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An explicit Nash equilibrium for a market share attraction game

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ABSTRACT

In competitive marketing, the speed of generating the best price has become as critical as its reliability. In this study, we aim to design a practical marketing management tool. We consider a non-cooperative marketing environment with multiple substitute products, where total market size is moderately price-sensitive. The price-demand relations are determined by a market share attraction model, where the attraction of each product is a linear function of its price. The product's brand image is reflected in the parameters of this linear function. For the general case of multiple substitute products, we derive explicit expressions for the best-response functions. For the specific case of two substitute products, we derive closed form expressions for the prices at Nash equilibrium. These expressions help managers in changing their marketing instruments other than price, so as to obtain substantial individual profits. We show how our closed form Nash equilibrium enables the examination of the profit loss due to competition. Relevant for practice is the fact that our model can be easily calibrated. We provide a simple procedure for estimating the model parameters.

1. Introduction

Over the past decades, research related to pricing strategies has expanded steadily. After all, a valid model that clearly explains the demand as a function of price will help organizations to maximize profits and customer satisfaction.

In case of competitive environments with substitute products, the problem is even more challenging, since the price of a certain product affects the demand of its substitutes, as well. The price competitions between the producers of mobile phones, beverages, and cars are examples of non-cooperative competitions where each producer tries to optimize his individual profit. In all these examples, it is crucial for a competitor to work with a reliable demand model that accurately estimates the best price, given the substitute product prices.

The relationship between demand and price of each substitute product can be modelled in various ways. A major determinant of this relationship is the behavior of the "total market size" with respect to the price changes of the substitute products. In many cases, the total market size is a decreasing function of the overall substitute prices. For instance, if air transportation is too costly in all airway companies, then people prefer to use ground transportation, and the total market size of air transportation decreases.

In some cases, the total market size is only moderately affected by the average market price. For instance, in the pharmaceutical industry, the total market size of a set of antibiotics is almost fixed, since a patient has to purchase one of the substitute medicines with all sacrifices. Similarly, obligatory insurances mandated by governmental laws have fixed total market sizes, i.e., a citizen has to insure his car or house to any of the insurance companies, even at high insurance costs. Likewise, in a single-buyer multiple-producers example, a municipality has to allocate a fixed number of maintenance activities among several companies that compete in price to maximize their individual profits. When buying is obligatory, an upper bound on the market price is usually placed by a market authority such as the government.

When there is no obligation to buy, then - in many cases - total market size is still moderately price-sensitive. Consider, for example, basic necessities like detergents. Usually, such markets are dominated by only a few brands. These brands compete on price. This will not keep the customer from buying the product. At most, he will switch between leading brands temporarily.

In marketing theory, market share attraction (MSA) models are frequently used to express the relationship between demand and marketing mix variables. Here, the market share of a product is modeled as the ratio of the attraction of that product to the sum of all attractions, where an attraction is a function of the marketing mix variables, mostly the price.

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We consider in this study the problem of pricing competing substitute products when total market size is moderately price-sensitive. The price-demand relations are determined by an MSA model. We model the attraction of each product as a linear function of its price.

In a non-cooperative environment with two products, the linear attraction allows us to obtain closed form characterizations for the Nash equilibrium that are hard to be found in literature. In e-commerce environments, pricing decisions are made far more frequently than in brick-and-mortar systems, since the updates can be made easily in online catalogues. Here, the speed of generating the best price becomes as critical as its reliability. Electronic reverse auctions (ERAs) are other areas where generating fast and reliable prices is key. A closed form solution is the fastest key for the price-setter to obtain the best price, which becomes critical in digital competition.

Furthermore, our linear attraction model helps to overcome computational complexity of the standard MNL model. Unlike numerous MSA models that use a different parameter for each marketing mix variable, our linear attraction model only uses a single parameter that reflects the overall demand-effect of all marketing mix variables other than price. So, the decision maker only needs to estimate this parameter.

In contrast to the MNL model, our model offers a natural means to incorporate upper bounds on product prices. This is a value adding property especially in those marketing environments where upper bounds are mandated by governmental regulations. Traffic insurances or ERAs organized by municipalities are examples of these.

The reliability of our model depends on its fitness to the empirical price and demand data collected from the competitive market. In this study, we generate a practical way to estimate the model parameters, so that reliability can be evaluated easily by goodness-of-fit tests.

The organization of the paper is as follows: In Section 2, a literature survey is provided on MSA models and their applications in competitive environments. Moreover, we specify our contribution. In Section 3, we develop a model for pricing products in a non-cooperative environment. For the case of two product types, we derive a closed form expression for the Nash equilibrium vector. Managerial consequences are discussed in Section 4. Section 5 provides our conclusion.

2. Literature review

Price-demand models are comprehensively reviewed from different perspectives by Chen and Simchi-Levi [9]. Our study focuses on a specific subset of these models: we consider price-demand relations determined by an MSA model. In this section, we review MSA literature from two perspectives. First, in Section 2.1, from the *cause-effect* perspective: how do MSA models generate market shares given the marketing inputs? Next, in Section 2.2, from a *decision-oriented* perspective: how do MSA models operate as decision enablers? In Section 2.3, we motivate our model and outline its contribution.

2.1. Market share models from a cause-effect perspective

Market share models are commonly used in marketing research to model the relationship between market share and the marketing mix variables like price, quality, advertising expense, distribution effort, etc. This relationship may be linear, multiplicative, exponential, etc. with the danger that market shares do not add up to one. An alternative approach is found by Ghosh et al. [16] in the form of an *attraction* model where the total market shares add up to one.

Based on Kotler [20] and several others [13,21,22,28], Bell et al. [5] present the first model where the term 'attraction' is used to define the relative attractiveness of a firm compared to the others. Attraction may be a function of the firm's advertising expenditure and effectiveness, the price, and the quality of the product, the reputation of the company, the service given during and after purchase, the location of retail stores and much more. Based on a set of intuitive assumptions, Bell et al. [5] arrive at the following: let A_i be the attraction of supplier i, $i = 1, 2, \ldots, n$, then the market share m_i of supplier i is a normalization of the attractions formulated as $m_i = A_i / \sum_{j=1}^{n} A_j$.

It is customary to express the attraction as a function of the marketing mix variables. The most frequently used forms nowadays are: (i) the multinomial logit (MNL) model, (ii) the multiplicative competitive interaction (MCI) model. In both models, the attraction is expressed as an exponential function. In the MNL model [27], the argument is a linear combination of the marketing mix variables. In the MCI model [20], the argument is a linear combination of the logarithms of the marketing mix variables.

In this paper, we follow Gallego et al. [14] and express the attraction as a linear function of only the price. We motivate our choice in Section 2.3.

2.2. MSA models as decision enablers in a non-cooperative marketing environment

This section presents a literature review of MSA models in non-cooperative marketing decision environments. Relevant studies are classified according to the i) structure of the MSA models considered, and ii) type of competition, i.e., being duopoly or oligopoly. These features help us to identify studies of problems similar to ours. Furthermore, the strength of the findings is discussed in terms of the iii) existence and iv) uniqueness of a Nash equilibrium and v) existence of a closed form characterization of an equilibrium solution. Our findings are summarized in Table 1.

Although MCI was frequently used in the early MSA studies, MNL - or generalizations thereof - has been more popular after 2000s. The scope of the models is quite wide, allowing oligopolies with many players. In all these studies, price is one of the variables of the attraction function. Gallego et al. [14] are exceptional in the fact that they consider attraction as a function of price only. Other studies incorporate variables such as marketing expenditures [3,8,18,19,23], quality [10], and service level standards [2,6,26]. Still others generalize the idea of attraction [1,7,11,12,15,25]. The practicality of these studies can be criticized due to the complexity of their attraction functions for the price-setters.

The findings in many of the studies in Table 1 are strong, in the sense that existence of a Nash equilibrium is demonstrated. However, the uniqueness of the Nash equilibrium is only derived under specific conditions or not even addressed. Nevertheless, in virtually all of these studies, closed form characterizations for the Nash equilibrium do not exist. Carpenter et al. [8], Basuroy and Nguyen [3], and Favory [11] provide closed form solutions for only special cases. In our literature survey, Gallego et al. [14] and Gallego and Wang [15] are the only ones where the linear attraction function is listed among other forms of attraction models. However, they do not provide closed form characterizations, but show how the Nash equilibrium can be approximated by a tâtonnement process. We believe that the current study provides a valuable basis to fill this void.

2.3. Motivation of our model and outline of its contribution

Ever since their introduction, the original elements of the marketing mix (the 4Ps: product, price, place, promotion) have been criticized. Many researchers propose alternative approaches, by adding new elements to the original mix [4,17]. However, in all these approaches, one element of the mix prevails: the price that the customer is willing to pay.

Motivated by the latter, we design a marketing management tool that is transparent in form and easily understandable for the practitioner, which primarily depends on the concept of pricing without ignoring the overall effect of other marketing instruments.

In a similar spirit, Gallego et al. [14] consider a broad class of MSA models, where attraction is a function of *price only*. In particular, they discuss linear attraction and establish existence, uniqueness, and stability of the Nash equilibrium. However, as stated in Section 2.2, they do not provide a closed form expression. Following Gallego et al. [14], we

Table 1
Studies in MSA Models in Non-Cooperative Marketing Environments.

		Problem Definition Type of MSA Model	Type of Competition	Strength of the Findings Existence of a Nash Equilibrium	Uniqueness of a Nash Equilibrium	Closed Form Solution
No:	**References in Chronological Order**	**MNL/MCI/Other**	**Cooperative/ Duopoly/ Oligopoly**	**YES/NO**	**YES-UCC (Under Certain Conditions) /NO**	**YES/NO**
1	Karnani [19]	MCI	Oligopoly	NO	NO	NO
2	Monahan [23]	MCI	Duopoly	YES	NOT ADDRESSED	NO
3	Carpenter et al. [8]	MCI	Oligopoly	YES	NOT ADDRESSED	YES
4	Choi et al. [10]	MNL	Duopoly and Oligopoly	YES	NO	NO
5	Gruca et al. [18]	MCI	Oligopoly	YES	YES-UCC	NO
6	Basuroy and Nguyen [3]	MNL	Oligopoly	YES	YES-UCC	YES
7	So [26]	Generalized MCI	Oligopoly	YES	YES-UCC	NO
8	Bernstein and Federgruen [6]	Generalized MNL	Oligopoly	YES	YES-UCC	NO
9	Rhim and Cooper [25]	MNL	Oligopoly	YES	YES-UCC	NO
10	Gallego et al. [14]	Generalized Attractions Including LINEAR Attraction	Oligopoly	YES	YES-UCC	NO
11	Allon and Federgruen [2]	Generalized Attractions Including MNL Extensions	Oligopoly	YES	YES-UCC	NO
12	Federgruen and Yang [12]	Generalized Attraction	Oligopoly	YES	YES-UCC	NO
13	Aksoy et al. (2013)	MNL	Oligopoly	YES	YES-UCC	NO
14	Wang (2013)	Generalized Attraction	Cooperative (Single Firm)	Not Applicable for Cooperative Competition	Not Applicable for Cooperative Competition	Not Applicable for Cooperative Competition
15	Gallego and Wang [15]	General Nested MNL Attraction and LINEAR Attraction	Cooperative (Single Firm) and Oligopoly	YES	YES-UCC	NO
16	Boonen et al. [7]	Generalized Attraction	Oligopoly	YES	NOT ADDRESSED	NO
17	Favory [11]	Generalized Attractions Including MNL Extensions	Oligopoly	YES	YES	YES

examine a marketing attraction model, where the attraction A_i is a linearly decreasing function of the price P_i, i.e., $A_i = 1 - \alpha_i P_i$. Here we refer to the positive parameter α_i as the *price sensitivity*.

Although the attraction is formulated as a function of price only, our model does not ignore other marketing instruments. We employ the price sensitivity α_i as an indirect way of introducing marketing instruments into the pricing decisions. A small α_i shows that the firm is in a robust or stable state in terms of its brand image, quality, distribution performance, etc., allowing the firm to set a higher price without causing significant losses in its attraction. Conversely, a large α_i shows that the attraction is highly price-sensitive, which produces obstacles for increasing the price. In finding a proper α_i, fine-tuning of appropriate marketing instruments is crucial. In a way, one may conceive α_i as well as P_i as variables. Here, P_i is a short-term variable. Conversely, α_i is a long-term variable, which can be treated as a constant during the period under consideration. In that light, our model is based on two pillars: price and brand image, represented by P_i and $1/\alpha_i$, respectively. Brand image determines the latitude that the price-setter has, since for the attraction to be non-negative, the price P_i may be no larger than $1/\alpha_i$.

Let us summarize the assumptions as well as the contributions of our model. We consider a non-cooperative marketing environment with multiple substitute products where total market size is moderately price-sensitive. We assume that each producer has sufficient production capacity to fulfill the total market demand. We focus on price-demand relations determined by a market share attraction (MSA) model. We assume the attraction A_i of product type i to be a linearly decreasing function of price. We assume prices to have upper bounds.

For the general case of multiple substitute products, we derive explicit expressions for the best-response functions. For the specific case of two substitute products, we derive the product prices in a unique Nash equilibrium. Closed form characterizations for the Nash equilibrium are hard to be found in literature. However, the specific linear form of the attraction allows us to obtain closed form expressions as a composition of elementary functions of the model parameters. These expressions help managers in changing their marketing instruments other than price, so as to obtain substantial individual profits. They also enable the examination of the profit loss due to competition. Relevant for practice is the fact that our model can be easily calibrated. We provide a simple procedure for estimating the model parameters.

Essentially, we propose an MSA model which we deem to be more practical and faster than the existing ones. Thus, it can be easily applied by the price-setters in competitive environments. This is due to three idiosyncrasies of the model. First, its linear attraction that provides computational advantages, such as simple and explicit best-response functions. Second, the closed form expression for the Nash equilibrium, providing each of the two firms with the profit increase due to improving its brand image. Third, the existence of a single parameter in the attraction model and a practical tool to estimate it.

3. Model development

In this section, we introduce a framework for optimizing pricing decisions in a supply chain with multiple products, multiple suppliers of these products, and a single market for these products. We assume the total market demand to be either fixed or (at most) moderately price-sensitive. Think, e.g., of insurance policies that are legally obliged (see Section 1).

We consider a *non-cooperative* environment. The objective of each product type is to maximize its own profit. Without harming generality, we assume that each of the competitive products under consideration has its own supplier, called *firm* in the sequel. Competition is therefore between the various firms that aim for conquering the market.

This section is organized as follows. Section 3.1 introduces our Market Share Attraction model as well as the notation. The focus of our analysis is on the case of only two product types. It is presented in Section 3.2. Section 3.3 briefly considers the case of more than two product types.

3.1. Introductory remarks and notation

Throughout, we use the following notation:

Indices

i: index of product type (and of the corresponding firm) ($i = 1, 2, \ldots, n$)

Parameters

c_i: production costs per unit for product type i

α_i: *price sensitivity*, coefficient indicating decrease of attraction with price for product type i

The above parameters are positive. For future reference, we assume that $0 < \alpha_i < \frac{1}{c_i}$. The latter restriction guarantees that the maximum obtainable attraction defined below is positive for prices that exceed production costs (see (2)). Incidentally, this restriction is identical to restriction (iii) in Proposition 1 of Gallego et al. [14], where it is shown to ensure existence, uniqueness, and stability of the Nash equilibrium.

Decision Variables

P_i = price per unit for product type i

Auxiliary Variables

Key to our analysis is the following Market Share Attraction model where the attraction is a linearly decreasing function of the price.

$A_i = 1 - \alpha_i P_i$ = attraction of product type i

$m_i = \frac{A_i}{\sum_{j=1}^{n} A_j}$ = market share of product type i

d = total market demand during the period under consideration

$Q_i = m_i d$ = quantity produced (to the market) of product type i

Throughout, we assume that for each product type, separately, there is enough capacity to fulfill the total market demand d. The coefficient α_i measures factors additional to price, such as quality, social awareness, sustainability, etc. The higher these factors score for the product type in question, the smaller the corresponding value of α_i. Note that, in this way, the parameter α_i has been fixed externally. On the other hand, c_i - the production cost per unit - has been fixed by the firm. Evidently, the total market demand d is depending on the prices P_i. This paper considers those markets where this dependence is only moderate (see the examples in Section 1). We assume that the parameters α_i and c_i remain fixed during the period under consideration. Crucial is the setting of the prices P_i.

Since both profit and attraction are assumed to be non-negative, we only consider values of the price P_i such that

$$c_i \leq P_i \leq \frac{1}{\alpha_i} \tag{1}$$

Consequently, m_i and Q_i are non-negative, whereas A_i satisfies

$$0 \leq A_i \leq a_i$$

where the *maximum obtainable attraction* a_i is given by

$$a_i \equiv 1 - \alpha_i c_i \ \epsilon \ (0, 1) \tag{2}$$

In the sequel, the parameters a_i play a crucial role. In particular, the Nash equilibrium vector is expressed solely in terms of the a_i (see Section 3.2.3).

Finally, to exclude that all attractions vanish, we require that

$$\sum_{i=1}^{n} A_i > 0. \tag{3}$$

We now define the *profit potential* π_i as the least upper bound on the profit that can be obtained per unit from a product type i. From (1) it is clear that

$$\pi_i \equiv \frac{1}{\alpha_i} - c_i = \frac{a_i}{\alpha_i} \tag{4}$$

Note that this profit potential can only be realized approximately as P_i approaches $\frac{1}{\alpha_i}$ from below, since putting P_i equal to $\frac{1}{\alpha_i}$ would wipe out the production quantity Q_i.

Central in our discussion is the profit per product type i given by $(P_i - c_i)Q_i = (P_i - c_i)m_i d$. As mentioned earlier, we assume that the total market demand d is moderately price-sensitive. Consequently, in a broad P_i range, d remains (nearly) constant. Hence, for purposes of profit analysis, d can be left out of the equation. Instead, let us focus henceforth on the profit made for product type i per overall unit demanded, which can conveniently be written in terms of the attractions as

$$\begin{aligned}(P_i - c_i)m_i &= \frac{1}{\sum_{j=1}^{n} A_j}(P_i - c_i)A_i \\ &= \frac{1}{\alpha_i}\ \frac{1}{\sum_{j=1}^{n} A_j}\ (1 - A_i - \alpha_i c_i)A_i \\ &= \frac{1}{\alpha_i}\ \frac{1}{\sum_{j=1}^{n} A_j}\ (a_i - A_i)A_i\end{aligned} \tag{5}$$

By changing the decision variables in the profit function from P_i to A_i we obtain several computational benefits, as will be shown in the subsequent analysis.

The next section shows that - in a non-cooperative case of two competing product types - competition is leading. Instead of dominance of one product type, there will be co-existence in the form of a Nash equilibrium. For the equilibrium vector we derive a closed form expression in terms of the maximum obtainable attractions.

3.2. Pricing two competing product types in a non-cooperative environment

In this section we consider a non-cooperative environment where the objective of each product type is to maximize its own profit. We confine our discussion to the case of only two competing products. We analyze the above price competition problem by modeling it as a strategic game in which:

(i) the players are the two firms
(ii) each player i can choose its attraction $A_i = 1 - \alpha_i P_i$ by specifying its price P_i
(iii) on the basis of the choices made in (ii), the payoff (profit) of player i is (cf (5))

$$\Pi_i(A_1, A_2) \equiv \frac{1}{\alpha_i}\ \frac{1}{A_1 + A_2}\ (a_i - A_i)A_i \tag{6}$$

The competition game starts as soon as one of the two firms decides to change its price setting. The other firm will react by adapting its own prices. And so on.

Section 3.2 is organized as follows. In Section 3.2.1, it is shown that our model can be calibrated in an effortless way. In Sections 3.2.2 and 3.2.3, we use our Market Share Attraction approach, introduced in Section 3.1, to analyze the price setting process. In Section 3.2.2, we demonstrate the existence of a unique non-trivial Nash-equilibrium and indicate how to approximate it iteratively. Remarkably, our model permits us to derive a closed form expression for the Nash equilibrium. We do so in Section 3.2.3. In Section 3.2.4, we explore monotonicity and magnitude of the equilibrium attractions, as well as the equilibrium prices.

3.2.1. Calibrating the model

Above, in (ii) and (iii), we tacitly assume that the players know all price sensitivities. In practice, this means that somehow the firms must have estimated these. Surprisingly, this estimation can be done in a straightforward way. To see this, note that $m_1 A_2 = m_2 A_1$. Inserting $A_i = 1 - \alpha_i P_i$, we find that $C_1\alpha_1 + C_2\alpha_2 = C_3$, where $C_1 = m_2 P_1$, $C_2 = -m_1 P_2$, and $C_3 = m_2 - m_1$. Since the current price levels and market shares are known to the firms, so are the coefficients C_k. Thus, we obtain a first linear equation for the price sensitivities. Undoubtedly, there have been price changes in the recent past, or else they still may be created. In both cases, the associated price levels and market shares yield the coefficients of a second linear equation for the α_i. Solving this simple system of two linear equations with two unknowns, we find values for the α_i which can be used in the competition game. Obviously, when further price changes occur, the associated price levels and market shares yield the coefficients of a series of linear equations, which are approximately fulfilled and yield an estimate of the α_i by linear regression (cf. Section 4.1).

To illustrate, suppose the current prices set by firm 1 and firm 2 are 44 and 50, leading to market shares 0.4 and 0.6, respectively. Since the cheaper firm has the lower market share, it is to be expected that $\alpha_2 < \alpha_1$. Suppose firm 1 lowers its price to 42, while firm 2 retains its price. Suppose this leads to market shares 0.45 and 0.55, respectively. Solving the system of linear equations, we find $\alpha_1 = 0.0189$ and $\alpha_2 = 0.0150$.

3.2.2. Pricing two competing product types

In this section, let us analyze the price competition problem for two product types for which there is ample capacity. Each firm i can perform an action A_i, which in practice is given implicitly, since it refers to the attraction $A_i = 1 - \alpha_i P_i$ evoked by setting the price P_i.

Let us recall the following two important concepts from game theory. A vector $A^\infty = (A_1^\infty, A_2^\infty)$ is a Nash equilibrium iff

$$\Pi_I\left(A_1^\infty, A_2^\infty\right) \geq \Pi_I\left(A_1, A_2^\infty\right) \ \forall A_1$$

$$\Pi_2\left(A_1^\infty, A_2^\infty\right) \geq \Pi_I\left(A_1^\infty, A_2\right) \ \forall A_2$$

Thus, at a Nash equilibrium, no one firm can do better by unilaterally changing its strategy.

The *best-response function* of a firm specifies the action that maximizes its payoff for any given action of the other firm. Denote the best-response function of firm i by r_i. Clearly, a Nash equilibrium (A_1^∞, A_2^∞) satisfies: (i) $A_1^\infty = r_1(A_2^\infty)$ and (ii) $A_2^\infty = r_2(A_1^\infty)$.

We now have the following:

Proposition 1:

For the price competition problem for two product types with ample capacity, the best-response functions r_i, $i = 1, 2$, reacting on an opposite action x, are given by

$$r_i(x) = -x + \sqrt{(a_i + x)x} \text{ for } 0 \leq x \leq 1 \tag{7}$$

Proof:

Assume firm 1 sets a price P_1^0, giving $A_1^0 = 1 - \alpha_1 P_1^0 > 0$. In view of (6), the profit objective to be maximized by product type 2 then becomes

$$\frac{1}{\alpha_2} \frac{1}{A_1^0 + A_2} (a_2 - A_2)A_2 \tag{8}$$

Now, write $A \equiv A_1^0 + A_2$, then (8) can be rewritten as

$$\frac{1}{\alpha_2} \frac{\left(a_2 - A + A_1^0\right)\left(A - A_1^0\right)}{A} \equiv \frac{1}{\alpha_2} G(A)$$

Introducing $B = a_2 + A_1^0$ yields:

$$G(A) = \frac{(B - A)\left(A - A_1^0\right)}{A} = B + A_1^0 - \frac{BA_1^0}{A} - A$$

Differentiating with respect to A gives: $\frac{dG}{dA} = \frac{BA_1^0}{A^2} - 1$. Hence, G is maximal for $A^* = \sqrt{BA_1^0} > A_1^0$. Thus firm 2 sets a price $P_2^0 = \frac{1}{\alpha_2}(1 - A_2^0)$ with $A_2^0 \ \epsilon \ (0, 1)$ given by

$$A_2^0 = A^* - A_1^0 = \sqrt{BA_1^0} - A_1^0 = -A_1^0 + \sqrt{(a_2 + A_1^0)A_1^0} = r_2\left(A_1^0\right)$$

Next, analogously to the above, firm 1 sets a price corresponding with

$$A_1^1 = -A_2^0 + \sqrt{\left(a_1 + A_2^0\right)A_2^0} = r_1\left(A_2^0\right). \ \blacksquare$$

From (7) we see that the best-response function $r_i(x)$ is monotonically increasing, showing that the best-response attraction of firm i increases as a result of a higher attraction provided by the other firm. The process where both firms keep on setting prices - by iterative application of the best-response strategy - is called *tâtonnement.* In Gallego et al. [14] it is proven - for a broad class of attraction models including ours - that the sequence of prices obtained by tâtonnement globally converges to a unique Nash equilibrium vector. Confined to our case, a proof requires far less abstraction. Let us therefore present the proof below, also to keep our paper self-contained.

Proposition 2:

Consider the tâtonnement price sequence for firm 1:

$$A_1^{n+1} = \varphi\left(A_1^n\right), \ n = 0, 1, 2, \cdots \quad A_1^0 \in (0, 1] \textit{ given} \tag{9}$$

where $\varphi \equiv r_1 \circ r_2$. The sequence (9) *converges monotonically to a positive fixed point A_1^∞ of φ. Moreover, the vector $(A_1^\infty, r_2(A_1^\infty))$ is the unique non-trivial Nash equilibrium.*

Proof:

Clearly r_i: $[0, 1] \rightarrow [0, 1]$. Hence φ: $[0, 1] \rightarrow [0, 1]$. Let us prove that the price setting sequence (9) converges. It is readily verified that r_i in (7) has the four properties:

1) r_i is continuous
2) r_i is monotonically increasing
3) r_i is concave
4) $r_i(0) = 0$

Taking into account that an increasing concave function of a concave function is concave, we find that $\varphi = r_1 \circ r_2$ has - like r_i - the properties 1) up to 4).

Note that, for small values of x, one has that $r_i(x) > x$. Hence $\varphi(x) > x$ for small x. Since φ: $[0, 1] \rightarrow [0, 1]$ is concave and since $\varphi(x) > x$ for small x, the function φ has exactly one fixed point on the interval $(0, 1]$.

Now consider the sequence (9). A simple induction proof gives us that the sequence (9) is monotonic. Since the sequence is also bounded $(0 \leq A_1^n \leq 1)$, it converges, say to A_1^∞. Then continuity of φ yields: $A_1^{n+1} = \varphi(A_1^n) \rightarrow \varphi(A_1^\infty)$ and so $\varphi(A_1^\infty) = A_1^\infty$ which means that A_1^n converges to a fixed point of φ.

In view of the fact that $\varphi(x) > x$ for small x, the sequence A_1^n does not converge to the fixed point $\varphi(0) = 0$, but instead to the unique positive fixed point of φ.

Conclusion: The sequence (9) converges monotonically to a positive fixed point A_1^∞ of φ. Now, define $A_2^\infty = r_2(A_1^\infty)$. Then $A_1^\infty = r_1 r_2(A_1^\infty) = r_1(A_2^\infty)$. Hence, $(A_1^\infty, r_2(A_1^\infty))$ is the unique non-trivial Nash equilibrium. $\blacksquare$

Note that the sequence A_1^n decreases monotonically (and hence the associated profit increases monotonically) iff firm 1 sets a price P_1^0 low enough to guarantee that $A_1^0 > A_1^\infty$. In that case also the sequence $A_2^n = r_2(A_1^n)$ decreases monotonically (by monotony of r_2).

The present section tells us how to approximate the Nash equilibrium iteratively. Closed form representations for the Nash equilibrium are rarely obtained in literature (see Section 2.2). Therefore, the closed form expression for the Nash equilibrium that emerges in Section 3.2.3 comes as a surprise.

3.2.3. A closed form expression for the Nash equilibrium

The next theorem gives a closed form expression for the Nash equilibrium.

Theorem 1:

For the price competition problem for two product types, with ample capacity, the unique non-trivial Nash equilibrium is the vector $(A_1^\infty, A_2^\infty) = (A_1^\infty, r_2(A_1^\infty))$ with

$$A_1^\infty = \frac{2}{9}a_1 - \frac{4}{9}a_2 + 2\sqrt{\frac{p}{3}}\cos\left(\frac{1}{3}\arccos\left(-\frac{3q}{2p}\sqrt{\frac{3}{p}}\right) - \frac{2\pi}{3}\right) \tag{10}$$

where

$$p = \frac{1}{27}\left(4a_1^2 + 20a_1a_2 + 16a_2^2\right)$$

$$q = \left(\frac{1}{27}\right)^2\left(-16a_1^3 + 123a_1^2a_2 + 240a_1a_2^2 + 128a_2^3\right)$$

Here, as before, $a_i = 1 - \alpha_i c_i$ denotes the maximum obtainable attraction. Moreover, A_2^∞ is given by (10) with the indices $i = 1$ and $i = 2$ interchanged. As for magnitude, the value of A_1^∞ is bounded by:

$$0 < A_1^\infty < \frac{1}{2}a_1 - \frac{a_1^2}{20a_1 + 32a_2} \tag{11}$$

The profit in equilibrium is given by:

$$\Pi_i\left(A_1^\infty, A_2^\infty\right) = \frac{1}{\alpha_i}\left(a_i - 2A_i^\infty\right) \tag{12}$$

The equilibrium market share m_i^∞ is given by:

$$m_i^\infty = \frac{a_i - 2A_i^\infty}{a_i - A_i^\infty} \tag{13}$$

In the special case that $a_1 = a_2$ one has that $A_1^\infty = A_2^\infty = \frac{1}{3}a_1$.

Proof:

The non-trivial Nash equilibrium (A_1^∞, A_2^∞) is characterized by (i) $A_1^\infty > 0$ (ii) $A_1^\infty = r_1\ (A_2^\infty)$ and (iii)$A_2^\infty = r_2\ (A_1^\infty)$. So

$$A_1^\infty = -A_2^\infty + \sqrt{\left(a_1 + A_2^\infty\right)A_2^\infty} \tag{14}$$

$$A_2^\infty = -A_1^\infty + \sqrt{\left(a_2 + A_1^\infty\right)A_1^\infty} \tag{15}$$

In (14), put $w = \sqrt{(a_1 + A_2^\infty)A_2^\infty}$. Then, (14) and (15) imply that $w = A_1^\infty + A_2^\infty = \sqrt{(a_2 + A_1^\infty)A_1^\infty}$. So

$$w^2 = \left(a_1 + A_2^\infty\right)A_2^\infty = \left(a_2 + A_1^\infty\right)A_1^\infty$$

Hence

$$\left(a_2 + A_1^\infty\right)A_1^\infty = \left(a_1 + w - A_1^\infty\right)\left(w - A_1^\infty\right)$$

i.e.,

$$w^2 = w^2 + w\left(a_1 - 2A_1^\infty\right) - A_1^\infty\left(a_1 - A_1^\infty\right)$$

Hence

$$w\left(a_1 - 2A_1^\infty\right) = A_1^\infty\left(a_1 - A_1^\infty\right) \tag{16}$$

Recall that

$$0 < A_1^\infty \le a_1 = 1 - \alpha_1 c_1 \tag{17}$$

Combining (16) and (17) we find that $a_1 - 2A_1^\infty \ge 0$, so $0 < A_1^\infty \le \frac{1}{2}a_1$. Squaring (16) yields:

$$\left(a_2 + A_1^\infty\right)A_1^\infty\left(a_1 - 2A_1^\infty\right)^2 = \left(A_1^\infty\right)^2\left(a_1 - A_1^\infty\right)^2$$

So, A_1^∞ is root of the equation $f(x) = 0$ with $f(x) = (a_2 + x)(2x - a_1)^2 - x(x - a_1)^2$.

For future reference, note that $f(0) = a_1^2 a_2 > 0$ and $f\left(\frac{1}{2}a_1\right) < 0$. Let us write out the third-degree polynomial f in full: $f(x) = ax^3 + bx^2 + cx + d$ with $a = 3, b = -2a_1 + 4a_2$, $c = -4a_1a_2$, and $d = a_1^2a_2$.

Since $b^2 - 3ac > 0$, the derivative $f'(x) = 3ax^2 + 2bx + c$ has two distinct zeros x_1 and x_2 satisfying

$$x_1 = \frac{-b - \sqrt{b^2 - 3ac}}{3a} < \frac{-b - \sqrt{b^2}}{3a} \le 0,$$

and

$$x_2 = \frac{-b + \sqrt{b^2 - 3ac}}{3a} > \frac{-b + \sqrt{b^2}}{3a} \ge 0.$$

To obtain an upper bound for x_2, note that

$$b^2 - 3ac = B - A \text{ with } B = \left(b + \frac{9}{2}a_1\right)^2 \text{ and } A = \frac{9}{4}a_1^2.$$

Since $A < B$, one has

$$\sqrt{b^2 - 3ac} \le \sqrt{B}\left(1 - \left(\frac{A}{2B}\right)\right)$$

Thus

$$x_2 = \frac{-b + \sqrt{b^2 - 3ac}}{3a} \le \frac{1}{2}a_1 - \frac{a_1^2}{20a_1 + 32a_2}$$

Consequently, the sign of $f(x)$ is as follows:

f increases from $-\infty$ to $f(x_1) > f(0) > 0$. Hence f has its smallest zero $\bar{x}_1$ bounded by $\bar{x}_1 < x_1 < 0$. Next, f decreases from $f(x_1)$ to $f(x_2)$ where $0 < x_2 < \frac{1}{2}a_1$ and f increases for $x > x_2$. Thus $f(x_2) < f\left(\frac{1}{2}a_1\right) < 0$. Hence, f has its second zero $\bar{x}_2$ between 0 and x_2 and its third zero $\bar{x}_3 > \frac{1}{2}a_1$.

So, since $0 < A_1^\infty \le \frac{1}{2}a_1$, the root corresponding with a Nash equilibrium is $A_1^\infty = \bar{x}_2$ which is found to be as stated in the theorem by using the classical formula of Viète (see [24]). From the fact that $0 < \bar{x}_2 < x_2 < \frac{1}{2}a_1$ we infer the inequality (11).

Furthermore, combining (6) with (16) we find:

$$\Pi_i\left(A_1^\infty, A_2^\infty\right) = \frac{1}{\alpha_i w}\left(a_i - A_i^\infty\right)A_i^\infty = \frac{1}{\alpha_i}\left(a_i - 2A_i^\infty\right)$$

Expression (13) for the market shares is a direct consequence of (16). Finally, when $a_1 = a_2$, then by combining (14) and (15) we find that $A_1^\infty = A_2^\infty = \frac{1}{3}a_1$. ■

Remarkably, all relevant entities in Theorem 1 are expressed in explicit form in only two parameters, namely $a_i = 1 - \alpha_i c_i$. Once production cost $c_i$and price sensitivity α_i are known, then all equilibrium entities are known as well. To illustrate this phenomenon, we present in Table 2 the equilibrium market share m_1^∞ for some key values of a_i.

Table 2 shows that enlarging the maximum obtainable attraction a_1 - in order to obtain a larger equilibrium market share m_1^∞ - is most effective in the lower a_1 region. One way to enlarge a_1 is by reducing the production cost c_1. Reducing c_1 with 22.2% will enlarge a_1 from 0.1 to 0.3.

The explicit expressions for the equilibrium entities in Theorem 1 enable us to derive monotonicity and magnitude properties which will be useful in discussing managerial model consequences in Section 4.

Table 2
The equilibrium market share m_1^∞ as a function of $a_i = 1 - \alpha_i c_i$.

a_1 \ a_2	**0.1**	**0.3**	**0.5**	**0.7**	**0.9**
0.1	0.50	0.34	0.28	0.24	0.22
0.3	0.66	0.50	0.42	0.38	0.34
0.5	0.72	0.58	0.50	0.45	0.41
0.7	0.76	0.62	0.55	0.50	0.46
0.9	0.78	0.66	0.59	0.54	0.50

3.2.4. Monotonicity and magnitude properties for equilibrium entities

Although it may not be immediately apparent from the explicit expression (10), we have the following monotonicity properties, which will be useful further on.

Proposition 3:

For fixed $a_2 \in (0,1)$, the equilibrium attraction A_1^∞ is a monotonically increasing function of a_1. In particular

$$0 < \frac{\partial A_1^\infty}{\partial a_1} < \frac{1}{2} \ \forall a_1, a_2 \in (0,1) \tag{18}$$

For fixed $a_1 \in (0,1)$, the equilibrium attraction A_1^∞ is a monotonically increasing function of a_2, i.e.,

$$\frac{\partial A_1^\infty}{\partial a_2} > 0 \ \forall a_1, a_2 \in (0,1)$$

Proof:

Since $A_2^\infty < a_2$ and since r_1 is monotonically increasing, we find that $A_1^\infty = r_1(A_2^\infty) \ < \ r_1(a_2) = -a_2 + \sqrt{(a_1+a_2)a_2}$. Hence

$$\left(A_1^\infty + a_2\right)^2 < (a_1+a_2)a_2 \tag{19}$$

From the proof of Theorem 1 we recall that A_1^∞ is a root of the equation $f(x) = 0$ with $f(x) = (a_2 + x)(2x - a_1)^2 - x(x - a_1)^2$. Moreover, it was shown that $f'(A_1^\infty) < 0$. Differentiating $f(A_1^\infty) = 0$ with respect to a_1 and using (19) we obtain

$$f'\left(A_1^\infty\right)\frac{\partial A_1^\infty}{\partial a_1} = 2\left[\left(A_1^\infty + a_2\right)^2 - (a_1+a_2)a_2\right] < 0 \tag{20}$$

Hence $\frac{\partial A_1^\infty}{\partial a_1} > 0$. From (11) we know that $0 < A_1^\infty < \frac{1}{2}a_1$.

Consequently $f'(A_1^\infty) - 4[(A_1^\infty + a_2)^2 - (a_1 + a_2)a_2] = A_1^\infty(5A_1^\infty - 4a_1) < 0$. Combining the latter with (20), we find that $\frac{\partial A_1^\infty}{\partial a_1} < \frac{1}{2}$.

Finally, differentiating $f(A_1^\infty) = 0$ with respect to a_2 yields:

$$f'\left(A_1^\infty\right)\frac{\partial A_1^\infty}{\partial a_2} = -\left(2A_1^\infty - a_1\right)^2 < 0$$

whence $\frac{\partial A_1^\infty}{\partial a_2} > 0$. ■

Intuitively, one would say: the better the brand image, the larger the market share and the larger the price that can be asked for the product. In other words, when α_1 gets small, market share and price will be rising, when at Nash equilibrium. The next proposition confirms this intuition. It will play a key role in our managerial analysis in Section 4.

Proposition 4:

For fixed α_2, the equilibrium market share m_1^∞ and the equilibrium price P_1^∞ are monotonically decreasing functions of α_1, i.e.,

$$\frac{\partial m_1^\infty}{\partial \alpha_1} < 0 \ \textit{and} \ \frac{\partial P_1^\infty}{\partial \alpha_1} < 0 \tag{21}$$

In addition

$$\frac{\partial m_I^\infty}{\partial \alpha_2} > 0 \text{ and } \frac{\partial P_I^\infty}{\partial \alpha_2} > 0 \tag{22}$$

Proof:

Since $\frac{\partial m_1^\infty}{\partial \alpha_1} = -c_1\frac{\partial m_1^\infty}{\partial a_1}$, it suffices to show that $\frac{\partial m_1^\infty}{\partial a_1} > 0$. By (13) we have $m_1^\infty = \frac{a_1 - 2A_1^\infty}{a_1 - A_1^\infty}$. Differentiating with respect to a_1 we find $(a_1 - A_1^\infty)^2\frac{\partial m_1^\infty}{\partial a_1} = A_1^\infty - a_1\frac{\partial A_1^\infty}{\partial a_1}$, which is positive since by (20)

$$\mathrm{f}'\left(\mathrm{A}_1^\infty\right)\left(\mathrm{A}_1^\infty - \mathrm{a}_1\frac{\partial \mathrm{A}_1^\infty}{\partial \mathrm{a}_1}\right)$$

$$= \mathrm{A}_1^\infty \mathrm{f}'\left(\mathrm{A}_1^\infty\right) - 2\mathrm{a}_1\left[\left(\mathrm{A}_1^\infty + \mathrm{a}_2\right)^2 - (\mathrm{a}_1+\mathrm{a}_2)\mathrm{a}_2\right] = 3\mathrm{f}\left(\mathrm{A}_1^\infty\right) - \mathrm{a}_2\left(2\mathrm{A}_1^\infty - \mathrm{a}_1\right)^2$$
$$= -\mathrm{a}_2\left(2\mathrm{A}_1^\infty - \mathrm{a}_1\right)^2 < 0.$$

Differentiating $P_1^\infty = (1 - A_1^\infty)/\ \alpha_1$ with respect to α_1 and using (18), we obtain

$$\alpha_1^2\frac{\partial P_1^\infty}{\partial \alpha_1} = A_1^\infty - 1 - \alpha_1\frac{\partial A_1^\infty}{\partial \alpha_1} = A_1^\infty - 1 + (1-a_1)\frac{\partial A_1^\infty}{\partial a_1} < A_1^\infty - a_1 < 0$$

As for the α_2 derivatives, note that $\frac{\partial P_1^\infty}{\partial \alpha_2} = \frac{c_2}{\alpha_1}\frac{\partial A_1^\infty}{\partial a_2} > 0$, whereas $\frac{\partial m_1^\infty}{\partial \alpha_2} = -\frac{\partial m_2^\infty}{\partial \alpha_2} > 0$. ■

Now that we have explored monotonicity, let us consider the magnitude of attraction and price in equilibrium.

Proposition 5:

The equilibrium attraction A_i^∞ satisfies:

$$0 < A_i^\infty < \frac{1}{3} \tag{23}$$

The equilibrium price P_i^∞ satisfies:

$$\frac{2}{3\alpha_i} < P_i^\infty < \frac{1}{\alpha_i} \tag{24}$$

Proof:

For symmetry reasons, we may confine ourselves to $i = 1$. Let *RHS* (a_1, a_2) denote the right-hand side of (10). Since $\frac{\partial A_1^\infty}{\partial a_1}$ and $\frac{\partial A_1^\infty}{\partial a_2}$ are both positive for $a_1, a_2 \in (0,1)$, it suffices to show that *RHS* $(1,1) = \frac{1}{3}$. In view of the proof of Theorem 1, we are done if we can show that $\frac{1}{3}$ is the second root of the equation $f(x) = 0$ with $f(x) = (1 + x)(2x-1)^2 - x(x-1)^2$. The latter follows directly by rewriting f as $f(x) = (3x - 1)(x^2 + x - 1)$. ■

Fig. 1 illustrates the spatial behavior of A_1^∞ as a function of a_1 and a_2. It shows that the attraction of product type 1 at Nash equilibrium increases with the maximum attainable attractions of both products and converges to its least upper bound of 1/3 given by (23). In Section 4, we attempt to modify the maximum attainable attractions $a_i = 1 - \alpha_i c_i$ by changing the parameters α_i and c_i. This provides further managerial insights on the dynamic structure of the game.

3.3. Pricing more than two competing product types in a non-cooperative environment

A practical enhancement of our model is to increase the number of firms in the non-cooperative case to $n > 2$. From Gallego et al. [14] we know that there exists a unique and stable Nash equilibrium. Let us discuss the possibility of obtaining closed form solutions for the Nash equilibrium.

For $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in [0,1]^n$, let $\boldsymbol{x}^{(i)} \equiv (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in [0,1]^{n-1}$ and let $\|\boldsymbol{x}^{(i)}\|_1 \equiv \sum_{j=1, j\neq i}^n x_j$. Reasoning as in the proof of Proposition 1, one readily finds:

Proposition 6:

For the price competition problem for n product types with ample capacity the best-response functions r_i, $i = 1, 2, \ldots, n$ reacting on an opposite action $\boldsymbol{x}^{(i)} \in [0,1]^{n-1}$ are given by

$$r_i\left(\boldsymbol{x}^{(i)}\right) = -\|\boldsymbol{x}^{(i)}\|_1 + \sqrt{\left(a_i + \|\boldsymbol{x}^{(i)}\|_1\right)\|\boldsymbol{x}^{(i)}\|_1}$$

Hence, a non-trivial Nash equilibrium $A^\infty = (A_1^\infty, A_2^\infty, \ldots, A_n^\infty)$ is characterized by

(i) $A_i^\infty > 0$ and (ii) $A_i^\infty = r_i(A^{\infty(i)})$ for $i = 1, 2, \ldots, n$

In view of Section 3.2.3, one may wonder whether in the $n > 2$ case there are closed form representations for the Nash equilibrium. Let us illustrate why we deem the latter unlikely.

Consider the case $n = 3$:

$$A_1^\infty = -A_2^\infty - A_3^\infty + \sqrt{\left(a_1 + A_2^\infty + A_3^\infty\right)\left(A_2^\infty + A_3^\infty\right)}$$

$$A_2^\infty = -A_1^\infty - A_3^\infty + \sqrt{\left(a_2 + A_1^\infty + A_3^\infty\right)\left(A_1^\infty + A_3^\infty\right)}$$

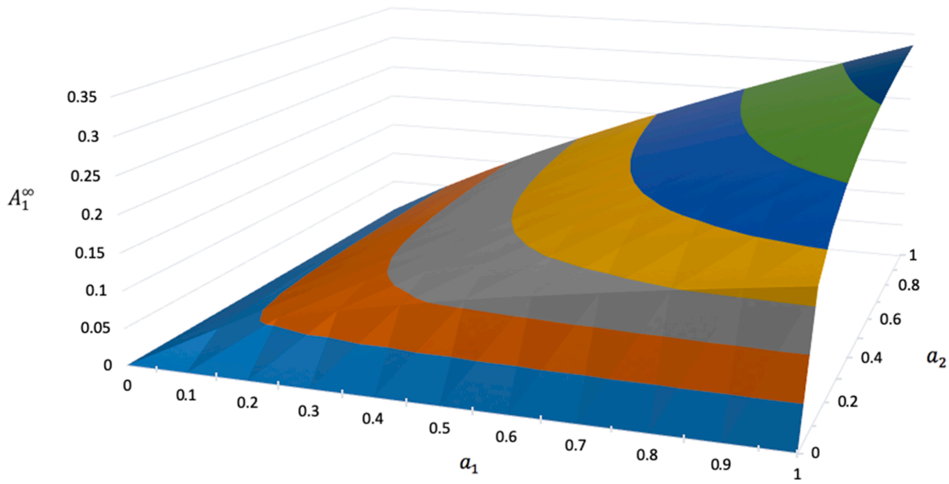

Fig. 1. The spatial behavior of A_1^∞.

$$A_3^\infty = -A_1^\infty - A_2^\infty + \sqrt{\left(a_3 + A_1^\infty + A_2^\infty\right)\left(A_1^\infty + A_2^\infty\right)}$$

Putting $w = A_1^\infty + A_2^\infty + A_3^\infty$, as in the proof of Theorem 1, we find that (cf. (16)) $w(a_i - 2A_i^\infty) = A_i^\infty(a_1 - A_i^\infty)$. To obtain an equation for A_1^∞, we previously considered this expression for $i = 1$. We squared both sides and took advantage of the fact that w^2 can be expressed as a quadratic function in terms of A_1^∞ (to be precisely as $(a_2 + A_1^\infty)A_1^\infty$). Next, A_1^∞ was obtained in closed form as a root of the resulting third-degree polynomial. Here, that approach does not work, since now w^2 can be expressed in either A_1^∞ and A_2^∞ or in A_1^∞ and A_3^∞, but not – in a simple way – in strictly A_1^∞. Ending up with a polynomial of degree higher than four - of which A_1^∞ is to be a root - will not yield a closed form expression for A_1^∞, as known from Galois theory. Consequently, we will have to resort to the best-response functions from Proposition 6 and approximate the Nash equilibrium iteratively (with linear convergence rate) by a tâtonnement process [14].

4. Managerial consequences of the model

Let us explore the managerial consequences of our model, thereby focusing on profits. In Section 4.1, we present a simple estimation procedure for the best pricing of a new product upon market entry, based on only a few pricing experiments in a multiple product environment. In the subsequent sections, we exploit the closed form expression for the Nash equilibrium. Section 4.2 analyzes how the profits at Nash equilibrium are influenced by brand image, as reflected in the price sensitivity. Section 4.3 examines the profit loss due to competition, by comparing the profit potential of a single monopolist firm with its equilibrium profit, resulting from the market entry of a second firm.

4.1. Estimating the best entry price of a new product in a multiple product environment

Consider firm 1, a manufacturer that recently started producing and selling a new vacuum cleaner. They have already explored the market by performing $T = 4$ pricing experiments (see Table 3). Unit production cost is $c_1 = 10$. The total market amounts to $d = 18{,}000$ per period.

Table 3
Market shares resulting from four pricing experiments.

Pricing experiment t	Unit selling price	Market share
1	55	0.04
2	50	0.05
3	45	0.07
4	40	0.10

Evidently, firm 1 is curious about the price they should set to maximize their profit. One may wonder whether this question can be answered at all, since no direct information about individual competitors is available. The only information available - furnished by the T pricing experiments - is the market share per experiment of the ensemble of competitors that results from the price setting of firm 1. The power of our model is that this limited information is sufficient to estimate the best price. Remarkably, we do not need a pricing model for more than two competing firms to achieve this.

Let us use the index $j = 1$ to denote firm 1. And let us denote 'them' by index $j = 2$. Here, 'them' is the ensemble of all competing substitute vacuum cleaner firms. Key to our analysis is the following Market Share Attraction model: $m_j = \frac{A_j}{A_1 + A_2}$, where we assume that $A_1 = 1 - \alpha_1 P_1$ with P_1 the price per vacuum cleaner from firm 1. We make no modelling assumption for A_2 except that it is positive.

Surprisingly, the data from Table 3 is sufficient to estimate the price sensitivity α_1 as well as the attraction A_2. We do not need any assumption about A_2, except the supposition that A_2 is invariant during the experiments, which is plausible, since A_2 models the attraction of the ensemble of all competitors. We have the following:

Claim

Suppose firm 1 conducted $T \geq 2$ pricing experiments. On each price setting $P_1^{(t)}$ the market responded with a market share of $m_1^{(t)}$, $t = 1, 2, \ldots, T$. Then, the price sensitivity α_1 and the attraction A_2 can be estimated through linear regression.

Proof:

Let us write: $A_1^{(t)} \equiv 1 - \alpha_1 P_1^{(t)}, m_2^{(t)} \equiv 1 - m_1^{(t)}, y_t \equiv \frac{m_2^{(t)}}{m_1^{(t)}}, x_t \equiv y_t P_1^{(t)}$.

According to the above model, we approximately have: $A_2 = y_t A_1^{(t)} = y_t(1 - \alpha_1 P_1^{(t)})$.

Hence $y_t = ax_t + b + \varepsilon_t$, with $a = \alpha_1$ and $b = A_2$, whereas ε_t represents the error term.

Since y_t and x_t are known, we can estimate a and b by linear regression. ■

Now, let us return to our example. Combining the above with Table 3, we find the regression data of Table 4.

Linear regression gives: $\alpha_1 = 0.0156$ and $A_2 = 3.7237$. Hence, the total profit for firm 1 for their new vacuum cleaner - as a function of price - can be estimated by:

$$(P_1 - c_1)m_1 d = (P_1 - 10)\frac{(1 - 0.0156P_1)}{(1 - 0.0156P_1) + 3.7237} 18{,}000$$

which is plotted in Fig. 2.

Table 4
Data for linear regression.

Experiment t	Unit selling price	Market share	y_t	x_t
1	55	0.04	24	1320
2	50	0.05	19	950
3	45	0.07	13.29	597.86
4	40	0.10	9	360

Using the best-response function (7), with $a_1 = 1 - \alpha_1 c_1 = 0.844$, we find for the best price: $P_1 = (1 - r_1(A_2))/\alpha_1 = 38.4317$. In conclusion, our model enables a simple estimation procedure for the entry pricing of a new product on the basis of only a few pricing experiments. The above analysis shows how our two-product model can be useful in a multiple product environment.

4.2. The influence of brand image on the profit at Nash equilibrium

Note that Theorem 1 expresses the Nash equilibrium vector - for two competing substitute product types - as a composition of elementary functions of the parameters α_i and c_i. From these, c_i - the production cost per unit - is determined by the firm itself, although there is usually limited room for improvement. The price sensitivity α_i, however, is inversely proportional to the brand image, which is determined externally. It is affected by factors related to past performance of the firm and/or product type, such as quality, social awareness, etc. The higher these factors score for the product type in question, the smaller the corresponding value of α_i. By means of an effective marketing strategy, brand image can be improved and thus α_i can be lowered. At Nash equilibrium, (24) tells us that $\frac{2}{3\alpha_i} < P_i^\infty < \frac{1}{\alpha_i}$. Hence, the equilibrium price P_i^∞ increases in a controlled way, when α_i tends to zero. In view of (21), this increase happens monotonically. Consequently, for fixed c_i, we find an increase - as α_i tends to zero - of the Nash profit $\Pi_i(A_1^\infty, A_2^\infty) = m_i^\infty(P_i^\infty - c_i) = \frac{1}{\alpha_i}(a_i - 2A_i^\infty)$ (see (12)). Fig. 3 illustrates the general behavior of the Nash profit of firm 1 as a function of the price sensitivity α_1. In fact, the increase in profit - for decreasing α_1 - is unbounded, since by (11) $\Pi_1(A_1^\infty, A_2^\infty) = \frac{1}{\alpha_1}(a_1 - 2A_1^\infty) > \frac{1}{\alpha_1}\left(\frac{a_1^2}{10a_1 + 16a_2}\right) > \frac{1}{\alpha_1}\left(\frac{a_1^2}{10a_1 + 16.}\right)$, which tends to infinity as α_1 tends to zero. Note that the latter inequality gives a lower bound for the Nash profit of firm 1 that is independent of any parameter of the competing firm 2. This is an important result. It suggests that by building up a solid brand image (i.e., a small α_1), any firm can be assured of sufficient profit, regardless of any action of any competitor.

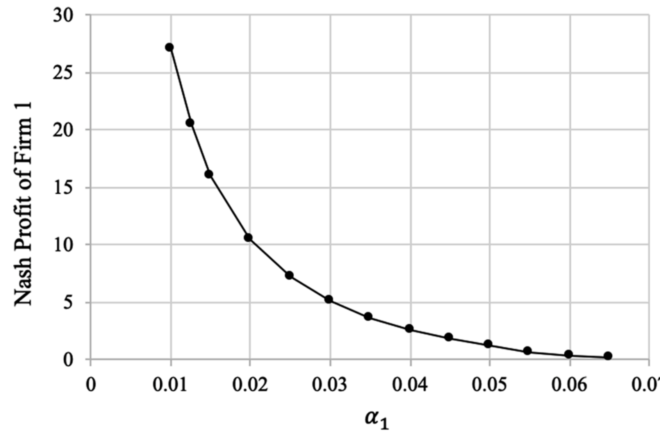

Fig. 3. Profit $\Pi_I(A_1^\infty, A_2^\infty)$ of firm 1 under Nash equilibrium as function of the price sensitivity α_1. The parameter values are: $\alpha_2 = 0.01$, $c_1 = 14$, $c_2 = 1$

The Nash profit of firm 1 in Fig. 3 reflects the single period profit obtained by subtracting the total manufacturing costs from the total revenue, as discussed in Section 3.2. This profit excludes fixed costs, such as marketing expenditure, facility, and machinery investments, etc. Recall from (5) that $\Pi_I(A_1^\infty, A_2^\infty)$ represents the profit made for product type *i* per overall unit demanded. Thus, an upper bound for the projected fixed costs attributed to any single period is given by $d\,\Pi_I(A_1^\infty, A_2^\infty)$, where d is the total market demand in that period.

As an example, in Fig. 3, the Nash profit of firm 1 for the price sensitivity $\alpha_1 = 0.02$ is - roughly - 10. Suppose d has the value 10,000. Then the firm experiences positive net profits, as long as the projected fixed costs attributed to the period under consideration are less than 100,000.

After evaluating the past period, firm 1 may allocate a budget for launching a marketing campaign to improve brand image. Our model enables an easy check in order to verify whether this campaign proceeds successfully, i.e., whether the value of α_1 indeed decreases: just execute the calibration procedure from Section 3.2.1 iteratively, and monitor - in this way - the value of α_1 during the campaign.

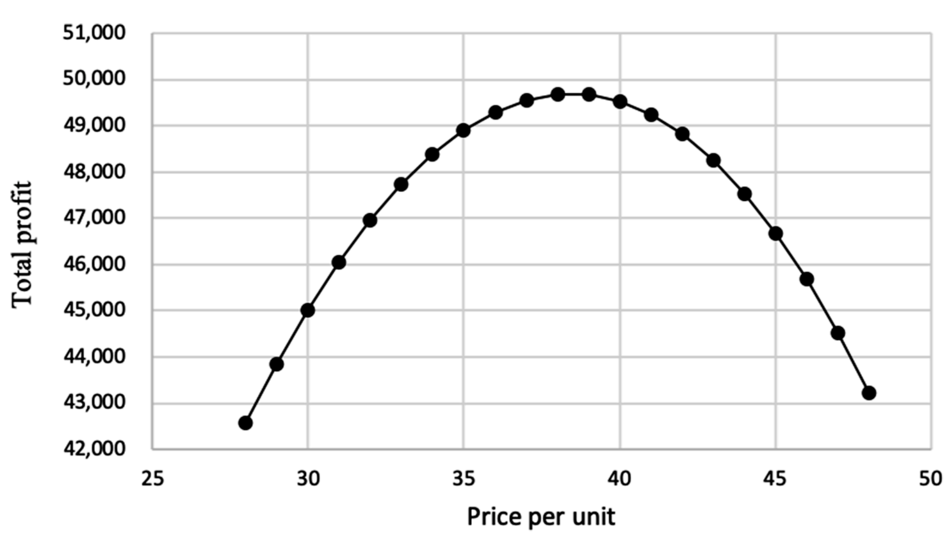

Fig. 2. Estimated profit for the new vacuum cleaner as a function of entry price.

Table 5
Relative profit loss of firm 1 as a function of $a_i = 1 - \alpha_i c_i$.

a_1 \ a_2	0.1	0.3	0.5	0.7	0.9
0.1	0.67	0.79	0.84	0.86	0.88
0.3	0.51	0.67	0.73	0.77	0.79
0.5	0.44	0.60	0.67	0.71	0.74
0.7	0.39	0.55	0.62	0.67	0.70
0.9	0.35	0.51	0.58	0.63	0.67

4.3. Profit loss due to competition

Let us explore the managerial consequences for a firm of having to compete in a non-cooperative environment rather than to be a monopolist. Suppose firm 1 is acting as a monopolist in the market. Suppose, furthermore, that a second firm - say firm 2 - enters the market. Then, let us examine the impact on the profit of firm 1 which used to serve the whole market.

As a monopolist, firm 1 has a profit potential of $\pi_1 = \frac{a_1}{\alpha_1}$ by (4). When firm 2 enters the market, the equilibrium profit of firm 1 amounts to $\Pi_I(A_1^\infty, A_2^\infty) = \frac{1}{\alpha_1}(a_1 - 2A_1^\infty) = \pi_1 - 2\frac{A_1^\infty}{\alpha_1}$. Hence, firm 1 suffers a profit loss of $2\frac{A_1^\infty}{\alpha_1}$ due to competition. In *relative* terms, this profit loss amounts to a fraction of magnitude $2\frac{A_1^\infty}{a_1}$ of the profit in the monopolist case. Note that this relative profit loss only depends on the two maximum obtainable attractions a_1 and a_2. Hence, we can tabulate it as in Table 5. As for the diagonal values: when $a_1 = a_2$, it holds that $A_1^\infty = A_2^\infty = \frac{1}{3}a_1$, yielding a relative profit loss of 2/3.

Table 5 shows row-wise, as well as column-wise monotonicity. As for row-wise, when a_2 increases, making the entering firm 2 potentially more attractive, then the relative profit loss of firm 1 increases if firm 1 does not change its potential attractivity. As for column-wise, if a_1 increases, whereas a_2 is fixed, then the relative profit loss of firm 1 decreases. In the latter case, the relative profit loss of firm 1 can be shown to be bounded from above in a uniform way. Let us illustrate this by an example. Suppose firm 1 has improved its price sensitivity α_1 in such a way that $a_1 > 0.9$. Then - in view of (23) - one has the following: for all a_2, the relative profit loss of firm 1 is smaller than $2/(3*0.9) = 0.741$. Hence, a strong brand image will keep the relative profit loss of a firm - due to a new entrant - manageable.

5. Conclusion and future research

This paper aims to design a practical marketing management tool. We consider a non-cooperative marketing environment with multiple substitute products, where total market size is moderately price-sensitive. The price-demand relations are determined by a market share attraction model, where the attraction of each product is a linearly decreasing function of its price. We assume that each producer has sufficient production capacity to fulfill the total market demand.

For the general case of multiple substitute products, we derive explicit expressions for the best-response functions. The latter enable us to approximate the unique Nash equilibrium iteratively by a tâtonnement process. For the specific case of two substitute products, we derive the product prices in a unique Nash equilibrium in closed form as a composition of elementary functions of the model parameters. These expressions help managers in changing their marketing instruments other than price, so as to obtain substantial individual profits. We show how our closed form Nash equilibrium enables the examination of the profit loss due to competition. Relevant for practice is the fact that our model can be easily calibrated. We provide a simple procedure for estimating the model parameters.

Key features of our model that enhance usability are the following. First, its linear attraction that provides computational advantages, such as simple and explicit best-response functions in a multiple product environment. Second, for the two-product case, the closed form expression for the Nash equilibrium, providing each of the two firms with the profit increase due to improving its brand image. Third, the existence of a single parameter in the attraction model and a practical tool to estimate it.

A limitation of this paper is the assumption of ample production capacity for each of the competing substitute products. For future work, we suggest exploring the influence of limited production capacity on the Nash equilibrium.

A second direction of future research is of a more practical nature: how accurate is our model in a real-life case? We experimented with emulating an electronic reverse auction. Sessions with university students showed promising results regarding the usefulness of our model in guiding the bidding decisions.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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