Modelling and estimation of traffic flow—a martingale approach

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A traffic situation is studied in which some counters are placed in the streets to count some combination of traffic flows in various directions. Based on these observations, individual traffic flows are estimated using recent results on filtering with counting observation. Simulation studies are given to demonstrate the workability of the approximate estimation equations obtained from theoretical consideration.

1. Introduction

Control of traffic flow is a problem of immense practical importance. At present, most interest is directed to the development of traffic-responsive control systems. Isaksen and Payne (1973) consider such a particular control problem. The general problem requires the knowledge of cars flowing in various directions in a given traffic profile. Often, however, the individual traffic flows may not be known; for example, when only a limited number of counters can be placed on the streets. Then we have to estimate individual traffic flows in various directions before any control action can be implemented. In this article we study this estimation problem using a model for traffic flows which is a multidimensional linear dynamical system with a martingale forcing term. We begin by modelling each subsystem consisting of the so called 'arrival' and 'departure processes' and then combine these subsystems to obtain the general model. This is an extension of the model first suggested in the work of van der Kallen (1975).

Our objective is to estimate, at the current time, all the arrival and departure processes separately based on the observation of the cars passing all the counters up to the current time. This is done using recent results on state estimation of systems with martingale forcing terms based on counting observations, due to Segall (1973) and van Schuppen (1977). For the general theory and applications of dynamical systems with martingale forcing terms we refer the reader to the works of Boel *et al.* (1975 a, b) and Varaiya (1975).

The exact solution to the estimation problem mentioned above involves an infinite set of equations for the conditional moments of the estimation error. We propose an approximation for the error covariance equation to obtain the estimates of the arrival and departure processes in a computable form suitable for practical applications. We also use the fact that the observation is a linear combination of the system states to obtain considerable reduction in the estimation equations. Simulation studies are performed in

Received 20 May 1979.

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‡ Supported partially by the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

two special traffic situations. The results justify the approximation proposed. All the definitions and proofs are given in the Appendices.

2. Modelling of traffic flow

2.1. Problem description

Traffic flows are counted by placing detectors in various streets. We consider pressure detectors for this purpose in which case the counts represent the number of car-axes and not the number of cars that passed the detector. In the following discussion we shall therefore only consider the car-axes for the purpose of estimation. In what follows, reference to 'cars' will always mean 'car-axes'. In our general model we consider the possibility that some counters might be placed over the complete two-way street. In principle one may, perhaps, position the detector so that it only detects one-way traffic. This, however, will not give us the complete information about the traffic profile. For example, in a two-way traffic intersection, even if all the eight detectors are placed on the streets, we would not be able to know exactly the evolution of the twelve possible traffic flows involved.

We consider a traffic situation in which m counters, numbered $1, 2, \dots, m$, are placed in various streets considered in the complete traffic profile. We take into account only those flows that cross at least one counter. Two flows are different if they pass different counters and/or the same set of counters in a different order. Let p be the total number of different flows.

We now define various counting processes that are of interest to us.

N_{jt} , $1 \leq j \leq m$, is the total number of cars that pass counter j , in the time interval $[0, t]$,

A_{kt} , $1 \leq k \leq p$, is the number of cars of traffic flow k that pass the first counter of the sequence of counters that flow k passes through, in the time interval $[0, t]$,

D_{ki}^t , $1 \leq k \leq p$, is the number of cars of traffic flow k that pass the $(i+1)$ st counter of the sequence of counters that flow k passes through in the time interval $[0, t]$. If flow k passes through M counters, the index i ranges over 1 to $M-1$. In case $M=2$ we write D_{ki} for the departure process D_{ki}^t .

The processes A_{kt} , $t \geq 0$, are called *arrival processes*. We call D_{ki}^t , $t \geq 0$, *departure processes* because they describe how cars in the traffic flow k leave the subsystem between i th and $(i+1)$ st counters. In the present paper, we investigate the problem of estimating A_i and D_i based on actual counting of the processes N_i . We illustrate all the various processes defined above by means of an example of a double T-crossing and possible flow pattern (Fig. 1).

2.2. Arrival processes

We take a very simple model for arrival processes. We assume that arrivals follow an inhomogeneous Poisson process. Thus

$$dA_{kt} = \lambda_{kt} dt + dm_{kt}, \quad 1 \leq k \leq p \quad (2.1)$$

where λ_{kt} , $1 \leq k \leq p$, is a strictly positive deterministic function and m_{kt} , $1 \leq k \leq p$, is a square integrable martingale (see Appendix A).

Figure 1.

The intensity function λ_{kt} , the so called 'rate process', is the frequency of arriving cars of traffic flow k . Furthermore, we assume that an *a priori* knowledge of λ_{kt} can be obtained on the basis of observation of previous days.

In practice, these *a priori* data may be difficult and expensive to obtain. In the absence of knowledge about the rate-processes, we can always incorporate them in a state model and estimate the rate processes along with the arrival processes. In the latter case, the arrival process model (2.1) can easily be extended to a doubly stochastic Poisson process model.

2.3. Departure processes

To clarify our analysis, we consider first the simplest subsystem in which departure actually occurs. This subsystem contains all the essential features of any subsystem involving a departure process. The subsystem is illustrated below in Fig. 2. We analyse in detail the departure process involved.

Figure 2.

Let $\{\mathcal{B}_t^k, t \geq 0\}$ denote the family of σ -algebras generated by the arrival and departure processes of flow k ; thus

$$\mathcal{B}_t^k = \sigma(A_{ks}, D_{ks}; 0 \leq s \leq t) \text{ for each } t \geq 0 \quad (2.2)$$

Assuming the existence of the rate process, we know that the departure process, being a counting process, can be uniquely represented by

$$dD_{kt} = \mu_{kt} dt + dn_{kt} \quad (2.3)$$

where the rate process μ_{kt} is a $\mathcal{B}_t^k$ -adapted process satisfying $\mu_{kt} \geq 0$ a.s. for all $t \geq 0$, and n_{kt} is a square integrable $\mathcal{B}_t^k$ -martingale. Modelling the departure process D_{kt} , therefore, reduces to finding an expression for the rate process μ_{kt} .

Following van der Kallen (1975), we make the following assumptions :

- (i) the time needed by different cars of the flow k to drive from counter i_1 to counter i_2 are independent random variables with the same distribution function F_t and corresponding density function f_t ,
- (ii) at $t=0$, there are no cars between the counters.

Based on these assumptions the departure process can be expressed in terms of arrival process and the distribution function F_t . Given the history of the arrival process up to time s , we have for all $t \geq s$.

$$\begin{aligned} E[D_{kt} | \mathcal{A}_s^k] &= E\left[\sum_{\theta \leq t} F_{t-\theta} \Delta A_{k\theta} | \mathcal{A}_s^k\right] \\ &= \int_0^s F_{t-\theta} dA_{k\theta} + \int_s^t F_{t-\theta} \lambda_{k\theta} d\theta \end{aligned} \quad (2.4)$$

where $\{\mathcal{A}_t^k, t \geq 0\}$ is the family of σ -algebras generated by the arrival process A_{kt} ; thus

$$\mathcal{A}_t^k = \sigma(A_{ks}; 0 \leq s \leq t) \quad \text{for each } t \geq 0 \quad (2.5)$$

From (2.4) we derive by straightforward calculation

$$dE[D_{kt} | \mathcal{A}_t^k] - E[dD_{kt} | \mathcal{A}_t^k] = \int_0^t F_{t-\theta} d\left(A_{k,t+\theta} - \int_0^{t+\theta} \lambda_{k\theta} ds\right) \quad (2.6)$$

Now, before specifying the distribution function F_t we note that there exists a minimum time to drive from counter i_1 to counter i_2 . This means that $F \equiv 0$ for all $t \leq \tau$, where τ is a strictly positive lower bound for the driving time. This fact yields two results ; from (2.4) it follows that

$$\begin{aligned} E[D_{kt} | \mathcal{A}_t^k] &= \int_0^t F_{t-\theta} dA_{k\theta} \\ &= - \int_0^t A_{k\theta} dF_{t-\theta} = \int_0^t f_{t-\theta} A_{k\theta} d\theta \end{aligned} \quad (2.7)$$

and (2.6) becomes

$$dE[D_{kt} | \mathcal{A}_t^k] = E[dD_{kt} | \mathcal{A}_t^k] \quad (2.8)$$

Since an arriving car takes a positive time to reach the departure counter, the arrival and departure processes will be uncorrelated increments conditioned on the σ -algebra $\mathcal{B}_t^k$. This implies that

$$\begin{aligned} E[dm_{kt} n_{kt} | \mathcal{B}_t^k] &= E[dm_{kt} | \mathcal{B}_t^k] E[dn_{kt} | \mathcal{B}_t^k] \\ &= E[dm_{kt} | \mathcal{B}_t^k] E[dn_{kt} | \mathcal{B}_t^k] = 0 \end{aligned}$$

so that

$$\langle m_k, n_k \rangle_t \equiv 0 \quad (2.9)$$

To obtain models for departure processes, we have to specify the density function f_t . A realistic density function is the gamma distribution:

$$f_t = \begin{cases} 0 & \text{for } t < \tau, \tau > 0 \\ \frac{1}{\alpha^2} (t - \tau) \exp[-(t - \tau)/\alpha] & \text{for } t \geq \tau, \alpha > 0 \end{cases} \quad (2.10)$$

where α and τ are positive constants. This density function will lead to linear models for departure processes, but with *time delays* introduced through the dead time τ in f_t . This results in considerable complexity in the estimation problem. A common procedure is to approximate the delayed system by a suitable lumped system. The same approximation can, however, be achieved by considering at the start an approximation to the function f_t without the dead time τ . A collection of density functions reasonably approximating (2.10) is

$$f_t = \begin{cases} 0 & \text{for } t < 0 \\ \frac{1}{\beta^{\gamma+1}(\gamma!)} t^\gamma \exp[-t/\beta] & \text{for } t \geq 0, \beta > 0, \gamma \geq 2 \text{ is an integer} \end{cases} \quad (2.11)$$

It is obvious that the higher γ is, the better is the approximation. However, with increasing γ , the dimension of the state space model for departure process will also increase. In the following, we take $\gamma = 2$. Higher values of γ can be treated similarly without any basic change in the following procedure. With $\gamma = 2$, (2.7) becomes

$$E[D_{kt} | \mathcal{A}_t^k] = \frac{1}{2\beta^3} \int_0^t (t-s)^2 \exp[-(t-s)/\beta] A_{ks} ds$$

which is the solution of the following integral equation

$$\begin{aligned} E[D_{kt} | \mathcal{A}_t^k] &= \frac{1}{\beta^3} \int_0^t \int_0^s \int_0^\sigma A_{k\theta} d\theta d\sigma ds - \frac{3}{\beta} \int_0^t E[D_{ks} | \mathcal{A}_s^k] ds \\ &- \frac{3}{\beta^2} \int_0^t \int_0^s E[D_{k\sigma} | \mathcal{A}_\sigma^k] d\sigma ds - \frac{1}{\beta^3} \int_0^t \int_0^s \int_0^\sigma E[D_{k\theta} | \mathcal{A}_\theta^k] d\theta d\sigma ds \end{aligned} \quad (2.12)$$

Since arrivals of cars after time t do not influence the departure process up to time t , we have for $\theta \geq t$

$$E[D_{kt} | \mathcal{A}_\theta^k] = E[D_{kt} | \mathcal{A}_t^k]$$

Using this principle of causality we see from (2.12) that

$$dE[D_{kt} | \mathcal{A}_t^k] = E \left[\frac{1}{\beta^3} \int_0^t \int_0^s (A_{k\sigma} - D_{k\sigma}) d\sigma ds - \frac{3}{\beta} D_{kt} - \frac{3}{\beta^2} \int_0^t D_{ks} ds | \mathcal{A}_t^k \right] dt \quad (2.13)$$

Combining (2.8) and (2.13) we have

$$E[dD_{kt} | \mathcal{A}_t^k] = E \left[\frac{1}{\beta^3} \int_0^t \int_0^s (A_{k\sigma} - D_{k\sigma}) d\sigma ds - \frac{3}{\beta} D_{kt} - \frac{3}{\beta^2} \int_0^t D_{ks} ds | \mathcal{A}_t^k \right] dt \quad (2.14)$$

Returning to our departure process model (2.3) we establish that

$$E(dD_{kt}|\mathcal{A}_t^k) = E[\mu_{kt}|\mathcal{A}_t^k] dt$$

yielding with (2.14) the equality

$$E[\mu_{kt}|\mathcal{A}_t^k] = E\left[\frac{1}{\beta^3} \int_0^t \int_0^s (A_{ks} - D_{ks}) d\sigma ds - \frac{3}{\beta} D_{kt} - \frac{3}{\beta^2} \int_0^t D_{ks} ds | \mathcal{A}_t^k\right] \quad (2.15)$$

We, therefore, consider the following expression for rate process μ_{kt}

$$\mu_{kt} = \frac{1}{\beta^3} \int_0^t \int_0^s (A_{ks} - D_{ks}) d\sigma ds - \frac{3}{\beta} D_{kt} - \frac{3}{\beta^2} \int_0^t D_{ks} ds \quad (2.16)$$

because the right-hand side of (2.16) is a $\mathcal{B}_t^k$ -adapted process.

Introducing new augmented state variables

$$\begin{aligned} {}^0D_{kt} &= D_{kt}, & {}^1D_{kt} &= \int_0^t D_{ks} ds, & {}^2D_{kt} &= \int_0^t \int_0^s D_{ks} d\sigma ds \\ {}^0A_{kt} &= A_{kt}, & {}^1A_{kt} &= \int_0^t A_{ks} ds, & {}^2A_{kt} &= \int_0^t \int_0^s A_{ks} d\sigma ds \end{aligned}$$

we can rewrite the departure process model (2.3) and (2.16) as

$$\left. \begin{aligned} d^0D_{kt} &= \left(-\frac{3}{\beta} {}^0D_{kt} - \frac{3}{\beta^2} {}^1D_{kt} - \frac{1}{\beta^3} {}^2D_{kt} + \frac{1}{\beta^3} {}^2A_{kt} \right) dt + dn_{kt} \\ d^1D_{kt} &= {}^0D_{kt} dt, & d^2D_{kt} &= {}^1D_{kt} dt \\ d^1A_{kt} &= {}^0A_{kt} dt, & d^2A_{kt} &= {}^1A_{kt} dt \end{aligned} \right\} \quad (2.17)$$

We, therefore, get a linear third order approximate model for departure processes. This approximation eliminates the complication of studying a system model involving time delay.

Other departure processes D_{kt}^i , $i > 1$, can be similarly modelled with D_{kt}^{i-1} replacing A_{kt} and D_{kt}^i replacing D_{kt} in (2.17).

3. Estimation of arrival and departure processes

3.1. General nature of the model

Suppose that in the traffic situation considered, there are altogether n states, consisting of p arrival processes, q departure processes and $n - (p + q)$ additional components of the augmented state vector, while there are m observations. Let

$$X_t = \text{col} (X_{1t}, \dots, X_{pt}, X_{(p+1)t}, \dots, X_{(p+q)t}, \dots, X_{(p+q+1)t}, \dots, X_{nt})$$

where X_{it} , $1 \leq i \leq p$, are the arrival processes, X_{it} , $p+1 \leq i \leq p+q$, are the departure processes, and X_{it} , $p+q+1 \leq i \leq n$, are the additional state processes ;

$$N_t = \text{col} (N_{1t}, \dots, N_{mt})$$

where N_j , $1 \leq j \leq m$, are the processes counted at the j th detector. Combining all the arrival and departure processes with models of the form (2.1) and (2.17) we have the general form :

$$\left. \begin{aligned} dX_t &= (AX_t + \lambda_t) dt + dM_t \\ N_t &= BX_t \end{aligned} \right\} \quad (3.1)$$

where A , B are constant matrices of appropriate dimensions, strongly related to the particular traffic situation; $\lambda_t = \text{col}(\lambda_{1t}, \dots, \lambda_{nt})$, where λ_{it} , $1 \leq i \leq p$, are as defined in (2.1) and $\lambda_{it} = 0$, $p+1 \leq i \leq n$, for all t ; $M_t = \text{col}(M_{1t}, \dots, M_{nt})$, the components of M_t are $\mathcal{B}_t$ -martingales, where $\{\mathcal{B}_t, t \geq 0\}$ is the family of σ -algebras generated, for each t by all the arrival and departure processes X_{is} , $1 \leq i \leq p+q$, $0 \leq s \leq t$. Moreover, $M_{it} = 0$, $p+q+1 \leq i \leq n$, for all t ; N_t is a vector of counting processes, which records counts at the detectors, and X_{it} , $1 \leq i \leq p+q$, are also counting processes. Note also that elements of the matrix B are 0 or 1, each column of B has at most one element equal to 1 and each row has at least one element 1.

We have already seen that $\langle M_i, M_j \rangle_t \equiv 0$ if i and j correspond to the arrival and departure processes of the same flow. In the absence of any specific knowledge, we assume that $\langle M_i, M_j \rangle_t \equiv 0$ whenever $i \neq j$. This does not necessarily imply that different flows do not interact with each other. The gamma distribution assumed for f_t was intended to incorporate the interaction between different flows.

Remark

The model (3.1) describes the system for all values of γ in (2.11). The number of states will, however, increase with γ , because the order of departure processes will increase. If we increase γ by 1, the number of states in (3.1) will be increased by $p+q-r$, where r is the number of flows that pass only one counter, and the number of estimation equations in the following discussion will increase enormously.

3.2. Estimation equations

We now obtain the best minimum variance estimate of X_t based on the observation N_s , $0 \leq s \leq t$. Let $\mathcal{F}_t$ be the smallest σ -algebra generated by N_s , $0 \leq s \leq t$. Then the best estimate of X_t is given by

$$\hat{X}_t = E[X_t | \mathcal{F}_t]$$

Now X_t is given by eqn. (3.1), which can be written in the standard form of stochastic differential system

$$\left. \begin{aligned} dX_t &= (AX_t + \lambda_t) dt + dM_t \\ dN_t &= B(AX_t + \lambda_t) dt + dB M_t \end{aligned} \right\} \quad (3.1)$$

To derive the recursive equation for $\hat{X}_t$, define

$$\nu_t = N_t - \int_0^t B(A\hat{X}_s + \lambda_s) ds \quad (3.2)$$

$$\tilde{X}_t = X_t - \hat{X}_t \quad (3.3)$$

and use the notation $X_t \cong Y_t$ if and only if $X_t - Y_t$ is an $\mathcal{F}_t$ -martingale. It is easy to see (Segall 1973) that for counting process observations as in this problem,

$\mathcal{F}_t \equiv$ smallest σ -algebra generated by ν_s , $0 \leq s \leq t$

The processes X_{it} , N_{jt} have left-hand limits, are right continuous and square integrable. Define

$$d\mu_t = d\hat{X}_t - (A\hat{X}_t + \lambda_t) dt, \quad \mu_0 = 0 \quad (3.4)$$

If we assume that $\{B(A\hat{X}_t + \lambda_t)\}_j > 0$, $\{B(A\hat{X}_{t-} + \lambda_{t-})\}_j > 0 \forall t, \forall j$, we can use the 'innovation theorem' (Segall 1973) which tells the existence of a $n \times m$ matrix Φ_t , elements of which are $\mathcal{F}_t$ -predictable, such that

$$\mu_t = \int_0^t \Phi_s d\nu_s \quad (3.5)$$

This gives us

$$\hat{X}_t = \hat{X}_0 + \int_0^t (A\hat{X}_s + \lambda_s) ds + \int_0^t \Phi_s d\left[N_s - \int_0^s B(A\hat{X}_r + \lambda_r) dr\right]$$

that is,

$$\hat{X}_t = \hat{X}_0 + \int_0^t \Phi_s dN_s + \int_0^t (I_n - \Phi_s B)(A\hat{X}_s + \lambda_s) ds \quad (3.6)$$

where the matrix Φ_t is given by (see Appendix B)

$$(\Phi_t)_{ij} = \begin{cases} 0 & \text{for } p+q+1 \leq i \leq n, \quad 1 \leq j \leq m \\ [(\hat{\Delta}_{t-} + \hat{P}_{t-}^x A^T)B^T(B\hat{\Delta}_{t-}B^T)^{-1}]_{ij}, & \text{otherwise} \end{cases} \quad (3.7)$$

where $\hat{\Delta}_t$ is a diagonal matrix with

$$\text{diag}(\hat{\Delta}_t) = \text{col}[(A\hat{X}_t + \lambda_t)_1, \dots, (A\hat{X}_t + \lambda_t)_{p+q}, 0, \dots, 0] \quad (3.8)$$

and

$$\hat{P}_t^x = E[(X_t - \hat{X}_t)(X_t - \hat{X}_t)^T | \mathcal{F}_t]$$

In deriving (3.7) we need to use the result (see Appendix B)

$$\langle \nu, \nu \rangle_t = \int_0^t B\hat{\Delta}_s B^T ds \quad (3.9)$$

One can directly verify that $(B\hat{\Delta}_t B^T)$ is a diagonal matrix. We now need to obtain an equation for $\hat{P}_t^x$. We readily derive from (3.5) that

$$d\langle \mu, \mu \rangle_t = \Phi_t d\langle \nu, \nu \rangle_t \Phi_t^T \quad (3.10)$$

To obtain $\langle \mu, \mu \rangle_t$, we see that

$$\begin{aligned}
 E[\langle \mu, \mu \rangle_t - \langle \mu, \mu \rangle_s | \mathcal{F}_s] &= E[(\mu_t - \mu_s)(\mu_t - \mu_s)^T | \mathcal{F}_s] \\
 &= E\left[\left(M_t - M_s - \tilde{X}_t + \tilde{X}_s + \int_s^t A \tilde{X}_r dr\right) \left(M_t - M_s - \tilde{X}_t + \tilde{X}_s + \int_s^t A \tilde{X}_r dr\right)^T \middle| \mathcal{F}_s\right] \\
 &= E\left[\widehat{\langle M, M \rangle}_t - \widehat{\langle M, M \rangle}_s - \widehat{\tilde{X}_t \tilde{X}_t^T} + \widehat{\tilde{X}_s \tilde{X}_s^T} + \int_s^t (\widehat{\tilde{X}_r \tilde{X}_r^T} A^T + A \widehat{\tilde{X}_r \tilde{X}_r^T}) dr \middle| \mathcal{F}_s\right]
 \end{aligned}$$

the rest of the terms being identically zero, so that

$$\langle \mu, \mu \rangle_t \cong \widehat{\langle M, M \rangle}_t - \hat{P}_t^x + \int_0^t (\hat{P}_s^x A^T + A \hat{P}_s^x) ds$$

and (3.10) yields

$$\Phi_t B_t \hat{\Delta}_t B_t^T \Phi_t^T dt \cong \hat{\Delta}_t dt - d\hat{P}_t^x + (\hat{P}_t^x A^T + A \hat{P}_t^x) dt$$

Here we used the fact that $\widehat{\langle M, M \rangle}_t = \int_0^t \hat{\Delta}_s ds + \text{an } \mathcal{F}_t\text{-martingale}$, which has been proved in Appendix B. We, therefore, obtain

$$\hat{P}_t^x \cong \hat{P}_0^x + \int_0^t (\hat{\Delta}_s + \hat{P}_s^x A^T + A \hat{P}_s^x - \Phi_s B_s \hat{\Delta}_s B_s^T \Phi_s^T) ds \quad (3.11)$$

where as defined before, $\cong$ means that the difference between the left- and the right-hand sides of (3.11) is an $\mathcal{F}_t$ -martingale.

As mentioned before, the above derivations are meaningful if all the components of $B(A\tilde{X}_t + \lambda_t)$ are non-zero for t and $t-$, in the whole interval of interest. Furthermore, the expression for (3.7) needs the additional condition that $B\hat{\Delta}_t B^T$ has inverse for all t and $t-$ in the time interval of interest. We claim that these two conditions are equivalent. This follows from an alternate expression for $\langle \nu, \nu \rangle_t$. We have already noted in eqn. (3.9) that $\langle \nu, \nu \rangle_t$ is a diagonal matrix. To determine the components on the diagonal, we write the i th component.

$$\nu_{it} = N_{it} - \int_0^t [B(A\tilde{X}_s + \lambda_s)]_i ds$$

where N_{it} is an one-dimensional counting process, so that

$$\langle \nu_i, \nu_i \rangle_t = \int_0^t B[(A\tilde{X}_s + \lambda_s)]_i^2 ds$$

and

$$\text{diag}(\langle \nu, \nu \rangle_t) = \int_0^t B(A\tilde{X}_s + \lambda_s) ds \quad (3.12)$$

and our assertion follows. One can prove this equality also by direct calculation.

If we should try to obtain an exact equation for $\hat{P}_t^x$ we would need the third order moment for the conditional error, which could be evaluated only with the knowledge of the fourth order moment and so on. The optimal filter, therefore, requires an infinite set of equations for its solution and an approximation has to be made. One possible approximation is to replace $\cong$ by equality in (3.11). This amounts to neglecting the martingale term in that equation, which can be justified on account of the fact that a martingale has always zero expectation. The approximate filter equation we use for actual computation is

$$\hat{X}_t = \hat{X}_0 + \int_0^t \Phi_s dN_s + \int_0^t (I_n - \Phi_s B)(A \hat{X}_s + \lambda_s) ds \quad (3.6 a)$$

$$\hat{P}_t^x = \hat{P}_0^x + \int_0^t (\hat{\Delta}_s + \hat{P}_s^x A^T + A \hat{P}_s^x - \Phi_s B \hat{\Delta}_s B^T \Phi_s^T) ds \quad (3.11 a)$$

3.3. Reduction of estimation equations

Because of the special nature of the observations we are able to reduce significantly the number of estimation equations. Each component of the observation equation in (3.1) is a linear combination of components of the state vector. Therefore, if we estimate $(n-m)$ appropriate states of (3.1), the estimates of the remaining states can be easily obtained from the observation equations. In the preceding subsection, we have to solve $\frac{1}{2}n(n+3)$ equations to obtain the desired estimates of the states. If however, we make use of the structure of the observations as just noted, the number of equations to be solved reduces to $\frac{1}{2}(n-m)(n-m+3)$.

We denote by the vector Z_t the appropriate $(n-m)$ states we want to estimate and write

$$X_t = \begin{bmatrix} Y_t \\ \vdots \\ Z_t \end{bmatrix} \quad \text{with corresponding partition} \quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}$$

where we take B_1 to be identity. This can always be done by proper rearrangement of the components of X_t . We make the corresponding of A , λ_t and Δ_t as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \vdots & \vdots \\ A_{21} & A_{22} \end{bmatrix}, \quad \lambda_t = \begin{bmatrix} \lambda_{1t} \\ \vdots \\ \lambda_{2t} \end{bmatrix}, \quad \Delta_t = \begin{bmatrix} \Delta_{1t} & 0 \\ \vdots & \vdots \\ 0 & \Delta_{2t} \end{bmatrix}$$

and write

$$K_1 = A_{11}B_2 - A_{12}, \quad K_2 = A_{22} - A_{21}B_2$$

The estimation equations for Z_t can easily be derived from (3.6 a) and (3.11 a) and become

$$\begin{aligned} \hat{Z}_t = \hat{Z}_0 + \int_0^t \Phi_{2s} dN_s + \int_0^t \{ [A_{21} - \Phi_{2s}(A_{11} + B_2 A_{21})] \hat{N}_s \\ + [K_2 - \Phi_{2s}(B_2 K_2 - K_1)] \hat{Z}_s + [(I_{n-m} - \Phi_{2s} B_2) \lambda_{2s} - \Phi_{2s} \lambda_{1s}] \} ds \end{aligned} \quad (3.13)$$

$$\hat{P}_t^z = \hat{P}_0^z + \int_0^t (\hat{\Delta}_{2s} + \hat{P}_s^z K_2^T + K_2 \hat{P}_s^z - \Phi_{2s} B \hat{\Delta}_s B^T \Phi_{2s}^T) ds \quad (3.14)$$

where

$$(\Phi_{2t})_{ij} = \begin{cases} 0 & \text{for } (p+q+1)-m \leq i \leq n-m, \quad 1 \leq j \leq m \\ [(-\hat{P}_{t-}^z K_1^T + (\hat{\Delta}_{2t-} + \hat{P}_t^z K_2^T) B_2^T)(B \hat{\Delta}_{t-} B^T)^{-1}]_{ij}, & \text{otherwise} \end{cases}$$

and

$$(\hat{\Delta}_t)_{ii} = \begin{cases} (A_{11}N_t - K_1\hat{Z}_t + \lambda_{1t})_i & \text{for } 1 \leq i \leq m \\ (A_{21}N_t + K_2\hat{Z}_t + \lambda_{2t})_{i-m} & \text{for } m+1 \leq i \leq p+q \\ 0 & \text{for } p+q+1 \leq i \leq n \end{cases}$$

Finally we have

$$\hat{Y}_t = N_t - B_2\hat{Z}_t \quad (3.15)$$

4. Simulation results

For two special traffic situations we present some numerical estimation results using data simulated with the model in (3.1). The first example is more of academic interest. It is intended to test the validity of our model and the approximate filter used. The second example portrays a typical situation in which cars from a minor street join the mainstream traffic.

Example 1

Figure 3.

If we assume that cars are not allowed to turn between the two counters, then there are 2 traffic flows and 6 counting processes consisting of 2 arrival, 2 departure and 2 observation processes. Our objective is to estimate the flows in both directions, which is equivalent to estimating the processes A_{1t} , A_{2t} , D_{1t} and D_{2t} , based on the observations N_{1t} and N_{2t} . With the third order model for the departure processes the state vector X_t will have 12 components, resulting in 65 equations in reduced form. The idea, that whenever at each counter one process is estimated, the observation equations yield immediately estimates for the other processes, illustrates the reduction described in subsection 3.3. In the simulation, the following parameter values were taken: $\lambda_{1t} \equiv \lambda_{2t} \equiv 20$ and $\beta^{-1} = 45$. The numbers were chosen so as to make the mean arrival times and the mean travel time of the same order. Errors in estimating the two arrival processes along with the simulation data for all the arrival and departure processes are shown in Fig. 4.

Example 2

In this T-crossing with 4 half-road counters—implying that we have 4 observations—there are 6 traffic flows, 10 arrival and 2 departure processes. Instead of solving 90 estimation equations we note that in subsection 2.3 we implicitly assumed that between the counters there is no interference of cars

Figure 4.

Example 2:

Figure 5.

Figure 6.

Figure 7.

of different traffic flows. Therefore, in this special situation, the system can be split up into two independent subsystems with identical model structure and we have to consider only one of them (see Fig. 6).

The reason for splitting up the system is that, for each subsystem, we now have to solve only 27 estimation equations. In the simulation, the following parameter values were taken: $\lambda_{1i} \equiv 5$, $\lambda_{2i} \equiv 10$, $\lambda_{3i} \equiv 20$ and $\beta^{-1} = 15$. Some data given in Slater (1976) indicates plausible values for these parameters in practice. Errors in estimating the arrival and the departure for flow 3 along with the simulation data for all the flows are shown in Fig. 7.

Similar results have been obtained for other processes and with other parameter values as well. We considered only constant frequencies of arriving cars.

5. Conclusion

In this paper, we presented a simple model describing the dynamics in a traffic situation with counters and estimation equations for individual traffic flows based on the counting observations. Simulation studies in two special traffic situations give credibility to the approximation proposed.

There are various directions in which the results in this paper can be extended. An interesting possibility is to consider stochastic rates for the arrival processes, which may be tackled by augmenting the state space.

ACKNOWLEDGMENTS

We are grateful to Professor H. Kwakernaak for his keen interest and valuable suggestions throughout the preparation of this paper.

Appendix A

Details may be found in Kunita and Watanabe (1967). Let $(\Omega, \mathcal{B}, p)$ denote a probability space and let $\{\mathcal{B}_t, t \geq 0\}$ be an increasing family of sub- σ -algebra's of $\mathcal{B}$ with $\bigcap_{t>0} \mathcal{B}_t = \mathcal{B}_{0+} = \mathcal{B}_0$ for all $t \geq 0$.

Definition 1

A process $M_t, t \geq 0$, is called a $\mathcal{B}_t$ -martingale, if (i) M_t is $\mathcal{B}_t$ -adapted, (ii) $E|M_t| < \infty$ for all $t \geq 0$, (iii) $M_0 = 0$ a.s., (iv) $E[M_t | \mathcal{B}_s] = M_s$ a.s. for all $s \leq t$.

If the process M_t also satisfies $EM_t^2 < \infty$ for all $t \geq 0$, then it is called a square integrable martingale.

Definition 2

The predictable variation process $\langle M, M \rangle_t, t \geq 0$, of a square integrable $\mathcal{B}_t$ -martingale M_t is defined to be the unique, predictable increasing process such that $M_t^2 - \langle M, M \rangle_t$ is a $\mathcal{B}_t$ -martingale. The process $\langle M, M \rangle_t$ always exists (see Kunita and Watanabe 1967). If M_{1t} and M_{2t} are two integrable $\mathcal{B}_t$ -martingales, then we define the predictable covariation process $\langle M, M_2 \rangle_t, t \geq 0$, by $\langle M_1, M_2 \rangle_t = \frac{1}{4}[\langle M_1 + M_2, M_1 + M_2 \rangle_t - \langle M_1 - M_2, M_1 - M_2 \rangle_t]$.

Appendix B

Details may be found in van Maarseveen (1976). From (3.1), we see that

$$X_t = \int_0^t (AX_s + \lambda_s) ds + M_t$$

The first $(p+q)$ terms of X_t are counting processes and the rest are deterministic, so that $M_t \equiv 0$ for these remaining components. For counting processes, we know that, if M_{it} denotes the i th component of M_t for $1 \leq i \leq p+q$, then

$$\langle M_i, M_i \rangle_t = \int_0^t (AX_s + \lambda_s)_i ds$$

Further,

$$\begin{aligned} \langle M_i, M_j \rangle_t &= 0 \quad \text{for } i \neq j \quad \text{and} \\ \langle M_i, M_i \rangle_t &= 0 \quad \text{trivially, for } p+q+1 \leq i \leq n \end{aligned}$$

Thus $\langle M, M \rangle_t$ is a diagonal matrix with

$$\text{diag } \langle M, M \rangle_t = \left[\int_0^t (AX_s + \lambda_s)_1 ds, \dots, \int_0^t (AX_s + \lambda_s)_{p+q} ds, 0, \dots, 0 \right]$$

and we obtain the result

$$\widehat{\langle M, M \rangle}_t = \int_0^t \hat{\Delta}_s ds + a\mathcal{F}_t\text{-martingale} \quad (\text{B1})$$

To obtain $\langle \nu, \nu \rangle_t$, we see that

$$\begin{aligned} E(\langle \nu, \nu \rangle_t - \langle \nu, \nu \rangle_s | \mathcal{F}_s) &= E[(\nu_t - \nu_s)(\nu_t - \nu_s)^T | \mathcal{F}_s] \\ &= E \left[\left(BM_t - BM_s + \int_s^t BA\tilde{X}_r dr \right) \left(BM_t - BM_s + \int_s^t BA\tilde{X}_r dr \right)^T \middle| \mathcal{F}_s \right] \\ &= E[B\widehat{\langle M, M \rangle}_t B^T - B\widehat{\langle M, M \rangle}_s B^T | \mathcal{F}_s] \end{aligned}$$

the rest of the terms being identically zero, so that

$$\langle \nu, \nu \rangle_t \cong B\widehat{\langle M, M \rangle}_t B^T \cong \int_0^t B\hat{\Delta}_s B^T ds$$

that is,

$$\langle \nu, \nu \rangle_t - \int_0^t B\hat{\Delta}_s B^T ds$$

is a $\mathcal{F}_t$ -martingale. On the other hand, $\langle \nu, \nu \rangle_t$ is a diagonal matrix with $\text{diag } \langle \nu, \nu \rangle_t = \int_0^t B(AX_s + \lambda_s) ds$ and a martingale with continuous sample paths cannot be of bounded variation. Therefore,

$$\langle \nu, \nu \rangle_t = \int_0^t B\hat{\Delta}_s B^T ds \quad (\text{B2})$$

Finally, to obtain the expression for Φ_t , we use the relation derived from (3.5), that

$$d\langle \mu, \nu \rangle_t = \Phi_t d\langle \nu, \nu \rangle_t \quad (\text{B3})$$

Since Φ_t is predictable, it must be left continuous at the jumps of N_t and we can take

$$\Phi_t = \frac{d}{dt} \langle \mu, \nu \rangle_t (B \hat{\Delta}_{t-} B^T)^{-1} \quad (\text{B4})$$

provided, of course, that the inverse above exists.

To obtain $\langle \mu, \nu \rangle_t$, observe that

$$\begin{aligned} E[\langle \mu, \nu \rangle_t - \langle \mu, \nu \rangle_s | \mathcal{F}_s] &= E[(\mu_t - \mu_s)(\nu_t - \nu_s)^T | \mathcal{F}_s] \\ &= E\left[\left(M_t - M_s - \tilde{X}_t + \tilde{X}_s + \int_s^t A \tilde{X}_r dr\right) \times \left(BM_t - BM_s + \int_s^t BA \tilde{X}_r dr\right)^T \middle| \mathcal{F}_s\right] \\ &= E[\widehat{\langle M, M \rangle}_t B^T - \widehat{\langle M, M \rangle}_s B^T + \int_s^t \widehat{\tilde{X}_r \tilde{X}_r} A^T B^T dr | \mathcal{F}_s] \end{aligned}$$

the rest of the terms being identically zero, so that

$$\langle \mu, \nu \rangle_t = \widehat{\langle M, M \rangle}_t B^T + \int_0^t \hat{P}_s^z A^T B^T ds$$

(B 3) then yields

$$\int_0^t \Phi_s B \hat{\Delta}_s B^T ds - \int_0^t \hat{\Delta}_s B^T ds - \int_0^t \mathbf{p}_s^z A^T B^T ds$$

is a $\mathcal{F}_t$ -martingale. Again a martingale with continuous sample path cannot be of bounded variation, so that

$$d\langle \mu, \nu \rangle_t = \hat{\Delta}_t B^T dt + \mathbf{p}_t^z A^T B^T dt$$

and using the fact that the components μ_i of μ are zero for $p+q+1 \leq i \leq n$ and using (B 4), we get

$$(\Phi)_{ij} = \begin{cases} 0, & p+q+1 \leq i \leq n, \quad 1 \leq j \leq m \\ [(\hat{\Delta}_{t-} + \hat{P}_{t-}^z A^T) B^T (B \hat{\Delta}_{t-} B^T)^{-1}]_{ij}, & \text{otherwise} \end{cases}$$

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