- [34] Z.-P. Sun, G. Schubert, and G. A. Glatzmaier, “Banded surface flow maintained by convection in a model of the rapidly rotating giant planets”, *Science* **260**, 661–664 (1993).
- [35] J. Y. K. Cho and L. M. Polvani, “The morphogenesis of bands and zonal winds in the atmospheres on the giant outer planets”, *Science* **273**, 335–337 (1996).
- [36] J.-I. Yano, O. Talagrand, and P. Drossart, “Outer planets: Origins of atmospheric zonal winds”, *Nature* **421**, 36 (2003).
- [37] D. H. Atkinson, J. B. Pollack, and A. Seiff, “Galileo Doppler measurements of the deep zonal winds at Jupiter”, *Science* **272**, 842–843 (1996).
- [38] C. C. Porco, R. A. West, A. McEwen, A. D. Del G., A. P. Ingersoll, P. Thomas, S. Squyres, L. Dones, C. D. Murray, T. V. Johnson, *et al.*, “Cassini imaging of Jupiter’s atmosphere, satellites, and rings”, *Science* **299**, 1541–1547 (2003).
- [39] A. P. Ingersoll and J. N. Cuzzi, “Dynamics of Jupiter’s cloud bands”, *J. Atmos. Sci.* **26**, 981–985 (1969).
- [40] A. P. Showman, “Numerical simulations of forced shallow-water turbulence: Effects of moist convection on the large-scale circulation of Jupiter and Saturn”, *J. Atmos Sci* **64**, 3132–3157 (2007).
- [41] K. Zhang and G. Schubert, “Penetrative convection and zonal flow on Jupiter”, *Science* **273**, 941–943 (1996).
- [42] K. Zhang and G. Schubert, “Teleconvection: remotely driven thermal convection in rotating stratified spherical layers”, *Science* **290**, 1944–1947 (2000).
- [43] S. A. Stern, F. Bagenal, K. Ennico, G. R. Gladstone, W. M. Grundy, W. B. McKinnon, J. M. Moore, C. B. Olkin, J. R. Spencer, H. A. Weaver, *et al.*, “The Pluto system: Initial results from its exploration by New Horizons”, *Science* **350** (2015).

- [44] K. Vilella and F. Deschamps, “Thermal convection as a possible mechanism for the origin of polygonal structures on Pluto’s surface”, *J. Geophys. Res. Planets* **122**, 1056–1076 (2017).
- [45] F. P. Incropera, *Liquid cooling of electronic devices by single-phase convection*, volume 3 (Wiley-Interscience) (1999).
- [46] P. F. Linden, “The fluid mechanics of natural ventilation”, *Annu. Rev. Fluid Mech.* **31**, 201–238 (1999).
- [47] G. R. Hunt and P. P. Linden, “The fluid mechanics of natural ventilation—displacement ventilation by buoyancy-driven flows assisted by wind”, *Build. Environ.* **34**, 707–720 (1999).
- [48] L. R. Glicksman, “Energy efficiency in the built environment”, *Phys. Today* **91**, 35–40 (2008).
- [49] X. Zhang and O. Zikanov, “Mixed convection in a horizontal duct with bottom heating and strong transverse magnetic field”, *J. Fluid Mech.* **757**, 33–56 (2014).
- [50] X. Zhang and O. Zikanov, “Two-dimensional turbulent convection in a toroidal duct of a liquid metal blanket of a fusion reactor”, *J. Fluid Mech.* **779**, 36–52 (2015).
- [51] M. Lappa, “Thermal convection and related instabilities in models of crystal growth from the melt on Earth and in microgravity: Past history and current status”, *Cryst. Res. Technol.* **40**, 531–549 (2005).
- [52] M. A. Bees, “Advances in bioconvection”, *Annu. Rev. Fluid Mech.* **52**, 449–476 (2020).
- [53] G. Ahlers, S. Grossmann, and D. Lohse, “Heat transfer and large scale dynamics in turbulent Rayleigh-Bénard convection”, *Rev. Mod. Phys.* **81**, 503–537 (2009).
- [54] D. Lohse and K.-Q. Xia, “Small-scale properties of turbulent Rayleigh-Bénard convection”, *Annu. Rev. Fluid Mech.* **42**, 335–364 (2010).
- [55] F. Chillà and J. Schumacher, “New perspectives in turbulent Rayleigh-Bénard convection”, *Eur. Phys. J. E* **35**, 58 (2012).
- [56] K.-Q. Xia, “Current trends and future directions in turbulent thermal convection”, *Theor. Appl. Mech. Lett.* **3**, 052001 (2013).
- [57] C. S. Ng, A. Ooi, D. Lohse, and D. Chung, “Vertical natural convection: application of the unifying theory of thermal convection”, *J. Fluid Mech.* **764**, 349–361 (2015).
- [58] O. Shishkina, “Momentum and heat transport scalings in laminar vertical convection”, *Phys. Rev. E* **93**, 051102 (2016).
- [59] O. Shishkina and S. Horn, “Thermal convection in inclined cylindrical containers”, *J. Fluid Mech.* **790**, R3 (2016).

- [60] Bénard, H., “Les tourbillons cellulaires dans une nappe liquide”, *Rev. Gen. Sci. Pures Appl.* **11**, 1261–1271 (1900).
- [61] L. Rayleigh, “On convection currents in a horizontal layer of fluid, when the higher temperature is on the under side”, *Philos. Mag.* **32**, 529–546 (1916).
- [62] P. Drazin and W. H. Reid, *Hydrodynamic stability* (Cambridge University Press, Cambridge) (1981).
- [63] S. Chandrasekhar, *Hydrodynamic and hydromagnetic stability* (Courier Corporation) (2013).
- [64] A. V. Getling, *Rayleigh-Bénard Convection: Structures and Dynamics*, volume 11 (World Scientific) (1998).
- [65] E. Bodenschatz, W. Pesch, and G. Ahlers, “Recent developments in Rayleigh-Bénard convection”, *Annu. Rev. Fluid Mech.* **32**, 709–778 (2000).
- [66] M. Lappa, *Thermal convection: patterns, evolution and stability* (John Wiley & Sons) (2009).
- [67] G. O. Hughes and R. W. Griffiths, “Horizontal convection”, *Annu. Rev. Fluid Mech.* **40**, 185–208 (2008).
- [68] O. Shishkina, S. Grossmann, and D. Lohse, “Heat and momentum transport scalings in horizontal convection”, *Geophys. Res. Lett.* **43**, 1219–1225 (2016).
- [69] O. Shishkina and S. Wagner, “Prandtl-number dependence of heat transport in laminar horizontal convection”, *Phys. Rev. Lett.* **116**, 024302 (2016).
- [70] O. Shishkina, “Mean flow structure in horizontal convection”, *J. Fluid Mech.* **812**, 525–540 (2017).
- [71] P. Reiter and O. Shishkina, “Classical and symmetrical horizontal convection: detaching plumes and oscillations”, *J. Fluid Mech.* **892**, R1 (2020).
- [72] Y. Yang, R. Verzicco, and D. Lohse, “From convection rolls to finger convection in double-diffusive turbulence”, *Proc. N. Acad. Sci.* **113**, 69–73 (2016).
- [73] Y. Yang, R. Verzicco, and D. Lohse, “Vertically bounded double diffusive convection in the finger regime: Comparing no-slip versus free-slip boundary conditions”, *Phys. Rev. Lett.* **117**, 184501 (2016).
- [74] P. H. Roberts, “Convection in horizontal layers with internal heat generation. theory”, *J. Fluid. Mech.* **30**, 33–49 (1967).
- [75] D. Goluskin, *Internally heated convection and Rayleigh-Bénard convection* (Springer) (2016).
- [76] Q. Wang, D. Lohse, and O. Shishkina, “Scaling in internally heated convection: a unifying theory”, *arXiv:2010.05789* (2020).

- [77] A. Oberbeck, “Über die Wärmeleitung der Flüssigkeiten bei Berücksichtigung der Strömungen infolge von Temperaturdifferenzen”, *Annalen der Physik* **243**, 271–292 (1879).
- [78] J. Boussinesq, *Théorie analytique de la chaleur: mise en harmonie avec la thermodynamique et avec la théorie mécanique de la lumière*, volume 2 (Gauthier-Villars) (1903).
- [79] S.-N. Xia, Z.-H. Wan, S. Liu, Q. Wang, and D.-J. Sun, “Flow reversals in Rayleigh–Bénard convection with non-Oberbeck–Boussinesq effects”, *J. Fluid Mech.* **798**, 628–642 (2016).
- [80] E. P. van der Poel, R. Ostilla-Mónico, J. Donners, and R. Verzicco, “A pencil distributed finite difference code for strongly turbulent wall-bounded flows”, *Computers & Fluids* **116**, 10–16 (2015).
- [81] D. Goluskin, H. Johnston, G. R. Flierl, and E. A. Spiegel, “Convectively driven shear and decreased heat flux”, *J. Fluid Mech.* **759**, 360–385 (2014).
- [82] J. von Hardenberg, D. Goluskin, A. Provenzale, and E. A. Spiegel, “Generation of large-scale winds in horizontally anisotropic convection”, *Phys. Rev. Lett.* **115**, 134501 (2015).
- [83] R. B. Leighton, “The solar granulation”, *Annu. Rev. Astron. and Astrophys* **1**, 19–40 (1963).
- [84] B. Buffett, “Geomagnetic fluctuations reveal stable stratification at the top of the Earth’s core”, *Nature* **507**, 484 (2014).
- [85] B. N. Antar, “Penetrative double-diffusive convection”, *Phys. Fluids* **30**, 322–330 (1987).
- [86] D. Goluskin and E. P. van der Poel, “Penetrative internally heated convection in two and three dimensions”, *J. Fluid Mech.* **791**, R6 (2016).
- [87] S. Musman, “Penetrative convection”, *J. Fluid Mech.* **31**, 343–360 (1968).
- [88] D. Moore and N. Weiss, “Nonlinear penetrative convection”, *J. Fluid Mech.* **61**, 553–581 (1973).
- [89] E. Large and C. Andereck, “Penetrative Rayleigh–Bénard convection in water near its maximum density point”, *Phys. Fluids* **26**, 094101 (2014).
- [90] Y.-P. Hu, Y.-R. Li, and C.-M. Wu, “Rayleigh–Bénard convection of cold water near its density maximum in a cubical cavity”, *Phys. Fluids* **27**, 034102 (2015).
- [91] D. Lecoanet, M. Le Bars, K. J. Burns, G. M. Vasil, B. P. Brown, E. Quataert, and J. S. Oishi, “Numerical simulations of internal wave generation by convection in water”, *Phys. Rev. E* **91**, 063016 (2015).
- [92] S. Toppaladoddi and J. S. Wettlaufer, “Penetrative convection at high Rayleigh numbers”, *Phys. Rev. Fluids* **3**, 043501 (2018).

- [93] L.-A. Couston, D. Lecoanet, B. Favier, and M. Le Bars, “Order out of chaos: Slowly reversing mean flows emerge from turbulently generated internal waves”, *Phys. Rev. Lett.* **120**, 244505 (2018).
- [94] J. Zhang, S. Childress, and A. Libchaber, “Non-Boussinesq effect: Thermal convection with broken symmetry”, *Phys. Fluids* **9**, 1034–1042 (1997).
- [95] G. Ahlers, E. Brown, F. F. Araujo, D. Funfschilling, S. Grossmann, and D. Lohse, “Non-Oberbeck–Boussinesq effects in strongly turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **569**, 409–445 (2006).
- [96] G. Ahlers, F. F. Araujo, D. Funfschilling, S. Grossmann, and D. Lohse, “Non-Oberbeck-Boussinesq effects in gaseous Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **98**, 054501 (2007).
- [97] A. Sameen, R. Verzicco, and K. R. Sreenivasan, “Non-Boussinesq convection at moderate Rayleigh numbers in low temperature gaseous helium”, *Phys. Scr.* **2008**, 014053 (2008).
- [98] A. Sameen, R. Verzicco, and K. Sreenivasan, “Specific roles of fluid properties in non-Boussinesq thermal convection at the Rayleigh number of 2×10^8 ”, *Eur. Phys. Lett.* **86**, 14006 (2009).
- [99] K. Sugiyama, E. Calzavarini, S. Grossmann, and D. Lohse, “Flow organization in two-dimensional non-Oberbeck–Boussinesq Rayleigh–Bénard convection in water”, *J. Fluid Mech.* **637**, 105–135 (2009).
- [100] S. Horn, O. Shishkina, and C. Wagner, “On non-Oberbeck–Boussinesq effects in three-dimensional Rayleigh–Bénard convection in glycerol”, *J. Fluid Mech.* **724**, 175–202 (2013).
- [101] S. Horn and O. Shishkina, “Rotating non-Oberbeck–Boussinesq Rayleigh–Bénard convection in water”, *Phys. Fluids* **26**, 055111 (2014).
- [102] S. Weiss, X. He, G. Ahlers, E. Bodenschatz, and O. Shishkina, “Bulk temperature and heat transport in turbulent Rayleigh–Bénard convection of fluids with temperature-dependent properties”, *J. Fluid Mech.* **851**, 374–390 (2018).
- [103] B. Gebhart and J. C. Mollendorf, “A new density relation for pure and saline water”, *Deep-Sea Res.* **24**, 831–848 (1977).
- [104] Q. Wang, B.-L. Xu, S.-N. Xia, Z.-H. Wan, and D.-J. Sun, “Thermal convection in a tilted rectangular cell with aspect ratio 0.5”, *Chin. Phys. Lett.* **34**, 104401 (2017).
- [105] S. Liu, S.-N. Xia, R. Yan, Z.-H. Wan, and D.-J. Sun, “Linear and weakly nonlinear analysis of Rayleigh–Bénard convection of perfect gas with non-Oberbeck–Boussinesq effects”, *J. Fluid Mech.* **845**, 141–169 (2018).

- [106] Q. Wang, Z.-H. Wan, R. Yan, and D.-J. Sun, “Multiple states and heat transfer in two-dimensional tilted convection with large aspect ratios”, *Phys. Rev. Fluids* **3**, 113503 (2018).
- [107] Q. Wang, S.-N. Xia, B.-F. Wang, D.-J. Sun, Q. Zhou, and Z.-H. Wan, “Flow reversals in two-dimensional thermal convection in tilted cells”, *J. Fluid Mech.* **849**, 355–372 (2018).
- [108] Q. Wang, Z.-H. Wan, R. Yan, and D.-J. Sun, “Flow organization and heat transfer in two-dimensional tilted convection with aspect ratio 0.5”, *Phys. Fluids* **31**, 025102 (2019).
- [109] Q. Wang, S.-N. Xia, R. Yan, D.-J. Sun, and Z.-H. Wan, “Non-Oberbeck-Boussinesq effects due to large temperature differences in a differentially heated square cavity filled with air”, *Int. J. Heat Mass Transf.* **128**, 479–491 (2019).
- [110] O. Shishkina, R. Stevens, S. Grossmann, and D. Lohse, “Boundary layer structure in turbulent thermal convection and its consequences for the required numerical resolution”, *New J. Phys.* **12**, 075022 (2010).
- [111] D. P. M. Van Gils, S. G. Huisman, G.-W. Bruggert, C. Sun, and D. Lohse, “Torque scaling in turbulent Taylor-Couette flow with co-and counterrotating cylinders”, *Phys. Rev. Lett.* **106**, 024502 (2011).
- [112] R. Ostilla-Mónico, R. J. A. M. Stevens, S. Grossmann, R. Verzicco, and D. Lohse, “Optimal Taylor–Couette flow: direct numerical simulations”, *J. Fluid Mech.* **719**, 14–46 (2013).
- [113] R. Ostilla-Mónico, E. P. van der Poel, R. Verzicco, S. Grossmann, and D. Lohse, “Exploring the phase diagram of fully turbulent Taylor–Couette flow”, *J. Fluid Mech.* **761**, 1–26 (2014).
- [114] K. Sugiyama, R. Ni, R. J. A. M. Stevens, T. S. Chan, S.-Q. Zhou, H.-D. Xi, C. Sun, S. Grossmann, K.-Q. Xia, and D. Lohse, “Flow reversals in thermally driven turbulence”, *Phys. Rev. Lett.* **105**, 034503 (2010).
- [115] S.-D. Huang, M. Kaczorowski, R. Ni, and K.-Q. Xia, “Confinement-induced heat-transport enhancement in turbulent thermal convection”, *Phys. Rev. Lett.* **111**, 104501 (2013).
- [116] K.-L. Chong, S.-D. Huang, M. Kaczorowski, and K.-Q. Xia, “Condensation of coherent structures in turbulent flows”, *Phys. Rev. Lett.* **115**, 264503 (2015).
- [117] K. L. Chong, Y.-T. Yang, S.-D. Huang, J.-Q. Zhong, R. J. A. M. Stevens, R. Verzicco, D. Lohse, and K.-Q. Xia, “Confined Rayleigh–Bénard, Rotating Rayleigh–Bénard, and Double Diffusive Convection: A unifying view on turbulent transport enhancement through coherent structure manipulation”, *Phys. Rev. Lett.* **119**, 064501 (2017).

- [118] K.-L. Chong, S. Wagner, M. Kaczorowski, O. Shishkina, and K.-Q. Xia, “Effect of Prandtl number on heat transport enhancement in Rayleigh–Bénard convection under geometrical confinement”, *Phys. Rev. Fluids* **3**, 013501 (2018).
- [119] K.-L. Chong and K.-Q. Xia, “Exploring the severely confined regime in Rayleigh–Bénard convection”, *J. Fluid Mech.* **805**, R4 (2016).
- [120] H. Johnston and C. R. Doering, “Comparison of turbulent thermal convection between conditions of constant temperature and constant flux”, *Phys. Rev. Lett.* **102**, 064501 (2009).
- [121] Y.-X. Huang and Q. Zhou, “Counter-gradient heat transport in two-dimensional turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **737**, R3 (2013).
- [122] Y. Zhang, Q. Zhou, and C. Sun, “Statistics of kinetic and thermal energy dissipation rates in two-dimensional turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **814**, 165–184 (2017).
- [123] D. D. Gray and A. Giorgini, “The validity of the Boussinesq approximation for liquids and gases”, *Int. J. Heat Mass Transf.* **19**, 545–551 (1976).
- [124] X.-Z. Wu and A. Libchaber, “Non-Boussinesq effects in free thermal convection”, *Phys. Rev. A* **43**, 2833 (1991).
- [125] J. Zhang, S. Childress, and A. Libchaber, “Non-Boussinesq effect: Asymmetric velocity profiles in thermal convection”, *Phys. Fluids* **10**, 1534–1536 (1998).
- [126] K. Sugiyama, E. Calzavarini, S. Grossmann, and D. Lohse, “Non-Oberbeck-Boussinesq effects in two-dimensional Rayleigh–Bénard convection in glycerol”, *Eur. Phys. Lett.* **80**, 34002 (2007).
- [127] G. Ahlers, E. Calzavarini, F. F. Araujo, D. Funfschilling, S. Grossmann, D. Lohse, and K. Sugiyama, “Non-Oberbeck-Boussinesq effects in turbulent thermal convection in ethane close to the critical point”, *Phys. Rev. E* **77**, 046302 (2008).
- [128] A. Demou and D. E. Grigoriadis, “Direct numerical simulations of Rayleigh–Bénard convection in water with non-Oberbeck-Boussinesq effects”, *J. Fluid Mech.* **881**, 1073–1096 (2019).
- [129] Y. Burnishev, E. Segre, and V. Steinberg, “Strong symmetrical non-Oberbeck–Boussinesq turbulent convection and the role of compressibility”, *Phys. Fluids* **22**, 035108 (2010).
- [130] Y. Burnishev and V. Steinberg, “Statistics and scaling properties of temperature field in symmetrical non-Oberbeck-Boussinesq turbulent convection”, *Phys. Fluids* **24**, 045102 (2012).
- [131] V. Valori, G. Elsinga, M. Rohde, M. Tummers, J. Westerweel, and T. van der Hagen, “Experimental velocity study of non-Boussinesq Rayleigh–Bénard convection”, *Phys. Rev. E* **95**, 053113 (2017).

- [132] S. Paolucci, “Filtering of sound from the Navier–Stokes equations. NASA STI”, Technical Report, Recon Tech. Rep. N, 83, 26036 (1982).
- [133] D. Livescu, “Turbulence with large thermal and compositional density variations”, *Annu. Rev. Fluid Mech.* **52** (2020).
- [134] F. White, *Viscous fluid flow* (McGraw-Hill New York) (1974).
- [135] S. Suslov and S. Paolucci, “Nonlinear stability of mixed convection flow under non-Boussinesq conditions. Part 1. analysis and bifurcations”, *J. Fluid Mech.* **398**, 61–85 (1999).
- [136] S. Suslov, “Mechanism of nonlinear flow pattern selection in moderately non-boussinesq mixed convection”, *Phys. Rev. E* **81**, 026301 (2010).
- [137] W. Sutherland, “The viscosity of gases and molecular force”, *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **36**, 507–531 (1893).
- [138] L.-W. Chen, C.-Y. Xu, and X.-Y. Lu, “Numerical investigation of the compressible flow past an aerofoil”, *J. Fluid Mech.* **643**, 97–126 (2010).
- [139] S. Pirozzoli, M. Bernardini, and F. Grasso, “Direct numerical simulation of transonic shock/boundary layer interaction under conditions of incipient separation”, *J. Fluid Mech.* **657**, 361–393 (2010).
- [140] Z.-H. Wan, L. Zhou, H.-H. Yang, and D.-J. Sun, “Large eddy simulation of flow development and noise generation of free and swirling jets”, *Phys. Fluids* **25**, 126103 (2013).
- [141] R. Verzicco and P. Orlandi, “A finite-difference scheme for three-dimensional incompressible flows in cylindrical coordinates”, *J. Comput. Phys.* **123**, 402–414 (1996).
- [142] W. L. Briggs, V. E. Henson, and S. F. McCormick, *A multigrid tutorial*, volume 72 (Siam) (2000).
- [143] H.-D. Xi, S. Lam, and K.-Q. Xia, “From laminar plumes to organized flows: the onset of large-scale circulation in turbulent thermal convection”, *J. Fluid Mech.* **503**, 47–56 (2004).
- [144] M. Chandra and M. K. Verma, “Flow reversals in turbulent convection via vortex reconnections”, *Phys. Rev. Lett.* **110**, 114503 (2013).
- [145] E. P. van der Poel, R. J. A. M. Stevens, K. Sugiyama, and D. Lohse, “Flow states in two-dimensional Rayleigh–Bénard convection as a function of aspect-ratio and Rayleigh number”, *Phys. Fluids* **24**, 085104 (2012).
- [146] Q. Wang, Q. Zhou, Z.-H. Wan, and D.-J. Sun, “Penetrative turbulent Rayleigh–Bénard convection in two and three dimensions”, *J. Fluid Mech.* **870**, 718–734 (2019).

- [147] S. Wagner and O. Shishkina, “Aspect-ratio dependency of Rayleigh–Bénard convection in box-shaped containers”, *Phys. Fluids* **25**, 085110 (2013).
- [148] Y.-Z. Zhang, C. Sun, Y. Bao, and Q. Zhou, “How surface roughness reduces heat transport for small roughness heights in turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **836** (2018).
- [149] C. S. Ng, A. Ooi, D. Lohse, and D. Chung, “Bulk scaling in wall-bounded and homogeneous vertical natural convection”, *J. Fluid Mech.* **841**, 825–850 (2018).
- [150] J. J. Niemela, L. Skrbek, K. R. Sreenivasan, and R. J. Donnelly, “The wind in confined thermal convection”, *J. Fluid Mech.* **449**, 169–178 (2001).
- [151] X.-L. Qiu and P. Tong, “Large-scale velocity structures in turbulent thermal convection”, *Phys. Rev. E* **64**, 036304 (2001).
- [152] C. Sun and K.-Q. Xia, “Scaling of the Reynolds number in turbulent thermal convection”, *Phys. Rev. E* **72**, 067302 (2005).
- [153] E. Brown, D. Funfschilling, and G. Ahlers, “Anomalous Reynolds-number scaling in turbulent Rayleigh–Bénard convection”, *J. Stat. Mech.* **2007**, P10005 (2007).
- [154] C. Sun, Y.-H. Cheung, and K.-Q. Xia, “Experimental studies of the viscous boundary layer properties in turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **605**, 79–113 (2008).
- [155] P. Wei and K.-Q. Xia, “Viscous boundary layer properties in turbulent thermal convection in a cylindrical cell: the effect of cell tilting”, *J. Fluid Mech.* **720**, 140–168 (2013).
- [156] Q. Zhou and K.-Q. Xia, “Thermal boundary layer structure in turbulent Rayleigh–Bénard convection in a rectangular cell”, *J. Fluid Mech.* **721**, 199–224 (2013).
- [157] N. Shi, M. Emran, and J. Schumacher, “Boundary layer structure in turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **706**, 5–33 (2012).
- [158] H.-D. Xi and K.-Q. Xia, “Flow mode transitions in turbulent thermal convection”, *Phys. Fluids* **20**, 055104 (2008).
- [159] E. P. van der Poel, R. J. A. M. Stevens, and D. Lohse, “Connecting flow structures and heat flux in turbulent Rayleigh–Bénard convection”, *Phys. Rev. E* **84**, 045303 (2011).
- [160] S. Weiss and G. Ahlers, “Effect of tilting on turbulent convection: cylindrical samples with aspect ratio $\Gamma = 0.50$ ”, *J. Fluid Mech.* **715**, 314–334 (2013).
- [161] Y.-C. Xie, G. Ding, and K.-Q. Xia, “Flow topology transition via global bifurcation in thermally driven turbulence”, *Phys. Rev. Lett.* **120**, 214501 (2018).

- [162] B. Favier, C. Guervilly, and E. Knobloch, “Subcritical turbulent condensate in rapidly rotating Rayleigh–Bénard convection”, *J. Fluid Mech.* **864**, R1 (2019).
- [163] S. G. Huisman, R. C. Van Der Veen, C. Sun, and D. Lohse, “Multiple states in highly turbulent Taylor–Couette flow”, *Nat. Commun.* **5**, 3820 (2014).
- [164] R. C. A. van der Veen, S. G. Huisman, O.-Y. Dung, H. L. Tang, C. Sun, and D. Lohse, “Exploring the phase space of multiple states in highly turbulent Taylor–Couette flow”, *Phys. Rev. Fluids* **1**, 024401 (2016).
- [165] R. Ostilla-Mónico, R. Verzicco, S. Grossmann, and D. Lohse, “The near-wall region of highly turbulent Taylor–Couette flow”, *J. Fluid Mech.* **788**, 95–117 (2016).
- [166] Y. Yang, W. Chen, R. Verzicco, and D. Lohse, “Multiple states and transport properties of double-diffusive convection turbulence”, *Proc. N. Acad. Sci.* **117**, 14676–14681 (2020).
- [167] F. Ravelet, L. Marié, A. Chiffaudel, and F. Daviaud, “Multistability and memory effect in a highly turbulent flow: Experimental evidence for a global bifurcation”, *Phys. Rev. Lett.* **93**, 164501 (2004).
- [168] F. Ravelet, A. Chiffaudel, and F. Daviaud, “Supercritical transition to turbulence in an inertially driven von Kármán closed flow”, *J. Fluid Mech.* **601**, 339–364 (2008).
- [169] P. P. Cortet, A. Chiffaudel, F. Daviaud, and B. Dubrulle, “Experimental evidence of a phase transition in a closed turbulent flow”, *Phys. Rev. Lett.* **105**, 214501 (2010).
- [170] D. Faranda, Y. Sato, B. Saint-Michel, C. Wiertel, V. Padilla, B. Dubrulle, and F. Daviaud, “Stochastic chaos in a turbulent swirling flow”, *Phys. Rev. Lett.* **119**, 014502 (2017).
- [171] D. S. Zimmerman, S. A. Triana, and D. P. Lathrop, “Bi-stability in turbulent, rotating spherical couette flow”, *Phys. Fluids* **23**, 065104 (2011).
- [172] Z.-H. Xia, Y.-P. Shi, Q.-D. Cai, M.-P. Wan, and S.-Y. Chen, “Multiple states in turbulent plane Couette flow with spanwise rotation”, *J. Fluid Mech.* **837**, 477–490 (2018).
- [173] F. Bouchet and E. Simonnet, “Random changes of flow topology in two-dimensional and geophysical turbulence”, *Phys. Rev. Lett.* **102**, 094504 (2009).
- [174] F. Bouchet and A. Venaille, “Statistical mechanics of two-dimensional and geophysical flows”, *Physics reports* **515**, 227–295 (2012).
- [175] M. J. Schmeits and H. A. Dijkstra, “Bimodal behavior of the Kuroshio and the Gulf Stream”, *J. Phys. Oceanogr.* **31**, 3435–3456 (2001).

- [176] P. L. Olson, R. S. Coe, P. E. Driscoll, G. A. Glatzmaier, and P. H. Roberts, “Geodynamo reversal frequency and heterogeneous core–mantle boundary heat flow”, *Phys. Earth Planet. Inter.* **180**, 66–79 (2010).
- [177] A. Sheyko, C. C. Finlay, and A. Jackson, “Magnetic reversals from planetary dynamo waves”, *Nature* **539**, 551 (2016).
- [178] E. R. Weeks, Y. Tian, J. Urbach, K. Ide, H. L. Swinney, and M. Ghil, “Transitions between blocked and zonal flows in a rotating annulus with topography”, *Science* **278**, 1598–1601 (1997).
- [179] F. Bouchet, J. Rolland, and E. Simonnet, “Rare event algorithm links transitions in turbulent flows with activated nucleations”, *Phys. Rev. Lett.* **122**, 074502 (2019).
- [180] A. N. Kolmogorov, “On degeneration (decay) of isotropic turbulence in incompressible viscous liquid”, *Dokl. Akad. Nauk SSSR* **31**, 538–540 (1941).
- [181] E. P. van der Poel, R. J. A. M. Stevens, and D. Lohse, “Comparison between two- and three-dimensional Rayleigh–Bénard convection”, *J. Fluid Mech.* **736**, 177–194 (2013).
- [182] G. L. Kooij, M. A. Botchev, E. M. Frederix, B. J. Geurts, S. Horn, D. Lohse, E. P. van der Poel, O. Shishkina, R. J. Stevens, and R. Verzicco, “Comparison of computational codes for direct numerical simulations of turbulent Rayleigh–Bénard convection”, *Comput. Fluids* **166**, 1–8 (2018).
- [183] X. Zhu, V. Mathai, R. J. A. M. Stevens, R. Verzicco, and D. Lohse, “Transition to the ultimate regime in two-dimensional Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **120**, 144502 (2018).
- [184] X. Zhu, V. Mathai, R. J. A. M. Stevens, R. Verzicco, and D. Lohse, “Reply to “Absence of evidence for the ultimate regime in two-dimensional Rayleigh–Bénard convection.””, *Phys. Rev. Lett.* **123**, 259402 (2019).
- [185] “See the Supplemental Material at”, URL <https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.125.074501#supplemental>.
- [186] L. Zwirner, A. Tilgner, and O. Shishkina, “Elliptical instability and multi-roll flow modes of the large-scale circulation in confined turbulent rayleigh–bénard convection”, *Phys. Rev. Lett.* **125**, 054502 (2020).
- [187] M. Landman and P. Saffman, “The three-dimensional instability of strained vortices in a viscous fluid”, *Phys. Fluids* **30**, 2339–2342 (1987).
- [188] F. Waleffe, “On the three-dimensional instability of strained vortices”, *Phys. Fluids A* **2**, 76–80 (1990).
- [189] R. R. Kerswell, “Elliptical instability”, *Annu. Rev. Fluid Mech.* **34**, 83–113 (2002).

- [190] B. I. Shraiman and E. D. Siggia, “Heat transport in high-Rayleigh number convection”, *Phys. Rev. A* **42**, 3650–3653 (1990).
- [191] E. P. van der Poel, R. Ostilla-Mónico, R. Verzicco, and D. Lohse, “Effect of velocity boundary conditions on the heat transfer and flow topology in two-dimensional Rayleigh-Bénard convection”, *Phys. Rev. E* **90**, 013017 (2014).
- [192] Q. Wang, K.-L. Chong, R. J. A. M. Stevens, R. Verzicco, and D. Lohse, “From zonal flow to convection rolls in Rayleigh-Bénard convection with free-slip plates”, *J. Fluid. Mech.* (In Press) (2020).
- [193] N. A. Maximenko, B. Bang, and H. Sasaki, “Observational evidence of alternating zonal jets in the world ocean”, *Geophys. Res. Lett.* **32** (2005).
- [194] K. J. Richards, N. A. Maximenko, F. O. Bryan, and H. Sasaki, “Zonal jets in the Pacific Ocean”, *Geophys. Res. Lett.* **33** (2006).
- [195] B. T. Nadiga, “On zonal jets in oceans”, *Geophys. Res. Lett.* **33** (2006).
- [196] T. Miyagoshi, A. Kageyama, and T. Sato, “Zonal flow formation in the Earth’s core”, *Nature* **463**, 793 (2010).
- [197] P. H. Diamond, S. I. Itoh, K. Itoh, and T. S. Hahm, “Zonal flows in plasma—a review”, *Plasma Phys. Control. Fusion* **47**, R35 (2005).
- [198] L. Novi, J. von H., D. W. H., A. Provenzale, and E. A. Spiegel, “Rapidly rotating Rayleigh-Bénard convection with a tilted axis”, *Phys. Rev. E* **99**, 053116 (2019).
- [199] Q. Wang, R. Verzicco, D. Lohse, and O. Shishkina, “Multiple states in turbulent large-aspect ratio thermal convection: What determines the number of convection rolls?”, *Phys. Rev. Lett.* **125**, 074501 (2020).
- [200] F. Seychelles, Y. Amarouchene, M. Bessafi, and H. Kellay, “Thermal convection and emergence of isolated vortices in soap bubbles”, *Phys. Rev. Lett.* **100**, 144501 (2008).
- [201] F. Seychelles, F. Ingremau, C. Pradère, and H. Kellay, “From intermittent to nonintermittent behavior in two dimensional thermal convection in a soap bubble”, *Phys. Rev. Lett.* **105**, 264502 (2010).
- [202] I. Grooms, K. Julien, J. B. Weiss, and E. Knobloch, “Model of convective Taylor columns in rotating Rayleigh-Bénard convection”, *Phys. Rev. Lett.* **104**, 224501 (2010).
- [203] K. L. Chong, J.-Q. Shi, S. Ding, G.-Y. Ding, H.-Y. Lu, J.-Q. Zhong, and K.-Q. Xia, “Crossover from ballistic to diffusive vortex motion in convection”, *Sci. Adv.* **6**, eaaz1110 (2020).
- [204] van der Poel, E. P. and Ostilla-Mónico, R. and Verzicco, R. and Grossmann, S. and Lohse, D., “Logarithmic mean temperature profiles and their connection to plume emissions in turbulent Rayleigh-Bénard convection”, *Phys. Rev. Lett.* **115**, 154501 (2015).

- [205] X. Zhu, R. J. A. M. Stevens, O. Shishkina, R. Verzicco, and D. Lohse, “ $Nu \sim Ra^{1/2}$ scaling enabled by multiscale wall roughness in Rayleigh–Bénard turbulence”, *J. Fluid Mech.* **869**, R4 (2019).
- [206] Z.-H. Wan, Q. Wang, B. Wang, S.-N. Xia, Q. Zhou, and D.-J. Sun, “On non-Oberbeck–Boussinesq effects in Rayleigh–Bénard convection of air for large temperature differences”, *J. Fluid Mech.* **889** (2020).
- [207] H.-R. Liu, K. L. Chong, Q. Wang, C. S. Ng, R. Verzicco, and D. Lohse, “Two-layer thermally driven turbulence: Mechanisms for interface breakup”, *arXiv:2005.05633* (2020).
- [208] R. Yang, K. L. Chong, Q. Wang, R. Verzicco, O. Shishkina, and D. Lohse, “Periodically modulated thermal convection”, *Phys. Rev. Lett.* **125** (2020).
- [209] K. L. Chong, R. Yang, Q. Wang, R. Verzicco, and D. Lohse, “Café Latte: Spontaneous layer formation in laterally cooled double diffusive convection”, *J. Fluid Mech.* **900**, R6 (2020).
- [210] B.-F. Wang, Q. Zhou, and C. Sun, “Vibration-induced boundary-layer destabilization achieves massive heat-transport enhancement”, *Sci. Adv.* **6**, eaaz8239 (2020).
- [211] K. Petschel, S. Stellmach, M. Wilczek, J. Lülff, and U. Hansen, “Dissipation layers in Rayleigh–Bénard convection: A unifying view”, *Phys. Rev. Lett.* **110**, 114502 (2013).
- [212] R. P. J. Kunnen, R. Ostilla-Mónico, E. P. van der Poel, R. Verzicco, and D. Lohse, “Transition to geostrophic convection: the role of the boundary conditions”, *J. Fluid Mech.* **799**, 413–432 (2016).
- [213] T. Hartlep, A. Tilgner, and F. H. Busse, “Large scale structures in Rayleigh–Bénard convection at high Rayleigh numbers”, *Phys. Rev. Lett.* **91**, 064501 (2003).
- [214] A. Parodi, J. von Hardenberg, G. Passoni, A. Provenzale, and E. A. Spiegel, “Clustering of plumes in turbulent convection”, *Phys. Rev. Lett.* **92**, 194503 (2004).
- [215] A. Pandey, J. D. Scheel, and J. Schumacher, “Turbulent superstructures in Rayleigh–Bénard convection”, *Nat. Commun.* **9**, 2118 (2018).
- [216] R. J. A. M. Stevens, A. Blass, X.-J. Zhu, R. Verzicco, and D. Lohse, “Turbulent thermal superstructures in Rayleigh–Bénard convection”, *Phys. Rev. Fluids* **3**, 041501 (2018).
- [217] D. Krug, D. Lohse, and R. J. A. M. Stevens, “Coherence of temperature and velocity superstructures in turbulent Rayleigh–Bénard flow”, *J. Fluid Mech.* **887**, A2 (2020).

- [218] G. Green, D. G. Vlaykov, J. P. Mellado, and M. Wilczek, “Resolved energy budget of superstructures in Rayleigh–Bénard convection”, *J. Fluid Mech.* **887**, A21 (2020).
- [219] X. Zhu, E. Phillips, V. Spandan, J. Donners, G. Ruetsch, J. Romero, R. Ostilla-Mónico, Y.-T. Yang, D. Lohse, M. Verzicco, Fatica, and R. Stevens, “AFiD-GPU: a versatile Navier–Stokes solver for wall-bounded turbulent flows on GPU clusters”, *Comput. Phys. Commun.* **229**, 199–210 (2018).
- [220] M. Chandra and M. K. Verma, “Dynamics and symmetries of flow reversals in turbulent convection”, *Phys. Rev. E* **83**, 067303 (2011).
- [221] X. Chen, S.-D. Huang, K.-Q. Xia, and H.-D. Xi, “Emergence of substructures inside the large-scale circulation induces transition in flow reversals in turbulent thermal convection”, *J. Fluid Mech.* **877**, R1 (2019).
- [222] S. Grossmann and D. Lohse, “Scaling in thermal convection: a unifying theory”, *J. Fluid Mech.* **407**, 27–56 (2000).
- [223] S. Grossmann and D. Lohse, “Thermal convection for large prandtl numbers”, *Phys. Rev. Lett.* **86**, 3316 (2001).
- [224] O. Shishkina, M. S. Emran, S. Grossmann, and D. Lohse, “Scaling relations in large-Prandtl-number natural thermal convection”, *Phys. Rev. Fluids* **2**, 103502 (2017).
- [225] G. Ahlers and X. Xu, “Prandtl-number dependence of heat transport in turbulent Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **86**, 3320 (2001).
- [226] K.-Q. Xia, S. Lam, and S.-Q. Zhou, “Heat-flux measurement in high-Prandtl-number turbulent Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **88**, 064501 (2002).
- [227] R. J. Stevens, D. Lohse, and R. Verzicco, “Prandtl and Rayleigh number dependence of heat transport in high Rayleigh number thermal convection”, *J. Fluid Mech.* **688**, 31–43 (2011).
- [228] A. Tilgner, A. Belmonte, and A. Libchaber, “Temperature and velocity profiles of turbulent convection in water”, *Phys. Rev. E* **47**, R2253 (1993).
- [229] E. Brown and G. Ahlers, “Temperature gradients, and search for non-Boussinesq effects, in the interior of turbulent Rayleigh–Bénard convection”, *Eur. Phys. Lett.* **80**, 14001 (2007).
- [230] Z.-H. Wan, P. Wei, R. Verzicco, D. Lohse, G. Ahlers, and R. J. A. M. Stevens, “Effect of sidewall on heat transfer and flow structure in Rayleigh–Bénard convection”, *J. Fluid Mech.* **881**, 218–243 (2019).
- [231] J.-Q. Zhong, R. J. A. M. Stevens, H. J. H. Clercx, R. Verzicco, D. Lohse, and G. Ahlers, “Prandtl-, Rayleigh-, and Rossby-number dependence of heat transport in turbulent rotating Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **102**, 044502 (2009).

- [232] A. Blass, X. Zhu, R. Verzicco, D. Lohse, and R. J. A. M. Stevens, “Flow organization and heat transfer in turbulent wall sheared thermal convection”, *J. Fluid Mech.* **897** (2020).
- [233] M. Lee and R. D. Moser, “Extreme-scale motions in turbulent plane couette flows”, *J. Fluid Mech.* **842**, 128–145 (2018).
- [234] C. D. Andereck, S. S. Liu, and H. L. Swinney, “Flow regimes in a circular Couette system with independently rotating cylinders”, *J. Fluid Mech.* **164**, 155–183 (1986).
- [235] E. van Doorn, B. Dhruva, K. R. Sreenivasan, and V. Cassella, “Statistics of wind direction and its increments”, *Phys. Fluids* **12**, 1529–1534 (2000).
- [236] M. K. Verma, A. Kumar, and A. Pandey, “Phenomenology of buoyancy-driven turbulence: recent results”, *New J. Phys.* **19**, 025012 (2017).
- [237] S.-D. Huang, F. Wang, H.-D. Xi, and K.-Q. Xia, “Comparative Experimental Study of Fixed Temperature and Fixed Heat Flux Boundary Conditions in Turbulent Thermal Convection”, *Phys. Rev. Lett.* **115**, 154502 (2015).
- [238] E. Brown, A. Nikolaenko, and G. Ahlers, “Reorientation of the large-scale circulation in turbulent Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **95**, 084503 (2005).
- [239] E. Brown and G. Ahlers, “Rotations and cessations of the large-scale circulation in turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **568**, 351–386 (2006).
- [240] H.-D. Xi, Q. Zhou, and K.-Q. Xia, “Azimuthal motion of the mean wind in turbulent thermal convection”, *Phys. Rev. E* **73**, 056312 (2006).
- [241] H.-D. Xi and K.-Q. Xia, “Cessations and reversals of the large-scale circulation in turbulent thermal convection”, *Phys. Rev. E* **75**, 066307 (2007).
- [242] H.-D. Xi and K.-Q. Xia, “Azimuthal motion, reorientation, cessation, and reversal of the large-scale circulation in turbulent thermal convection: a comparative study in aspect ratio one and one-half geometries”, *Phys. Rev. E* **78**, 036326 (2008).
- [243] H.-D. Xi, Y.-B. Zhang, J.-T. Hao, and K.-Q. Xia, “Higher-order flow modes in turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **805**, 31–51 (2016).
- [244] P. K. Mishra, A. K. De, M. K. Verma, and V. Eswaran, “Dynamics of reorientations and reversals of large-scale flow in Rayleigh–Bénard convection”, *J. Fluid Mech.* **668**, 480–499 (2011).
- [245] R. Benzi, “Flow reversal in a simple dynamical model of turbulence”, *Phys. Rev. Lett.* **95**, 024502 (2005).
- [246] E. Brown and G. Ahlers, “Large-scale circulation model for turbulent Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **98**, 134501 (2007).

- [247] F. F. Araujo, S. Grossmann, and D. Lohse, “Wind reversals in turbulent Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **95**, 084502 (2005).
- [248] N. Foroozani, J. J. Niemela, V. Armenio, and K. R. Sreenivasan, “Reorientations of the large-scale flow in turbulent convection in a cube”, *Phys. Rev. E* **95**, 033107 (2017).
- [249] K. Petschel, M. Wilczek, M. Breuer, R. Friedrich, and U. Hansen, “Statistical analysis of global wind dynamics in vigorous Rayleigh–Bénard convection”, *Phys. Rev. E* **84**, 026309 (2011).
- [250] B. Podvin and A. Sergent, “A large-scale investigation of wind reversal in a square Rayleigh–Bénard cell”, *J. Fluid Mech.* **766**, 172–201 (2015).
- [251] M. K. Verma, S. C. Ambhire, and A. Pandey, “Flow reversals in turbulent convection with free-slip walls”, *Phys. Fluids* **27**, 047102 (2015).
- [252] A. Castillo-Castellanos, A. Sergent, and M. Rossi, “Reversal cycle in square Rayleigh–Bénard cells in turbulent regime”, *J. Fluid Mech.* **808**, 614–640 (2016).
- [253] B. Podvin and A. Sergent, “Precursor for wind reversal in a square Rayleigh–Bénard cell”, *Phys. Rev. E* **95**, 013112 (2017).
- [254] M. Mannattil, A. Pandey, M. K. Verma, and S. Chakraborty, “On the applicability of low-dimensional models for convective flow reversals at extreme prandtl numbers”, *Eur. Phys. J. B* **90**, 259 (2017).
- [255] R. Ni, S.-D. Huang, and K.-Q. Xia, “Reversals of the large-scale circulation in quasi-2D Rayleigh–Bénard convection”, *J. Fluid Mech.* **778**, R5 (2015).
- [256] S.-D. Huang and K.-Q. Xia, “Effects of geometric confinement in quasi-2-D turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **794**, 639–654 (2016).
- [257] Y.-C. Xie, P. Wei, and K.-Q. Xia, “Dynamics of the large-scale circulation in high-prandtl-number turbulent thermal convection”, *J. Fluid Mech.* **717**, 322–346 (2013).
- [258] D. Gubbins, B. Sreenivasan, J. Mound, and S. Rost, “Melting of the Earth’s inner core”, *Nature* **473**, 361–363 (2011).
- [259] R. Howard and B.-J. LaBonte, “The sun is observed to be a torsional oscillator with a period of 11 years”, *Astrophys. J.* **239**, L33–L36 (1980).
- [260] E. Brown and G. Ahlers, “Azimuthal asymmetries of the large-scale circulation in turbulent Rayleigh–Bénard convection”, *Phys. Fluids* **20**, 105105 (2008).
- [261] C. Sun, H.-D. Xi, and K.-Q. Xia, “Azimuthal symmetry, flow dynamics, and heat transport in turbulent thermal convection in a cylinder with an aspect ratio of 0.5”, *Phys. Rev. Lett.* **95**, 074502 (2005).
- [262] Y. Bao, J. Chen, B.-F. Liu, Z.-S. She, J. Zhang, and Q. Zhou, “Enhanced heat transport in partitioned thermal convection”, *J. Fluid Mech.* **784** (2015).

- [263] Q. Zhou, R. J. A. M. Stevens, K. Sugiyama, S. Grossmann, D. Lohse, and K.-Q. Xia, “Prandtl–Blasius temperature and velocity boundary-layer profiles in turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **664**, 297–312 (2010).
- [264] Q. Zhou, K. Sugiyama, R. J. A. M. Stevens, S. Grossmann, D. Lohse, and K.-Q. Xia, “Horizontal structures of velocity and temperature boundary layers in two-dimensional numerical turbulent Rayleigh–Bénard convection”, *Phys. Fluids* **23**, 125104 (2011).
- [265] O. Shishkina and C. Wagner, “Modelling the influence of wall roughness on heat transfer in thermal convection”, *J. Fluid Mech.* **686**, 568–582 (2011).
- [266] S. Toppaladoddi, S. Succi, and J. S. Wettlaufer, “Roughness as a route to the ultimate regime of thermal convection”, *Phys. Rev. Lett.* **118**, 074503 (2017).
- [267] X. Zhu, R. J. A. M. Stevens, R. Verzicco, and D. Lohse, “Roughness-facilitated local $1/2$ scaling does not imply the onset of the ultimate regime of thermal convection”, *Phys. Rev. Lett.* **119**, 154501 (2017).
- [268] B.-L. Xu, Q. Wang, Z.-H. Wan, R. Yan, and D.-J. Sun, “Heat transport enhancement and scaling law transition in two-dimensional Rayleigh–Bénard convection with rectangular-type roughness”, *Int. J. Heat Mass Transf.* **121**, 872–883 (2018).
- [269] Y. Zhang, Q. Zhou, and C. Sun, “Statistics of kinetic and thermal energy dissipation rates in two-dimensional turbulent Rayleigh–Bénard convection”, *J. Fluid Mech.* **814**, 165–184 (2017).
- [270] S.-X. Guo, S.-Q. Zhou, X.-R. Cen, L. Qu, Y.-Z. Lu, L. Sun, and X.-D. Shang, “The effect of cell tilting on turbulent thermal convection in a rectangular cell”, *J. Fluid Mech.* **762**, 273–287 (2015).
- [271] S.-X. Guo, S.-Q. Zhou, L. Qu, Y.-Z. Lu, L. Sun, and X.-D. Shang, “Evolution and statistics of thermal plumes in tilted turbulent convection”, *Int. J. Heat Mass Transfer* **111**, 933–942 (2017).
- [272] J. Bailon-Cuba, M. S. Emran, and J. Schumacher, “Aspect ratio dependence of heat transfer and large-scale flow in turbulent convection”, *J. Fluid Mech.* **655**, 152–173 (2010).
- [273] S. Weiss and G. Ahlers, “Turbulent Rayleigh–Bénard convection in a cylindrical container with aspect ratio $\Gamma = 0.50$ and Prandtl number $Pr = 4.38$ ”, *J. Fluid Mech.* **676**, 5–40 (2011).
- [274] K.-Q. Xia, “How heat transfer efficiencies in turbulent thermal convection depend on internal flow modes”, *J. Fluid Mech.* **676**, 1–4 (2011).
- [275] J. P. Valet, A. Fournier, V. Courtillot, and E. Herrero-Bervera, “Dynamical similarity of geomagnetic field reversals”, *Nature* **490**, 89–93 (2012).

- [276] J. Schumacher, P. Götzfried, and J. D. Scheel, “Enhanced enstrophy generation for turbulent convection in low-Prandtl-number fluids”, *Proc. N. Acad. Sci.* **112**, 9530–9535 (2015).
- [277] J. D. Scheel and J. Schumacher, “Global and local statistics in turbulent convection at low Prandtl numbers”, *J. Fluid Mech.* **802**, 147–173 (2016).
- [278] C. S. Ng, A. Ooi, D. Lohse, and D. Chung, “Changes in the boundary-layer structure at the edge of the ultimate regime in vertical natural convection”, *J. Fluid Mech.* **825**, 550–572 (2017).
- [279] P. Frick, R. Khalilov, I. Kolesnichenko, A. Mamykin, V. Pakholkov, A. Pavlinov, and S. Rogozhkin, “Turbulent convective heat transfer in a long cylinder with liquid sodium”, *Eur. Phys. Lett.* **109**, 14002 (2015).
- [280] I. V. Kolesnichenko, A. D. Mamykin, A. M. Pavlinov, V. V. Pakholkov, S. A. Rogozhkin, P. G. Frick, R. I. Khalilov, and S. F. Shepelev, “Experimental study on free convection of sodium in a long cylinder”, *Therm. Engng* **62**, 414–422 (2015).
- [281] A. Y. Vasil’ev, I. V. Kolesnichenko, A. D. Mamykin, P. G. Frick, R. I. Khalilov, S. A. Rogozhkin, and V. V. Pakholkov, “Turbulent convective heat transfer in an inclined tube filled with sodium”, *Tech. Phys.* **60**, 1305–1309 (2015).
- [282] R. Khalilov, I. Kolesnichenko, A. Pavlinov, A. Mamykin, A. Shestakov, and P. Frick, “Thermal convection of liquid sodium in inclined cylinders”, *Phys. Rev. Fluids* **3**, 043503 (2018).
- [283] A. Teimurazov and P. Frick, “Thermal convection of liquid metal in a long inclined cylinder”, *Phys. Rev. Fluids* **2**, 113501 (2017).
- [284] S. Ciliberto, S. Cioni, and C. Laroche, “Large-scale flow properties of turbulent thermal convection”, *Phys. Rev. E* **54**, R5901 (1996).
- [285] S. Cioni, S. Ciliberto, and J. Sommeria, “Strongly turbulent Rayleigh–Bénard convection in mercury: comparison with results at moderate Prandtl number”, *J. Fluid Mech.* **335**, 111–140 (1997).
- [286] F. Chillà, M. Rastello, S. Chaumat, and B. Castaing, “Long relaxation times and tilt sensitivity in Rayleigh Bénard turbulence”, *Eur. Phys. J. B.* **40**, 223–227 (2004).
- [287] G. Ahlers, E. Brown, and A. Nikolaenko, “The search for slow transients, and the effect of imperfect vertical alignment, in turbulent Rayleigh–Bénard convection thermal boundary layer profiles”, *J. Fluid Mech.* **557**, 347–367 (2006).
- [288] H.-C. Jiang, X. Zhu, V. Mathai, R. Verzicco, D. Lohse, and C. Sun, “Controlling heat transport and flow structures in thermal turbulence using ratchet surfaces”, *Phys. Rev. Lett.* **120**, 044501 (2018).

- [289] M. Lappa and T. Gradinscak, “On the oscillatory modes of compressible thermal convection in inclined differentially heated cavities”, *Int. J. Heat Mass Transf.* **121**, 412–436 (2018).
- [290] L. Zwirner and O. Shishkina, “Confined inclined thermal convection in low-prandtl-number fluids”, *J. Fluid Mech.* **850**, 984–1008 (2018).
- [291] L. Jiang, C. Sun, and E. Calzavarini, “Robustness of heat transfer in confined inclined convection at high prandtl number”, *Phys. Rev. E* **99**, 013108 (2019).
- [292] X. Chavanne, F. Chilla, B. Castaing, B. Hebral, B. Chabaud, and J. Chaussy, “Observation of the ultimate regime in Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **79**, 3648 (1997).
- [293] X. He, D. Funfschilling, H. Nobach, E. Bodenschatz, and G. Ahlers, “Transition to the ultimate state of turbulent Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **108**, 024502 (2012).
- [294] R. Verzicco and R. Camussi, “Numerical experiments on strongly turbulent thermal convection in a slender cylindrical cell”, *J. Fluid Mech.* **477**, 19–49 (2003).
- [295] G. Amati, K. Koal, F. Massaioli, K. Sreenivasan, and R. Verzicco, “Turbulent thermal convection at high Rayleigh numbers for a Boussinesq fluid of constant Prandtl number”, *Phys. Fluids* **17**, 121701 (2005).
- [296] R. J. Stevens, R. Verzicco, and D. Lohse, “Radial boundary layer structure and Nusselt number in Rayleigh–Bénard convection”, *J. Fluid Mech.* **643**, 495–507 (2010).
- [297] P.-E. Roche, F. Gauthier, R. Kaiser, and J. Salort, “On the triggering of the ultimate regime of convection”, *New J. of Phys.* **12**, 085014 (2010).
- [298] A. Mazzino, “Two-dimensional turbulent convection”, *Phys. Fluids* **29**, 111102 (2017).
- [299] S. Grossmann and D. Lohse, “Thermal convection in small prandtl number liquids: strong but ineffective”, in *AIP Conference Proceedings*, volume 1076, 68–75 (AIP) (2008).
- [300] A. de la Torre and J. Burguete, “Slow dynamics in a turbulent von kármán swirling flow”, *Phys. Rev. Lett.* **99**, 054101 (2007).
- [301] R. J. A. M. Stevens, J.-Q. Zhong, H. J. H. Clercx, G. Ahlers, and D. Lohse, “Transitions between turbulent states in rotating Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **103**, 024503 (2009).
- [302] S. Weiss, R. J. A. M. Stevens, J.-Q. Zhong, H. J. H. Clercx, D. Lohse, and G. Ahlers, “Finite-size effects lead to supercritical bifurcations in turbulent rotating Rayleigh–Bénard convection”, *Phys. Rev. Lett.* **105**, 224501 (2010).

- [303] P. Wei, S. Weiss, and G. Ahlers, “Multiple transitions in rotating turbulent Rayleigh-Bénard convection”, *Phys. Rev. Lett.* **114**, 114506 (2015).
- [304] B. Podvin and A. Sergent, “Proper orthogonal decomposition investigation of turbulent rayleigh-bénard convection in a rectangular cavity”, *Phys. Fluids* **24**, 105106 (2012).
- [305] K. Bai, D. Ji, and E. Brown, “Ability of a low-dimensional model to predict geometry-dependent dynamics of large-scale coherent structures in turbulence”, *Phys. Rev. E* **93**, 023117 (2016).
- [306] D. Giannakis, A. Kolchinskaya, D. Krasnov, and J. Schumacher, “Koopman analysis of the long-term evolution in a turbulent convection cell”, *J. Fluid Mech.* **847**, 735–767 (2018).
- [307] R. J. A. M. Stevens, Q. Zhou, S. Grossmann, R. Verzicco, K.-Q. Xia, and D. Lohse, “Thermal boundary layer profiles in turbulent Rayleigh-Bénard convection in a cylindrical sample”, *Phys. Rev. E* **85**, 027301 (2012).
- [308] S. Paolucci and D. R. Chenoweth, “Transition to chaos in a differentially heated vertical cavity”, *J. Fluid Mech.* **201**, 379–410 (1989).
- [309] R. Janssen and R. Henkes, “Influence of prandtl number on instability mechanisms and transition in a differentially heated square cavity”, *J. Fluid Mech.* **290**, 319–344 (1995).
- [310] M. Net and J. S. Umbría, “Periodic orbits in tall laterally heated rectangular cavities”, *Phys. Rev. E* **95**, 023102 (2017).
- [311] C. Ng, D. Chung, and A. Ooi, “Turbulent natural convection scaling in a vertical channel”, *Int. J. Heat Fluid Flow* **44**, 554–562 (2013).
- [312] G. de Vahl Davis and I. Jones, “Natural convection in a square cavity: a comparison exercise”, *Int. J. Numer. Methods Fluids* **3**, 227–248 (1983).
- [313] P. Le Quéré, “Accurate solutions to the square thermally driven cavity at high Rayleigh number”, *Comput. Fluids* **20**, 29–41 (1991).
- [314] H. Ding, C. Shu, K. Yeo, and D. Xu, “Development of least-square-based two-dimensional finite-difference schemes and their application to simulate natural convection in a cavity”, *Comput. Fluids* **33**, 137–154 (2004).
- [315] K. Szewc, J. Pozorski, and A. Taniere, “Modeling of natural convection with smoothed particle hydrodynamics: non-Boussinesq formulation”, *Int. J. Heat Mass Transf.* **54**, 4807–4816 (2011).
- [316] A. Xu, L. Shi, and T. Zhao, “Accelerated lattice Boltzmann simulation using GPU and OpenACC with data management”, *Int. J. Heat Mass Transf.* **109**, 577–588 (2017).

- [317] G. Batchelor, “Heat transfer by free convection across a closed cavity between vertical boundaries at different temperatures”, *Quart. Appl. Math.* **12**, 209–233 (1954).
- [318] J. Patterson, “On the existence of an oscillatory approach to steady natural convection in cavities”, *J. Heat Transfer* **106**, 104 (1984).
- [319] J. C. Patterson and S. W. Armfield, “Transient features of natural convection in a cavity”, *J. Fluid Mech.* **219**, 469–497 (1990).
- [320] D. Chenoweth and S. Paolucci, “Natural convection in an enclosed vertical air layer with large horizontal temperature differences”, *J. Fluid Mech.* **169**, 173–210 (1986).
- [321] S. Paolucci, “Direct numerical simulation of two-dimensional turbulent natural convection in an enclosed cavity”, *J. Fluid Mech.* **215**, 229–262 (1990).
- [322] F. Trias, M. Soria, A. Oliva, and C. Pérez-Segarra, “Direct numerical simulations of two- and three-dimensional turbulent natural convection flows in a differentially heated cavity of aspect ratio 4”, *J. Fluid Mech.* **586**, 259–293 (2007).
- [323] F. Trias, A. Gorobets, M. Soria, and A. Oliva, “Direct numerical simulation of a differentially heated cavity of aspect ratio 4 with Rayleigh numbers up to 10^{11} —Part I: Numerical methods and time-averaged flow”, *Int. J. Heat Mass Transf.* **53**, 665–673 (2010).
- [324] F. Trias, A. Gorobets, M. Soria, and A. Oliva, “Direct numerical simulation of a differentially heated cavity of aspect ratio 4 with Rayleigh numbers up to 10^{11} —Part II: Heat transfer and flow dynamics”, *Int. J. Heat Mass Transf.* **53**, 674–683 (2010).
- [325] H. Yu, N. Li, and R. E. Ecke, “Scaling in laminar natural convection in laterally heated cavities: Is turbulence essential in the classical scaling of heat transfer?”, *Phys. Rev. E* **76**, 026303 (2007).
- [326] P. Kiš and H. Herwig, “The near wall physics and wall functions for turbulent natural convection”, *Int. J. Heat Mass Transf.* **55**, 2625–2635 (2012).
- [327] F. Xu, “Convective instability of a vertical thermal boundary layer in a differentially heated cavity”, *Int. J. Heat Mass Transf.* **52**, 8–14 (2014).
- [328] F. Xu, J. C. Patterson, and C. Lei, “Transition to a periodic flow induced by a thin fin on the sidewall of a differentially heated cavity”, *Int. J. Heat Mass Transf.* **52**, 620–628 (2009).
- [329] F. Xu, J. C. Patterson, and C. Lei, “Transient natural convection flows around a thin fin on the sidewall of a differentially heated cavity”, *J. Fluid Mech.* **639**, 261–290 (2009).

- [330] F. Xu and S. C. Saha, “Transition to an unsteady flow induced by a fin on the sidewall of a differentially heated air-filled square cavity and heat transfer”, *Int. J. Heat Mass Transf.* **71**, 236–244 (2014).
- [331] J. Ma and F. Xu, “Dynamic and heat transfer of a $Pr < 1$ thermal flow around a fin on the thermal wall”, *Int. J. Heat Mass Transf.* **111**, 696–708 (2017).
- [332] J. Ma, B. Nie, and F. Xu, “Transient flows on an evenly heated wall with a fin”, *Int. J. Heat Mass Transf.* **118**, 235–246 (2018).
- [333] D. Chenoweth and S. Paolucci, “Gas flow in vertical slots with large horizontal temperature differences”, *Phys. Fluids* **28**, 2365–2374 (1985).
- [334] D. Chenoweth and S. Paolucci, “Gas flow in open vertical slots with large horizontal temperature differences and arbitrary external temperature”, *Phys. Fluids* **29**, 3187–3198 (1986).
- [335] S. A. Suslov and S. Paolucci, “Stability of natural convection flow in a tall vertical enclosure under non-Boussinesq conditions”, *Int. J. Heat Mass Transf.* **38**, 2143–2157 (1995).
- [336] H. Mlaouah, T. Tsuji, and Y. Nagano, “A study of non-boussinesq effect on transition of thermally induced flow in a square cavity”, *Int. J. Heat Fluid Flow* **18**, 100–106 (1997).
- [337] S. A. Suslov and S. Paolucci, “Nonlinear analysis of convection flow in a tall vertical enclosure under non-Boussinesq conditions”, *J. Fluid Mech.* **344**, 1–41 (1997).
- [338] D. Kizildag, I. Rodríguez, A. Oliva, and O. Lehmkuhl, “Limits of the Oberbeck–Boussinesq approximation in a tall differentially heated cavity filled with water”, *Int. J. Heat Mass Transf.* **68**, 489–499 (2014).
- [339] J. Armengol, F. Bannwart, J. Xamán, and R. Santos, “Effects of variable air properties on transient natural convection for large temperature differences”, *Int. J. Therm Sci* **120**, 63–79 (2017).
- [340] J. Schmalzl, M. Breuer, and U. Hansen, “On the validity of two-dimensional numerical approaches to time-dependent thermal convection”, *Eur. Phys. Lett.* **67**, 390 (2004).
- [341] F. Nicoud, “Conservative high-order finite-difference schemes for low-Mach number flows”, *J. Comput. Phys.* **158**, 71–97 (2000).
- [342] F. Arpino, N. Massarotti, and A. Mauro, “High Rayleigh number laminar-free convection in cavities: new benchmark solutions”, *Numer. Heat Transf., Part B* **58**, 73–97 (2010).
- [343] R. A. W. M. Henkes, “Natural-convection boundary layers”, PhD thesis University of Delft (1990).

- [344] E. Fonda, A. Pandey, J. Schumacher, and K. R. Sreenivasan, “Deep learning in turbulent convection networks”, *Proc. N. Acad. Sci.* **116**, 8667–8672 (2019).
- [345] A. Blass, P. Tabak, R. Verzicco, R. J. A. M. Stevens, and D. Lohse, “The effect of prandtl number on turbulent sheared thermal convection”, *arXiv:2007.02825* (2020).
- [346] A. Blass, R. Verzicco, D. Lohse, R. J. A. M. Stevens, and D. Krug, “Flow organization in laterally unconfined Rayleigh-Bénard turbulence”, *arXiv:2006.01505* (2020).
- [347] P. Berghout, W. J. Baars, and D. Krug, “The large-scale footprint in small-scale Rayleigh-Bénard turbulence”, *arXiv:2007.09994* (2020).
- [348] L.-A. Couston, D. Lecoanet, B. Favier, and M. Le Bars, “Shape and size of large-scale vortices: A generic fluid pattern in geophysical fluid dynamics”, *Phys. Rev. Res.* **2**, 023143 (2020).
- [349] L. Zwirner, R. Khalilov, I. Kolesnichenko, A. Mamykin, S. Mandrykin, A. Pavlinov, A. Shestakov, A. Teimurazov, P. Frick, and O. Shishkina, “The influence of the cell inclination on the heat transport and large-scale circulation in liquid metal convection”, *J. Fluid Mech.* **884** (2020).
- [350] G. Houseman, “The dependence of convection planform on mode of heating”, *Nature* **332**, 346–349 (1988).
- [351] D. Bercovici, G. Schubert, and G. A. Glatzmaier, “Three-dimensional spherical models of convection in the Earth’s mantle”, *Science* **244**, 950–955 (1989).
- [352] P. J. Tackley, D. J. Stevenson, G. A. Glatzmaier, and G. Schubert, “Effects of an endothermic phase transition at 670 km depth in a spherical model of convection in the Earth’s mantle”, *Nature* **361**, 699–704 (1993).
- [353] H.-P. Bunge, M. A. Richards, and J. R. Baumgardner, “Effect of depth-dependent viscosity on the planform of mantle convection”, *Nature* **379**, 436–438 (1996).
- [354] T. Lay, J. Hernlund, and B. A. Buffett, “Core–mantle boundary heat flow”, *Nature Geosci.* **1**, 25 (2008).
- [355] W. B. Moore and A. A. G. Webb, “Heat-pipe earth”, *Nature* **501**, 501–505 (2013).
- [356] C. Mallard, N. Coltice, M. Seton, R. D. Müller, and P. J. Tackley, “Subduction controls the distribution and fragmentation of earth’s tectonic plates”, *Nature* **535**, 140–143 (2016).
- [357] G. Schubert, D. L. Turcotte, and P. Olson, *Mantle convection in the Earth and planets* (Cambridge University Press) (2001).

- [358] D. J. Tritton and M. N. Zarraga, “Convection in horizontal layers with internal heat generation. experiments”, *J. Fluid Mech.* **30**, 21–31 (1967).
- [359] D. Tritton, “Internally heated convection in the atmosphere of venus and in the laboratory”, *Nature* **257**, 110–112 (1975).
- [360] W. B. McKinnon, F. Nimmo, T. Wong, P. M. Schenk, O. L. White, J. H. Roberts, J. M. Moore, J. R. Spencer, A. D. Howard, O. M. Umurhan, *et al.*, “Convection in a volatile nitrogen-ice-rich layer drives Pluto’s geological vigour”, *Nature* **534**, 82–85 (2016).
- [361] H. Kim, D. A. Boysen, J. M. Newhouse, B. L. Spatocco, B. Chung, P. J. Burke, D. J. Bradwell, K. Jiang, A. A. Tomaszowska, K. Wang, *et al.*, “Liquid metal batteries: past, present, and future”, *Chem. Rev.* **113**, 2075–2099 (2013).
- [362] L. Xiang and O. Zikanov, “Subcritical convection in an internally heated layer”, *Phys. Rev. Fluids* **2**, 063501 (2017).
- [363] T. Koyaguchi, M. A. Hallworth, H. E. Huppert, and R. S. J. Sparks, “Sedimentation of particles from a convecting fluid”, *Nature* **343**, 447–450 (1990).
- [364] R. C. Kerr, “Erosion of a stable density gradient by sedimentation-driven convection”, *Nature* **353**, 423–425 (1991).
- [365] R. S. Sparks, H. E. Huppert, T. Koyaguchi, and M. A. Hallworth, “Origin of modal and rhythmic igneous layering by sedimentation in a convecting magma chamber”, *Nature* **361**, 246–249 (1993).
- [366] B. Liu and J. Zhang, “Self-induced cyclic reorganization of free bodies through thermal convection”, *Phys. Rev. Lett.* **100**, 244501 (2008).
- [367] L. Jiang, E. Calzavarini, and C. Sun, “Rotation of anisotropic particles in Rayleigh–Bénard turbulence”, *J. Fluid Mech.* **901**, A8 (2020).
- [368] R. Lakkaraju, R. J. A. M. Stevens, P. Oresta, R. Verzicco, D. Lohse, and A. Prosperetti, “Heat transport in bubbling turbulent convection”, *Proc. N. Acad. Sci.* **110**, 9237–9242 (2013).
- [369] R. Lakkaraju, F. Toschi, and D. Lohse, “Bubbling reduces intermittency in turbulent thermal convection”, *J. Fluid Mech.* **745**, 1 (2014).
- [370] Z. Wang, V. Mathai, and C. Sun, “Self-sustained biphasic catalytic particle turbulence”, *Nat. Commun.* **10**, 1–7 (2019).

Summary

Turbulent thermal convection widely exists in many geophysical and astrophysical flows, and it also plays an important role in many industrial applications. In this thesis, we have studied turbulent thermal convection with three classic model problems, namely, Rayleigh-Bénard convection (Part I), tilted convection (Part II), and vertical convection (Part III).

In Part I we studied non-Oberbeck-Boussinesq (NOB) effects and multistability/metastability in turbulent RBC. In §2, we reported several universal properties in penetrative turbulent RBC of cold water near 4°C. We showed that the ratio of the top and bottom thermal boundary layer thickness ($F_\lambda = \lambda_t^\theta/\lambda_b^\theta$), the dimensionless center temperature θ_c , the normalized Nusselt number $Nu(\theta_m)/Nu(0)$, and the normalized Reynolds number $Re(\theta_m)/Re(0)$ all have universal relationships with the density inversion parameter θ_m , which seem to be independent of Ra . We then focused on NOB effects due to a large temperature difference in air (§3). The center temperature T_c is found to be increased compared to that of the OB case, and the shifts of T_c are strongly dependent on Ra and temperature differential ϵ . We also identified the insensitivity of Nu and Re towards NOB effects. Furthermore, the extended laminar boundary layer equations were developed based on the low-Mach-number Navier-Stokes equations, which qualitatively predicts the NOB effects on velocity profiles. §4 and §5 are devoted to multistability/metastability in 2D turbulent RBC with no-slip (§4) or free-slip (§5) boundary conditions. In §4, by making use of a generalized Friedrichs inequality, we showed that the elliptical shape of the rolls and viscous damping determine the mean convection roll size Γ_r -window for the realizable turbulent states. The theoretical results are in excellent agreement with our numerical finding $2/3 \leq \Gamma_r \leq 4/3$, where the lower threshold is approached for the larger Ra . We also showed that the theoretical approach to frame Γ_r also works for free-slip boundary conditions. In §5, we found that zonal flow, which was observed previously in 2D RBC and spanwise rotating 3D RBC, is only stable when the aspect ratio Γ is smaller than a critical value. With increasing Γ , we first find a second regime in which both zonal flow and different convection roll states can be statistically stable. For even larger Γ , in a third regime, only convection roll states are statistically stable and zonal flow is not sustained.

In Part II we investigated global flow organization and heat transport properties

in tilted convection. §6-§8 explored two-dimensional tilted convection over a wide parameter space spanned by (Ra, Pr, Γ, β) . In §6, we have interestingly found that for $\Gamma = 2$, flow reversals characterized by two main rolls are promoted by tilt, which is quite different from the $\Gamma = 1$ case where flow reversals are significantly suppressed. The mode decomposition analysis shows that the appearance of the intermediate single-roll mode is crucial for reversals, and flow reversals in a tilted cell with $\Gamma = 2$ can be viewed as a mode competition process between single-roll mode and horizontally adjacent double-roll mode. Similar phenomenon was also found in a $\Gamma = 1/2$ cell (§7), where we showed that for small Pr , flow reversals of the vertically stacked two-roll state can happen when the tilt angle is larger than a critical value. Tilted convection with much larger Γ was addressed in §8, where we reported multiple stable states for small β with $\Gamma \geq 2$. Moreover, flow mode transitions between different roll states were observed for large $\Gamma \geq 8$ when β is larger than a critical value. In §9, we studied 3D tilted convection with aspect ratio $\Gamma = 1/2$ both for OB cases and considering NOB effects. We found that the influence of NOB effects on the Nusselt number Nu , Reynolds number Re and central temperature T_c is different for different β ranges. NOB effects can either increase or decrease Nu , Re , and T_c when β is varied.

In part III we focused on vertical convection and we particularly investigated NOB effects in vertical convection in §10. We found that under NOB conditions, similar to RBC, the center temperature is enhanced in comparison with the OB case. We also demonstrated robustness of Nu and Re to NOB effects even for very large temperature difference up to 360K.

Summary (Dutch)

Turbulente thermische convectie komt op grote schaal voor in veel geofysische en astrofysische stromen, en het speelt ook een belangrijke rol in veel industriële toepassingen. In dit proefschrift hebben we turbulente thermische convectie bestudeerd met drie klassieke modelproblemen, namelijk Rayleigh-Bénard convectie (deel I), gekantelde convectie (deel II) en verticale convectie (deel III).

In deel I hebben we NOB-effecten en multistabiliteit/metastabiliteit in turbulente RBC bestudeerd. In §2 rapporteerden we verschillende universele eigenschappen in penetrerende turbulente RBC van koud water nabij 4°C. We toonden aan dat de verhouding van de bovenste en onderste thermische grenslaagdikte ($F_\lambda = \lambda_t^\theta / \lambda_b^\theta$), de dimensieloze middentemperatuur θ_c , het genormaliseerde Nusseltgetal $Nu(\theta_m)/Nu(0)$, en het genormaliseerde Reynoldsgetal $Re(\theta_m)/Re(0)$ allemaal universele relaties hebben met de dichtheidsinversieparameter θ_m , die onafhankelijk lijken te zijn van Ra . Vervolgens hebben we ons gericht op NOB-effecten veroorzaakt door een groot temperatuurverschil in de lucht (§3). De middelste temperatuur T_c blijkt verhoogd te zijn in vergelijking met die van het OB-geval, en de verschuivingen van T_c zijn sterk afhankelijk van Ra en temperatuurverschil ϵ . We hebben ook de ongevoeligheid van Nu en Re voor NOB-effecten vastgesteld. Bovendien werden de uitgebreide laminaire grenslaagvergelijkingen ontwikkeld op basis van de Navier-Stokes-vergelijkingen met een laag Mach-nummer, die kwalitatief de NOB-effecten op snelheidsprofielen voorspellen. §4 en §5 zijn gewijd aan multistabiliteit/metastabiliteit in 2D turbulente RBC met antislip- (§4) of free-slip (§5) randvoorwaarden. In §4 hebben we, door gebruik te maken van een gegeneraliseerde Friedrichs-ongelijkheid, aangetoond dat de elliptische vorm van de rollen en de viskeuze damping de gemiddelde convectie-rolmaat Γ_r -venster bepalen voor de realiseerbare turbulente toestanden. De theoretische resultaten zijn in uitstekende overeenstemming met onze numerieke bevinding $2/3 \leq \Gamma_r \leq 4/3$, waarbij de lagere drempel wordt benaderd voor de grotere Ra . We hebben ook laten zien dat de theoretische benadering van frame Γ_r ook werkt voor free-slip randvoorwaarden. In §5 ontdekten we dat zonale stroming, die eerder werd waargenomen in 2D RBC en spangewijs roterende 3D RBC, alleen stabiel is als de hoogte-breedte verhouding Γ kleiner is dan een kritische waarde. Met toenemende Γ vinden we eerst een tweede regime waarin zowel zonale stroming als verschillende convectieroltoestanden statistisch stabiel kunnen zijn. Voor nog grotere Γ , in een

derde regime, zijn alleen convectieroltoestanden statistisch stabiel en wordt de zonale stroming niet ondersteund.

In deel II hebben we de globale stromingsorganisatie en warmtetransporteigenschappen bij gekantelde convectie onderzocht. §6-§8 onderzocht tweedimensionale gekantelde convectie over een brede parameterruimte omspannen door (Ra, Pr, Γ, β) . In §6 hebben we op interessante wijze ontdekt dat voor $\Gamma = 2$ stroomomkeringen die worden gekenmerkt door twee hoofdrollen, worden bevorderd door kanteling, wat nogal verschilt van het geval $\Gamma = 1$ waarin stroomomkeringen significant worden onderdrukt. De analyse van de decompositie van de modus laat zien dat het uiterlijk van de tussenliggende modus met enkele rol cruciaal is voor omkeringen, en stroomomkeringen in een gekantelde cel met $\Gamma = 2$ kunnen worden gezien als een moduscompetitie tussen de modus met enkele rol en horizontaal aangrenzende dubbele -rolmodus. Een soortgelijk fenomeen werd ook gevonden in een $\Gamma = 1/2$ cel (§7), waar we aantoonen dat voor kleine Pr stroomomkeringen van de verticaal gestapelde toestand met twee rollen kunnen optreden wanneer de kantelhoek groter is dan een kritische waarde. Gekantelde convectie met veel grotere Γ werd behandeld in §8, waar we meerdere stabiele toestanden rapporteerden voor kleine β met $\Gamma \geq 2$. Bovendien werden stromingsmodusovergangen tussen verschillende roltoestanden waargenomen voor grote $\Gamma \geq 8$ wanneer β groter is dan een kritische waarde. In §9 hebben we gekantelde 3D-convectie met hoogte- breedteverhouding $\Gamma = 1/2$ onderzocht, zowel voor OB-gevallen als rekening houdend met NOB-effecten. We ontdekten dat de invloed van NOB-effecten op het Nusseltgetal Nu , Reynoldsgetal Re en centrale temperatuur T_c verschillend is voor verschillende β bereiken. NOB-effecten kunnen Nu , Re en T_c verhogen of verlagen wanneer β wordt gevarieerd.

In deel III concentreerden we ons op verticale convectie en onderzochten we in het bijzonder NOB-effecten bij verticale convectie in §10. We ontdekten dat onder NOB-omstandigheden, vergelijkbaar met RBC, de middelste temperatuur verhoogd is in vergelijking met de OB-casus. We hebben ook de robuustheid van Nu en Re tot NOB-effecten aangetoond, zelfs bij zeer grote temperatuurverschillen tot 360K.

Scientific output

Publications

2020

- **Q. Wang**, R. Verzicco, D. Lohse, and O. Shishkina, Multiple states in turbulent large-aspect ratio thermal convection: What determines the number of convection rolls? *Phys. Rev. Lett.*, 125, 07451 (2020) – see §4
- **Q. Wang**, K. L. Chong, R. J. A. M. Stevens, R. Verzicco, and D. Lohse, From zonal flow to convection rolls in Rayleigh-Bénard convection with free-slip plates, *J. Fluid Mech.*, 905, A21 (2020) – see §5
- R. Yang, K. L. Chong, **Q. Wang**, R. Verzicco, O. Shishkina, D. Lohse, Periodically driven thermal convection. *Phys. Rev. Lett.*, 125, 154502 (2020)
- K. L. Chong, R. Yang, **Q. Wang**, R. Verzicco, D. Lohse, Càfe Lattee: Spontaneous layer formation in laterally cooled double diffusive convection, *J. Fluid Mech.*, 900, R6 (2020)
- Z. H. Wan, **Q. Wang**, B. Wang, S. N. Xia, Q. Zhou, D. J. Sun, On non-Oberbeck-Boussinesq effects in Rayleigh-Bénard convection of air for large temperature differences, *J. Fluid Mech.*, 889, A10 (2020) – see §3
- **Q. Wang**, D. Lohse, O. Shishkina, Scaling in internally heated convection: a unifying theory. (under review)
- H. R. Liu, K. L. Chong, **Q. Wang**, C. S. Ng, R. Verzicco, D. Lohse, Two-layer thermally driven turbulence: Mechanisms for interface breakup. (under review)
- **Q. Wang**, H. R. Liu, R. Verzicco, O. Shishkina, and D. Lohse, Regime transitions in high-Rayleigh number vertical convection. (In preparation)

2019

- **Q. Wang**, Q. Zhou, Z. H. Wan, D. J. Sun, Penetrative turbulent Rayleigh-Bénard convection in two and three dimensions, *J. Fluid Mech.*, 870, 718-734 (2019). – see §2

- **Q. Wang**, Z. H. Wan, R. Yan, D. J. Sun, Flow organization and heat transfer in two-dimensional tilted convection with aspect ratio 0.5, *Phys. Fluids.*, 31, 025102 (2019). – see §7
- **Q. Wang**, S. N. Xia, R. Yan, D. J. Sun, Z. H. Wan, Non-Oberbeck-Boussinesq effects due to large temperature differences in a differentially heated square cavity filled with air, *Int. J. Heat Mass Trans.*, 128, 479-491 (2019). – see §10
- C. S. Sun, S. Liu, **Q. Wang**, Z. H. Wan, D. J. Sun, Bifurcations in penetrative Rayleigh-Bénard convection in a cylindrical container, *Appl. Math. Mech.* 40, 695-704 (2019).

2018

- **Q. Wang**, S. N. Xia, B. F. Wang, D. J. Sun, Q. Zhou, Z. H. Wan, Flow reversals in two-dimensional thermal convection in tilted cells, *J. Fluid Mech.*, 849, 355-372 (2018). – see §6
- **Q. Wang**, Z. H. Wan, R. Yan, D. J. Sun, Multiple states and heat transfer in two-dimensional tilted convection with large aspect ratios, *Phys. Rev. Fluids.*, 3, 113503 (2018). – see §8
- B. L. Xu, **Q. Wang**, Z. H. Wan, R. Yan, D. J. Sun, Heat transport enhancement and scaling law transition in two-dimensional Rayleigh-Bénard convection with rectangular-type roughness, *Int. J. Heat Mass Trans.*, 121, 872-883 (2018).

2017

- **Q. Wang**, B. L. Xu, S. N. Xia, Z. H. Wan, D. J. Sun, Thermal convection in a tilted rectangular cell with aspect ratio 0.5, *Chin. Phys. Lett.*, 34, 104401 (2017). – see §9

2016

- S. N. Xia, Z. H. Wan, S. Liu, **Q. Wang**, D. J. Sun. Flow reversals in Rayleigh-Bénard convection with non-Oberbeck-Boussinesq effects, *J. Fluid Mech.* 798, 628-642 (2016).

Acknowledgements

Almost two years have passed since I joined the Physics of Fluids group in November 2018. I still remember three years ago, my Chinese co-supervisor Zhenhua Wan introduced me to Detlef, and then I had a Skype interview with Detlef. Although I felt I behaved very badly during the interview because of my poor English and terrible presentation skills, Detlef still generously accepted me, for which I always have much gratitude. Detlef knew there is a double-PhD degree programme between the University of Science and Technology of China (USTC, my Chinese university) and the University of Twente, and he then suggested me to follow this programme. There is no doubt that I have benefitted tremendously from Detlef's suggestion and the good friendship between the two universities. Looking back to the past years of my PhD, numerous people have helped and supported me in the completion of this thesis. I apologize in advance if I have forgotten mentioning some names due to my bad memory.

First and foremost, I would like to thank my supervisors *Detlef Lohse* and *Roberto Verzicco*, and my main collaborator *Olga Shishkina*.

Dear *Detlef*, I cannot thank you enough for accepting me as a joint PhD student in your group, and more importantly, for introducing me with this wonderful double-PhD degree programme. I really admire your comprehensive and thorough knowledge on the field of fluid mechanics, endless passion for science, infinite patience towards students, high efficiency during work, friendliness and humour towards people, exceptional ability to manage such a large group, and consideration and skills to make students happy. I have learned a great deal from you through our various meetings, either in your office or online, many email exchanges, iPad annotated PDFs of drafts, and many other occasions. You have a great taste of physical problems, and I have benefitted a lot from you on choosing scientific questions to work on. Thank you also for initializing various collaborations with different people, which enormously helped me in the completion of several chapters in this thesis. I will always look up and learn from you, no matter where I will go and whatever I will work on in the future.

Dear *Roberto*, thank you for all the meetings and discussions over the years. You are really a great expert on numerics, every time we met problems on numerical details, you always gave us some insightful suggestions. The weekly video meeting

with you was both enlightening and encouraging for learning numerical methods and physics of fluids. Thank you for the valuable suggestions on my projects and careful revisions to my manuscripts, both of which significantly improved the quality of my work. I also appreciate your numerous useful comments during our meetings, which significantly helped me on interpreting my research results.

Dear *Olga*, you are a world-class scientist on the theory of turbulent thermal convection. I cannot thank you enough for your significant contribution to the theoretical part in our multiple turbulent states paper. I really benefitted a lot from our discussions in Göttingen and several online meetings, through which I indeed felt your endless passion towards science. It is unbelievable and enlightening that you still work so hard despite such remarkable achievements you have already obtained. Besides, you are also like a great supervisor to me. I really enjoy working with you on various aspects of internally heated convection, vertical convection, and may other projects. Hope we can have more collaborations in the future.

I would also like to thank my Chinese supervisor *Dejun Sun*, co-supervisor *Zhenhua Wan*, and my predecessor *Shuning Xia*.

Dear *Prof. Sun*, thank you for all your meticulous guidance and thought-provoking discussions over the years. I admire your tremendous knowledge on fluid mechanics, especially on the hydrodynamic instability. It has also been such a great experience to work as your teaching assistant for the bachelor course on fluid mechanics, through which I have also learned a lot. We also had many discussions and you taught me a lot on how to do research and choose physical problems. Moreover, thank you for giving me considerable trust and freedom during my research, so that I could work on several projects I was interested in. I am happy that most of our work finally got published.

Dear *Dr. Wan*, I am indebted to you for introducing me to Detlef. Your passion and dedication toward science have influenced me both academically and personally. In particular, you taught me to have a good taste of physical problems. Thanks a lot for your illuminating feedback on my various manuscripts. Every time when I sent you my initial version of a manuscript, I could always quickly get your responses. Your suggestions and comments are crucial to guarantee the quality of these works.

Dear *Dr. Xia*, you are the person who guided me to the exciting field of thermal convection. Thanks a lot for generously providing me with your lMn2d/3d code, which made several chapters in this thesis possible. Thanks for your guidance during my research for bachelor thesis. I am very happy that this thesis was awarded as “Excellent Bachelor Thesis in USTC”, which would be impossible without your help. I wish you all the best in the future.

I wish to express my gratitude to several collaborators over the years.

Dear *Richard*, thank you for all the constructive discussions about turbulent thermal convection. I have learned a lot from you over the past years. You are such a careful person that you could point out almost every tiny mistake in a manuscript or

a proposal I have sent to you. Thanks a lot for helping me carefully revise the zonal flow paper, through which I have learned how to present research results precisely and concisely. I also appreciate your outstanding ability to handel very large-scale simulations, which have significantly contributed to the community.

Dear *Steven*, we have interacted quite often over the years and I always feel so lucky to be one of your colleagues. We almost discuss every day, both scientifically and personally. Over the past years, we have exchanged lots of ideas, discussed about many interesting papers, worked together on various projects and proposals. I am happy that some of these collaborations have led to publications. You are truly a great friend, collaborator, and teacher. I really have learned a lot from you.

Dear *David*, you are a pioneer on the study of zonal flow in RB convection. Thank you for various critical and constructive suggestions on our zonal flow manuscript, it has been such a great pleasure to discuss with you on this topic. Thanks a lot for initializing the collaboration about “lifetime of zonal flow”, which I think is a really interesting topic to explore. Hope this collaboration can finally lead to a good publication.

Dear *Quan*, thanks a lot for your insightful suggestions on our penetrative convection work and careful revisions to several of our joint papers. You have much physical insight on RB turbulence, and I was inspired and encouraged a lot from various discussions with you.

Dear *Haoran*, I am so lucky to be one of your officemates. I deeply admire your extraordinary ability on numerical methods and coding. It is unbelievable that you could implement the phase-field method to the AFiD code within such a short time. Whenever I had numerical problems, I could always turn to you. It is also such a great pleasure to discuss with you on multiphase RB convection. Thanks for your various help and suggestions on our vertical convection project.

Dear *Rui*, we are from the same university, you did excellent work during your Master. I really enjoy our discussions on modulated RB convection, flow reversals, melting, and so on. I wish you all the best for your life of PhD, and I believe you will have more amazing research outputs in the coming years.

I have also been inspired from discussions with several professors/predecessors who work on thermal convection, and whose excellent work has greatly influenced my research. Millions of thanks go to *Prof. Chao Sun*, for lighting up my passion towards science, and for several illuminating conversations; to *Prof. Hengdong Xi*, for clarifying some of my questions on reorientations and flow reversals of large-scale circulation in RB convection; to *Dr. Yantao Yang*, for many illuminating discussions on different kinds of thermal convection; to *Dr. Xiaojue Zhu*, for addressing some of my questions on the use of the AFiD code during my initial research; to *Dr. Yichao Xie*, for some discussions on roughness in RB convection; and to *Dr. Bofu Wang*, for many insightful discussions on RB convection;

I would also like to extend my gratitude to all the committee members who took

their time to read the thesis and participated in the defense. Thanks to my highly learned opponents *S. Luding*, *T. H. van der Meer*, and *Z. Xia* for all the great questions during my defense.

Moreover, special thanks go to the kind help I received from *Joanita* and *Bas*. Dear *Joanita*, you have made such a great contribution to the group. Thanks for all kinds of generous help over the years which made my life easier. Dear *Bas*, thank you for all technical help on my computer and provision of enough hard disk to store my research data.

I owe my thanks to my other two officemates *Saeed* and *Nathan*. I am so lucky to have both of you as my officemates. Thank you both for your sharing of various experiences in China with me and all the kind help you have offered.

I would also like to thank my numerical colleagues with whom I have had group meetings every week. I have learned a lot from all of you. Dear *C. S. Ng*, I deeply admire not only your ability on numerical methods and coding, but also your physical insight into thermal convection and multiphase flows. You are a generous, easy-going man who is always ready to help others. I really benefitted a lot from various (non)-scientific discussions with you. Dear *Guiquan*, you have really done an excellent work on particle-laden turbulent flows. Thanks for your patient explanations on some general knowledge of particle-laden turbulence. I wish that you will have a wonderful journey in your research of turbulent spherical RB convection. Dear *Pieter*, you did an excellent job on numerical Taylor-Couette (TC) turbulence. I am always amazed at your productivity and efficiency. Working with you on the PRACE 20 proposal was a great experience. I have learned a lot from our discussions on RB and TC turbulence. Dear *Alexander*, your visualizations on sheared RB turbulence are incredible, thanks for your instructions on drawing nice plots. Moreover, thank you so much for your careful revisions of my thesis and many enlightening comments and suggestions. I really appreciate your profound knowledge of the English language. Dear *Yibo*, we keep a close friendship, and your help has made my life in Enschede much easier. Traveling to various countries with you has been and will always be a precious memory in my life. Dear *Martin A.*, you are really a friendly person. Talking with you was of great fun, and thanks for some help on the code during my initial research. Thanks also for translating my English summary into the Dutch one. Dear *Farzan*, thanks for sharing me with many stories about your country. Our time in Jülich during the parallel programming course was pleasant and unforgettable. Dear *Sreevanshu*, *Robert*, and *Naoki*, I wish you all the best in your following PhD life.

Further, I would like to thank several of my friends from PoF group for their interesting academic interactions, close friendship and tremendous help. Dear *Luo-qin*, your knowledge about fluid mechanics seems endless. I really have no idea how you could be so knowledgeable and expertized at many theoretical aspects of fluid

mechanics. You have taught me a lot about fluid mechanics and I hope I can learn more from you in the future. Thanks for several nice dinners your family prepared! Dear *Shuai*, you are a man always with smile, which always makes me comfortable. Thanks for all kinds of illuminating conversations we have had. Travelling with you to several different countries was extraordinarily interesting. I will treasure our delighted laughter during dinners, card games, travels, etc. Dear *Yaxing*, we are from the same university. You are a man always ready to help others. I still clearly remember your warm welcome on the first day I came to PoF. Thanks for diverse delicious dinners you prepared. Congratulations on having such great publications during your PhD!

I would like to express my gratitude to the other Chinese folks from the group. Dear *Xiaolai*, *Binglin*, *Xiaoyan*, *Yanshen*, *Jiaming*, various parties and dinners with all of you are unforgettable memories. I am so lucky to have all of you in my life during these years.

Moreover, my heartfelt thanks go to all members from the PoF group. Dear *Devaraj*, *Jacco*, *Michel*, *Alvaro*, *Sander*, *Dominik*, *Gert-Wim*, *Dennis*, *Youssef*, *Lijun*, *Charu*, *Pierre*, *Pallav*, *Mathieu*, *Walter*, *Pim*, *Marvin*, *Peter*, *Martin E.*, *Utkarsh*, *Yogesh*, *Minkush*, *Srinath*, *Yoan*, *Ricardo*, *Ali*, *Vatsal*, *Anja*, *Jessica*, and *Jelle*, thanks for all the seminars, meetings and many other things we have had together.

During my stay in Enschede I have met many Chinese friends, whose company made me never feel lonely. Dear *Danqi Li*, lovely *Sipei Liu*, *Lingna Yang*, *Xin Chen*, *Zhuoyun Kan*, *Miao Yu*, *Mijun Zou*, *Chao Gao*, *Zhicheng Yang*, and *Jin Han*, thanks for all the happy time we have had together.

I would like to extend my sincere thanks to many professors from the Department of Modern Mechanics of USTC, whose diverse courses have greatly broadened my knowledge on physics of fluids. Dear *Prof. Xiyun Lu*, *Prof. Jiming Yang*, *Prof. Xisheng Luo*, *Prof. Ding Hang*, *Prof. Nansheng Liu*, and *Prof. Peng Gao*, I really have learned a lot through your insightful lectures and discussions. Special thanks also go to *Prof. Rui Yan*, we interacted frequently since you joined USTC. I really enjoyed various (non)-scientific discussions with you.

I owe my thanks to the colleagues from my Chinese group: *Xingchen Zhang*, *Chuangchao Ye*, *Shuang Liu*, *Liang Wang*, *Lingke Ran*, *Bo Li*, *Bolun Xu*, *Chuanishi Sun*, *Jiandong Yang*, *Jinfei Xin*, *Penjunyi Zhang*, *Hongji Yan*, *Guangchun Xu*, *Yasen Hu*, *Ben Wang*, and *Miaomiao Liu*. Dear *Chuangchao*, thanks for all of your great contributions to the group, I was impressed a lot by your extraordinary abilities on developing high-order GPU-based code for compressible flows and managing the local clusters. Dear *Penjunyi*, thanks for interesting and insightful discussions over the years.

Furthermore, I would like to thank my three bachelor roommates: *Han Chen*, *Shen Chen*, *Huanhuan Ma*. We have fought together, and are now all in pursuit for the PhD degree. Your accompany makes me never feel lonely during my PhD journey.

I am particularly grateful to two members from Olga's group: *Xuan* and *Lucas*. Dear *Xuan* and *Lucas*, thanks for inviting me to your home for the wonderful dinner. I really enjoyed all kinds of conversations with both of you.

Last but definitely not least, I express my profound gratitude to my beloved family: my parents and my brother. This thesis would be impossible without your unreserved love and complete support all the time. I hope the finishing of this thesis can make you happy.

Qi
October 2020

About the author

Qi Wang was born on June 08th, 1994 in Gansu, China. He spent the first 15 years of his life in a small, beautiful and peaceful town Xihe, where he completed his primary and secondary education. After that, he moved to Lanzhou, where he finished his high school education in the High School Attached to Northwest Normal University in 2012. In the same year, he went to Hefei where he studied at the Department of Modern Mechanics in the University of Science and Technology of China (USTC). He obtained his Bachelor degree in June 2016 and continued as a Master student under the supervision of Prof. Dejun Sun and Dr. Zhenhua Wan in the same university. After one year, in September 2017, he began his PhD study in the same group. In November 2018, he joined the Physics of Fluids group as a joint PhD student of USTC and the University of Twente (UT). For this collaboration, he followed a double PhD degree agreement between the two universities which was signed in 2017. For his two years stay in PoF, he mainly worked with Prof. Detlef Lohse and Prof. Roberto Verzicco. He will go back to China after his PhD defense in the UT, and he is supposed to finish his Chinese defense in 2021.