Convexity of Functions Which Are Generalizations of the Erlang Loss Function and the Erlang Delay Function: Problem 90-8

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Also solved by MANSOUR ESLAMI (University of Illinois at Chicago), who refers to his paper and book, respectively, *Robust nonlinear dynamic systems—The old question with a new answer*, Systems Control Lett. 14 (1990), pp. 445–451; *An Introduction to the Theory of Sensitivity of Dynamic Systems*, Springer-Verlag, New York, 1990. His solution of (ii) is not an explicit solution but an iterative one. Part (i) also solved by the proposer.

Convexity of Functions Which Are Generalizations of the Erlang Loss Function and the Erlang Delay Function

Problem 90-8, by A. A. JAGERS AND E. A. VAN DOORN (Universiteit Twente, Enschede, the Netherlands).

For real λ , a , x with λ , $a > 0$, let

$$f_{\lambda}(x, a) = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} e^{-at} t^{\lambda-1} (1+t)^{x-\lambda+1} dt,$$

a confluent hypergeometric or Kummer function. Prove that, for $\lambda \geq 1$, $\log f_{\lambda}(x, a)$ and $f_{\lambda}(x, a)^{-1}$ are convex functions of x , for $x \geq 0$. Or, equivalently, prove that for $\lambda \geq 1$, $x \geq 0$

$$(f'_{\lambda}(x, a))^2 \leq f_{\lambda}(x, a) f''_{\lambda}(x, a) \leq 2(f'_{\lambda}(x, a))^2,$$

where a prime denotes differentiation with respect to x .

For $\lambda = 1, 2$ two celebrated functions from teletraffic theory are obtained, viz., for x a nonnegative integer

$$af_1(x, a) = \left(\frac{a^x}{x!}\right)^{-1} \sum_{i=0}^x \frac{a^i}{i!} = B(x, a)^{-1},$$

the reciprocal of the Erlang loss function, while for x a nonnegative integer and $x > a$

$$af_2(x, a) = 1 + \left(\frac{a^x}{x!(1-a/x)}\right)^{-1} \sum_{i=0}^{x-1} \frac{a^i}{i!} = C(x, a)^{-1},$$

the reciprocal of the Erlang delay function (cf. [1]).

For $\lambda = 1$ the problem has been solved in [3] (the error on line 4 of p. 45 may be corrected by replacing a^2 by $(d/dt)^2$). For $\lambda = 2$ only a weaker result is available in the literature. It has been shown in [2], and in a more general context in [4], that $C(x, a)/(x - a)$ is a convex function of x for $x > a$ and x a nonnegative integer. Clearly, the latter result is implied by the convexity of $f_2(x, a)^{-1}$ since $C(x, a)$ is decreasing in x , $x > a$.

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Solution by the proposers.

We extend the analysis in [3] by considering the function

$$f(x, a) \equiv \int_0^\infty \exp(-at + xh(t) + w(t)) dt \quad (a > 0, x \geq 0),$$

with h an increasing, concave and continuously differentiable function on $[0, \infty)$ satisfying $h(0) = 0$ and $t^{-1}h(t) \rightarrow 0$ ($t \rightarrow \infty$), and with w a continuous concave function on $(0, \infty)$ satisfying $t^{-1}w(t) \rightarrow 0$ ($t \rightarrow \infty$) and $t^{1-\varepsilon}e^{w(t)} \rightarrow 0$ ($t \downarrow 0$) for some $\varepsilon > 0$. Propositions 1 and 2 of [3] pertain to the case $w \equiv 0$ and state that

$$\phi(x, a) \equiv 2(f'(x, a))^2 - f(x, a)f''(x, a)$$

and

$$\psi(x, a) \equiv f(x, a)f''(x, a) - (f'(x, a))^2,$$

respectively, are completely monotone (and hence nonnegative). The extension of Proposition 2 to the present context is obvious (as is Proposition 2 itself). To prove the extension of Proposition 1, we proceed as in the proof in [3] and apply Lemma 1 of [3], but this time with $p(u) \equiv \exp\{x(h(u) + h(t-u)) + w(u) + w(t-u)\}$, which is a (positive) increasing function of u for $0 \leq u \leq t/2$, since both h and w are concave. What remains to be shown, precisely as in [3], is that, for $t \geq 0$,

$$G(t) \equiv \int_0^t \{2h(u)h(t-u) - h^2(u)\} du \geq 0.$$

The following proof differs slightly from the proof in [3]. We note that

$$\begin{aligned} G'(t) &= 2 \int_0^t h(u)h'(t-u) du - h^2(t) \\ &= 2 \int_0^t h(t-u)h'(u) du - h^2(t), \end{aligned}$$

since h is continuously differentiable on $(0, \infty)$ with $h(0) = 0$. Hence $G'(0) = 0$ ($=G(0)$) and

$$\begin{aligned} G''(t) &= 2 \int_0^t h'(t-u)h'(u) du - 2h'(t)h(t) \\ &= 2 \int_0^t \{h'(t-u) - h'(t)\} h'(u) du \\ &\geq 0, \end{aligned}$$

since h is increasing and concave. It follows that G is convex, increasing and nonnegative. The desired inequalities now follow by choosing

$$h(t) = \log(1+t), \quad w(t) = (\lambda-1) \log(t/(1+t)),$$

and checking that the required conditions are satisfied.