ONE-DIMENSIONAL FLOW OF LIQUIDS CONTAINING SMALL GAS BUBBLES

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1. INTRODUCTION

Early papers on the subject matter of this review, such as Mallock's (1910) "The Damping of Sound in Frothy Liquids" or Minnaert's (1933) "Musical Air Bubbles and the Sound of Running Water," were written out of interest in physical phenomena. In papers written during and after World War II this interest was mixed with motives of a more grim nature because it was sought to take advantage of the acoustical properties of bubbly liquids to control the sound produced by propellers, both of surface ships and submerged ships. The literature on the subject deals therefore for a large part with theoretical and experimental studies of the various aspects of propagation of pressure waves of small amplitude in bubbly liquids. A major part of this review will be devoted to this topic, Sections 2-4. Apart from this, various schemes have been envisaged in which the dynamic properties of bubbly liquids could be profitably used in the design of devices to propel high-speed waterborne craft, such as hydrofoil and aircushion boats. This made it valuable to investigate the flow of bubbly liquids through Laval nozzles and similar configurations. This type of flow will be discussed in Section 5. In recent years research not directed by the need for data on acoustical properties has been carried out on waves in bubbly fluids beyond the acoustical regime, where disturbances to a given quiescent state are no longer of infinitesimal smallness. The effects of nonlinearity, dispersion, and dissipation, discussed in Section 6, lead as in other areas, such as those of plasma waves and water waves, to mathematical description by Burgers' equation and that of Korteweg and de Vries. Nonlinearity, dispersion, and dissipation cooperate in the formation of shock waves in bubbly liquids, the structure of which has been investigated to some extent, as reviewed in Section 6.

The present review is restricted to flows described along the above lines. This means that bubbly flow forming the coolant of nuclear reactors is not included. There the bubbles are mostly large vapor bubbles and the interest is primarily in heat transfer.

2. ACOUSTICAL WAVES IN A HOMOGENEOUS MIXTURE

The simplest model for a bubbly liquid is that of a homogeneous fluid in which any effect connected with the bubbly character of the gas content is

disregarded except for the compressibility. When, further, isothermal behavior of the gas is assumed, the calculation of the velocity of sound is straightforward. In the following, we indicate quantities belonging to the gas phase with subscript g , those belonging to the fluid with f . So, ρ_f and ρ_g denote the densities of fluid and gas respectively. We define β as the volume occupied by the gas in a unit volume of the mixture, the density ρ of which is

$$\rho = \rho_f(1 - \beta) + \rho_g\beta \quad (2.1)$$

Under the further assumption that gas and fluid move at the same velocity, the mass of gas in a unit mass of the mixture is constant,

$$\frac{\rho_g\beta}{\rho_f(1 - \beta)} = \text{constant} \quad (2.2)$$

In a homogeneous mixture, as envisaged here, the pressure p in the mixture equals the pressure p_g in the gas, which is at constant temperature T proportional to ρ_g . Therefore we have also

$$\frac{p\beta}{\rho_f(1 - \beta)} = \text{constant} \quad (2.3)$$

For the sound velocity c we have, from Equation (2.1),

$$\frac{1}{c^2} = \frac{d\rho}{dp} = (1 - \beta) \frac{d\rho_f}{dp} + (\rho_g - \rho_f) \frac{d\beta}{dp} + \beta \frac{d\rho_g}{dp}$$

which gives, with the help of Equation (2.3), writing c_f^2 for $d\rho_f/dp$ and c_g^2 for $d\rho_g/dp$, and neglecting $\rho_g^2 c_g^2 / \rho_f^2 c_f^2$ with respect to unity,

$$\frac{1}{c^2} = \frac{(1 - \beta)^2}{c_f^2} + \frac{\beta^2}{c_g^2} + \frac{\rho_f\beta(1 - \beta)}{p} \quad (2.4)$$

This relation can be found in the textbook by Wood (1941) and contains the interesting result that, unless β is very close either to zero or to unity, the speed of sound in a bubbly fluid is lower than the speed of sound in pure gas. The condition of β neither very close to zero nor to unity is almost always satisfied in practical cases, which allows the approximation

$$c^2 = \frac{p}{\rho_f\beta(1 - \beta)} \quad (2.5)$$

A minimum exists for $\beta = 0.5$, in which case, at a pressure of 1 bar, a mixture of air and water has a sound velocity of 20 m/s. But also, for β as small as 4 percent, c is only 50 m/s at 1 bar, which is substantially lower than either c_f or c_g .

How far the assumption of isothermal behavior is justified was investi-

gated by Hsieh & Plesset (1961), who, always assuming equal pressure in mixture and gas phase, included the equations for conservation of momentum and energy in their analysis, leaving, however, viscous dissipation and viscous forces as well as a velocity difference between the phases out of account. They concluded, provided $\beta p / C_{pf} \rho_f T \ll 1$, C_p denoting specific heat, that wave propagation is isothermal.

Later, Crespo (1969) allowed for relative motion and viscosity. A typical length scale for viscous effects is $\nu_f / \omega R$, where ν is the kinematic viscosity, ω the angular frequency of the wave, and R the bubble radius. A representative length for thermal diffusion is $K_g / \rho_g \omega R C_{pg}$ or $a_g / \omega R$ where $a_g = K_g / \rho_g C_{pg}$, K_g being the thermal conductivity and C_{pg} the specific heat in the gas. As could be expected, Crespo finds that Equation (2.5) holds when

$$R \ll \frac{\nu_f}{\omega R} \ll \frac{a_g}{\omega R} \quad (2.6)$$

When R is still small with respect to the thermal length but large with respect to the viscous length,

$$\frac{\nu_f}{\omega R} \ll R \ll \frac{a_g}{\omega R}$$

the bubbles move with respect to the fluid, and for values of β small with respect to unity,

$$c^2 = \frac{(1 + 2\beta)p}{\rho_f(1 - \beta)\beta} \quad (2.7)$$

When

$$\frac{\nu_f}{\omega R} \ll \frac{a_g}{\omega R} \ll R$$

the waves are isentropic and there is a relative velocity, the speed of sound being given by

$$c^2 = \frac{(1 + 2\beta)\gamma p}{\rho_f\beta(1 - \beta)} \quad (2.8)$$

in which expression γ indicates the ratio of specific heats in the gas. The range of validity of Equation (2.7) is rather limited, because the frequency should be below the bubble-resonance frequency, to be discussed in the next section. In experiments, β is usually not larger than a few percent and then the difference between the values given by Equations (2.5) and (2.7) is within the experimental error. Therefore when, in the context of homogeneous mixtures, comparison with experiment is made, Equation (2.5) is the appropriate relation to use for the sound velocity.

Silberman (1957) carried out experiments to measure the sound velocity in bubbly fluids at frequencies well below the resonance frequency and found very good agreement with the relation (2.5). Note that, because the waves are attenuated by various mechanisms, to be discussed in section 4, the relations given here essentially represent the real part of the complex propagation velocity. The high value of $\rho_f c_f / \rho c$ makes a bubble-liquid mixture interesting for underwater acoustics. In particular, the screening of the sound produced by ship propellers, by means of a cylindrical bubble screen around the propeller and its wake, has found wide application. For a theoretical treatment of such a configuration see Le Grand (1970). In this and similar calculations no consideration is given to the sound generated by turbulence in the fluid. Crighton & Ffowcs Williams (1969) treated the problem of sound generation by a bubbly turbulent flow and showed that the presence of bubbles leads to an increase of acoustic power output by a factor $(c_f/c)^4$ compared with the output in pure fluid. This increase is due to resonance between pressure fluctuations in the fluid and bubble oscillations, which will be discussed in what follows.

3. DYNAMIC BEHAVIOR OF A BUBBLE SUBMERGED IN AN INFINITE INCOMPRESSIBLE FLUID

Due to fluid inertia and gas compressibility, an isolated spherical bubble is capable of executing radially symmetrical oscillations, during which the gas in the bubble expands and is compressed in turn. A velocity potential ϕ in the fluid is connected with the radial velocity dR/dt , t denoting time. When r is the distance from the bubble center this potential is at any point of the fluid

$$\phi = -R^2 \frac{dR/dt}{r} \quad (3.1)$$

The kinetic energy E_{kin} of the fluid is given by the well-known integral

$$E_{\text{kin}} = -\frac{\rho_f}{2} \int_A \phi \frac{\partial \phi}{\partial n} dA$$

over the surface A of the bubble, where $\partial \phi / \partial n$ is the derivative of ϕ in the direction of the outward normal to the bubble surface. Carrying out the integration with the help of Equation (3.1) gives

$$E_{\text{kin}} = 2\pi\rho_f R^3 \left(\frac{dR}{dt} \right)^2 \quad (3.2)$$

With an instantaneous volume V of the bubble and equilibrium volume V_0 , the potential energy $\int_{V_0}^V p dV$ is, assuming isothermal behavior,

$$E_{\text{pot}} = \frac{4\pi}{3} p_0 (R^3 - R_0^3) - 4\pi R_0^3 \ln \frac{R}{R_0} \quad (3.3)$$

In harmonic oscillation, E_{kin} and E_{pot} are equal, when averaged over a cycle. Taking the average E_{kin} and E_{pot} , as given by Equations (3.2) and (3.3), leads for the angular frequency ω_B of free oscillations to

$$\omega_B = \frac{1}{R_0} \left(\frac{3p_0}{\rho_f} \right)^{1/2} \quad (3.4)$$

where equilibrium values have the subscript 0. A bubble of equilibrium radius 10^{-4} m oscillates at 1 bar in water with $\omega_B \simeq 10^5 \text{ sec}^{-1}$. Waves in a mixture with this kind of bubbles can, at frequencies ω below 10^4 sec^{-1} , be treated along the lines of Section 2. However, larger bubbles are, at acoustic frequencies, excited near their resonance frequency and the dynamic response of the bubbles to pressure changes in the fluid can no longer be neglected in considering wave propagation. The derivation of Equation (3.4) is given here in the spirit of Minnaert (1933), who assumed, however, adiabatic behavior, which results for ω_B in

$$\omega_B = \frac{1}{R_0} \left(\frac{3\gamma p_0}{\rho_f} \right)^{1/2} \quad (3.5)$$

where γ is the ratio of specific heats of the gas. Which of the relations (3.4) and (3.5) is appropriate has been investigated for bubble oscillations at arbitrary frequencies by Plesset & Hsieh (1960) and Plesset (1964), where the results are given in terms of the characteristic lengths R_0 and $a_g/\omega R_0$ and the acoustic wavelength in the gas phase λ_g . As long as the latter is large with respect to the former, pressure and temperature within the bubble are uniform and the bubble oscillations are isothermal for $a_g/\omega R_0 \gg R_0$ and adiabatic for $a_g/\omega R_0 \ll R_0$. At very high frequencies the oscillations are again isothermal because then the acoustic wavelength becomes less than the bubble radius. The thermal length has to be compared not with R_0 but with λ_g , and if the latter is smaller than $a_g/\omega R_0$ the behavior is isothermal. These results are represented in Table 1, taken from Plesset (1964).

Minnaert (1933) made measurements on air bubbles of the order of 1 mm radius in water at 1 bar. For air, $a_g = 2 \times 10^{-5} \text{ m}^2/\text{s}$, whence $a_g/\omega_B R_0 < R_0$ and Equation (3.5) applies.

In these calculations, viscosity and surface tension are not considered. Viscosity effects will be discussed when dealing with wave attenuation. Surface tension will be disregarded throughout, its effect, apart from keeping the bubbles spherical, becoming important only for bubbles of the order of 1 micron, at atmospheric conditions.

4. ACOUSTIC WAVE PROPAGATION INCLUDING BUBBLE RESPONSE

Wave propagation as a multiple scattering problem.—The wave-propagation problem becomes much more complicated, compared with the analysis in Section 2, when the effects of bubble oscillations become important. Consider a sound wave incident on a bubbly fluid. The incident wave is scattered by the bubbles and at a point in the mixture the signal consists of the incident

TABLE 1. Thermodynamic behavior of oscillating bubbles

Frequency range	Comparison of lengths	Thermodynamic behavior
Very high	$\lambda_g \ll \frac{a_g}{\omega R_0} \ll R_0$	isothermal
High	$\frac{a_g}{\omega R_0} < \lambda_g < R_0$	adiabatic
Moderately high	$\frac{a_g}{\omega R_0} < R_0 < \lambda_g$	adiabatic
Low	$R_0 < \frac{a_g}{\omega R_0} < \lambda_g$	isothermal

wave and the signals from the individual scattering bubbles, randomly distributed in the mixture. When measuring the pressure p at a particular location for a long time, one may assume that the mixture passes through all its possible realizations. In practice, the bubbles move about with such small velocities that Doppler effects can be neglected. Then the time average $\bar{p}$ equals the ensemble average $\langle p \rangle$ —in other words, we may assume the system to be ergodic. The instantaneously measured pressure differs from $\langle p \rangle$. The mean square difference is $\langle |p - \langle p \rangle|^2 \rangle$, which is equal to $\langle |p|^2 \rangle - |\langle p \rangle|^2$. In the theory of multiple scattering the latter quantity is called the incoherent part of the wave, whereas $\langle p \rangle$ is called the coherent part. Foldy (1945), in an important paper, dealt with the problem of multiple scattering by randomly distributed point scatterers and showed that the coherent part $\langle p \rangle$ satisfies a wave equation. He also obtained an approximate expression for the incoherent part which, however, has been criticized in later years (McCloskey 1967). Carstensen & Foldy (1947) applied the general theory of Foldy (1945) to the scattering of sound waves by a liquid-bubble mixture, where the wave equation satisfied by the coherent part of the wave is

$$\nabla^2 \langle p \rangle + \left\{ \frac{\omega^2}{c_f^2} + 4\pi G \right\} \langle p \rangle = 0 \quad (4.1)$$

The function G is connected with the probability distribution $n(\mathbf{r}, R_0)$ of the bubbles as a function of position $\mathbf{r}$ and equilibrium radius R_0 , and the scattering coefficient $g(\mathbf{r}, R_0)$ of an individual bubble, by

$$G = \int g(\mathbf{r}, R_0) n(\mathbf{r}, R_0) dR_0 \quad (4.2)$$

The scattering coefficient is defined such that

$$g A p_0 \frac{e^{-i\omega r/c_f}}{r} e^{i\omega t} \quad (4.3)$$

is the wave emitted by a bubble when a wave with strength $A p_0 e^{i\omega t}$, with respect to the equilibrium pressure p_0 , is incident. Neglecting the compressibility of the fluid, we can calculate g with the help of the analysis in Section 3. If we write $R = R_0(1 + \tau e^{i\omega t})$, it follows from Equation (3.1) that

$$p = -\rho_f \frac{\partial \phi}{\partial t} = -\frac{\rho_f \omega^2 \tau R_0^3}{r} e^{i\omega t}$$

In the linear approximation, to which scattering theory is restricted, the pressure $p_0 R_0^3 / R^3$ in the bubble equals $p_0 - 3\tau p_0 e^{i\omega t}$. At $r = R_0$ the sum of the pressures in the incident and scattered wave must equal this value, giving

$$p_0 + A p_0 e^{i\omega t} - \rho_f R_0^2 \omega^2 \tau e^{i\omega t} = -3p_0 \tau e^{i\omega t} + p_0$$

From this follows

$$\tau = \frac{A p_0}{\rho_f R_0^2 \omega^2 - 3p_0}$$

and hence

$$p = -\frac{\rho_f R_0^3 A p_0 \omega^2}{\rho_f R_0^2 \omega^2 - 3p_0} \frac{e^{i\omega t}}{r}$$

If we use the definition (3.4) of ω_B , comparison with Equation (4.3) shows that for the bubble considered as a point scatterer,

$$g = \frac{R_0}{(\omega_B^2 / \omega^2) - 1} \quad (4.4)$$

When all the bubbles have the same equilibrium size R_0 and there is equal probability of finding a bubble anywhere in the fluid, we have, taking into account that the volume available for the bubbles in a unit volume of the mixture is $(1 - \beta)$,¹

$$G = ng(1 - \beta)$$

n being the number density of the bubbles.

Inserting this into Equation (4.1) and using Equation (4.4) gives for plane waves with wavenumber k the dispersion equation

$$k^2 = \frac{4\pi n R_0 (1 - \beta)}{(\omega_B^2 / \omega^2) - 1} + \frac{\omega^2}{c_f^2} \quad (4.5)$$

We may write this, using Equation (2.5) for the isothermal sound speed c and Equation (3.4) for the isothermal resonance frequency ω_B , in addition taking account of dissipation by introducing a dissipation coefficient δ , as

¹ The factor $1 - \beta$ is absent in the original analysis since the bubbles were treated as point scatterers throughout.

$$\frac{k^2}{\omega^2} = \frac{1}{c^2} \frac{1}{1 - i\delta\omega/\omega_B - \omega^2/\omega_B^2} + \frac{1}{c_f^2} \quad (4.6)$$

Only for viscous forces connected with radial pulsations of a bubble is the effect on the dispersion relation, as we shall show later on, represented exactly by a constant factor δ as in Equation (4.6). For other dissipative processes this representation, used by most writers on the subject, is only approximate. Apart from some minor differences, the relation (4.6) was derived along the lines given here by Carstensen & Foldy (1947) and in a similar, although more heuristic, way by Meyer & Skudrzyk (1953). It follows from Equation (4.6) that scattering by the bubbles makes the wave propagation dispersive, because the reciprocal phase velocity k/ω is a function of the frequency ω . Carstensen & Foldy (1947) also made an approximate calculation of the contribution of the incoherent part of the wave to p , showing that its effect on the phase velocity (the real part of ω/k) is negligibly small.

Continuum theory.—The analysis of Foldy (1945) starts with individual scatterers and involves subsequent ensemble averaging. Alternatively, we may start with averaging and introduce average pressure $\bar{p}$, velocity u , and density ρ , where the averaging is over a volume element of the mixture containing many bubbles, but of linear dimensions small with respect to the characteristic length of the motion. If we neglect the contribution of the gas, the mass density is [cf Equation (2.1)] $\rho = \rho_f(1 - \beta)$, and in terms of the quantities thus defined the equations for one-dimensional motion are

$$\frac{d\rho}{dt} + \rho \frac{\partial u}{\partial x} = 0 \quad (4.7)$$

$$\rho \frac{du}{dt} + \frac{\partial p}{\partial x} = 0 \quad (4.8)$$

In the homogeneous flow theory of Section 2 it is assumed that p equals the pressure p_g in the gas phase. We now drop this assumption and suppose that the relation between p and p_g is the same as exists between the pressure in an isolated oscillating bubble in an infinite fluid and the pressure far away. By using Bernoulli's theorem, we can calculate the pressure in the fluid at the bubble wall, with the help of the potential (3.1). The continuity of stresses at the bubble wall, including the viscous stress $2\mu_f \partial^2\phi/\partial r^2$, where μ_f is the dynamic viscosity of the fluid, gives

$$p_g - p_\infty = \rho_f R \frac{d^2 R}{dt^2} + \frac{3}{2} \rho_f \left(\frac{dR}{dt} \right)^2 + \frac{4\mu_f}{R} \frac{dR}{dt} \quad (4.9)$$

The assumption that in a mixture the same relation exists between p and p_g is introduced now. In Foldy's (1945) theory, the inertia effect of fluid

surrounding a bubble is expressed in the scattering coefficient of a single bubble, followed by ensemble averaging. Here the averaging is done at the outset but in a more heuristic way, the accuracy of which is hard to estimate, whereas in the scattering theory, by a more precise definition of the averaging procedure, the approximation involved is more clear.

Before proceeding, it is worthwhile to consider the term $4(\mu_f/R)dR/dt$, associated with radial viscous stress in the fluid. Since $\beta = (4\pi/3)nR^3$ this term is equal to $(4/3)\mu_f d/dt \ln(\beta/n)$. The continuity equation (4.7) enables us to write this as $(4/3)(\mu_f/\beta)\partial u/\partial x$, whence, by viscosity alone, it follows from Equation (4.9) that

$$p_0 - p = \frac{4}{3} \frac{\mu_f}{\beta} \frac{\partial u}{\partial x} \quad (4.10)$$

In fluid mechanics, the difference between the mechanically defined pressure and the thermodynamic pressure is expressed by means of a coefficient of bulk viscosity. The relation (4.10) is of this kind and its interpretation therefore is that the bubbles give the mixture an apparent bulk viscosity of magnitude $(4/3)\mu_f/\beta$, as shown by Taylor (1954).

We now linearize Equations (4.7)–(4.9) and obtain for the velocity u or any other first-order quantity, under isothermal conditions,

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} = & \frac{p_0}{\rho_f(1-\beta_0)\beta_0} \frac{\partial^2 u}{\partial x^2} + \frac{1}{3} \frac{R_0^2}{\beta_0(1-\beta_0)} \frac{\partial^4 u}{\partial x^2 \partial t^2} \\ & + \frac{4}{3} \frac{\mu_f}{\rho_f\beta_0(1-\beta_0)} \frac{\partial^3 u}{\partial x^2 \partial t} \end{aligned} \quad (4.11)$$

This equation was, without the viscous term and for $\beta_0 \ll 1$, given by van Wijngaarden (1966). With $R_0 \rightarrow 0$ and $\mu_f = 0$, the speed of sound c , calculated in Section 2 (Equation 2.5), is obtained again. Writing $R_0^2/3\beta_0(1-\beta_0)$ as c^2/ω_B^2 we find, substituting $u = \exp i(kx - \omega t)$ in Equation (4.11), as the dispersion equation

$$\frac{k^2}{\omega^2} = \frac{1}{c^2} \frac{1}{1 - (\omega/\omega_B)^2 - 4\nu_f i\omega/\omega_B^2 R_0^2} \quad (4.12)$$

which is, for $c_f \rightarrow \infty$ and the special case $\delta = 4\nu_f/\omega_B R_0^2$, identical with the result (4.6) of scattering theory. This shows that the continuum approach employed in this section gives the coherent part of the wave. It does not, however, enable us to obtain the incoherent part. The calculation of the latter from scattering theory, although possible in principle, is very difficult and complicated (Foldy 1945, McCloskey 1967). Recently, Howe (1971a,b) has proposed a continuum theory for wave propagation in random media. It is suggested here that Howe's theory is perhaps better suited for the calculation of the incoherent part of the wave, which, for bubbly suspensions,

does not appreciably affect the signal velocity but may contribute to the wave attenuation.

Attenuation mechanisms considered in the literature.—The damping coefficient δ in Equation (4.6) is small and its effect on the wavespeed, the real part of ω/k , is therefore important only in the neighborhood of $\omega/\omega_B = 1$. In a first approximation, the contributions to δ of various dissipation mechanisms may be added to obtain the resulting damping coefficient. The contribution δ_{vr} due to relative radial motion of the bubbles has been discussed already and is equal to $4\nu/\omega_B R_0^2$. The logarithmic decrement of free oscillating bubbles is just $\pi\delta_{vr}$. Mallock (1910) discussed dissipation by radial motion and suggested this mechanism as the main cause of damping of sound waves in bubble-liquid mixtures. Other mechanisms, however, are generally of more importance.

The logarithmic decrement Λ_{ac} associated with acoustic radiation of free oscillating bubbles into the fluid is $\pi\omega_B R_0/c_f$ (Chapman & Plesset 1970). Meyer & Skudrzyk (1953) give for the contribution δ_{ac} of acoustic radiation $\delta_{ac} = 0.0144(\omega/\omega_B)^3$ for air bubbles in water at atmospheric conditions. This shows that δ_{ac} is a function of frequency and bubble radius and not a constant, as in the case of dissipation by radial motion. In most theoretical work one takes δ constant and uses the value at resonance. For small bubbles the most important damping mechanism stems from thermal conduction. During expansion and compression, heat is conducted from the fluid to the bubble and vice versa, causing a difference in phase between the pressure in the bubble and the external pressure. To calculate the associated damping, the heat equation in the fluid has to be solved together with the dynamic equation for the bubble and its heat equation, taking into account the continuity of pressure and temperature at the interface.

The contribution δ_{th} to δ by thermal conduction has been calculated under a number of assumptions by Pfriem (1940), who finds

$$\delta_{th} = \frac{3(\gamma - 1)}{2(\omega/2a_0)^{1/2}R_0}$$

For air bubbles in water at room temperature and at a pressure of 1 bar this amounts to (Meyer & Skudrzyk 1953)

$$\delta_{th} = 4.45 \times 10^{-4} \frac{f_B}{f^{1/2}} \quad (4.13)$$

where f_B and f indicate the frequencies $\omega_B/2\pi$ and $\omega/2\pi$ respectively. Devin (1959), in a survey of damping coefficients for bubble oscillations at resonance, reconsiders Pfriem's work and finds the slightly different expression

$$\delta_{th} = 4.41 \times 10^{-4} f_B^{1/2} \quad (4.14)$$

FIGURE 1. Logarithmic decrement of freely oscillating air bubbles in water due to various mechanisms. After Chapman & Plesset (1970).

Chapman & Plesset (1970) relaxed Pfriem's assumptions in computing the logarithmic decrement of free oscillating bubbles. Their results cannot be given as a simple analytic expression, but are represented graphically (see Figure 1). In this figure the contribution to the logarithmic decrement by thermal conduction is represented together with the contributions due to radial viscous stresses and acoustic radiation. It is clear that for bubbles with a radius larger than 5 microns, thermal conduction dominates over other dissipation mechanisms discussed so far.

Comparison with experiment.—The dispersion equation (4.12), which may be written as

$$\frac{k^2}{\omega^2} = \frac{1}{c_f^2} + \frac{1 - \omega^2/\omega_B^2 + i\delta\omega/\omega_B}{c^2\{(1 - \omega^2/\omega_B^2)^2 + \delta^2\omega^2/\omega_B^2\}} \quad (4.15)$$

and from which the phase speed of a wave can be obtained as the real part of ω/k and the wave attenuation as the imaginary part, is given here for one bubble size. When the mixture contains a distribution of bubble sizes the dispersion equation is more complicated, involving integrals over the bubble distribution, and requires a sufficiently accurate mathematical description of this bubble distribution, which is difficult to obtain. Generally, there are

FIGURE 2. Sound speed in gas bubble-water mixture as a function of frequency. $R_0 = 1.2 \times 10^{-4} \text{ m} \pm 4.10^{-5} \text{ m}$, $\beta = 2.10^{-4}$. The solid curve represents the theoretical results, with $\delta = 0.5$, for the actual bubble distribution. The broken curve would hold if all the bubbles had the same size $R_0 = 1.1 \times 10^{-4} \text{ m}$. The dots represent measured values (Fox, Curley & Larson 1955).

many difficulties in the experiments, such as the production of bubbles within a reasonably narrow range of diameters and the measurement of bubble size. Also, measuring equipment such as hydrophones and pressure recorders generally present problems for the experimenter. A comparison with theory is therefore very hard to obtain in the case where the experiments are done with a mixture containing bubbles of many different sizes. In this category are the experiments by Carstensen & Foldy (1947), Laird & Kendig (1952), and Meyer & Skudrzyk (1953), all done with air bubbles in water. They compared their experimental results with Equation (4.15), allowing for the experimental bubble distribution but using different values for δ . Carstensen & Foldy (1947) made independent measurements to obtain δ and found a value that is larger than the sum of δ_{ac} , δ_{vr} , and δ_{th} , given before.

For δ they gave the empirical relation

$$\delta = 0.057(1 + 10^{-4}f_B)$$

where f_B is the bubble resonance frequency in kilocycles. Using this value for δ , they came to reasonable agreement with measured wave attenuation in

mixtures with $\beta \simeq 10^{-3}$ and bubbles with radius of order 10^{-4} m. Laird & Kendig (1952) used the above relation of Carstensen & Foldy (1947) for attenuation and found good agreement with experiment. Meyer & Skudrzyk (1953) used for δ the sum of δ_{ac} , δ_{vr} , and δ_{th} and found order-of-magnitude agreement of measured wave attenuation with theory. The authors of this category could not obtain reliable measurements of wavespeed in their mixtures.

Bubble distributions within a very narrow range of bubble sizes were produced in the experiments of a second series of investigations by Fox, Curley & Larson (1955), Macpherson (1957), and Silberman (1957), whose measurements on low-frequency waves were mentioned earlier. The mixture obtained by Fox, Curley & Larson (1955) contained bubbles with radii $1.2 \times 10^{-4} \text{ m} \pm 4.10^{-5} \text{ m}$. Their value of β was 2.10^{-4} . Their attenuation measurement fitted with theory, provided the empirical value 0.5 for δ was used. This is considerably larger than the sum of δ_{ac} and δ_{th} , which would give $\delta = 0.093$ (not 0.128 as given in their paper). They also used this value of δ in comparing measured wavespeed with theory. In Figure 2 and Figure 3 their results are reproduced. The solid curve in Figure 2 represents the theory for the actual experimental distribution, whereas the broken line represents the theoretical result if all the bubbles were of the same radius $R_0 = 1.1 \times 10^{-4} \text{ m}$. The difference between these curves clearly demonstrates the

FIGURE 3. Attenuation of pressure waves in a water-air bubble mixture. Further details are the same as in Figure 2.

influence of a small spread in bubble sizes. With regard to wavespeed, the agreement with theory is reasonable. Note in particular that above resonance the speed of propagation of a wave reaches values larger than the sound speed in pure water. Above resonance the predicted wavespeed is on the average somewhat larger than the measured speed. This may be caused by an incomplete representation of the dissipation (remember that a constant value $\delta=0.5$ is used in the theory).

In the experiments of Macpherson (1957), bubble size was controlled with great accuracy, a standard deviation as low as 7 percent being achieved. Macpherson (1957) did not measure wavespeed but obtained accurate data on wave attenuation. He reports a measured value $\delta=0.08$, rightly pointing out the very close agreement with the theoretical value 0.079. He did not mention the source from which he obtained this theoretical value, but we find just this value upon inserting Macpherson's data in Pfriem's formula (4.13) for δ_{th} . Silberman's (1957) measurements are perhaps the most conclusive ones of all. He obtained air bubble-water mixtures with all the bubbles of practically the same size and reported measurements at various magnitudes of this uniform size. His theoretical value of δ was based on work by Spitzer (1943), which results essentially in $\delta=\delta_{ac}+\delta_{vr}+\delta_{th}$, as given before. His measurements of both wavespeed and wave attenuation agree very well with the corresponding theoretical values. In Figure 4 some typical results obtained by Silberman (1957) are shown. Also here it appears that the theoretical attenuation is on the average somewhat lower than the actually measured one. However, especially when looking at this second class of experimental results, one can say that on the whole the experiments agree quite well with theory. The theoretical prediction of wave attenuation is somewhat below the actual one, but the situation is not as bad as it appeared to Batchelor (1969) upon looking at the results of Fox, Curley & Larson (1955). In his search for other dissipation mechanisms, Batchelor (1969) drew attention to viscous dissipation associated with relative translation of the bubbles with respect to the fluid. He showed that the relative magnitude of this type of dissipation to that by relative radial motion is for sound waves equal to $9\beta(1-\beta)^{-1}(R\delta^2\omega/2\nu)^{1/2}(\omega_B^2/\omega^2-1)$ in the case where, owing to surface-active agents, the fluid sticks to the bubble surface. When at the interface only the relative normal velocity vanishes and as a consequence the dissipation takes place in the (potential) flow around the bubble, this ratio equals $9\beta(1-\beta)^{-1}(\omega_B^2/\omega^2-1)$. For sound waves with $\omega\ll\omega_B$, both quantities are at least equal to and mostly larger than unity, but with the small value of β used by Fox, Curley & Larson (1955), the dissipation by relative translational motion hardly contributes to wave attenuation, as Batchelor (1969) points out.

It would be worthwhile to consider the dissipation associated with enhanced heat transfer from bubble to fluid caused by relative translational motion, which leads to convection within the bubble. Further causes of attenuation, not yet investigated, are vaporization and condensation taking place during passage of a sound wave. Fox, Curley & Larson (1955) men-

FIGURE 4. Sound speed and attenuation in water-air bubble mixtures, as measured by Silberman (1957); $\beta = 5.84 \times 10^{-4}$, $R_0 \approx 1.8 \times 10^{-4}$ m. The solid lines and broken lines represent the theoretical results. The experimental values are indicated by $\diamond$, $\circ$, $\square$, and $\triangle$, and belong to various values of R_0 , as indicated in the figure.

tioned this but no estimate is given. Finally, more detailed study of the effect of the incoherent part of the wave, discussed earlier in this section, might be a useful subject for further research.

5. FLOW IN LAVAL NOZZLES

Bubble-liquid mixtures have been considered for use in propulsion devices for captured air-bubble ships and other high-speed ships. The underlying idea is that expansion of a compressed gas bubble-liquid mixture might

be an efficient way to produce the momentum necessary for thrust. It is natural to investigate in this connection the flow of a bubble-liquid mixture through Laval nozzles. Among the first to publish on this were Tangren, Dodge & Seifert (1949), who discussed some theoretical aspects and also reported results of experimental work. They used the homogeneous flow model, discussed in Section 2, where gas and fluid move at the same velocity, the gas behaving isothermally. When nozzle area is denoted with A , the pertinent equations for steady flow through the nozzle are, with the fluid considered as incompressible,

$$\rho_f u A (1 - \beta) = \text{constant} \quad (5.1)$$

$$\frac{dp}{dx} + \rho_f (1 - \beta) u \frac{du}{dx} = 0 \quad (5.2)$$

$$\frac{p\beta}{1 - \beta} = \frac{p_0\beta_0}{1 - \beta_0} \quad (5.3)$$

In Equation (5.3) and also in what follows, reservoir conditions are indicated with the subscript 0. From Equations (5.2)–(5.3), Tangren, Dodge & Seifert (1949) deduced that

$$\frac{1}{2} \frac{\rho_f u^2}{p_0} = \left(1 - \frac{p}{p_0}\right) - \frac{\beta_0}{1 - \beta_0} \ln \frac{p}{p_0} \quad (5.4)$$

For $\beta_0 \rightarrow 0$ this relation reduces to Bernoulli's law. The last term on the right-hand side of Equation (5.4) gives the contribution to $\frac{1}{2}\rho_f u^2$ due to the bubbles. According to Equation (5.4), this quantity tends to infinity for $p \rightarrow 0$. Indeed, for $p \rightarrow 0$ the gas bubbles grow indefinitely large and the fluid must obtain infinitely large velocity to maintain the mass flow. This limit has, however, no practical significance because long before p gets very small several assumptions on which the analysis rests break down, notably the assumption of no relative motion between fluid and bubbles and that of equal pressure in bubbles and fluid. Another useful relation, which can be derived from Equations (5.1)–(5.3), is

$$\frac{1}{A} \frac{dA}{dx} + \frac{\beta}{p} \frac{dp}{dx} \left\{ 1 - \frac{p}{\rho_f \beta (1 - \beta) u^2} \right\} = 0$$

or, using Equation (2.5) for the velocity of sound c ,

$$\frac{1}{A} \frac{dA}{dx} + \frac{\beta}{p} \frac{dp}{dx} \left\{ 1 - \frac{c^2}{u^2} \right\} = 0 \quad (5.5)$$

This shows the familiar result that in the throat either the pressure reaches a minimum or the velocity equals the sound velocity. The value of the pres-

sure ratio in the latter case, the critical pressure ratio, follows from inserting $u=c$ in Equation (5.4) and is given by

$$\frac{1}{2} \left\{ 1 + \frac{1 - \beta_0}{\beta_0} \frac{p}{p_0} \right\}^2 = \frac{1 - \beta_0}{\beta_0} \left(1 - \frac{p}{p_0} \right) - \ln \frac{p}{p_0} \quad (5.6)$$

The critical pressure ratio, a constant in the flow of a gas, depends here on the gas content in the reservoir. The above relation was tested by Tangren, Dodge & Seifert (1949) in a limited number of experiments. They found in the throat of converging-diverging nozzles values of p/p_0 above those predicted by Equation (5.6). Supersonic flow was convincingly shown to exist in the diverging part of the nozzle because at low enough exit pressure the expansion of the mixture took place outside the nozzle. Flow in converging-diverging nozzles was investigated in more detail by Muir & Eichhorn (1963), who found many discrepancies with homogeneous flow theory, including the one mentioned above. Muir & Eichhorn (1963) attributed these discrepancies to the neglect, in the homogeneous flow theory, of relative motion between bubbles and fluid. They gave, albeit incorrectly in some places, the equations involving this relative motion, but did not carry out further analysis with these. Although Muir & Eichhorn (1963) rightly pointed out that inertia effects associated with relative radial motion are important (but nevertheless wrote $\rho_g = p/RT_g$, which should be $\rho_g = p_g/RT_g$, R being the gas constant), we shall restrict ourselves to equations containing the effects of relative translational motion, often called "slip" by workers in two-phase flow. The equations for mass conservation are, when no mass transfer takes place between the phases and the velocity of the gas phase is indicated by v ,

$$\frac{d}{dx} \rho_g \beta v A = 0 \quad (5.7)$$

$$\frac{d}{dx} \rho_f (1 - \beta) u A = 0 \quad (5.8)$$

When relative radial motion is left out of account and isothermal behavior of the gas in the bubbles is assumed, we have

$$\frac{p}{\rho_g} = \text{constant} \quad (5.9)$$

To establish the equation of momentum is more difficult. First, consider a single bubble moving with velocity v through a fluid. Let the volume of the bubble be V and let F be the frictional force exerted by the fluid on the bubble. Then in steady flow, that is, $u = u(x)$, we have

$$v \frac{d}{dx} \left\{ \frac{1}{2} \rho_f V (v - u) \right\} = - V \frac{dp}{dx} + F \quad (5.10)$$

This equation is valid either when there is no velocity boundary layer along the bubble or when the boundary layer, if present, does not separate. The first case occurs when at the bubble wall only the relative normal velocity vanishes. A boundary layer exists at the bubble wall when an apparent no-slip condition applies there because of the presence of surface-active agents. Provided this boundary layer does not separate, we can use Equation (5.10). In other cases its use is questionable because the virtual mass of a sphere takes the value $\frac{1}{2}\rho_f V$ only when potential flow exists around the sphere. Multiplying each term by nA , n being as before the number density of the bubbles, we obtain, bearing in mind that nvA is constant,

$$\frac{d}{dx} \left\{ \frac{1}{2} \rho_f v A \beta (v - u) \right\} = -\beta A \frac{dp}{dx} + nAF \quad (5.11)$$

The momentum equation for the fluid, with friction neglected at the walls of the nozzle, is

$$(1 - \beta) A \frac{dp}{dx} + \rho_f (1 - \beta) A u \frac{du}{dx} + nAF = 0 \quad (5.12)$$

Addition of Equations (5.11) and (5.12) results in the momentum equation for the mixture as a whole,

$$A \frac{dp}{dx} + \rho_f A u (1 - \beta) \frac{du}{dx} + \frac{d}{dx} \left\{ \frac{1}{2} \rho_f \beta v (v - u) \right\} = 0 \quad (5.13)$$

which contains, apart from the rate of change of mean momentum, the rate of change of the Kelvin impulse associated with the relative translational motion of the bubbles, as expressed by the last term on the left-hand side. In most expositions on the equations of motion, including Muir & Eichhorn (1963) and Witte (1969a), the Kelvin impulse is lacking. Equations (5.7)–(5.13) have not been analyzed in the literature and an attempt to do so is beyond the scope of a review. We shall restrict ourselves therefore to some qualitative conclusions. From Equations (5.9) and (5.10), together with the constancy of the mass of one bubble, it follows that

$$\frac{1}{2} \rho_f v \frac{d}{dx} (v - u) = -\frac{dp}{dx} \left\{ 1 - \frac{\frac{1}{2} \rho_f v (v - u)}{p} \right\} + \frac{F}{V} \quad (5.14)$$

In nozzle flow the pressure gradient is negative. Initially, when $v \simeq u$, the gas is, according to Equation (5.14), accelerated with respect to the fluid: $d/dx(v - u) > 0$. For certain values of v , u , and p , however, the quantity $1 - \frac{1}{2} \rho_f v (v - u)/p$ becomes negative and, because $F < 0$ also, the relative velocity decreases accordingly. This can indeed be observed in the experimental results of Muir & Eichhorn (1963), from which Figure 5 is taken. The solid line represents the bubble velocity v and the broken curve the mixture velocity u as follows from homogeneous flow theory. The latter

FIGURE 5. Relative velocity of air bubbles (solid line) with respect to fluid (broken line) in flow of a bubbly fluid through a nozzle. Reservoir pressure 6.3×10^5 N/m². Volumetric gas content in throat 0.1. Size of air bubbles in mixing section (that is, the region upstream from the throat) $\approx 10^{-4}$ m. After Muir & Eichhorn (1963).

may, roughly speaking, be equated to the fluid velocity. It appears that along the nozzle $v-u$ increases up to a location near the throat and subsequently decreases. This might, as suggested above, be explained by the behavior of the Kelvin impulse. Its growth is first expressed in an increase of $v-u$, but later the increase of the bubble volume consumes the growth of Kelvin impulse even at the expense of $v-u$.

The increase of the Kelvin impulse means that the increase of mean momentum is less than it would be in the case when $u=v$. Muir & Eichhorn (1963) found that in the throat of the nozzle the velocity is lower and the pressure higher compared with the predictions of homogeneous flow theory. These effects could in principle be investigated theoretically by analysis of the equations given before. This has not been done so far, so that there is room for future work here. In order to see what kind of results could be obtained from these equations, let us take β small, for convenience. When we write down two momentum equations and two equations for mass conservation as equations in the unknown quantities du/dx , $d\beta/dx$, dv/dx , and dp/dx , it follows that the solution breaks down, which indicates the occurrence of critical flow when

$$u^2 = \frac{p}{\rho_f \beta} \frac{v/u}{2v/u - 1}$$

For $v/u > 1$ this indicates a value for the critical velocity that is lower than follows from homogeneous flow theory. Qualitatively this is in agreement with the experimental results of Muir & Eichhorn (1963).

Witte (1969a) investigated the efficiency of a propulsion device based on injection of compressed air bubbles in the throat section of a nozzle. He computed the flow characteristics numerically, using the equations given in this section but leaving Kelvin impulse out of account in the momentum equation for the mixture.

Since in Witte's (1969a) calculations compressed air bubbles are injected

at a temperature well above the fluid temperature, an equation describing heat transfer from the bubbles to the fluid was included. Witte (1969a) concluded that with a ramjet working in this way a propulsive efficiency, defined as the ratio (thrust $\times$ speed of advance)/(installed compressor power), of 50–60 percent could be obtained. Whether any device of this sort has been, or is, in operation is not known to the present writer.

Muir & Eichhorn (1963) also observed shock waves in the divergent part of the nozzle. Normal shock waves in a straight liquid-bubble shock tube were previously investigated experimentally and theoretically by Campbell & Pitcher (1958), who established the pertinent Hugoniot relations, in the context of the homogeneous flow theory of Section 2. When quantities in front of the shock are indicated by subscript 0, those behind the shock by subscript 1, and in addition the velocity of propagation of a steady shock wave by U , these relations are

$$\rho_f(1 - \beta_0)U = \rho_f(1 - \beta_1)(U + u_1) \quad (5.15)$$

(where u is, as before, the mixture velocity),

$$p_0 + \rho_f(1 - \beta_0)U^2 = p_1 + \rho_f(1 - \beta_1)(U + u_1)^2, \quad (5.16)$$

$$\frac{p_0\beta_0}{1 - \beta_0} = \frac{p_1\beta_1}{1 - \beta_1} \quad (5.17)$$

Defining the Mach number M_{is} of the shock as the ratio of U to the velocity of sound c_0 in front of the shock, as given by Equation (2.5), Campbell & Pitcher (1958) obtained the relation

$$M_{is}^2 = \frac{p_1}{p_0} \quad (5.18)$$

and found that their experimental results agreed very well with this simple relation. Eddington (1970) performed experiments on both normal shocks and oblique shocks and also found good agreement with Equation (5.18). In Eddington's experiments, the topology of the air-water mixture varied. In some of his experiments the flow consisted of droplets dispersed in air rather than bubbles in water. Since the relations (5.15)–(5.17) make no distinction between these cases (only the gas content per unit volume occurs in the equations), they hold for a droplet-air mixture too, as long as there is no relative velocity between the phases. Therefore the structure of the shock waves investigated by Eddington (1970) may be quite different from the structure of the shocks in bubble-liquid mixtures, the discussion of which we postpone to the next section.

Witte (1969b) found shock waves in air-water flow that displayed a different topology behind the shock than in front. In front of these shocks, produced in a straight circular tube with controlled exit pressure, the mixture consisted of thin jets of water issuing from nozzles and entraining air, whereas behind the shock the mixture consisted of water with small air bubbles. The

pertinent Hugoniot relations are not (5.15)–(5.17), because there is, especially in front of the shock, a considerable velocity difference between the phases. Witte (1969b), who called these shocks “mixing shocks,” included this relative velocity in his theoretical analysis.

6. WAVES OF FINITE AMPLITUDE

Equations for nonlinear waves.—In Section 4 the propagation of acoustic waves was considered, for which purpose only the linearized equations of motion were needed. We need the complete equations now and write these down once more:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \quad (6.1)$$

$$\rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0 \quad (6.2)$$

$$\rho = \rho_f(1 - \beta) \quad (6.3)$$

$$\frac{p_g \beta}{1 - \beta} = \text{constant} \quad (6.4)$$

$$p_g R^3 = \text{constant} \quad (6.5)$$

$$p_g - p = \rho_f \left\{ R \frac{d^2 R}{dt^2} + \frac{3}{2} \left(\frac{dR}{dt} \right)^2 + \frac{4\nu_f}{R} \frac{dR}{dt} \right\} \quad (6.6)$$

It is assumed again that the gas in the bubbles behaves isothermally and that the bubbles move locally with the fluid. Upon linearization, these equations lead to the wave equation (4.11), as we have seen. Benjamin (1966), in a comment on van Wijngaarden (1966), where this equation was given, pointed out that for waves of finite amplitude the interaction between nonlinearity and dispersion is of the same kind, mathematically, as for long gravity waves on a fluid of finite depth. His idea was worked out in van Wijngaarden (1968), from which we borrow the following details.

Consider a pressure disturbance propagating through a mixture otherwise at rest. In the undisturbed state all quantities have the subscript 0. The magnitude of the pressure disturbance is specified by writing the pressure p_g in the gas bubbles, which are again supposed to be all of the same radius R_0 in the undisturbed state, as

$$\frac{p_g}{p_0} = 1 + \epsilon \xi$$

where ϵ is a small number.

The independent variables are rendered dimensionless by choosing a typical wavelength λ as length scale and the time λ/c_0 as time scale. The dimensionless velocity in the mixture is $w = u/\epsilon c_0 \beta_0$. When written in terms

of the dimensionless variables, equations (6.1)–(6.6) reduce, in the approximation one order in ϵ beyond the linear, to

$$\frac{\partial \xi}{\partial t} + \epsilon \beta_0 w \frac{\partial \xi}{\partial x} + \{1 + (2 - \beta_0) \epsilon \xi\} \frac{\partial w}{\partial x} = 0 \quad (6.7)$$

$$\frac{\partial w}{\partial t} + \epsilon \beta_0 w \frac{\partial w}{\partial x} + (1 - \beta_0 \epsilon \xi) \frac{\partial \xi}{\partial x} + \frac{\sigma}{3} \frac{\partial^3 \xi}{\partial x \partial t^2} + \frac{\nu_f c_0}{\lambda \omega_B^2 R_0^2} \frac{\partial^2 \xi}{\partial x \partial t} = 0 \quad (6.8)$$

The parameters in the last two terms on the left-hand side of Equation (6.8) are the same as in the linear equation (4.11). They represent the dispersion effect and viscous effect respectively, σ being given by

$$\sigma = \frac{3c_0^2}{\omega_B^2 \lambda^2} \quad (6.9)$$

It is assumed that σ and the viscous parameter $\nu_f c_0 / \lambda \omega_B^2 R_0^2$ are of order ϵ . Without the viscous term, which was not considered in van Wijngaarden (1968) and is included here for our purposes later, these equations are of the same form as the well-known Boussinesq equations for long gravity waves on a fluid of finite depth. In that case only two parameters occur, describing nonlinearity and dispersion respectively. With bubbly mixtures ϵ and σ fulfill these roles, but apart from these there is a third parameter, namely β_0 , reflecting the fact that there are four basic length scales—the amplitude of the bubble oscillations, the distance between the bubbles, the radius of the bubbles, and the wavelength. When $\epsilon \rightarrow 0$, the linear equation (4.11) may be reobtained from equations (6.7) and (6.8). On the other hand, the equations of one-dimensional gas dynamics remain when σ and ν_f are zero. In particular, a wave progressing into an undisturbed region, a simple wave, is governed by

$$\frac{\partial \xi}{\partial t} + (1 + \epsilon \xi) \frac{\partial \xi}{\partial x} = 0 \quad (6.10)$$

which clearly displays the steepening of a compression wave, sometimes denoted by amplitude dispersion. As is well known, this stems from two effects, the first being that the sound velocity is larger in a compressed part than in an expanding part, the second that in a compressed part the mass velocity is in the direction of propagation of the wave, in an expanding part against this. This latter effect, dominant in the case of gases, is relatively unimportant here because of the low value that β_0 usually has.

The equation for waves progressing in one direction only becomes, when the effect of dispersion is included,

$$\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} + \epsilon \xi \frac{\partial \xi}{\partial x} + \frac{\sigma}{6} \frac{\partial^3 \xi}{\partial x^3} = 0 \quad (6.11)$$

which is of the form of the famous Korteweg-de Vries equation, widely studied in recent years in connection with nonlinear wave propagation of various kinds (see Lick 1970). This equation has solutions representing uniform waves. These are known as cnoidal waves and solitary waves. Waves of this type may therefore occur in bubbly flows. Kogarko (1969) investigated the equation for uniform stationary waves where dispersion is not small. There is no analytical solution possible in that case. Kogarko (1969) could by analytical means show that, just as for the solitary wave solution of the Korteweg-de Vries equation, the speed of propagation exceeds the speed of sound in the undisturbed state. The Korteweg-de Vries equation has no solution that represents a shock wave, that is, a solution starting at a given pressure level and asymptotically reaching a higher pressure. When we take also the viscous term in Equation (6.8) into account and write $\tilde{\mu} = \nu_f c_0 / 2\lambda\omega_B^2 R_0^2$, we obtain

$$\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x} + \epsilon \xi \frac{\partial \xi}{\partial x} + \frac{\sigma}{6} \frac{\partial^3 \xi}{\partial x^3} + \tilde{\mu} \frac{\partial^2 \xi}{\partial x^2} = 0 \quad (6.12)$$

This equation, a combination of the Korteweg-de Vries equation and Burgers's equation (both discussed by Lick 1970), has solutions of the type of a shock wave. This suggests that, in contrast to what happens in gases, shock waves in liquid-bubble mixtures are structured by three mechanisms, namely convection, dispersion, and dissipation.

Structure of shock waves.—Hugoniot relations for normal shocks were derived, as mentioned in Section 5, by Campbell & Pitcher (1958), in whose experiments the bubbles were small enough, of radius about 10^{-4} m, to justify the assumption of temperature equilibrium between fluid and gas. Since in view of the great difference in heat capacity between these media the temperature rise of the fluid is very small, the mixture could be considered as isothermal, except for the calculation of the entropy increase across the shock. What might happen under conditions different from those in the experiments of Campbell & Pitcher (1958) was discussed by Parkin, Gilmore & Brode (1961), who established the Hugoniot relations for the case in which the bubbles are larger and can be regarded as thermally insulated.

For low values of β and incompressible fluid, the expression for the corresponding Mach number M_{ad} for air bubbles is

$$M_{ad}^2 = \frac{1}{1.4} \frac{p_1/p_0 - 1}{1 - (p_0/p_1)^{5/7}} \quad (6.13)$$

Parkin, Gilmore & Brode (1961) also discussed several other aspects of shock waves. They considered the effect of diffusion of the gas from the bubbles into the fluid, by which process the bubbles are dissolved. The time necessary for an air bubble to dissolve in water is of the order of magnitude R^2/D , where $D = 2 \times 10^{-9}$ m²/s is the diffusion coefficient of air in water.

Only for $R \simeq 10^{-7}$ m is this comparable with ω_B^{-1} , which is the time necessary for a bubble to change appreciably in volume. This led Parkin, Gilmore and Brode to the conclusion that the amount of dissolved air is negligible except perhaps for very strong shocks where bubbles break up into much smaller ones. Parkin, Gilmore and Brode were also the first to point out the importance for shock waves of the inertia effect discussed in Section 5. Basing their conclusions on order-of-magnitude arguments, they estimated the shock thickness as the shock speed multiplied by a time of order ω_B^{-1} and qualitatively predicted the structure of the shock as follows: "First comes the steep shock front with a thickness of a few bubble diameters and a pressure rise corresponding to the insulated bubble theory. This is followed by a slower pressure rise of 20 or 30 percent as the bubbles cool and the pressure approaches the thermal equilibrium for the same value of the shock velocity." This picture of the shock is not always the correct one, as we shall see. The present writer, unaware of the work of Parkin, Gilmore & Brode (1961), made an estimate (van Wijngaarden 1969) of the shock-thickness d along the same lines, arguing that the time of passage of a shock wave along a bubble, of order d/c , should at least equal ω_B^{-1} , from which follows

$$d > \frac{R_0}{\beta_0^{1/2}}$$

In van Wijngaarden (1969) the equations governing the structure of the steady shock wave were subsequently formulated. These are obtained from Equations (6.1)–(6.6). For $\beta \ll 1$ the velocities u and v of fluid and gas will be small with respect to the shock velocity U , so that d/dt may be approximated by Ud/dx , x running along the shock from the low-pressure side, where pertinent quantities have the subscript 0, to the high-pressure side, indicated with subscript 1. Including in the momentum equation the Kelvin impulse (which in Equations (6.1)–(6.6) is left out of account) and linearizing the right-hand side of Equation (6.6) with respect to average quantities $\bar{\beta}$ and $\bar{R}$, one obtains

$$u = U(\beta - \beta_0) \quad (6.15)$$

$$\frac{d}{dx} \left\{ p + \rho_f(1 - \beta)(U + u)^2 + \frac{1}{2} \rho_f \beta U(v - u) \right\} = 0 \quad (6.16)$$

$$p = p_0 - \frac{\rho_f \bar{R}^2 U^2}{3\bar{\beta}} \frac{d^2 \beta}{dx^2} - \frac{4\mu_f U}{3\bar{\beta}} \frac{d\beta}{dx} \quad (6.17)$$

In van Wijngaarden (1969) it is in addition assumed that the bubbles behave isothermally. It appears that the effect of relative translational motion is negligible when $d^2 \beta_0^2 / R_0^2$ is a small quantity, which may, in view of Equation (6.14), be expected to be the case for $\beta \ll 1$. The variation of β across the shock is under these circumstances governed by

$$(\beta_0 - \beta)(\beta_1 - \beta) = \frac{\bar{R}^2}{3} \frac{d^2\beta}{dx^2} + \frac{4\nu_f}{3U} \frac{d\beta}{dx} \quad (6.18)$$

This equation, which upon differentiation with respect to x is essentially the steady-state form of Equation (6.12), cannot be solved analytically but important information is gained from considering the outskirts of the wave. At the shockfront, linearizing with respect to $\beta_0 - \beta$ and putting $\bar{R}$ equal to R_0 gives the solution

$$\beta_0 - \beta \sim \exp \left[-\frac{2\nu_f x}{UR_0^2} + \left\{ \left(\frac{2\nu_f}{UR_0^2} \right)^2 + \frac{3(\beta_0 - \beta_1)}{R_0^2} \right\}^{1/2} x \right] \quad (6.19)$$

Since except for very low pressure ratios $\nu_f/UR_0 \ll (\beta_0 - \beta_1)^{1/2}$, the shock front is determined by the balance between convection of momentum and dispersion of momentum. It follows from Equation (6.19) that the thickness d of the shock, that is, the distance in which β rises from β_0 to β_1 , is of order $R_0/(\beta_0 - \beta_1)^{1/2}$, or in terms of the Mach number M_{is} defined by Equation (5.18), is

$$d \sim \frac{R_0}{\beta_0^{1/2}(1 - 1/M_{is}^2)^{1/2}} \quad (6.20)$$

For adiabatic behavior of the air bubbles, the same relation holds; however, M_{is} is replaced by M_{ad} , as given by Equation (6.13) (Noordzij 1971, van Wijngaarden 1971).

Upon linearization with respect to $\beta - \beta_1$, it follows from Equation (6.18) that at the back side of the shock wave

$$\beta - \beta_1 \sim \exp \left[-\frac{2\nu_f x}{UR_1^2} + \left\{ \left(\frac{2\nu_f}{UR_1^2} \right)^2 - \frac{3(\beta_0 - \beta_1)}{R_1^2} \right\}^{1/2} x \right] \quad (6.21)$$

When $2\nu_f/UR_1 < 3^{1/2}(\beta_0 - \beta_1)^{1/2}$, this represents waves with wavelength

$$\lambda = \frac{2\pi R_1}{\{3(\beta_0 - \beta_1) - (2\nu_f/UR_1)^2\}^{1/2}} \quad (6.22)$$

and logarithmic decrement $UR_1^2/2\nu_f$.

The corresponding expression for adiabatic behavior (Noordzij 1971, van Wijngaarden 1971), disregarding the small effect of viscosity, is

$$\lambda = \frac{2\pi R_1}{3^{1/2}\{\gamma(\beta_0 - \beta_1) - \beta_0(\beta_1/\beta_0 - M_{ad}^{-2})\}^{1/2}} \quad (6.23)$$

Relations similar to Equations (6.19) and (6.21) were independently obtained by Crespo (1969), who, apart from viscous effects, also took heat transfer between bubbles and fluid into account.

FIGURE 6. Pressure profile across a shock wave in a bubble-liquid mixture, from Noordzij (1971). Gas: air; fluid: aqueous solution of glycerine. $p_1/p_0 = 2.07$; $\beta_0 = 4.07 \times 10^{-2}$; $R_0 = 1.57 \times 10^{-3}$ m; $d = 2.8 \times 10^{-2}$ m; $\lambda = 3.6 \times 10^{-2}$ m.

A photograph in Campbell & Pitcher (1958) of the pressure profile across a shock with a pressure ratio 6, the bubble radii being of order 10^{-4} m, in a mixture of 5 percent gas content, does not show very much of the waves at the back side. Apparently, dissipation dominates over dispersion in such a way as to make the dispersion term in Equation (6.18) relatively unimportant. In that case Equation (6.18) has a hyperbolic-tangent type of solution, to which form Campbell's & Pitcher's (1958) profiles bear strong resemblance. Measurements with larger bubbles, with radius of order 10^{-3} m, carried out to verify the theory outlined above, were reported in Noordzij (1971) and van Wijngaarden (1971).

A typical pressure profile taken from Noordzij (1971) is shown in Figure 6. The shock waves, all of moderate strength $p_1/p_0 < 5$ and running in a mixture of a few percent volumetric gas fraction, display indeed an exponential steepening at the front together with waves at the back side.

The experimental results in Noordzij (1971) and van Wijngaarden (1971) show that the shock speed agrees more closely with Equation (6.13) for adiabatic bubbles than with Equation (5.18) for isothermal behavior, in accordance with the prediction by Parkin, Gilmore & Brode (1961), who, however, expected a monotonic and not a wavy rise of the pressure to its equilibrium value. The shock thickness, defined as the distance between the point

where the pressure starts to rise and the crest of the first wave, agrees quite well with Equation (6.20), with M_{ad} instead of M_{is} , the constant of proportionality being 3.1.

Noordzij (1971), using theoretical work by Johnson (1970) on an equation of the type of Equation (6.18), made a theoretical estimate for this constant, resulting in the value 3.9. Whereas the measured values of shock thickness, shock speed, and wavelength agree satisfactorily with theory, the experiments show an attenuation that is much stronger than the attenuation following from the theory in van Wijngaarden (1969). Heat flow from bubbles to liquid is dominated by heat resistance at the gas side, and not, as Crespo (1969) assumes, at the liquid side. Its contribution to attenuation was found large enough by Noordzij (1971) to account for the damping observed in his experiments. It also appears (Noordzij 1971) that the shock is, at low pressure ratios, smeared out in a more significant way by dissipation than suggested by Equation (6.19), so that Equation (6.20) needs correction at low pressure ratio.

It can be concluded that both types of shock waves, the one predicted by Parkin, Gilmore & Brode (1961) and the one described by Crespo (1969) and van Wijngaarden (1969), have been observed in experiments. With which range of governing parameters, such as gas content, bubble radius, and pressure ratio, these types are associated, remains to be investigated.

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