Scale-free Protocol Design for H_∞ Almost Output and Regulated Output Synchronization of Heterogeneous Multi-agent Systems

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Abstract: This paper studies scale-free protocol design for H_∞ almost output and regulated output synchronization of heterogeneous multi-agent systems with linear, right-invertible and introspective agents in presence of external disturbances. The collaborative linear dynamic protocol designs are based on localized information exchange over the same communication network, which do not require any knowledge of the directed network topology and spectrum of the associated Laplacian matrix. Moreover, the proposed scale-free protocols achieve H_∞ almost synchronization with a given arbitrary degree of accuracy for any size of the network.

Key Words: Heterogeneous multi-agent systems, H_∞ almost synchronization, Scale-free collaborative protocols

1 Introduction

In recent decades, the synchronization problem for multi-agent systems (MAS) has attracted substantial attention due to the wide potential for applications in several areas such as automotive vehicle control, satellites/robots formation, sensor networks, and so on. See the books [9, 13, 4] and references therein.

State synchronization inherently requires homogeneous networks (i.e. agents which have identical models). For heterogeneous network it is more reasonable to consider output synchronization since the dimensions of states and their physical interpretation may be different. If the agents have absolute measurements of their own dynamics in addition to relative information from the network, they are said to be introspective, otherwise, they are called non-introspective. For heterogeneous MAS with non-introspective agents, it is well-known that one needs to regulate outputs of the agents to a priori given trajectory generated by a so-called exosystem (see [12, 1]). Other works on synchronization of MAS with non-introspective agents can be found in the literature as [3, 2].

Synchronization and almost synchronization in presence of external disturbances are studied in the literature, where three classes of disturbances have been considered namely:

1) Disturbances and measurement noise with known frequencies.
2) Deterministic disturbances with finite power.
3) Stochastic disturbances with bounded variance.

For disturbances and measurement noise with known frequencies, it is shown in [16] and [17] that actually exact synchronization is achievable. This is shown in [16] for heterogeneous MAS with minimum-phase and non-introspective agents and networks with time-varying directed communi-

cation graphs. Then, [17] extended this results for non-minimum phase agents utilizing localized information exchange.

For deterministic disturbances with finite power, the notion of H_∞ almost synchronization is introduced by Peymani et.al for homogeneous MAS with non-introspective agents utilizing additional communication exchange [7]. The goal of H_∞ almost synchronization is to reduce the impact of disturbances on the synchronization error to an arbitrarily degree of accuracy (expressed in the H_∞ norm). This work was extended later in [8, 15, 18] to heterogeneous MAS with non-introspective agents and without the additional communication and for network with time-varying graphs. H_∞ almost synchronization via static protocols is studied in [10] for MAS with passive and passifiable agents. Recently, necessary and sufficient conditions are provided in [11] for solvability H_∞ almost synchronization of homogeneous networks with non-introspective agents and without additional communication exchange. Finally, we developed a scale-free framework for H_∞ almost state synchronization for homogeneous network [5] utilizing suitably designed localized information exchange.

In the case of stochastic disturbances with bounded variance, the concept of stochastic almost synchronization is introduced by [19] where both stochastic disturbance and disturbance with known frequency are considered. The idea of stochastic almost synchronization is to reduce the stochastic RMS norm of synchronization error arbitrary small in the presence of colored stochastic disturbances that can be modeled as the output of linear time invariant systems driven by white noise with unit power spectral intensities. By augmenting this model with agent model one can essentially assume that stochastic disturbance is white noise with unit power spectral intensities. In this case under linear protocols the stochastic RMS norm of synchronization error is the H_2 norm of the transfer function from disturbance to the synchronization error. As such one can formulate the stochastic almost

synchronization equivalently in a deterministic framework requiring to reduce the H_2 norm of the transfer function from disturbance to synchronization error arbitrary small. This deterministic approach is referred to as almost H_2 synchronization problem which is equivalent to stochastic almost synchronization problem. Recent work on H_2 almost synchronization problem is [11] which provided necessary and sufficient conditions for solvability of H_∞ almost synchronization for homogeneous networks with non-introspective agents and without additional communication exchange. Finally, H_2 almost synchronization via static protocols is studied in [10] for MAS with passive and passifiable agents.

In this paper, we develop scale-free protocol design for H_∞ almost synchronization of heterogeneous MAS in presence of external disturbances. A collaborative linear parameterized dynamic protocols with localized information exchange is proposed which can work for heterogeneous MAS with any size of the network. The main contribution of this work is that the protocol design does not require any information about the communication network such as a lower bound of non-zero eigenvalues of the associated Laplacian matrix and the number of agents. Moreover, the scalable protocol achieves H_∞ almost synchronization with a given arbitrary degree of accuracy for heterogeneous MAS with any number of agents.

Notations and definitions

Given a matrix $A \in \mathbb{R}^{m \times n}$, A^{T} and A^* denote transpose and conjugate transpose of A respectively while $\|A\|_2$ denotes the induced 2-norm (which has submultiplicative property). The $\mathrm{im}(\cdot)$ denote the image of matrix (vector). For a continuous-time system having a $q \times l$ stable transfer function $G(s)$, $\|G\|_{H_\infty}$ denotes the H_∞ norm of $G(s)$. A square matrix A is said to be Hurwitz stable if all its eigenvalues are in the open left half complex plane. $A \otimes B$ depicts the Kronecker product between A and B. A block diagonal matrix constructed by A_i's is denoted by $\mathrm{diag}\{A_i\}$ for $i = 1, \ldots, n$. I_n denotes the n-dimensional identity matrix and 0_n denotes $n \times n$ zero matrix; sometimes we drop the subscript if the dimension is clear from the context.

A *weighted graph* $\mathcal{G}$ is defined by a triple $(\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \ldots, N\}$ is a node set, $\mathcal{E}$ is a set of pairs of nodes indicating connections among nodes, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ is the weighting matrix. Each pair in $\mathcal{E}$ is called an *edge*, where $a_{ij} > 0$ denotes an edge $(j, i) \in \mathcal{E}$ from node j to node i with weight a_{ij}. Moreover, $a_{ij} = 0$ if there is no edge from node j to node i. We assume there are no self-loops, i.e. we have $a_{ii} = 0$. A *path* from node i_1 to i_k is a sequence of nodes $\{i_1, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k - 1$. A *directed tree* with root r is a subgraph of the graph $\mathcal{G}$ in which there exists a unique path from node r to each node in this subgraph. A *directed spanning tree* is a directed tree containing all the nodes of the graph.

For a weighted graph $\mathcal{G}$, the matrix $L = [\ell_{ij}]$ with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^{N} a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph $\mathcal{G}$. The Laplacian matrix L has all its eigenvalues in the closed right half plane and at least one eigenvalue at zero associated with right eigenvector $\mathbf{1}$, i.e. a vector with all entries equal to

1. When graph contains a spanning tree, then it is known that the Laplacian matrix L has a simple eigenvalue at the origin, with the corresponding right eigenvector $\mathbf{1}$, and all the other eigenvalues are in the open right-half complex plane.

2 Problem formulation

Consider a heterogeneous MAS composed of N nonidentical linear time-invariant agents of the form,

$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i + E_i \omega_i, \\ y_i &= C_i x_i, \end{aligned} \qquad (i = 1, \ldots, N) \qquad (1)$$

where $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{m_i}$, $y_i \in \mathbb{R}^p$ are respectively the state, input, and output vectors of agent i, and $\omega_i \in \mathbb{R}^{w_i}$ is the external disturbance. $n_{q0} \geqslant 1$ is the upper bound on infinite zeros of triples (C_i, A_i, B_i) for $i \in \{1, \ldots, N\}$.

The agents are introspective, meaning that each agent has access to its own local information. Specifically, each agent has access to part of its state, i.e.,

$$z_i = C_i^m x_i, \quad z_i \in \mathbb{R}^{q_i} \qquad (2)$$

The communication network provides each agent with a linear combination of its own outputs relative to that of other neighboring agents. In particular, each agent $i \in \{1, \ldots, N\}$ has access to the quantity

$$\zeta_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) \qquad (3)$$

where $a_{ij} \geqslant 0$ and $a_{ii} = 0$ indicate the communication among agents. This communication topology of the network can be described by a weighted and directed graph $\mathcal{G}$ with nodes corresponding to the agents in the network and the weight of edges given by the coefficient a_{ij}. In terms of the coefficient of the associated Laplacian matrix L, (3) can be rewritten as

$$\zeta_i = \sum_{j=1}^{N} \ell_{ij} y_j. \qquad (4)$$

The goal of this paper is to design scale-free protocols which can be achieved by utilizing localized information exchange among neighbors, as such each agent $i \in \{1, \ldots, N\}$ has access to localized information, denoted by $\hat{\zeta}_i$, of the form

$$\hat{\zeta}_i = \sum_{j=1}^{N} a_{ij}(\xi_i - \xi_j) \qquad (5)$$

where ξ_j is a variable produced internally by agent j which will be appropriately chosen in the coming sections.

For almost output synchronization, we define the set of graphs $\mathbb{G}^N$ for the network communication topology as following.

Definition 1 *Let* $\mathbb{G}^N$ *denote the set of directed graphs of N agents which contains a directed spanning tree.*

We formulate the scale-free H_∞ almost output synchronization problem of a heterogeneous MAS with localized information exchange as following.
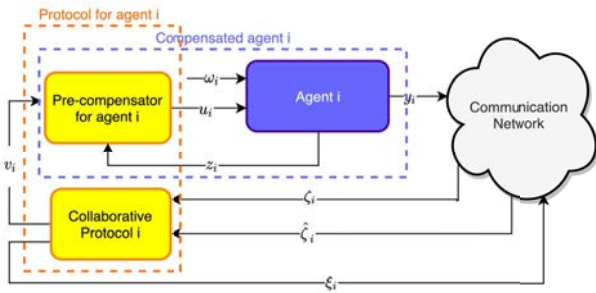
Figure 1: Architecture of scale-free collaborative protocol for H_∞ almost output synchronization

Problem 1 *The **scale-free** H_∞ **almost output synchronization problem with localized information exchange (scale-free H_∞-AOSWLIE)** for MAS* (1) *and* (4) *is to find, if possible, a scalar parameter $\varepsilon^* > 0$ and a fixed linear protocol parameterized in scalar ε, **only** based on the knowledge of agent models (C_i, A_i, B_i), of the form*

$$\begin{cases} \dot{x}_{c_i} = A_{c_i}(\varepsilon)x_{c_i} + B_{c_i}(\varepsilon)\zeta_i + C_{c_i}(\varepsilon)\hat{\zeta}_i + D_{c_i}(\varepsilon)z_i \\ u_i = E_{c_i}(\varepsilon)x_{c_i} + F_{c_i}(\varepsilon)\zeta_i + G_{c_i}(\varepsilon)\hat{\zeta}_i + H_{c_i}(\varepsilon)z_i \end{cases} \quad (6)$$

where $\hat{\zeta}_i$ is defined by (5), *with $\xi_i = M_{c_i}x_{c_i}$ with $x_{c_i} \in \mathbb{R}^{n_{c_i}}$ such that for any number of agents N, and any communication graph $\mathcal{G} \in \mathbb{G}^N$ we have:*

- *in the absence of the disturbance $\omega = \begin{pmatrix} \omega_1^T & \dots & \omega_N^T \end{pmatrix}^T$, for all initial conditions the output synchronization*

$$\lim_{t \to \infty}(y_i - y_j) = 0 \quad \text{for all } i, j \in \{1, ..., N\} \quad (7)$$

  is achieved for any $\varepsilon \in (0, \varepsilon^*]$.

- *in the presence of the disturbance ω, for any $\gamma > 0$, one can render the H_∞ norm from ω to $y_i - y_j$ less than γ by choosing ε sufficiently small.*

The architecture of the protocol (6) *is shown in Figure 1.*

In the case of H_∞ almost regulated output synchronization, we consider a reference trajectory y_r generated by a so-called exosystem as

$$\begin{aligned} \dot{x}_r &= A_r x_r, \quad x_r(0) = x_{r0}, \\ y_r &= C_r x_r, \end{aligned} \quad (8)$$

where $x_r \in \mathbb{R}^r$ and $y_r \in \mathbb{R}^p$.

We assume a nonempty subset $\mathcal{C}$ of the agents which have access to their output relative to the output of the exosystem. In other word, each agent i has access to the quantity

$$\Psi_i = \iota_i(y_i - y_r), \qquad \iota_i = \begin{cases} 1, & i \in \mathcal{C}, \\ 0, & i \notin \mathcal{C}. \end{cases} \quad (9)$$

By combining the above equation with (3), the information exchange among agents is given by

$$\bar{\zeta}_i = \sum_{j=1}^{N} a_{ij}(y_i - y_j) + \iota_i(y_i - y_r). \quad (10)$$

The exchanging information $\bar{\zeta}_i$ as defined in above, can be rewritten in terms of the coefficients of so-called expanded Laplacian matrix $\bar{L} = L + diag\{\iota_i\} = [\bar{\ell}_{ij}]_{N \times N}$ as

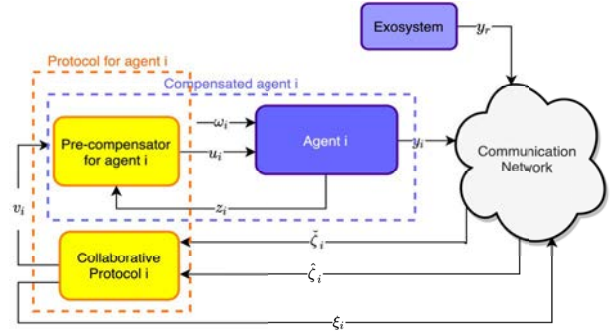$$\bar{\zeta}_i = \sum_{j=1}^{N} \bar{\ell}_{ij}(y_j - y_r). \quad (11)$$

Figure 2: Architecture of scale-free collaborative protocol for H_∞ almost regulated output synchronization

We know that all the eigenvalues of $\bar{L}$, have positive real parts. In particular matrix $\bar{L}$ is invertible.

We define the following set of graphs.

Definition 2 *Given a node set $\mathcal{C}$, we denote by $\mathbb{G}_{\mathcal{C}}^N$ the set of all graphs with N nodes containing the node set $\mathcal{C}$, such that every node of the network graph $\mathcal{G} \in \mathbb{G}_{\mathcal{C}}^N$ is a member of a directed tree which has its root contained in the node set $\mathcal{C}$. We will refer to the node set $\mathcal{C}$ as root set.*

Remark 1 *Note that Definition 2 does not require necessarily the existence of directed spanning tree.*

Now, we formulate scale-free H_∞ almost regulated output synchronization problem.

Problem 2 *The **scale-free** H_∞ **almost regulated output synchronization problem with localized information exchange (scale-free H_∞-AROSWLIE)** for MAS* (1) *and* (11) *and the associated exosystem* (8) *is to find, if possible, a scalar parameter $\varepsilon^* > 0$ and a fixed linear protocol parameterized in scalar ε, **only** based on the knowledge of agent models (C_i, A_i, B_i), of the form*

$$\begin{cases} \dot{x}_{c_i} = A_{c_i}(\varepsilon)x_{c_i} + B_{c_i}(\varepsilon)\bar{\zeta}_i + C_{c_i}(\varepsilon)\hat{\zeta}_i + D_{c_i}(\varepsilon)z_i \\ u_i = E_{c_i}(\varepsilon)x_{c_i} + F_{c_i}(\varepsilon)\bar{\zeta}_i + G_{c_i}(\varepsilon)\hat{\zeta}_i + H_{c_i}(\varepsilon)z_i \end{cases} \quad (12)$$

where $\hat{\zeta}_i$ is defined by (5), *with $\xi_i = M_{c_i}x_{c_i}$ with $x_{c_i} \in \mathbb{R}^{n_{c_i}}$ such that for any number of agents N, and any communication graph $\mathcal{G} \in \mathbb{G}_{\mathcal{C}}^N$ we have:*

- *in the absence of the disturbance $\omega = \begin{pmatrix} \omega_1^T & \dots & \omega_N^T \end{pmatrix}^T$, for all initial conditions the regulated output synchronization*

$$\lim_{t \to \infty}(y_i - y_r) = 0 \quad \text{for all } i \in \{1, ..., N\} \quad (13)$$

  is achieved for any $\varepsilon \in (0, \varepsilon^*]$.

- *in the presence of the disturbance ω, for any $\gamma > 0$, one can render the H_∞ norm from ω to $y_i - y_r$ less than γ by choosing ε sufficiently small.*

The architecture of the protocol (12) *is shown in Figure 2.*

We make the following assumptions for agents and the exosystem.

Assumption 1 *For agents $i \in \{1, ..., N\}$,*
1) (C_i, A_i, B_i) is stabilizable and detectable.

2) (C_i, A_i, B_i) is right-invertible.
3) (C_i^m, A_i) is detectable.

Assumption 2 *For exosystem*

1) (C_r, A_r) is observable.
2) All the eigenvalues of A_r are in the closed left half complex plane.

3 Scale-free H_∞ almost output synchronization

According to Figure 1, the design for scale-free H_∞ almost output synchronization includes two major modules. First, we design precompensators for homogenizing agents, then we design collaborative protocols for the compensated agents to achieve H_∞ almost output synchronization.

Step I: choosing target model First we choose suitable target model, i.e. (C_d, A_d, B_d) such that the following conditions are satisfied.

1) rank $C_d = p$,
2) (C_d, A_d, B_d) is invertible, of uniform rank $n_q \geqslant n_{q0}$ and has no invariant zero. $n_{q0} \geqslant 1$ is the maximum order of infinite zeros of triples (C_i, A_i, B_i) for $i \in \{1, ..., N\}$.
3) eigenvalues of A_d are in the closed left half plane.

Remark 2 *The choice of A_d is designer choice and obviously the eigenvalues of A_d determines the desired trajectory. Without loss of generality, we assume that the triple (C_d, A_d, B_d) has the following form:*

$$A_d = \bar{A} + \bar{B}\Gamma, \bar{A} := \begin{pmatrix} 0 & I_{p(n_q-1)} \\ 0 & 0 \end{pmatrix}, B_d = \bar{B} := \begin{pmatrix} 0 \\ I_p \end{pmatrix},$$
$$C_d = \bar{C} := \begin{pmatrix} I_p & 0 \end{pmatrix}.$$

Step II: designing precompensators Following the instruction given in [14, Appendix B], we design the following precompensator for each agent of MAS (1) to homogenize the agents to the target model chosen in the previous step.

$$\begin{aligned} \dot{p}_i &= G_i p_i + H_{1i} v_i + H_{2i} z_i, \\ u_i &= Q_i p_i + R_{1i} v_i + R_{2i} z_i, \end{aligned} \tag{14}$$

where v_i is the input of the precompensator.

We obtain the compensated agents by combining (1) and (14) as

$$\begin{aligned} \dot{\bar{x}}_i &= \bar{A}\bar{x}_i + \bar{B}(v_i + \Gamma\bar{x}_i) + \bar{E}_i\bar{\omega}_i, \\ y_i &= \bar{C}\bar{x}_i, \end{aligned} \tag{15}$$

where $\bar{\omega}_i = \begin{pmatrix} \omega_i^\mathsf{T} & \rho_i^\mathsf{T} \end{pmatrix}^\mathsf{T}$. Meanwhile, ρ_i is generated by the following system

$$\begin{aligned} \dot{\theta}_i &= S_i \theta_i + E_{0i}\bar{\omega}_i, \\ \rho_i &= W_i \theta_i, \end{aligned}$$

where S_i is Hurwitz stable.

Step III: designing collaborative protocols for the compensated agents In this step, the following linear dynamic protocol is designed for the compensated agents (15)

$$\begin{cases} \dot{\hat{x}}_i = \bar{A}\hat{x}_i - \varepsilon^{-n_q}\bar{B}F\Delta\hat{\zeta}_i + \bar{B}\Gamma\hat{x}_i + \varepsilon^{-1}K(\zeta_i - \bar{C}\hat{x}_i) \\ \dot{\chi}_i = \bar{A}\chi_i + \bar{B}v_i + \bar{B}\Gamma\chi_i + \varepsilon^{-1}(\hat{x}_i - \hat{\zeta}_i) \\ v_i = -\varepsilon^{-n_q}F\Delta\chi_i, \end{cases} \tag{16}$$

where $\varepsilon \in (0, \varepsilon^*]$ is the tuning parameter. We define the scaling matrix $\Delta = \text{diag}\{I_p, \varepsilon I_p, \ldots, \varepsilon^{n_q-1}I_p\} \in \mathbb{R}^{pn_q \times pn_q}$ and we choose matrix F such that $\bar{A} - \bar{B}F$ is Hurwitz stable. Meanwhile, we partition $\bar{A}, \bar{B}, \Gamma$ as

$$\bar{A} = \begin{pmatrix} 0_{p\times p} & \bar{C}_1 \\ 0_{p(n_q-1)\times p} & \bar{A}_1 \end{pmatrix}, \bar{B} = \begin{pmatrix} 0_{p\times p} \\ \bar{B}_1 \end{pmatrix}, \Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix}$$

with $\bar{C}_1 = \begin{pmatrix} I_p & 0 & \cdots & 0 \end{pmatrix}$. We choose $K = \begin{pmatrix} K_1 \\ K_2 K_1 \end{pmatrix}$ with K_2 such that $\bar{A}_1 + \bar{B}_1\Gamma_2 - K_2\bar{C}_1$ is Hurwitz stable. And then let $K_1 = K_1^\mathsf{T} > \frac{\alpha}{2}I \in \mathbb{R}^{p\times p}$, where the value of α depends on K_2 and is given explicitly in the proof of Theorem 1.

In this protocol, each agent communicate χ_i with its neighbors, (i.e., $\xi_i = \chi_i$). Therefore, each agent has access to the localized information exchange appeared in (16) represented by

$$\hat{\zeta}_i = \sum_{j=1}^{N} \ell_{ij} \chi_j, \tag{17}$$

which can be simply obtained by $\xi_i = \chi_i$ in (5).

We have the following theorem.

Theorem 1 *Consider a MAS described by (1) and (4) satisfying Assumption 1. Let $\mathbb{G}^N$ be the set of network graphs as defined in Definition 1.Then, the scalable H_∞-AOSWLIE problem as stated in Problem 1 is solvable. More specifically,*

1) *in the absence of the disturbance ω, protocol (16) and (14) achieves output synchronization (7), for any graph $\mathcal{G} \in \mathbb{G}^N$ with any number of agents N, and for any $\varepsilon \in (0, \varepsilon^*]$ with $\varepsilon^* < 1$ where the value of ε^* only depends on $\bar{A}, \bar{B}, \Gamma, F$,*
2) *in the presence of the disturbance ω, for any $\gamma > 0$, the H_∞ norm from ω to $y_i - y_j$ is less that γ for all $i, j \in \{1, \ldots, N\}$ by choosing ε sufficiently small.*

Proof: Due to space limitation the proof is omitted and provided in [6]. ∎

4 Scale-free H_∞ almost regulated output synchronization

According to Figure 2, the design for scale-free H_∞ almost regulated output synchronization includes two major modules. First, we design precompensators for homogenizing agents, then we design collaborative protocols for the compensated agents to achieve H_∞ almost regulated output synchronization. The design consists of the following steps.

Step I: remodeling the exosystem and choosing target model Following the design procedure in [14, Appendix C] we remodel the exosystem (8) to arrive at a suitable choice for the target model. There exists another exosystem given by

$$\begin{aligned} \dot{\check{x}}_r &= \check{A}_r \check{x}_r, \quad \check{x}_r(0) = \check{x}_{r0} \\ y_r &= \check{C}_r \check{x}_r, \end{aligned} \tag{18}$$

such that for all $x_{r0} \in \mathbb{R}^r$, there exists $\check{x}_{r0} \in \mathbb{R}^{\check{r}}$ for which (18) generate exactly the same output y_r as the original exosystem

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(8). Furthermore, we can find a matrix $\check{B}_r$ such that the triple $(\check{C}_r, \check{A}_r, \check{B}_r)$ is invertible, of uniform rank n_q, and has no invariant zero, where n_q is an integer greater than or equal to maximal order of infinite zeros of $(C_i, A_i, B_i), i \in \{1, ..., N\}$ and all the observability indices of (C_r, A_r). Note that the eigenvalues of $\check{A}_r$ consists of all eigenvalues of A_r and additional zero eigenvalues as such eigenvalues of $\check{A}_r$ are in the closed left have plane. We choose our target model as $(\check{C}_r, \check{A}_r, \check{B}_r)$.

Remark 3 *Similar to Remark 2, we assume the $(\check{C}_r, \check{A}_r, \check{B}_r)$ has the following form.*

$$\check{A}_r = \bar{A} + \bar{B}\Gamma, \bar{A} := \begin{pmatrix} 0 & I_{p(n_q-1)} \\ 0 & 0 \end{pmatrix}, \check{B}_r = \bar{B} := \begin{pmatrix} 0 \\ I_p \end{pmatrix},$$
$$\check{C}_r = \bar{C} := \begin{pmatrix} I_p & 0 \end{pmatrix}.$$

Step II: designing precompensators Following the instruction given in [14, Appendix B], we use the following precompensator for each agent of MAS (1) to homogenize the agents to the target model chosen in the previous step.

$$\begin{aligned} \dot{p}_i &= G_i p_i + H_{1i} v_i + H_{2i} z_i, \\ u_i &= Q_i p_i + R_{1i} v_i + R_{2i} z_i, \end{aligned} \quad (19)$$

where v_i is the input of the precompensator.

We obtain the compensated agents by combining (1) and (19) as

$$\begin{aligned} \dot{\bar{x}}_i &= \bar{A}\bar{x}_i + \bar{B}(v_i + \Gamma\bar{x}_i) + \bar{E}_i \bar{\omega}_i \\ y_i &= \bar{C}\bar{x}_i. \end{aligned} \quad (20)$$

Step III: designing collaborative protocols for the compensated agents Finally, the following linear dynamic protocol is designed for the compensated agents (20) as

$$\begin{cases} \dot{\hat{x}}_i = \bar{A}\hat{x}_i - \varepsilon^{-n_q}\bar{B}F\Delta\hat{\zeta}_i + \bar{B}\Gamma\hat{x}_i \\ \qquad\qquad + \varepsilon^{-1}K(\zeta_i - C\hat{x}_i) + \iota_i\bar{B}v_i \\ \dot{\chi}_i = \bar{A}\chi_i + \bar{B}v_i + \bar{B}\Gamma\chi_i + \varepsilon^{-1}(\hat{x}_i - \hat{\zeta}_i) - \varepsilon^{-1}\iota_i\chi_i \\ v_i = -\varepsilon^{-n_q}F\Delta\chi_i, \end{cases} \quad (21)$$

where $\varepsilon \in (0, \varepsilon^*]$ is the tuning parameter. We define the scaling matrix $\Delta = \mathrm{diag}\{I_p, \varepsilon I_p, \ldots, \varepsilon^{n_q-1}I_p\} \in \mathbb{R}^{pn_q \times pn_q}$ and we choose matrix F such that $\bar{A} - \bar{B}F$ is Hurwitz stable. Meanwhile, we partition $\bar{A}, \bar{B}, \Gamma$ as

$$\bar{A} = \begin{pmatrix} 0_{p\times p} & \bar{C}_1 \\ 0_{p(n_q-1)\times p} & \bar{A}_1 \end{pmatrix}, \bar{B} = \begin{pmatrix} 0_{p\times p} \\ \bar{B}_1 \end{pmatrix}, \Gamma = \begin{pmatrix} \Gamma_1 & \Gamma_2 \end{pmatrix}$$

with $\bar{C}_1 = \begin{pmatrix} I_p & 0 & \cdots & 0 \end{pmatrix}$. We choose $K = \begin{pmatrix} K_1 \\ K_2 K_1 \end{pmatrix}$ with K_2 such that $\bar{A}_1 + \bar{B}_1\Gamma_2 - K_2\bar{C}_1$ is Hurwitz stable. And then let $K_1 = K_1^{\mathrm{T}} > \frac{\alpha}{2}I \in \mathbb{R}^{p\times p}$, where the value of α depends on K_2 and is given explicitly in the proof of Theorem 1.

In this protocol, each agent communicate χ_i with its neighbors, (i.e., $\xi_i = \chi_i$). Therefore, each agent has access to the localized information exchange appeared in (21) represented by (17), which can be simply obtained by $\xi_i = \chi_i$ in (5).

We have the following theorem.

Theorem 2 *Consider a MAS described by (1) and (11), and the exosystem (8) satisfying Assumption 1 and 2. Let $\mathbb{G}_{\mathscr{C}}^N$ be the set of network graphs as defined in Definition 2. Then, the scalable H_∞-AROSWLIE problem as stated in Problem 2 is solvable. More specifically,*

1) *in the absence of the disturbance ω protocol (21) and (19) achieves scalable regulated output synchronization (7), for any graph $\mathcal{G} \in \mathbb{G}_{\mathscr{C}}^N$ with any number of agents N, and for any $\varepsilon \in (0, \varepsilon^*]$ with $\varepsilon^* < 1$ where the value of ε^* only depends on $\bar{A}, \bar{B}, \Gamma, F$,*

2) *in the presence of the disturbance ω, for any $\gamma > 0$, the H_∞ norm from ω to $y_i - y_r$ is less that γ for all $i \in \{1, \ldots, N\}$ by choosing ε sufficiently small.*

Proof: Due to space limitation the proof is omitted and provided in [6]. ∎

Remark 4 *It is worth noting that the formulations in Problem 1 and 2 do not specify how the matrices of the protocols should evolve with parameter ε. However, our designs, as given in Protocol (16) and (21), provide an explicit solution from which the protocol matrices can be derived. The structure of the protocols are independent of the parameter ε; thus, one may develop the structure at one stage and tune the parameter ε later so as to obtain the desired degree of accuracy. Due to continuity in ε, tuning may be even carried out online. Hence, the method is a one-shot design and is not iterative.*

5 Simulation Results

In this section, we will illustrate the effectiveness of our protocols with numerical examples for H_∞ almost output synchronization of heterogeneous MAS. We show that our one-shot-designed protocol (16) is scale-free and works for any MAS with any communication graph and any number of agents. Consider the agents models (1) with

$$A_i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B_i = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}, C_i^{\mathrm{T}} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, E_i = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

and $C_i^m = I$, for $i = 1, 6$, and

$$A_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, B_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, C_i^{\mathrm{T}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, E_i = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, C_i^m = I,$$

for $i = 2, 7$, and

$$A_i = \begin{pmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 & 1 \end{pmatrix}, B_i = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}, C_i^{\mathrm{T}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

and $E_i = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$, $C_i^m = I$, for $i = 3, 4, 8, 9$. Finally

$$A_i = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, B_i = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, C_i^{\mathrm{T}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, E_i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C_i^m = I,$$

for $i = 5, 10$. The agents are subject to disturbances $\omega_i = \sin(t)$, for $i = 2, 5, 7, 10$, $\omega_i = \sin(2t)$ for $i = 3, 4, 8, 9$, and
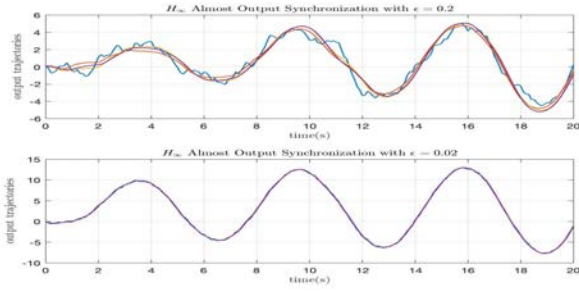
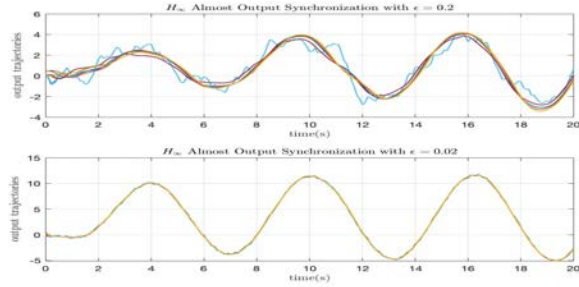Figure 3: H_∞ Almost output synchronization for MAS with $N = 4$

Figure 4: H_∞ Almost output synchronization for MAS with $N = 10$

$\|\omega_i\| \leqslant 5$ for $i = 1, 6$ which is a random number normally distributed. To show the scalability of our protocols, we consider two heterogeneous MAS with different number of agents and different communication topologies.

Case 1: Consider a MAS with 4 agents with agent models (C_i, A_i, B_i) for $i \in \{1, \ldots, 4\}$, and directed communication topology with adjacency matrix $\mathscr{A}_1$ where $a_{14} = a_{21} = a_{31} = a_{42} = 1$.

Case 2: Next, consider a MAS with 10 agents with agent models (C_i, A_i, B_i) for $i \in \{1, \ldots, 10\}$ and directed communication topology with adjacency matrix $\mathscr{A}_2$, where $a_{21} = a_{5,10} = a_{32} = a_{43} = a_{54} = a_{65} = a_{76} = a_{87} = a_{98} = a_{10,9} = a_{15} = 1$.

Note that for both cases $p = 1$ and $\bar{n}_d = 3$, which is the degree of infinite zeros of (C_2, A_2, B_2). It is obtained that $n_q = 3$. We choose the target model as

$$A_d = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, B_d = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, C_d = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}.$$

By designing precompensators as stated in *step II*, we obtain the compensated agents (15) with $\Gamma = \begin{pmatrix} 0 & -1 & 0 \end{pmatrix}$. We choose $F = \begin{pmatrix} 30 & 31 & 10 \end{pmatrix}$ and $K^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$. The simulation results show that by decreasing the value of ε, H_∞ almost output synchronization is achieved with higher degree of accuracy.

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