Various identities for iterated integrals of a semimartingale

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Abstract. This paper derives several identities for iterated integrals of a semimartingale. They involve powers, brackets, exponential and the stochastic exponential. Some, like for counting or finite-variation processes, are apparently new. Others, like two of the three formulae for continuous semimartingales, are generalizations of well-known formulae.

1. Introduction and summary of results

This paper derives several formulae involving iterated integrals of a semimartingale.

To motivate the results, let $t \mapsto X_t$ be a smooth function of time $t \geq 0$. Then, $dX^n = nX^{n-1}dX$ for $n \in \mathbb{N}$. So, $X^n = n \int X^{n-1}dX$ if $X_0 = 0$. Applied to $m = n - 1$ in place of n, we get $X^{n-1} = (n-1) \int X^{n-2}dX$. Substituting gives $X^n = n(n-1) \int \int X^{n-2}dXdX$. Continuing in this manner, we see that $X^n = n!X^{\{n\}}$, where $X^{\{n\}} = \int \cdots \int dX \cdots dX$ with n-iterated integrals. This now implies $e^X = \sum_{n=0}^{\infty} X^{\{n\}}$, no doubt a long-known result.

The above formulae for X^n and e^X remain valid more generally (with Stieltjes integrals) if X is a continuous and finite variation function of $t \geq 0$, because again, $dX^n = nX^{n-1}dX$ (as measures on $[0, \infty)$). Hence, applied pathwise, they are also valid when X is a continuous process of finite variation on some probability space. Our aim is to investigate them (with the stochastic integral) when X is a semimartingale on some stochastic basis.

Let X be a semimartingale with $X_0 = 0$. For any integer $n \geq 0$, the n-th iterated integral $X^{\{n\}}$ of X is defined inductively by $X^{\{0\}} = 1$ and $X^{\{n+1\}} = \int X_-^{\{n\}}dX$. So, $X^{\{1\}} = X$, $X^{\{2\}} = \int X_-dX = \int \int^- dXdX$, etc. When X is a continuous, we show that for any $n \in \mathbb{N}$,

$$(1.1) \qquad X^n = \sum_{i=0}^{[\frac{n}{2}]} \frac{n!}{i!2^i}[X]^i X^{\{n-2i\}}.$$

(Here $[X] := [X, X]$ denotes the quadratic variation of X). Linearly inverting this equation to express for $X^{\{n\}}$ in terms of the powers X^j, yields readily the more familiar result

$$(1.2) \qquad X^{\{n\}} = \sum_{i=0}^{[\frac{n}{2}]} \frac{a_{n,i}}{n!} X^{n-2i}[X]^i,$$

where $a_{n,i}$ are the nonzero coefficients of the Hermite polynomials, meaning, $\sum_{i=0}^{[\frac{n}{2}]} a_{n,i} x^{n-2i}$ is the Hermite polynomial of order n. This is a well-known result of Itö (1951) when X is a Brownian motion. Previously, it was certainly also known when X equals $\int g dB$, with g deterministic and B a Brownian motion. We refer to Oertel(2003) for an alternative development in this case and its application to chaotic representation of martingales under Brownian filtrations. Either of the above two formulae leads easily to the formula

$$(1.3) \qquad \mathcal{E}(X) = \sum_{n=0}^{\infty} X^{\{n\}}$$

for the stochastic exponential $\mathcal{E}(X)$ of X (which here equals $e^{X-\frac{1}{2}[X]}$, as X is continuous), with the infinite sum absolutely convergent (except possibly on an evanescent set).

We prove that (1.3) holds if more generally X is the sum of a finite variation and a continuous semimartingale, and conjecture that it holds for all semimartingales X.

We show that the iterated integral of sum of two (general) semimartingales X and Y satisfying $[X, Y] = 0$ is given by

$$(1.4) \qquad (X + Y)^{\{n\}} = \sum_{i=0}^{n} X^{\{i\}} Y^{\{n-i\}}.$$

This easily implies that (1.3) holds for $X + Y$ if it holds for X and Y. From this and the continuous case, we deduce that (1.3) holds when X is the sum of a finite variation and a continuous semimartingale, by showing that (1.3) holds when X is the (absolutely convergent) sum of its jumps, i.e., $X = \sum_{s \leq t} \Delta X_s$. In this special case, we find that

$$(1.5) \qquad X_t^{\{n\}} = \sum_{i_1 < \cdots < i_n, \, i_j \in \mathbb{N}_m} \Delta X_{T_{i_1}} \cdots \Delta X_{T_{i_n}},$$

where $(T_i)_{i=1}^m$, $m \leq \infty$, is a sequence stopping times exhausting the jumps of X on $[0, t]$, and $\mathbb{N}_m$ equals $\mathbb{N}$ if $m = \infty$ and $\{1, \cdots, m\}$ otherwise. We further show $\sum_{n=0}^{\infty} |X^{\{n\}}| \leq \exp(\sum_{s \leq .} |\Delta X_s|)$, from which we conclude that $\sum_{n=0}^{\infty} X^{\{n\}}$ is absolutely convergent, and using (1.5), equals $\prod_{s \leq .}(1 + \Delta X_s)$, which in turn equals $\mathcal{E}(X)$ in this special case.

An interesting instance is the case of a Poisson process, or more generally any (counting) semimartingale N with $N_0 = 0$ satisfying $[N] = N$ (equivalently, N equals sum of its jumps, all of which equal 1), e.g., a Cox process. In this case, (1.5) simplifies to

$$(1.6) \qquad N^{\{n\}} = 1_{N \geq n} \binom{N}{n}.$$

This and the binomial theorem imply that again $\sum_{n=0}^{\infty} N^{\{n\}} = 2^N = \mathcal{E}(N)$. It also implies

$$(1.7) \qquad N^n = \sum_{i=1}^{n} c_{n,i} N^{\{i\}}, \quad c_{n,i} := \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^n.$$

The numbers $\frac{c_{n,i}}{i!}$ are the Stirling numbers of the second kind. We provide an alternative derivation of (1.7) directly by induction, and another one that follows immediately from an iterated integral formula for powers X^n of a general semimartingale X in Jamshidian (2005). Eq. (1.7) implies (1.6). It also yields $e^{aN} = \sum_{i=0}^{\infty} N^{\{i\}}(e^a - 1)^i$ for $a \in \mathbb{R}$.

2. ITERATED INTEGRALS OF CONTINUOUS SEMIMARTINGALES

Let X be a continuous semimartingale with $X_0 = 0$ and $n \in \mathbb{N}$. Define $X^{\{n\}}$ inductively by $X^{\{1\}} = X$ and $X^{\{n+1\}} = \int X^{\{n\}} dX$. So, $X^{\{2\}} = \int X dX = \int \int dX dX$, and $X^{\{n\}} = \int \cdots \int dX \cdots dX$ with n iterated integrals. We also set $X^{\{0\}} := 1$.

Let $[\frac{n}{2}]$ denote the integer part of $n/2$, i.e., $n/2$ if n is even, and $(n-1)/2$ if n is odd.

2.1. **Formula for integer powers X^n.** When X is continuous and of finite variation, as we saw, $X^n = n! X^{\{n\}}$. For general continuous semimartingales, our main result is

Theorem 2.1. *Let X be a continuous semimartingale with $X_0 = 0$, and $n \in \mathbb{N}$. Then*

$$X^n = \sum_{i=0}^{[\frac{n}{2}]} \frac{n!}{i! 2^i} [X]^i X^{\{n-2i\}}.$$

Proof. For $n = 1$ this is trivial and for $n = 2$ it follows since $X^2 = 2 \int X dX + [X]$. Assume $n \geq 3$. By Itô's formula (and continuity assumption),

$$X^n = n \int X^{n-1} dX + \frac{1}{2} n(n-1) \int X^{n-2} d[X].$$

Hence, substituting for X^{n-1} and X^{n-2} by induction, we get,

$$X^n = \sum_{i=0}^{[\frac{n-1}{2}]} \frac{n!}{i! 2^i} \int [X]^i X^{\{n-1-2i\}} dX + \sum_{i=0}^{[\frac{n-2}{2}]} \frac{n!}{i! 2^{i+1}} \int X^{\{n-2-2i\}} [X]^i d[X] =: \text{I} + \text{II}.$$

By definition $dX^{\{n-2i\}} = X^{\{n-1-2i\}} dX$. Substituting in the first sum I, we get

$$\text{I} = \sum_{i=0}^{[\frac{n-1}{2}]} \frac{n!}{i! 2^i} \int [X]^i dX^{\{n-2i\}}.$$

The summand corresponding to $i = 0$ above is just $n! X^{\{n\}}$. Moreover, we can replace $[\frac{n-1}{2}]$ in the upper sum limit by $[\frac{n}{2}]$, for if n is odd, these two are equal, and if n is even, then we are only adding a zero term, as $dX^{\{n-2i\}}$ equals zero for $i = \frac{n}{2}$. Hence

$$I = n!X^{\{n\}} + \sum_{i=1}^{[\frac{n}{2}]} \frac{n!}{i!2^i} \int [X]^i dX^{\{n-2i\}}.$$

Concentrating next on the second sum II , we get by shifting the dummy index i by 1,

$$II = \sum_{i=1}^{[\frac{n}{2}]} \frac{n!\, i}{i!2^i} \int X^{\{n-2i\}}[X]^{i-1}d[X].$$

But, using $d[X]^i = i[X]^{i-1}d[X]$, followed integration by parts, we have,

$$i \int X^{\{n-2i\}}[X]^{i-1}d[X] = \int X^{\{n-2i\}}d[X]^i$$

$$= X^{\{n-2i\}}[X]^i - \int [X]^i dX^{\{n-2i\}}.$$

Substituting in the sum II above, we get

$$II = \sum_{i=1}^{[\frac{n}{2}]} \frac{n!}{i!2^i}[X]^i X^{\{n-2i\}} - \sum_{i=1}^{[\frac{n}{2}]} \frac{n!}{i!2^i} \int [X]^i dX^{\{n-2i\}}.$$

Thus, adding I and II, the sum in RHS of I cancels the (negative) second sum in RHS of II above, resulting in $X^n = I + II = n!X^{\{n\}} + \sum_{i=1}^{[\frac{n}{2}]} \frac{n!}{i!2^i}[X]^i X^{\{n-2i\}}$, as required.   □

Note, the leading term $(i = 0)$ is $n!X^{\{n\}}$, while the last term $(i = [n/2])$ equals $\frac{n!}{i!2^{\frac{n}{2}}}[X]^{\frac{n}{2}}$ if n is even, and equals $\frac{n!}{i!2^{\frac{n-1}{2}}}[X]^{\frac{n-1}{2}}X$ if n is odd. The low powers are given by

$$X^2 = 2X^{\{2\}} + [X].$$

$$X^3 = 6X^{\{3\}} + 3[X]X.$$

$$X^4 = 24X^{\{4\}} + 12[X]X^{\{2\}} + 3[X]^2.$$

$$X^5 = 120X^{\{5\}} + 60[X]X^{\{3\}} + 15[X]^2X.$$

$$X^6 = 720X^{\{6\}} + 360[X]X^{\{4\}} + 90[X]^2X^{\{2\}} + 15[X]^3.$$

2.2. Formula for iterated integrals $X^{\{n\}}$ in terms of Hermite polynomials.

Inverting by hand the above linear system of five equations for $X^2 \cdots, X^6$, we find

$$2X^{\{2\}} = X^2 - [X].$$

$$6X^{\{3\}} = X^3 - 3[X]X.$$

$$24X^{\{4\}} = X^4 - 6[X]X^2 + 3[X]^2.$$

$$120X^{\{5\}} = X^5 - 10[X]X^3 + 15[X]^2X.$$

$$720X^{\{6\}} = X^6 - 15[X]X^4 + 45[X]^2X^2 - 15[X]^3.$$

We recognize the (modified) Hermite polynomials on the RHS (with $[X]$ replaced by 1).

In general, it is clear by a linear inversion that for each $n \in \mathbb{N}$ and integer $0 \leq i \leq [n/2]$, there exists a unique real number (in fact integer) $a_{n,i}$ with $a_{n,0} = 1$ such that

$$n!X^{\{n\}} = \sum_{i=0}^{[\frac{n}{2}]} a_{n,i} X^{n-2i}[X]^i.$$

Moreover, using the recursive relation for the coefficients of Hermite polynomials and induction, one can easily show by that $\sum_{i=0}^{[\frac{n}{2}]} a_{n,i} x^{n-2i}$ is the Hermite polynomial of order n, normalized with (leading) coefficient $a_{n,0}$ of x^n equal to 1. We conclude

Corollary 2.2. *Let X be a continuous semimartingale with $X_0 = 0$, and $n \in \mathbb{N}$. Let $\sum_{i=0}^{[\frac{n}{2}]} a_{n,i} x^{n-2i}$ be the Hermite polynomial of order n with $a_{n,0} = 1$. Then*

$$X^{\{n\}} = \frac{1}{n!} \sum_{i=0}^{[\frac{n}{2}]} a_{n,i} X^{n-2i}[X]^i.$$

2.3. Chaotic expansion of the stochastic exponential.

Substituting the above formula in a well-known property of Hermite polynomials, namely, the absolutely convergent series $\exp(x - y/2) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{i=0}^{[\frac{n}{2}]} a_{n,i} x^{n-2i} y^i$, $x, y \in \mathbb{R}$, yields $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{\{n\}}$.[1] We can also show this by substituting the formula for X^n in the (convergent) power series $e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!}$. By index manipulation into odd and even terms and an interchange of order of summation, we have pathwise,

$$e^X = \sum_{n=0}^{\infty} \frac{X^n}{n!} X^n = \sum_{n=0}^{\infty} \sum_{i=0}^{[\frac{n}{2}]} \frac{[X]^i}{i!2^i} X^{\{n-2i\}}$$

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[1]I thank Frank Oertel for suggesting this method to me.

$$= \sum_{m=0}^{\infty} \sum_{i=0}^{m} \frac{[X]^i}{i! 2^i} (X^{\{2m-2i\}} + X^{\{2m+1-2i\}})$$

$$= \sum_{i=0}^{\infty} \sum_{m=i}^{\infty} \frac{[X]^i}{i! 2^i} (X^{\{2m-2i\}} + X^{\{2m+1-2i\}})$$

$$= \sum_{i=0}^{\infty} \frac{[X]^i}{i! 2^i} \sum_{j=0}^{\infty} (X^{\{2j\}} + X^{\{2j+1\}}) = e^{\frac{1}{2}[X]} \sum_{n=0}^{\infty} X^{\{n\}}.$$

Either of the above two methods therefore yield

Corollary 2.3. *Let X be a continuous semimartingale with $X_0 = 0$, and $n \in \mathbb{N}$. Then $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{\{n\}}$, with the series absolutely convergent (outside an evanescent set).*

3. ITERATED INTEGRAL OF A SUM OF SEMIMARTINGALES

Let X be a (general) semimartingale X with $X_0 = 0$. The iterated integral $X^{\{n\}}$ are defined by induction similarly $X^{\{0\}} = 1$ and $X^{\{n+1\}} = \int X_-^{\{n\}} dX$.

Proposition 3.1. *Let X and Y be semimartingales satisfying $X_0 = Y_0 = 0$ and $[X, Y] = 0$. Then for all $n \in \mathbb{N}$,*

$$(X + Y)^{\{n\}} = \sum_{i=0}^{n} X^{\{i\}} Y^{\{n-i\}}.$$

Proof. We use induction, cases $n = 0$ and $n = 1$ being clear. Assume $n \geq 2$. We have, first using the definition of iterated integral and induction, next definition again and some index manipulations, then integration by parts using $[X, Y] = 0$ and recombining terms,

$$(X + Y)^{\{n\}} = \int (X + Y)^{\{n-1\}} dX + \int (X + Y)^{\{n-1\}} dY$$

$$= \sum_{i=0}^{n-1} \int X_-^{\{i\}} Y_-^{\{n-1-i\}} dX + \sum_{i=0}^{n-1} \int X_-^{\{i\}} Y_-^{\{n-1-i\}} dY$$

$$= \sum_{i=0}^{n-1} \int Y_-^{\{n-1-i\}} dX^{\{i+1\}} + \sum_{i=0}^{n-1} \int X_-^{\{i\}} dY^{\{n-i\}}$$

$$= \sum_{i=1}^{n} \int Y_-^{\{n-i\}} dX^{\{i\}} + \sum_{i=0}^{n-1} \int X_-^{\{i\}} dY^{\{n-i\}}$$

$$= X^{\{n\}} + \sum_{i=1}^{n-1} \int Y_-^{\{n-i\}} dX^{\{i\}} + \sum_{i=1}^{n-1} \int X_-^{\{i\}} dY^{\{n-i\}} + Y^{\{n\}}$$

$$= X^{\{n\}} + \sum_{i=1}^{n-1} X^{\{i\}} Y^{\{n-i\}} + Y^{\{n\}} = \sum_{i=0}^{n} X^{\{i\}} Y^{\{n-i\}}.$$

$\square$

Since the continuous part X^c and discontinuous part X^d of a semimartingale X have zero bracket, the proposition yields

Corollary 3.2. *Let X be a semimartingale with $X_0 = 0$ and $X = X^c + X^d$ be its continuous-discontinuous decomposition. Then, for all $n \in \mathbb{N}$,*

$$X^{\{n\}} = \sum_{i=0}^{n} X^{c\{i\}} X^{d\{n-i\}}.$$

We next use Proposition 3.1 to deduce the chaotic iterated integral representation of the stochastic integral of a sum from the corresponding property of each summand.

Lemma 3.3. *Let X and Y be semimartingales satisfying $X_0 = Y_0 = 0$ and $[X, Y] = 0$. Suppose $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{\{n\}}$ and $\mathcal{E}(Y) = \sum_{n=0}^{\infty} Y^{\{n\}}$ with both sums absolutely convergent. Then, $\mathcal{E}(X + Y) = \sum_{n=0}^{\infty} (X + Y)^{\{n\}}$, with the sum absolutely convergent.*

Proof. Using $[X, Y] = 0$ and the assumption on $\mathcal{E}(X)$ and $\mathcal{E}(Y)$, we have

$$\mathcal{E}(X + Y) = \mathcal{E}(X)\mathcal{E}(Y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} X^{\{n\}} Y^{\{m\}}$$

with the double sum is absolutely convergent. Hence, we can rearrange the order of double summation, to get

$$\mathcal{E}(X + Y) = \sum_{i=0}^{\infty} \sum_{n=0}^{i} X^{\{n\}} Y^{\{i-n\}} = \sum_{i=0}^{\infty} (X + Y)^{\{i\}},$$

with the last equality following from Proposition 3.1 $\qquad\square$

4. Iterated integrals of a finite-variation semimartingale

Proposition 4.1. *Let X be a finite variation semimartingale satisfying $X = \sum_{s \leq \cdot} \Delta X_s$. Let $t > 0$. Let stopping times $(t_i)_{i=1}^{m}$, $m \leq \infty$, be any enumeration of jumps of X on $[0, t]$. Set (along each path), $\mathbb{N}_m = \mathbb{N}$ if $m = \infty$ and $\{1, \cdots, m\}$ otherwise. Then, for $n \in \mathbb{N}$,*

$$X_t^{\{n\}} = \sum_{i_1 < \cdots < i_n, \, i_j \in \mathbb{N}_m} \Delta X_{t_{i_1}} \cdots \Delta X_{t_{i_n}}.$$

Moreover, we have, $\sum_{n=0}^{\infty} |X^{\{n\}}| \leq \exp(\sum_{s \leq \cdot} |\Delta X_s|) < \infty$, and

$$\mathcal{E}(X) = \prod_{s \leq \cdot} (1 + \Delta X_s) = \sum_{n=0}^{\infty} X^{\{n\}}.$$

Proof. Let $\mathbb{N}_{m,n} := \{(i_1, \cdots, i_n) \in \mathbb{N}_m^n : i_j \neq i_k \text{ if } j \neq k\}$. Then,

$$\sum_{i_1 < \cdots < i_n, \, i_j \in \mathbb{N}_m} \Delta X_{t_{i_1}} \cdots \Delta X_{t_{i_n}} = \frac{1}{n!} \sum_{(i_1, \cdots, i_n) \in \mathbb{N}_{m,n}} \Delta X_{t_{i_1}} \cdots \Delta X_{t_{i_n}}.$$

This is because if σ is any permutation of $\{1, \cdots, n\}$, then, by commutativity of product, $\Delta X_{t_{i_1}} \cdots \Delta X_{t_{i_n}} = \Delta X_{t_{i_{\sigma(1)}}} \cdots \Delta X_{t_{i_{\sigma(n)}}}$, and moreover, given $(i_1, \cdots, i_n) \in \mathbb{N}_{m,n}$, there exists a unique permutation σ such that $i_{\sigma(1)} < \cdots < i_{\sigma(n)}$. Next, using the definition of iterated integral, we have $\Delta X^{\{n\}} = X_-^{\{n-1\}} \Delta X$. Since X is the sum of its jump, this implies by induction that so is $X^{\{n\}}$ and

$$X_t^{\{n\}} = \sum_{t_n \leq t} \sum_{t_{n-1} < t_n} \cdots \sum_{t_2 < t_1} \Delta X_{t_1} \cdots \Delta X_{t_{n-1}} \Delta X_{t_n}.$$

Using commutativity and the fact that because if $(i_1, \cdots, i_n) \in \mathbb{N}_{m,n}$, then there exist a unique permutation σ such that $t_{i_{\sigma(1)}} < \cdots < t_{i_{\sigma(n)}}$, we see that the right hand side equals $\frac{1}{n!} \sum_{(i_1, \cdots, i_n) \in \mathbb{N}_{m,n}}^\infty \Delta X_{t_{i_1}} \cdots \Delta X_{t_{i_n}}$. Combined with the earlier formula, this yields the first statement. Moreover, it also shows

$$|X_t^{\{n\}}| = \frac{1}{n!} \Big| \sum_{(i_1, \cdots, i_n) \in \mathbb{N}_{m,n}}^\infty \Delta X_{t_{i_1}} \cdots \Delta X_{t_{i_n}} \Big|$$

$$\leq \frac{1}{n!} \sum_{(i_1, \cdots, i_n) \in \mathbb{N}_{m,n}}^\infty |\Delta X_{t_{i_1}}| \cdots |\Delta X_{t_{i_n}}|$$

$$\leq \frac{1}{n!} \sum_{(i_1, \cdots, i_n) \in \mathbb{N}_m^n}^\infty |\Delta X_{t_{i_1}}| \cdots |\Delta X_{t_{i_n}}|$$

$$= \frac{1}{n!} \prod_{j=1}^n \sum_{t_j=1}^\infty |\Delta X_{t_{i_j}}| = \frac{1}{n!} \Big(\sum_{s \leq t} |\Delta X_s| \Big)^n.$$

Therefore,

$$\sum_{n=0}^\infty |X^{\{n\}}| \leq \sum_{n=0}^\infty \frac{1}{n!} \Big(\sum_{s \leq \cdot} |\Delta X_s| \Big)^n = \exp\Big(\sum_{s \leq \cdot} |\Delta X_s| \Big),$$

which is finite since $\sum_{s \leq \cdot} |\Delta X_s| < \infty$, as X is of finite variation. We further have

$$\mathcal{E}(X)_t = \prod_{s \leq t} (1 + \Delta X_s) = \prod_{i=1}^\infty (1 + \Delta X_{t_i})$$

$$= 1 + \sum_{n=1}^\infty \sum_{i_1 < \cdots < i_n, \, i_j \in \mathbb{N}_m} \Delta X_{t_{i_1}} \cdots \Delta X_{t_{i_n}} = \sum_{n=0}^\infty X_t^{\{n\}},$$

where the last equality follows by the first statement. $\square$

Combining with Corollary 2.3 and Lemma 3.3 leads to our second main result.

Theorem 4.2. *Let X be a sum of a continuous semimartingale and a finite-variation semimartingale, and $X_0 = 0$. Then, $\mathcal{E}(X) = \sum_{n=0}^\infty X^{\{n\}}$, with the sum absolutely convergent.*

Proof. By Lemma 3.3 and Corollary 2.3, it suffices to prove this when X is of finite variation. Set $Y := \sum_{s \leq} \Delta X_s$ and $A := X - Y$. Note, $\Delta Y = \Delta X$, so Y equals the sum of its jumps. Therefore, by Proposition 4.1, we have $\mathcal{E}(Y) = \sum_{n=0}^{\infty} Y^{\{n\}}$. We also have $\mathcal{E}(A) = \sum_{n=0}^{\infty} A^{\{n\}}$ because A is a continuous (and finite variation) semimartingale. Since $[A, Y] = 0$, using Lemma 3.3 again, it follows that $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{\{n\}}$, as desired. $\qquad\square$

We conjecture that $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{\{n\}}$ for all purely discontinuous semimartingales X of infinite variation with $X_0 = 0$. If true, the conjecture would imply in view of the above theorem and Lemma 3.3 that $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{\{n\}}$ for all semimartingales X with $X_0 = 0$.

The following heuristic argument lends credence to the conjecture:

$$d \sum_{n=0}^{\infty} X^{\{n\}} = \sum_{n=1}^{\infty} dX^{\{n\}} = \sum_{n=1}^{\infty} X_-^{\{n-1\}} dX = \sum_{n=0}^{\infty} X_-^{\{n\}} dX.$$

This and uniqueness of solution of SDE $d\mathcal{E}(X) = \mathcal{E}(X)_- dX$ indicate $\mathcal{E}(X) = \sum_{n=0}^{\infty} X^{\{n\}}$.

5. Special case of counting semimartingales

We call a semimartingale N with $N_0 = 0$ a *counting semimartingale* if $[N] = N$, or equivalently, N is the sum of its jumps all which equal 1, implying N is piecewise constant, increasing, and integer valued. Examples are Poisson processes, or more generally, Cox processes. Proposition 4.1 simplifies nicely for counting semimartingales.

Proposition 5.1. *Let N be a counting semimartingale. Then, for $n \in \mathbb{N}$, we have*

$$(5.1) \qquad\qquad N^{\{n\}} = 1_{N \geq n} \binom{N}{n}.$$

Moreover, $\mathcal{E}(aN) = (1+a)^N = \sum_{i=0}^{\infty} a^i N^{\{i\}}$ for any real number a. In particular,

$$\mathcal{E}(N) = 2^N = \sum_{i=0}^{\infty} N^{\{i\}}; \qquad \mathcal{E}(-N) = 1_{\{N=0\}} = \sum_{i=0}^{\infty} (-1)^i N^{\{i\}}.$$

Proof. Let T_i denote the i-th jump time of N. Let $t > 0$. Then $(T_i)_{i=1}^{N_t}$, is an enumeration of jumps of N on $[0, t]$. By Proposition 4.1, we have $N_t^{\{n\}} = \sum_{1 \leq i_1 < \cdots < i_n \leq N_t} \Delta N_{T_{i_1}} \cdots \Delta N_{T_{i_n}}$. If $N_t < n$, then the sum is taken over the empty set, and is zero. Assume $N_t \geq n$. As all jumps equal 1, we get $N_t^{\{n\}} = \sum_{1 \leq i_1 < \cdots < i_n \leq N_t}$. But, this sum clearly equals $\binom{N_t}{n}$, for the number of indices $1 \leq i_1 < \cdots < i_n \leq N_t$ equals the number of subsets of $\{1, \cdots, N_t\}$ with n elements. The second statement follows from the second statement of Proposition 4.1 applied to $X = aN$, because, as jumps of N are 1, we have $\prod_{s \leq .} (1 + \Delta X_s) = (1+a)^N$. $\quad\square$

Eq. (5.1) can be inverted to express powers of N in terms of $N^{\{i\}}$. For this purpose we set

$$c_{n,i} := \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^n, \qquad n, i = 0, 1, 2 \cdots.$$

The numbers $\frac{c_{n,i}}{i!}$ are the Stirling numbers of the second kind.[2] (In particular, there are natural numbers if $n \geq i$ and zero otherwise, except when $n = i = 0$, as $c_{0,0} = 0^0 := 1$.)

Proposition 5.2. *Let N be a counting semimartingale. Then, for $n \in \mathbb{N}$, we have*

$$(5.2) \qquad\qquad N^n = \sum_{i=1}^{n} c_{n,i} N^{\{i\}}.$$

Proof. By a well-known property of Stirling numbers, given $N, N_1, N_2, \cdots \in \mathbb{N}$, one has $N^n = \sum_{i=1}^{n} c_{n,i} N_i$ for all n if and only if $N_n = 1_{N \geq n} \binom{N}{n}$ for all n. Thus, Eq. (5.1) implies (5.2) (and vice versa). $\qquad\square$

As a consequence of (5.1) and simple property of Sterling numbers we obtain

Corollary 5.3. *Let N be a counting semimartingale and a be a real number. Then,*

$$(5.3) \qquad\qquad e^{aN} = \sum_{i=0}^{\infty} N^{\{i\}} (e^a - 1)^i.$$

Proof. Using (5.1), we have

$$e^{aN} = 1 + \sum_{n=1}^{\infty} \frac{a^n}{n!} N^n = 1 + \sum_{n=1}^{\infty} \sum_{i=1}^{n} \frac{a^n}{n!} c_{n,i} X^{\{i\}}$$

$$= 1 + \sum_{i=1}^{\infty} N^{\{i\}} \sum_{n=i}^{\infty} \frac{a^n}{n!} c_{n,i} = \sum_{i=0}^{\infty} N^{\{i\}} (e^a - 1)^i.$$

In the last equality we used the identity $\sum_{n=i}^{\infty} \frac{a^n}{n!} c_{n,i} = (e^a - 1)^i$ for $i \in \mathbb{N}$. This identity follows by using the fact that $c_{n,i} = 0$ for $n < i$, and (obviously) $c_{j,0} = c_{0,j} = 0$ for $j \geq 1$ while $c_{0,0} = 1$ (as usual 0^0 is set to 1), and the following calculation

$$\sum_{n=i}^{\infty} \frac{a^n}{n!} c_{n,i} = \sum_{n=0}^{\infty} \frac{a^n}{n!} c_{n,i} = \sum_{n=0}^{\infty} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} \frac{(aj)^n}{n!}$$

$$= \sum_{j=0}^{i} \sum_{n=0}^{\infty} (-1)^{i-j} \binom{i}{j} \frac{(aj)^n}{n!} = \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} e^{aj} = (e^a - 1)^i.$$

$$\square$$

Note, setting $a = \log(2)$ in (5.3) yields $2^N = \sum_{i=0}^{\infty} N^{\{i\}}$, as before.

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[2]We refer to the online-encyclopedia Wikipedia, for both facts, and the other properties of Stirling numbers (and by extension of $c_{n,i}$) which are used in what follows.

5.1. **Alternative direct derivations.** It is instructive to give two alternative derivations of Eq. (5.2) (and by implication of Eq. (5.1)) which do not use Proposition 4.1.

The first derivation is directly by induction. Using Ito's formula, one easily gets

$$N^n = \sum_{i=0}^{n-1} \binom{n}{i} \int N_-^i dN.$$

Hence, substituting on the right hand side for N_-^i using induction, we get

$$N^n = N + \sum_{i=1}^{n-1} \binom{n}{i} \sum_{j=1}^{i} c_{i,j} \int N_-^{\{j\}} dN.$$

$$= N + \sum_{i=1}^{n-1} \binom{n}{i} \sum_{j=1}^{i} c_{i,j} N^{\{j+1\}}$$

$$= N + \sum_{j=1}^{n-1} N^{\{j+1\}} \sum_{i=j}^{n-1} \binom{n}{i} c_{i,j}$$

$$= N + \sum_{j=2}^{n} N^{\{j\}} \sum_{i=j-1}^{n-1} \binom{n}{i} c_{i,j-1}$$

$$= \sum_{j=1}^{n} N^{\{j\}} c_{n,j},$$

where, in the last step we used the easily verified fact that $\sum_{i=j-1}^{n-1} \binom{n}{i} c_{i,j-1} = c_{n,j}$.

Our next derivation uses the following identity from Jamshidian (2005) for any semimartingale X with $X_0 = 0$. For $n \in \mathbb{N}$, we have

$$X^n = \sum_{p=1}^{n} \sum_{I \in \mathbb{N}^{p,n}} \frac{n!}{i_1! \cdots i_p!} \int \int^- \cdots \int^- d[X]^{(i_1)} \cdots d[X]^{(i_{p-1})} d[X]^{(i_p)}.$$

Above, for all integers $1 \le p \le n$, we have set

$$\mathbb{N}^{p,n} := \{I = (i_1, \cdots, i_p) \in \mathbb{N}^p : i_1 + \cdots + i_p = n\},$$

and $[X]^{(n)}$ is defined inductively by $[X]^{(1)} = X$ and $[X]^{(n+1)} = [X, [X]^{(n)}]$. So, $[X]^{(2)} = [X]$ and $[X]_t^{(n)} = \sum_{s \le t} (\Delta X_s)^n$ for $n \ge 3$. For a counting process N, we clearly have $[N]^{(n)} = N$ for all $n \in \mathbb{N}$. Hence in this case, we get

$$N^n = \sum_{p=1}^{n} \sum_{I \in \mathbb{N}^{p,n}} \frac{n!}{i_1! \cdots i_p!} N^{\{i\}} = \sum_{i=1}^{n} c_{n,i} N^{\{i\}},$$

as desired, where we used the readily verified identity that $c_{n,p} = \sum_{I \in \mathbb{N}^{p,n}} \frac{n!}{i_1! \cdots i_p!}$.

References

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[2] Jamshidian, F.: Chaotic expansion of powers and martingales representation, working paper (2005)

[3] Oertel, F.: A Hilbert space approach to Wiener Chaos Decomposition and applications to finance, working paper (2003).

[4] Wikipedia, online encyclopedia, http://en.wikipedia.org/wiki/Stirling_number.