A General Framework for Coloring Problems: Old Results, New Results, and Open Problems

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Abstract. In this survey paper we present a general framework for coloring problems that was introduced in a joint paper which the author presented at WG2003. We show how a number of different types of coloring problems, most of which have been motivated from frequency assignment, fit into this framework. We give a survey of the existing results, mainly based on and strongly biased by joint work of the author with several different groups of coauthors, include some new results, and discuss several open problems for each of the variants.

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1 General Introduction

In the application area of frequency assignment graphs are used to model the topology and mutual interference between transmitters (receivers, base stations): the vertices of the graph represent the transmitters; two vertices are adjacent in the graph if the corresponding transmitters are so close (or so strong) that they are likely to interfere if they broadcast on the same or 'similar' frequency channels. The problem in practice is to assign a limited number of frequency channels in an economical way to the transmitters in such a way that interference is kept at an 'acceptable level'. This has led to various different types of coloring problems in graphs, depending on different ways to model the level of interference, the notion of similar frequency channels, and the definition of acceptable level of interference (See e.g. [25],[32]).

In [10] an attempt was made to capture a number of different coloring problems in a unifying model. This general framework that we will consider here too is as follows:

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> Given two graphs G_1 and G_2 with the property that G_1 is a (spanning) subgraph of G_2, one considers the following type of coloring problems: Determine a coloring of (G_1 and) G_2 that satisfies certain restrictions of type 1 in G_1, and restrictions of type 2 in G_2, using a limited number of colors.

Many known coloring problems related to frequency assignment fit into this general framework. We will discuss the following types of problems.

1.1 Distant-2 Coloring

First of all suppose that $G_2 = G_1^2$, i.e. G_2 is obtained from G_1 by adding edges between all pairs of vertices that are at distance 2 in G_1. If one just asks for a proper vertex coloring of G_2 (and G_1), this is known as the distant-2 coloring problem. Much of the research has been concentrated on the case that G_1 is a planar graph, and on obtaining good upper bounds in terms of the maximum degree of G_1 for the minimum number of colors needed in this case. In Section 2 we will survey some of the existing results, and discuss the proof techniques and open problems in this subarea.

1.2 Radio Coloring

In some versions of the previous problem one puts the additional restriction on G_1 that the colors should be sufficiently separated. One way to model this is to use positive integers for the colors (modeling certain frequency channels) and to ask for a coloring of G_1 and G_2 such that the colors on adjacent vertices in G_2 are different, whereas they differ by at least 2 on adjacent vertices in G_1. This problem is known as the radio coloring problem and has been studied under various names. In Section 3 we will briefly survey some of the existing results in this subarea.

1.3 Radio Labeling

The so-called radio labeling problem models a practical setting in which all assigned frequency channels should be distinct, with the additional restriction that adjacent transmitters should use sufficiently separated frequency channels. Within the above framework this can be modeled by considering the graph G_1 that models the adjacencies of n transmitters, and taking $G_2 = K_n$, the complete graph on n vertices. The restrictions are clear: one asks for a proper vertex coloring of G_2 such that adjacent vertices in G_1 receive colors that differ by at least 2. In Section 4 we will discuss some of the existing results in this subarea, with an emphasis on recent results concerning a prelabeled version of this problem.

1.4 Backbone Coloring

The last type of coloring is the recently in [10] introduced notion of backbone coloring. In this variant one models the situation that the transmitters form a network in which a certain substructure of adjacent transmitters (called the backbone) is more crucial for the communication than the rest of the network. This means one should put more restrictions on the assignment of frequency channels along the backbone than on the assignment of frequency channels to other adjacent transmitters. The backbone could e.g. model so-called hot spots in the network where a very busy pattern of communications takes place, whereas the other adjacent transmitters supply a more moderate service. This leads to the problem of coloring the graph G_2 (that models the whole network) with a proper vertex coloring such that the colors on adjacent vertices in G_1 (that model the backbone) differ by at least 2. So far three types of backbones have been considered: (perfect) matchings, spanning trees and a special type of spanning trees also known as Hamiltonian paths. In Section 5 we will discuss the existing results and many open problems in this subarea. Note that the notion of backbone coloring in fact generalizes both radio coloring and radio labeling: radio coloring is the special case of backbone coloring in which G_1 is the backbone of $G_2 = G_1^2$, while radio labeling is the special case in which G_1 is the backbone of K_n.

2 Distant-2 Coloring

In this section we will survey some of the existing results on distant-2 coloring, and discuss the proof techniques and open problems in this subarea. We refer to [1], [7], [8], [29], [33], and [38] for more details.

2.1 Introduction and Main Results

Throughout Section 2, G is a plane graph (i.e., a representation in the plane of a planar graph), that is simple (i.e., without loops and multiple edges) and with vertex set V and edge set E. The *distance* between two vertices u and v is the length of a shortest path joining them.

A *distant-2-coloring* of G is a coloring of the vertices such that vertices at distance one or two have different colors. The least number for which a distant-2-coloring exists is called the *distant-2 chromatic number* of G, denoted by $\chi_2(G)$. We recall that a distant-2-coloring of G is equivalent to an ordinary vertex coloring of the square G^2 of G. (The *square* of a graph G, denoted G^2, is the graph with the same vertex set and in which two vertices are joined by an edge if and only if they have distance one or two in G.) And hence the distant-2 chromatic number $\chi_2(G)$ equals the ordinary *chromatic number* $\chi(G^2)$ of G^2.

The following conjecture was formulated in Wegner [38]. (See also Jensen & Toft [[30], Section 2.18].)

Conjecture 1. *If G is a planar graph with maximum degree Δ, then*

$$\chi_2(G) \le \begin{cases} \Delta + 5, & \text{if } 4 \le \Delta \le 7; \\ \lfloor \frac{3}{2}\Delta \rfloor + 1, & \text{if } \Delta \ge 8. \end{cases}$$

A first result towards a proof of this conjecture can be found in work of Jonas [31]. From one of the results in [31] it follows directly that $\chi_2(G) \leq 8\,\Delta - 22$ for a planar graph G with maximum degree $\Delta \geq 7$. This bound was significantly improved in Van den Heuvel & McGuinness [29] to $\chi_2(G) \leq 2\,\Delta + 25$. Independently, a result with a smaller factor in front of the Δ was proved by Agnarsson & Halldórsson [1] who showed that, provided $\Delta \geq 749$, for a planar graph G with maximum degree Δ we have $\chi_2(G) \leq \lfloor \frac{9}{5}\,\Delta \rfloor + 2$.

In [8], the lower bound on Δ for this last bound has been reduced. In fact, the following result was proved there.

Theorem 2. *If G is a planar graph with maximum degree Δ, then*

$$\chi_2(G) \leq \begin{cases} 59, & \text{if } \Delta \leq 20; \\ \max\{\,\Delta + 39,\ \lceil \frac{9}{5}\,\Delta \rceil + 1\,\}, & \text{if } \Delta \geq 21. \end{cases}$$

In particular, if $\Delta \geq 47$, then $\chi_2(G) \leq \lceil \frac{9}{5}\,\Delta \rceil + 1$.

The proof of Theorem 2 in [8] involves the establishment of the existence of certain *unavoidable configurations* in a planar graph, first of all in [7] for maximal planar graphs. The existence of these configurations is proved by the *discharging technique.* This approach goes back to Heawood's proof of the 5-Color Theorem [27], and the old and new proofs of the 4-Color Theorem ([3], [4], [35]).

To give the general idea, let us repeat the structure of the unavoidable configurations in [7] and [8]. They are defined in terms of "bunches" and "stars".

We say that G has a *bunch* of *length* $m \geq 3$ with as *poles* the vertices p and q, where $p \neq q$, if G contains a sequence of paths $Q_1, Q_2, \ldots, Q_m$ with the following properties. Each Q_i has length 1 or 2 and joins p with q. Furthermore, for each $i = 1, \ldots, m-1$, the cycle formed by Q_i and Q_{i+1} is not separating in G (i.e., has no vertex of G inside). Moreover, this sequence of paths is maximal in the sense that there is no path Q_0 (or Q_{m+1}) that could be added to the bunch, preserving the above properties. If the cycle bounded by Q_1 and Q_m separates G, then the internal vertices of Q_1 and Q_m (if they exist) are the *end vertices* of the bunch. A path $Q_i = pq$ of length 1 in the bunch will be referred to as a *parental edge.*

A *d-vertex* in G is a vertex of degree d. The *big vertices* in G are those of degree at least 26, and *minor vertices* those of degree at most 5.

Let u be a d-vertex, and let $v_1, \ldots, v_k$ be adjacent to u for some integer k with $1 \leq k \leq d$. We say that the vertices $u, v_1, \ldots, v_k$ and edges $uv_1, \ldots, uv_k$ form a *k-star at u*, defined by $v_1, \ldots, v_k$, of *weight* $\sum_{i=1}^{k} d(v_i)$. A $(d-1)$-star at a d-vertex is called *precomplete*, and a d-star at a d-vertex is *complete.*

The following result describes the unavoidable configurations used in [8] to prove Theorem 2 on distant-2-colorings. We omit the proof.

Theorem 3. *For each plane graph G at least one of the following holds:*

(a) G has a precomplete star of weight at most 38 that does not contain big vertices and is centred at a minor vertex.

(b) G has a big vertex b that satisfies at least one of the following conditions:
- *b is a pole for a bunch of length greater than $d(b)/5$;*
- *b is a pole for a bunch of length precisely $d(b)/5$ with a parental edge;*
- *b is a pole for 5 bunches of length $d(b)/5$ without parental edges and with pairwise different end vertices. Moreover, among the end vertices there is a vertex v_0 of degree at most 11, and each other end vertex has degree at most 5. Furthermore, if v_i and v_{i+1} are consecutive in the vicinity of b and are end vertices of two bunches such that $v_i \neq v_0$ and $d(v_i) = 5$, then v_i and v_{i+1} are adjacent in G.*

As proved in Borodin & Woodall [9], each plane graph with minimum degree 5 has a precomplete star of weight at most 25 centred at a 5-vertex. On the other hand, planar graphs with vertices of degree less than 5 may have arbitrarily large weight of the precomplete stars at all minor vertices, as follows from the n-bipyramid. Theorem 3 shows that this is only possible if there are long enough bunches at big vertices.

This structural result of Theorem 3 can be used to prove Theorem 2 by induction (contracting an edge in a suitable star or bunch). In fact, the structural result is used to first prove a best possible upper bound on the minimum degree of the square of a planar graph, and hence on a best possible bound for the number of colors needed in a greedy coloring of it.

Using different unavoidable configurations and a proof which is also based on the discharging method, Molloy & Salavatipour were able to prove the following considerable strengthening of Theorem 2 in [33].

Theorem 4. *If G is a planar graph with maximum degree Δ, then*

$$\chi_2(G) \le \begin{cases} \lfloor \frac{5}{3}\Delta \rfloor + 78, & \\ \lfloor \frac{5}{3}\Delta \rfloor + 24, & \text{if } \Delta \ge 241. \end{cases}$$

Based on the length and depth of the proofs of the previous results one is likely to think that further improvements should be based on a different proof approach, avoiding complicated discharging and unavoidable configurations.

Recently, Andreou & Spirakis announced an “almost proof” of Conjecture 1 in [2].

3 Radio Coloring

In some versions of the previous problem one puts the additional restriction on G_1 that the colors should be sufficiently separated, in order to model practical frequency assignment problems in which interference should be kept at an acceptable level. One way to model this is to use positive integers for the colors (modeling certain frequency channels) and to ask for a coloring of G_1 and

$G_2 = G_1^2$ such that the colors on adjacent vertices in G_2 are different, whereas they differ by at least 2 on adjacent vertices in G_1. This problem is known as the radio coloring problem and has been studied (under various names, e.g. $L(2,1)$-labeling, $\lambda_{2,1}$-coloring and $\chi_{2,1}$-labeling) in [6], [12], [15], [16], [17], [18], and [31].

A *radio coloring* of a graph $G = (V, E)$ is a function $f: V \to \mathbf{N}^+$ such that $|f(u) - f(v)| \geq 2$ if $\{u, v\} \in E$ and $|f(u) - f(v)| \geq 1$ if the distance between u and v in G is 2. The notion of radio coloring was introduced by Griggs & Yeh [23] under the name $L(2,1)$-labeling. The *span* of a radio coloring f of G is $\max_{v \in V} f(v)$.

The problem of determining a radio coloring with minimum span has received a lot of attention. For various graph classes the problem was studied by Sakai [36], Bodlaender, Kloks, Tan & van Leeuwen [6], Van den Heuvel, Leese & Shepherd [28], and others. NP-hardness results for this Radio Coloring problem (RC) restricted to planar, split, or cobipartite graphs were obtained by Bodlaender, Kloks, Tan & van Leeuwen [6]. Fixed-parameter tractability properties of RC are discussed by Fiala, Kratochvíl and Kloks [16]. Fiala, Fishkin & Fomin [15] study on-line algorithms for RC. For only very few graph classes the problem is known to be polynomially solvable. Chang & Kuo [12] obtained a polynomial time algorithm for RC restricted to trees and cographs. The complexity of RC even for graphs of treewidth 2 is a long standing open question. An interesting direction of research was initiated by Fiala, Kratochvíl & Proskurowski [17]. They consider a precolored version of RC, i.e. a version in which some colors are pre-assigned to some vertices. They proved that RC with a given precoloring can be solved in polynomial time for trees. Recently Golovach [21] proved that RC is NP-hard for graphs of treewidth 2.

Due to page limitations we omit the details and confine ourselves to referring the interested reader to the cited papers.

4 Radio Labeling

The so-called radio labeling problem models a practical setting in which all assigned frequency channels should be distinct, with the additional restriction that adjacent transmitters should use sufficiently separated frequency channels. Within the above framework this can be modeled by considering the graph G_1 that models the adjacencies of n transmitters, and taking $G_2 = K_n$, the complete graph on n vertices. The restrictions are clear: one asks for a proper vertex coloring of G_2 such that adjacent vertices in G_1 receive colors that differ by at least 2.

4.1 Definitions and Preliminary Observations

The *girth* of a graph G is the length of a shortest cycle in G. A graph G is *t-degenerate* if each of its subgraphs has a vertex of degree at most t. A *labeling* of the (vertex set of the) graph $G = (V, E)$ is an injective mapping $L: V \to \mathbf{N}^+$. A labeling L of G is called a *radio labeling* of G if for any edge $\{u, v\} \in E$ the inequality $|L(u) - L(v)| \geq 2$ holds; the *span* of such a labeling L is $\max_{v \in V} L(v)$.

The RADIO LABELING problem (RL) is defined as follows: "For a given graph G, find a radio labeling L with the smallest span." The name radio labeling was suggested by FOTAKIS & SPIRAKIS in [19] but the same notion (under different names) has been introduced independently and earlier by other researchers (see, e.g. CHANG & KUO [12]). Problem RL is equivalent to the special case of the TRAVELING SALESMAN problem TSP(2,1) in which all edge weights (distances) are either one or two. The relation is as follows. For a graph $G = (V, E)$ let K_G be the complete weighted graph on V with edge weights 1 and 2 defined according to E: for every $\{u, v\} \in E$ the weight $w(\{u, v\})$ in K_G is 2 and for $\{u, v\} \notin E$ the weight $w(\{u, v\}) = 1$. The *weight* of a path in K_G is the sum of the weights of its edges. The following proposition can be found in [19, 18].

Proposition 5. *There is a radio labeling of G with span k if and only if there is a Hamiltonian path (i.e. a path on $|V|$ vertices) of weight $k-1$ in K_G.*

Another equivalent formulation of this problem, which was extensively studied in the literature, is the HAMILTONIAN PATH COMPLETION problem (HPC), i.e. the problem of partitioning the vertex set of a graph G into the smallest possible number of sets which are spanned by paths in G. This equivalence is expressed in the following well-known proposition. Here $\overline{G}$ denotes the *complement* of G, i.e. the graph obtained from a complete graph on $|V(G)|$ vertices by deleting the edges of G.

Proposition 6. *There is a radio labeling of G with span $\leq k$ if and only if there is a partition of V into $\leq k$ sets, such that each of these sets induces a subgraph in $\overline{G}$ that contains a Hamiltonian path.*

As we mentioned above the TRAVELING SALESMAN problem TSP(2,1) (which is equivalent to RL) is a well-studied problem. PAPADIMITRIOU & YANNAKAKIS [34] proved that this problem is MAX SNP-hard, but gave an approximation algorithm for TSP(2,1) which finds a solution not worse than 7/6 times the optimum solution. Later ENGEBRETSEN [14] improved their result by showing that the problem is not approximable within $5381/5380 - \varepsilon$ for any $\varepsilon > 0$.

DAMASCHKE, DEOGUN, KRATSCH & STEINER [13] proved that the HAMILTONIAN PATH COMPLETION problem HPC can be solved in polynomial time on cocomparability graphs (complements of comparability graphs). To obtain this result they used a reduction to the problem of finding the bump number of a partial order. (The *bump number* of a poset P and its linear extension L is the number of neighbors in L which are comparable in P.) It was proved by HABIB, MÖHRING & STEINER [24] and by SCHÄFFER & SIMONS [37] that the BUMP NUMBER problem can be solved in polynomial time. By Proposition 6, the result of DAMASCHKE, DEOGUN, KRATSCH & STEINER yields that RL is polynomial time solvable for comparability graphs. Later, this result was rediscovered by CHANG & KUO [12] but under the name of $L'(2,1)$-labeling and only for cographs, a subclass of the class of comparability graphs. Notice that RL is NP-hard for cocomparability graphs because the HAMILTONIAN PATH problem is known to be NP-hard for bipartite graphs which form a subclass of comparability graphs. Recently, Fotakis & Spirakis [19] proved that RL can be solved in polynomial time within the class of graphs for which a k-coloring can be obtained in polynomial time (for some fixed k). Note that, for example, this class of graphs includes the well-studied classes of planar graphs and graphs with bounded treewidth.

4.2 Radio Labeling with Prelabeling

Here we focus briefly on a recently studied prelabeling version of the radio labeling problem ([5]).

For a graph $G = (V, E)$ a *pre-labeling* L' of a subset $V' \subset V$ is an injective mapping $L' : V' \to \mathbf{N}^+$. We say that a labeling L of G *extends* the pre-labeling L' if $L(u) = L'(u)$ for every $u \in V'$. We consider the following two problems that were introduced in [5]:

- p-RL($*$): RL with an arbitrary number of pre-labeled vertices.
  For a given graph G and a given pre-labeling L' of G, determine a radio labeling of G extending L' with the smallest span.
- p-RL(l): RL with a fixed number of pre-labeled vertices.
  For a given graph $G = (V, E)$, a subset $V' \subseteq V$ with $|V| \leq l$, and a pre-labeling $L' : V' \to \mathbf{N}^+$, determine a radio labeling of G extending L' with the smallest span.

In [5], the authors studied algorithmical, complexity-theoretical, and combinatorial aspects of radio labeling with pre-labeled vertices. We will briefly summarize these results in Section 4.3, and list some open problems in Section 4.4.

4.3 Upper Bounds for the Minimum Span

Let $G = (V, E)$ denote a graph on n vertices, and let $V' \subseteq V$ and $L' : V' \to \mathbf{N}^+$ be a fixed subset of V and a pre-labeling for V', respectively. Let

$$M \ := \ \max\{n, \ \max_{v \in V'} L'(v)\}.$$

Clearly, M is straightforward to compute if G and L' are known. And clearly, M is a lower bound on the span of any radio labeling in G extending the pre-labeling L' of G. A natural question is how far M can be away from the minimum span of such a labeling. In [5] it is shown that the answer to this question heavily relies on the girth of the graph G:

Theorem 7. *Consider a graph G on $n \geq 7$ vertices, and a pre-labeling L' of G. Then there is a radio labeling in G extending L'*

(a) with span $\leq \lfloor (7M - 2)/3 \rfloor$
(b) with span $\leq \lfloor (5M + 2)/3 \rfloor$ if G has girth at least 4
(c) with span $\leq M + 3$ if G has girth at least 5.

All these bounds are best possible. The third bound is even best possible for the class of paths.

For graphs with bounded degeneracy the following results are proved in [5].

Lemma 8 *If G is a t-degenerate graph on n vertices, then it has a radio labeling with span $\leq n + 2t$.*

Theorem 9. *If G is a t-degenerate graph and L' is a pre-labeling of G, then there exists a radio labeling extending L' with span $\leq M + (4 + \sqrt{3})t + 1$.*

The above results imply a polynomial time approximation algorithm for solving the radio labeling problem (with pre-labeling) in t-degenerate graphs. The bound in Theorem 9 can possibly be improved considerably. In [5] the following conjecture is posed.

Conjecture 10. *If G is a t-degenerate graph and L' is a pre-labeling of G, then there exists a radio labeling extending L' with span $\leq M + 3t$.*

The upper bound in the above conjecture cannot be improved.

We now turn to graphs with a bounded maximum degree.

Theorem 11. *Let $G = (V, E)$ be a t-degenerate graph with maximum degree Δ and let $V' \subseteq V$ be the set of vertices that is pre-labeled by L'. If the number of unlabeled vertices $p = |V \setminus V'| \geq 4\Delta(t + 1)$, then L' can be extended to a radio labeling of G with span M.*

In [5] Theorem 11 is used to obtain the following complexity result for graphs with a bounded maximum degree.

Corollary 12. *Let k be a fixed positive integer. For every graph G with maximum degree $\Delta \leq k$ and pre-labeling L',* P-RL($*$) *can be solved in polynomial time.*

The above corollary shows that P-RL(l) and P-RL($*$) have the same complexity behavior as RL for graphs with a bounded maximum degree, i.e. all three of the problems can be solved in polynomial time. This picture changes if we restrict ourselves to graphs which are k-colorable and for which a k-coloring is given (as part of the input) for some fixed positive integer k.

Related to Proposition 5 we discussed the useful equivalence between RL and the TRAVELING SALESMAN problem TSP(2,1). In [5] this equivalence is adapted as follows to capture the restrictions of the pre-labeling problem. Let L be a labeling of a graph $G = (V, E)$ on n vertices. The path $P = (v_1, v_2, \ldots, v_n)$ *corresponding* to L visits the vertices by increasing labels, i.e. for all $1 \leq a < b \leq n$ we have $L(v_a) < L(v_b)$. P is a path in the complete graph K_G; its *weight* $w(P)$ is measured according to the edge weights w in K_G. In [5] the following result is proved and used to obtain the next corollary.

Theorem 13. *Let $G = (V, E)$ be a graph with a given k-coloring with color classes $I_1, I_2, \ldots, I_k$. Let L' be a pre-labeling of a subset $V' \subseteq V$ with $|V'| = l$. Then a radio labeling L of G extending L' with the smallest possible span can be computed in time $O(n^{4(l+1)k(k-1)})$.*

For each of the graph classes in the following corollary, it is possible to construct a vertex coloring with a constant number of colors in polynomial time. Hence:

Corollary 14. *The radio labeling problem* P-RL(l) *is polynomially solvable*

– *on the class of planar graphs,*
– *on any class of graphs of bounded treewidth,*
– *on the class of bipartite graphs.*

The above results show that P-RL(l) is solvable in polynomial time for graphs with a bounded chromatic number and a given coloring. As shown in [5] this result does not carry over to the more general labeling problem P-RL($*$) where the number of pre-labeled vertices is part of the input. It is shown there that P-RL($*$) is NP-hard even when restricted to 3-colorable graphs with a given 3-coloring by a transformation from PARTITION INTO TRIANGLES; this result is then easily generalized in [5] to k-colorable graphs ($k \geq 4$) with a given k-coloring.

Theorem 15. *For any fixed $k \geq 3$, problem* P-RL($*$) *is NP-hard even when the input is restricted to graphs with a given k-coloring.*

The final results in [5] refer to another class of graphs for which RL is known to be polynomially solvable, namely the class of cographs, i.e. graphs without an induced path on four vertices. Using an easy reduction from 3-PARTITION it is shown in [5] that P-RL($*$) is NP-hard for cographs.

The complexity of P-RL(l) for cographs is left in [5] as one of the open problems.

4.4 Open Problems

In this section we focussed on two versions of the radio labeling problem in which a pre-labeling is assumed. The known results are summarized in the following table.

	graphs with a bounded Δ	graphs with a given k-coloring	Cographs
RL	P [19]	P [19]	P [13, 12]
P-RL(l)	P [5]	P [5]	???
P-RL($*$)	P [5]	NP for $k \geq 3$ [5]	NP [5]

In this table, an entry P denotes solvable in polynomial time, NP denotes NP-hard, and the sign ??? marks an open problem.

For the results in the middle column, we assume that k is a fixed integer that is not part of the input. Note that the class of graphs with a given k-coloring contains important and well-studied graph classes such as the class of planar graphs and the class of graphs with bounded treewidth.

Many questions remain open, a few of which are listed in [5] and repeated below:

– The complexity of any of the variants of Radio Labeling (RL, p-RL(l) and p-RL($*$)) for interval graphs are open problems.
– As mentioned earlier another open problem concerns the computational complexity of p-RL(l) for cographs.
– The results in [5] imply that p-RL(l) is polynomial for bipartite graphs. On the other hand, it is proved there that p-RL($*$) is NP-hard for 3-partite graphs even if a 3-coloring of the graph is given. The complexity of p-RL($*$) for bipartite graphs is open.
– As shown in [5] p-RL(l) is polynomial for planar graphs and graphs of bounded treewidth. The complexity of p-RL($*$) for these graph classes is open.

5 Backbone Coloring

5.1 Introduction and Terminology

In this last section we consider backbone colorings, a variation on classical vertex colorings that was introduced in [10]: Given a graph $G = (V, E)$ and a spanning subgraph H of G (the backbone of G), a *backbone coloring* for G and H is a proper vertex coloring $V \to \{1, 2, \dots\}$ of G in which the colors assigned to adjacent vertices in H differ by at least two. We recall that the chromatic number $\chi(G)$ is the smallest integer k for which there exists a proper coloring $f : V \to \{1, \dots, k\}$. The *backbone coloring number* $\textsc{bbc}(G, H)$ of (G, H) is the smallest integer ℓ for which there exists a backbone coloring $f : V \to \{1, \dots, \ell\}$.

In [10] the results are concentrated on cases where the backbone is either a spanning tree or a spanning path, in [11] the backbone is a perfect matching. In both [10] and [11] combinatorial and algorithmic aspects are treated. We summarize the main results from [10] and [11] in the next two subsections, but first introduce some additional terminology and notation.

A *Hamiltonian path* of the graph $G = (V, E)$ is a path containing all vertices of G, i.e. a sequence $(v_1, v_2, \dots, v_n)$ such that $V = \{v_1, v_2, \dots, v_n\}$, all v_i are distinct, and $\{v_i, v_{i+1}\} \in E$ for all $i = 1, 2, \dots, n-1$. A *perfect matching* is a subset of $|V|/2$ edges from E in which none of the edges share a common end vertex. A *split graph* is a graph whose vertex set can be partitioned into a *clique* (i.e. a set of mutually adjacent vertices) and an *independent set* (i.e. a set of mutually nonadjacent vertices), with possibly edges in between. The size of a largest clique in G is denoted by $\omega(G)$. Split graphs are perfect graphs, and hence satisfy $\chi(G) = \omega(G)$.

5.2 Relations with the Chromatic Number

Part of the results in [10] and [11] are motivated by the following question: How far away from $\chi(G)$ can $\textsc{bbc}(G, H)$ be in the worst case? To answer this question, in [10] the authors introduced, for integers $k \geq 1$, the values

$$\mathcal{T}(k) = \max\{\textsc{bbc}(G, T) :\ G \text{ with spanning tree } T, \text{ and } \chi(G) = k\} \qquad (1)$$

As shown in [10], it turns out that this function $\mathcal{T}(k)$ behaves quite primitively:

Theorem 16. *$\mathcal{T}(k) = 2k-1$ for all $k \geq 1$.*

The upper bound $\mathcal{T}(k) \leq 2k-1$ in this theorem in fact is straightforward to see. Indeed, consider a proper coloring of G with colors $1, \dots, \chi(G)$, and replace every color i by a new color $2i-1$. The resulting coloring uses only odd colors, and hence constitutes a 'universal' backbone coloring for any spanning tree T of G. The proof (in [10]) of the matching lower bound $\mathcal{T}(k) \geq 2k-1$ is more involved and is omitted.

Next, let us discuss the situation where the backbone tree is a Hamiltonian path. Similarly as in (1), in [10] the authors introduced, for integers $k \geq 1$, the values

$$\mathcal{P}(k) = \max\{\text{BBC}(G,P):\ G \text{ with Hamiltonian path } P, \text{ and } \chi(G)=k\} \quad (2)$$

In [10] all the values of $\mathcal{P}(k)$ were exactly determined. They roughly grow like $3k/2$. Their precise behavior is summarized in the following theorem.

Theorem 17. *For $k \geq 1$ the function $\mathcal{P}(k)$ takes the following values:*

(a) For $1 \leq k \leq 4$: $\mathcal{P}(k) = 2k-1$;
(b) $\mathcal{P}(5) = 8$ and $\mathcal{P}(6) = 10$;
(c) For $k \geq 7$ and $k = 4t$: $\mathcal{P}(4t) = 6t$;
(d) For $k \geq 7$ and $k = 4t+1$: $\mathcal{P}(4t+1) = 6t+1$;
(e) For $k \geq 7$ and $k = 4t+2$: $\mathcal{P}(4t+2) = 6t+3$;
(f) For $k \geq 7$ and $k = 4t+3$: $\mathcal{P}(4t+3) = 6t+5$;

In [10] the authors also discuss the special case of backbone colorings on split graphs. Split graphs were introduced by Hammer & Földes [26]; see also the book [22] by Golumbic. They form an interesting subclass of the class of perfect graphs. The combinatorics of most graph problems becomes easier when the problem is restricted to split graphs. The following theorem from [10] is a strengthening of Theorems 16 and 17 for the special case of split graphs.

Theorem 18. *Let $G = (V,E)$ be a split graph.*

(a) In G is connected, then for every spanning tree T in G, $\text{BBC}(G,T) \leq \chi(G)+2$.
(b) If $\omega(G) \neq 3$ and G contains a Hamiltonian path, then for every Hamiltonian path P in G, $\text{BBC}(G,P) \leq \chi(G)+1$.

Both bounds are tight.

An example in [10] shows why for split graphs with clique number 3 the statement in Theorem 18(b) does not work.

In [11], similar as in (1) and (2), for integers $k \geq 1$, the values

$$\mathcal{M}(k) = \max\{\text{BBC}(G,M):\ G \text{ with perfect matching } M, \text{ and } \chi(G)=k\} \quad (3)$$

were introduced and exactly determined. These values roughly grow like $4k/3$. Their precise behavior is summarized in the following theorem,

Theorem 19. *For $k \geq 1$ the function $\mathcal{M}(k)$ takes the following values:*

(a) $\mathcal{M}(4) = 6$*;*
(b) For $k = 3t$*:* $\mathcal{M}(3t) = 4t$*;*
(c) For $k \neq 4$ *and* $k = 3t + 1$*:* $\mathcal{M}(3t + 1) = 4t + 1$*;*
(d) For $k = 3t + 2$*:* $\mathcal{M}(3t + 2) = 4t + 3$*.*

5.3 Complexity of Backbone Coloring

In [10] and [11] the authors also discussed the computational complexity of computing the backbone coloring number: *"Given a graph G, a spanning subgraph H, and an integer ℓ, is* $\mathrm{BBC}(G, H) \leq \ell$*?"* Of course, this general problem is NP-complete. It turns out that in case H is a spanning tree for this problem the complexity jump occurs between $\ell = 4$ (easy for all spanning trees) and $\ell = 5$ (difficult even for Hamiltonian paths). This is proved in [10].

Theorem 20.

(a) The following problem is polynomially solvable for any $\ell \leq 4$*: Given a graph G and a spanning tree T of G, decide whether* $\mathrm{BBC}(G, T) \leq \ell$*.*
(b) The following problem is NP-complete for all $\ell \geq 5$*: Given a graph G and a Hamiltonian path P of G, decide whether* $\mathrm{BBC}(G, P) \leq \ell$*.*

As shown in [11], in case H is a perfect matching, the complexity jump occurs between $\ell = 3$ and $\ell = 4$.

Theorem 21.

(a) The following problem is polynomially solvable for any $\ell \leq 3$*: Given a graph G and a perfect matching M of G, decide whether* $\mathrm{BBC}(G, M) \leq \ell$*.*
(b) The following problem is NP-complete for all $\ell \geq 4$*: Given a graph G and a perfect matching M of G, decide whether* $\mathrm{BBC}(G, M) \leq \ell$*.*

5.4 Discussion

In [10] and [11] the combinatorics and the complexity of backbone colorings of graphs have been analyzed, where the backbone is formed by a Hamiltonian path, by a spanning tree, or by a matching.

Since this area is new, it contains many open problems. For *arbitrary* graphs G with spanning tree T, the backbone coloring number $\mathrm{BBC}(G, T)$ can be as large as $2\chi(G) - 1$. What about *triangle-free* graphs G? Does there exist a small constant c such that $\mathrm{BBC}(G, T) \leq \chi(G) + c$ holds for all triangle-free graphs G? And what about *chordal* graphs? It can be shown that $\mathrm{BBC}(G, P) \leq \chi(G) + 4$ whenever G is chordal and P is a Hamiltonian path of G. Does this result carry over to arbitrary spanning trees, i.e., does $\mathrm{BBC}(G, T) \leq \chi(G) + c$ hold for any chordal graph G with spanning tree T?

Finally, what about *planar* graphs? The 4-Color Theorem together with Theorem 16 implies that $\mathrm{BBC}(G, T) \leq 7$ holds for any planar graph G with spanning tree T. However, this bound 7 is probably not best possible. Can it be improved to 6? There are planar graphs that demonstrate that this bound can not be improved to 5, even for Hamiltonian path backbones. What about perfect matching backbones? The 4-Color Theorem together with Theorem 19 implies that $\text{BBC}(G, M) \leq 6$ holds for any planar graph G with perfect matching M. It seems that this bound 6 is not best possible, but there are planar graphs showing that we cannot improve this bound to 4.

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