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Clique polynomials and independent set polynomials of graphs

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Abstract

This paper introduces two kinds of graph polynomials, clique polynomial and independent set polynomial. The paper focuses on expansions of these polynomials. Some open problems are mentioned.

1. Introduction

Many kinds of graph polynomials have been introduced and extensively studied in the literature. We mention characteristic polynomial, chromatic polynomial, Tutte polynomial [2], matching polynomial [6–9]. For terminology and notations, we refer to Biggs [2]. In this paper we introduce two new graph polynomials, clique polynomial and independent set polynomial.

These concepts came up in connection with our research on the Maximum Clique Problem and the Independent Set Problem [12, 13]. Recently it came to our notice that Fisher and Solow [5] have studied dependence polynomials that are essentially the same as our clique polynomials. These dependence polynomials have been introduced by Fisher [4] in a counting problem for words. Our first, easy, results are the same as those in [5], but extended to independent set polynomials. These polynomials too turned out to have been studied before by Gutman and Harary [10], who called them independence polynomials.

The main part of this paper focuses on expansions. These heavily rest on the principle of inclusion and exclusion.

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C_i will denote a condition satisfied by some elements of a set S and $N(C_1, C_2, \ldots, C_t)$ will denote the number of elements of S that satisfy the conditions $C_1, C_2, \ldots, C_t$.

As an example we prove the following lemma, needed in Section 3.

Lemma 1.1. *Let K_p be the complete graph on p vertices. Then the number $f(p,q)$ of spanning subgraphs of K_p with q edges is given by*

$$f(p,q)=\sum_{i=0}^{p-2}(-1)^i\binom{p}{i}\left[\begin{matrix}\binom{p-i}{2}\\ q\end{matrix}\right].$$

Proof. Let N_q denote the set of all subgraphs of K_p with q edges. Then

$$|N_q|=\left[\begin{matrix}\binom{p}{2}\\ q\end{matrix}\right].$$

Let condition C_i be 'does not contain vertex i of K_p', $(i=1,2,\ldots,p)$. By the principle of inclusion and exclusion we have

$$\begin{aligned} f(p,q)&=\left[\begin{matrix}\binom{p}{2}\\ q\end{matrix}\right]-\sum_{1\leqslant i\leqslant p}\left[\begin{matrix}\binom{p-1}{2}\\ q\end{matrix}\right]+\sum_{1\leqslant i<j\leqslant p}\left[\begin{matrix}\binom{p-2}{2}\\ q\end{matrix}\right]-\cdots\\ &=\left[\begin{matrix}\binom{p}{2}\\ q\end{matrix}\right]-\binom{p}{1}\left[\begin{matrix}\binom{p-1}{2}\\ q\end{matrix}\right]+\binom{p}{2}\left[\begin{matrix}\binom{p-2}{2}\\ q\end{matrix}\right]+\cdots+(-1)^{p-2}\binom{p}{p-2}\left[\begin{matrix}\binom{2}{2}\\ q\end{matrix}\right]\\ &\quad+(-1)^{p-1}\binom{p}{p-1}\left[\begin{matrix}\binom{1}{2}\\ q\end{matrix}\right]+(-1)^p\binom{p}{p}\left[\begin{matrix}\binom{0}{2}\\ q\end{matrix}\right]\\ &=\sum_{i=0}^{p-2}(-1)^i\binom{p}{i}\left[\begin{matrix}\binom{p-i}{2}\\ q\end{matrix}\right]. \qquad \square \end{aligned}$$

We also need the following result that is easily derived.

Lemma 1.2. *$f(p,q)$ satisfies*

$$\sum_{q=1}^{\binom{p}{2}}(-1)^q f(p,q)=(-1)^{p-1}(p-1).$$

2. Definitions and examples

Throughout this paper we assume that G is a simple graph. A k-clique of G is a complete subgraph of G with k vertices and a k-independent set of G is a k-subset of the vertex set $V(G)$ that induces a subgraph without any edges (n denotes the order and m the size of G).

Induced subgraphs of a graph (V, E) are denoted by brackets. $\langle A \rangle$ denotes the subgraph induced by a vertex set $A \subseteq V(G)$ or an edge set $A \subseteq E(G)$. $\langle\langle A \rangle\rangle$ denotes the subgraph induced by the set of vertices incident with one or more edges of the edge set A.

Definition 2.1. The *clique polynomial* of a graph G, denoted by $C(G; x)$, is defined by

$$C(G; x) = \sum_{k=0}^{n} a_k(G) x^k,$$

where $a_0(G) = 1$ and $a_k(G)$ is the number of k-cliques of G. For $V(G) = \emptyset$ we define $C(G; x) = 1$.

We will write a_k for $a_k(G)$. Obviously $a_1 = |V(G)|$, the number of vertices of G, and $a_2 = |E(G)|$, the number of edges of G. If $a_p \neq 0$ and $a_{p+1} = 0$, then p is the cardinality of a maximum clique. If we put $x = -z$, the clique polynomial turns into Fisher's dependence polynomial.

Definition 2.2. The independent set polynomial of a graph G, denoted by $I(G; x)$, is defined by

$$I(G; x) = \sum_{k=0}^{n} b_k(G) x^k,$$

where $b_0(G) = 1$ and $b_k(G)$ is the number of k-independent sets of G.

Again we define $I(G; x) = 1$ for $V(G) = \emptyset$. b_k denotes $b_k(G)$. Then $b_1 = |V(G)|$, the number of vertices of G, $b_2 = |E(\bar{G})|$, the number of edges of $\bar{G}$, the complement of G. Since $a_k(G) = b_k(\bar{G})$ we have the identity

$$C(G; x) = I(\bar{G}; x).$$

Obviously, we also have

$$C(G; x) + C(\bar{G}; x) = I(G; x) + I(\bar{G}; x).$$

The following results are easily obtained.

Theorem 2.3. *Let G_1 and G_2 be two vertex-disjoint graphs. Then*

(a) $$C(G_1 \cup G_2; x) = C(G_1; x) + C(G_2; x) - 1,$$

$$I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x);$$

(b) $$C(G_1 + G_2; x) = C(G_1; x) \cdot C(G_2; x),$$

$$I(G_1 + G_2; x) = I(G_1; x) + I(G_2; x) - 1.$$

Examples

- Let M_r be an r-matching of K_n, a set of r independent edges, then

$$C(K_n - M_r; x) = I(rK_2 \cup (n-2r)K_1; x)$$
$$= I(rK_2; x) \cdot I((n-2r)K_1; x) = (1+2x)^r(1+x)^{n-2r}.$$

- $I(K_{p,q}; x) = C(K_p \cup K_q; x) = (1+x)^p + (1+x)^q - 1.$

Theorem 2.4. *Let G_1 and G_2 be two vertex-disjoint graphs with $|V(G_1)| = n_1$ and $|V(G_2)| = n_2$, then*

$$C(G_1 \times G_2; x) = n_2 C(G_1; x) + n_1 C(G_2; x) - (n_1 + n_2 + n_1 n_2 x) + 1.$$

Proof. It is not difficult to prove that H is a clique of $G_1 \times G_2$ if and only if there exists a vertex $u \in V(G_1)$ and a clique H_2 of G_2 such that $H = \langle u \rangle \times M_2$, or there exists a vertex $v \in V(G_2)$ and a clique H_1 of G_1 such that $H = H_1 \times \langle v \rangle$. Note that $\langle u \rangle$ denotes the graph induced by the one vertex u.

The result then follows from the fact that, for each of the n_2 vertices of G_2, $C(G_1; x)$ counts the cliques in G_1, and, for each of the n_1 vertices of G_1, $C(G_2; x)$ counts the cliques in G_2. The third term corrects the constant and the double counting of the vertices. □

3. Subgraph expansions

For $u \in V(G)$, $N(u)$ denotes the neighborhood of u in G. Let $T \subseteq V(G)$. We use G_T to denote $\langle \bigcap_{u \in T} N(u) \rangle$. Similarly, for $e \in E(G)$ and $e = uv$, let $N(e) = N(u) \cap N(v)$. For $S \subseteq E(G)$, we use G_S to denote $\langle \bigcap_{e \in S} N(e) \rangle$.

Theorem 3.1 (Vertex-subgraph expansion). *Let U be a nonempty subset of the vertex set of a graph G. Then we have*

$$C(G; x) = C(G - U; x) + \sum_{r=1}^{|U|} (-1)^{r-1} x^r \sum_{\substack{T \subseteq U \\ |T| = r}} C^*(G_T; x),$$

where

$$C^*(G_T; x) = \begin{cases} C(G_T; x) & \text{if } \langle T \rangle \text{ is a clique of } G, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The proof is by the principle of inclusion and exclusion. Let k be any integer and let H_k denote the set of k-cliques of G. Then $|H_k| = a_k(G)$. Let $U = \{u_1, u_2, \ldots, u_h\}$.

For each u_i, $i=1,\dots,h$, condition C_i reads 'contains vertex u_i'. The number $N(C_{i_1},\dots,C_{i_r})$ of k-cliques in H_k satisfying conditions $C_{i_1},\dots,C_{i_r}$ is

$$a^*_{k-r}(G_{\{u_{i_1},\dots,u_{i_r}\}})=\begin{cases} a_{k-r}(G_{\{u_{i_1},\dots,u_{i_r}\}}) & \text{if } \langle\{u_{i_1},\dots,u_{i_r}\}\rangle \text{ is a clique of } G,\\ 0, & \text{otherwise.}\end{cases}$$

By the principle we have

$$a_k(G-U)=a_k(G)-\sum_{1\leqslant i\leqslant h} a^*_{k-1}(G_{\{u_i\}})+\sum_{1\leqslant i<j\leqslant h} a^*_{k-2}(G_{\{u_i,u_j\}})-\cdots$$

$$+(-1)^k\sum_{1\leqslant i_1<\cdots<i_k\leqslant h} a^*_0(G_{\{u_{i_1},\dots,u_{i_k}\}}).$$

For the clique polynomial of $G-U$, we find

$$\sum_k a_k(G-U)x^k=\sum_k a_k(G)x^k-\sum_k x^k\sum_{1\leqslant i\leqslant h} a^*_{k-1}(G_{\{u_i\}})$$

$$+\sum_k x^k\sum_{1\leqslant i<j\leqslant h} a^*_{k-2}(G_{\{u_i,u_j\}})-\cdots$$

$$=\sum_k a_k(G)x^k-x\sum_k\sum_{1\leqslant i\leqslant h} a^*_{k-1}(G_{\{u_i\}})x^{k-1}$$

$$+x^2\sum_{1\leqslant i<j\leqslant h} a^*_{k-2}(G_{\{u_iu_j\}})x^{k-2}-\cdots.$$

Thus

$$C(G-U;x)=C(G;x)-x\sum_{1\leqslant i\leqslant h} C^*(G_{\{u_i\}};x)+x^2\sum_{1\leqslant i<j\leqslant h} C^*(G_{\{u_iu_j\}};x)-\cdots,$$

or

$$C(G;x)=C(G-U;x)+\sum_{r=1}^{h}(-1)^{r-1}x^r\sum_{\substack{T\subseteq U\\|T|=r}} C^*(G_T;x),$$

where $C^*(G_T;x)=\sum_k a^*_k(G_T)x^k$. Obviously,

$$C^*(G_T;x)=\begin{cases} C(G_T;x) & \text{if } \langle T\rangle \text{ is a clique of } G,\\ 0, & \text{otherwise.}\end{cases} \qquad \square$$

Corollary 3.2. *If $\langle U\rangle$ is a clique of G, then*

$$C(G;x)=C(G-U;x)+\sum_{r=1}^{|U|}(-1)^{r-1}x^r\sum_{\substack{T\subseteq U\\|T|=r}} C(G_T;x).$$

Corollary 3.3. *If U is an independent set of G, then*

$$C(G;x)=C(G-U;x)+\sum_{u\in U} C(G_u;x), \quad \textit{where } G_u=G_{\{u\}}.$$

Particularly for any $u\in V(G)$

$$C(G;x)=C(G-u;x)+xC(G_u;x).$$

Examples

• Arnborg and Proskurowski [1] consider k-trees. These graphs yield particularly explicit polynomials.

If G is a k-tree with n vertices, i.e. a graph built from the complete graph K_k by repeated addition of vertices, each adjacent to the vertices of a k-clique, then

$$C(G;x)=x(1+x)^k\,(1+(n-k)x).$$

• We define a generalized clique-tree with n vertices as a graph G built from the complete graph K_{k_1} by repeated addition of vertices, each adjacent to the vertices of a k_2-clique, a k_3-clique, etc. Then $C(G;x)=(1+x)^{k_m}+\sum_{i=1}^{m-1} x(1+x)^{k_i}$. The maximum of the numbers $k_1+1,\dots,k_{m+1}+1$, and k_m is, therefore, the cardinality of the maximum cliques of G.

Theorem 3.4 (Edge-subgraph expansion). *Let M be a nonempty subset of the edge set of a graph G. Then we have*

$$C(G;x)=C(G-M;x)+\sum_{r=2}^{|M|}(-1)^r \sum_{\substack{S\subseteq M\\ |S|=(1/2)r(r-1)}} (r-1)x^r C^*(G_S;x)$$

where

$$C^*(G_S;x)=\begin{cases} C(G_S;x) & \textit{if } \langle S\rangle \textit{ is a clique of } G,\\ 0, & \textit{otherwise.}\end{cases}$$

Proof. Let H_k be the set of k-cliques of G. Then $|H_k|=a_k(G)$. Let $M=\{e_1,e_2,\dots,e_h\}$. Condition C_i $(i=1,2,\dots,h)$, reads 'contains edge e_i'. Then the number of k-cliques in H_k that satisfy conditions $C_{i_1},\dots,C_{i_t}$ is given by

$$N(C_{i_1},C_{i_2},\dots,C_{i_t})=a^*_{k-|V(\langle\langle e_{i_1},\dots,e_{i_t}\rangle\rangle)|}(G_{\{e_{i_1},\dots,e_{i_t}\}})$$

$$=\begin{cases} a_{k-t^*}(G_{\{e_{i_1},\dots,e_{i_t}\}}) & \text{if } \langle\langle e_{i_1},\dots,e_{i_t}\rangle\rangle \text{ is a } t^*\text{-clique of } G;\\ 0, & \text{otherwise.}\end{cases}$$

By the principle of inclusion and exclusion again we have the same formula as in the proof of Theorem 3.1 (M replaces U and e's replace u's):

$$a_k(G-M)=a_k(G)-\sum_{1\leqslant i\leqslant h} a^*_{k-|V(\langle\langle e_i\rangle\rangle)|}(G_{\{e_i\}})$$

$$+\sum_{1\leqslant i<j\leqslant h} a^*_{k-|V(\langle\langle e_ie_j\rangle\rangle)|}(G_{\{e_i,e_j\}})$$

$$-\sum_{1\leqslant i<j<l\leqslant h} a^*_{k-|V(\langle\langle e_ie_je_l\rangle\rangle)|}(G_{\{e_i,e_j,e_l\}})+\cdots.$$

The difficulty in this formula stems from the fact of the following sort. A K_6 can be induced by the vertices of 3 independent edges, while $4,5,6,\dots,15$ edges may also induce a K_6. We want to rearrange the sums according to the orders of the graphs $\langle\langle e_{i_1},\dots,e_{i_t}\rangle\rangle$. We observe:

(1) If $\langle e_{i_1},\dots,e_{i_t}\rangle$ is a clique, then $t=\binom{p}{2}$ for some p, i.e. $|V(\langle e_{i_1},\dots,e_{i_t}\rangle)|=p$.

(2) A clique K_p can have a spanning subgraph with q edges for $\lceil \frac{p}{2}\rceil\leqslant q\leqslant\binom{p}{2}$, where $\lceil x\rceil$ is the smallest integer not smaller than x. Contributions to $a_p(G-M)$ for a specific value q stem from different terms in the expansion with a sign $(-1)^q$.

(3) By Lemma 1.1 there are $f(p,q)$ spanning subgraphs with q edges in K_p. These differ from the spanning subgraphs with q edges in another graph K_p, as at least one vertex is different. Summing the contributions of edge sets that induce a specific K_p gives

$$\sum_{q=1}^{\binom{p}{2}}(-1)^q f(p,q)$$

which, by Lemma 1.2, is $(-1)^p(p-1)$. The numbers of k-cliques containing a specific p-clique is

$$a^*_{k-p}(G_{\{e_{i_1},e_{i_2},\dots,e_{i_{\binom{p}{2}}}\}})=\begin{cases} a_{k-p}(G_{\{e_{i_1},\dots,e_{i_{\binom{p}{2}}}\}}) & \text{if } \langle e_{i_1},\dots,e_{i_{\binom{p}{2}}}\rangle \text{ is a } p\text{-clique},\\ 0, & \text{otherwise.}\end{cases}$$

So, finally, we have

$$a_k(G-M)=a_k(G)-1\sum_{1\leqslant i\leqslant h} a^*_{k-2}(G_{\{e_i\}})+2\sum_{1\leqslant i<j<l\leqslant h} a^*_{k-3}(G_{\{e_i,e_j,e_l\}})$$

$$-3\sum_{1\leqslant i_1\leqslant i_2<\cdots<i_6\leqslant h} a^*_{k-4}(G_{\{e_{i_1},\dots,e_{i_6}\}})+\cdots$$

By the same method as used in Theorem 3.1, we obtain

$$C(G;x)=C(G-M;x)+\sum_{r=2}^{|M|}(-1)^r(r-1)x^r\sum_{\substack{S\subseteq M\\ |S|=(1/2)r(r-1)}} C^*(G_S;x).$$

Moreover $C^*(G_S;x)$ satisfies the condition in Theorem 3.4. □

Corollary 3.5. *If M induces a clique, then*

$$C(G;x)=C(G-M;x)+\sum_{r=2}^{|M|}(-1)^r(r-1)x^r\sum_{\substack{S\subseteq M\\ |S|=(1/2)r(r-1)}}C(G_S;x).$$

Corollary 3.6. *If M induces a triangle-free subgraph of G, then*

$$C(G;x)=C(G-M;x)+x^2\sum_{e\in M}C(G_e;x),\quad \text{where } G_e=G_{\{e\}}.$$

Particularly, M can be a matching, a path P or a cycle C_m ($m\geqslant 4$). If $M=\{e\}$, then $C(G;x)=C(G-e;x)+x^2C(G_e;x)$.

In a similar way we can obtain expansions for the independent set polynomial.

Theorem 3.7. *Let U be a subset of the vertex set of a graph G. Then*

$$I(G;x)=I(G-U;x)+\sum_{r=1}^{|U|}(-1)^{r-1}x^r\sum_{\substack{T\subseteq U\\ |T|=r}}I^*\left(G-\bigcup_{u\in T}\overline{N(u)};x\right),$$

where $\overline{N(u)}=\{u\}\cup N(u)$ *and*

$$I^*\left(G-\bigcup_{u\in T}\overline{N(u)};x\right)=\begin{cases}I(G-\bigcup_{u\in T}\overline{N(u)};x) & \text{if } T \text{ is an independent set of } G,\\ 0, & \text{otherwise.}\end{cases}$$

Particularly, when U is an independent set of G,

$$I(G;x)=I(G-U;x)+\sum_{r=1}^{|U|}(-1)^{r-1}x^r\sum_{\substack{T\subseteq U\\ |T|=r}}I\left(G-\bigcup_{u\in T}\overline{N(u)};x\right).$$

Corollary 3.8. (i) *For any* $u\in V(G)$; $I(G;x)=I(G-u;x)+xI(G-\overline{N(u)};x)$.

(ii) *If U induces a clique of G, then*

$$I(G;x)=I(G-u;x)+x\sum_{u\in U}I(G-\overline{N(u)};x).$$

The edge-subgraph expansion of $I(G;x)$ is rather involved and therefore omitted. The following special case is easily proved.

Theorem 3.9. *Let* $e\in E(G)$ *and* $e=(u,v)$. *Then*

$$I(G;x)=I(G-e;x)-x^2I(G-(N(u)\cup N(v));x).$$

We can give many, simple, examples of results by applying the theorems in this section. The reader may try to investigate paths, cycles, cliques, cocliques, triangle-free graphs, etc.

4. Some open problems

There are still many problems that remain unsolved. We conclude this paper by raising some of them.

Problem 4.1. Determine graphs G such that if there is a graph H satisfying $C(H;x)=C(G;x)$, then $H\cong G$. Such kind of graphs are called *clique-unique graphs.* K_n is an example of a clique-unique graph.

Problem 4.2. Determine graphs G such that

$$C(G;x)=I(G;x), \quad \text{i.e. } C(G;x)=C(\bar{G};x).$$

Such kind of graphs are called *clique-independent set equivalent* graphs. Self-complementary graphs are clique-independent set equivalent graphs.

Problem 4.3. Determine graphs G such that if

$$C(G;x)=C(\bar{G};x), \quad \text{then } G\cong\bar{G}.$$

There is a graph G for which $C(G;x)=C(\bar{G};x)$, but $G\not\cong\bar{G}$, see Fig. 1. We have $C(G;x)=C(\bar{G};x)=1+8x+14x^2+6x^3$ but $G\not\cong\bar{G}$. Note that $C(G;x)=C(\bar{G};x)$ implies that $n(n-1)\equiv 0 \pmod 4$, where $n=|V(G)|$.

Problem 4.4. Characterize polynomials which are clique polynomials or independent set polynomials of graphs.

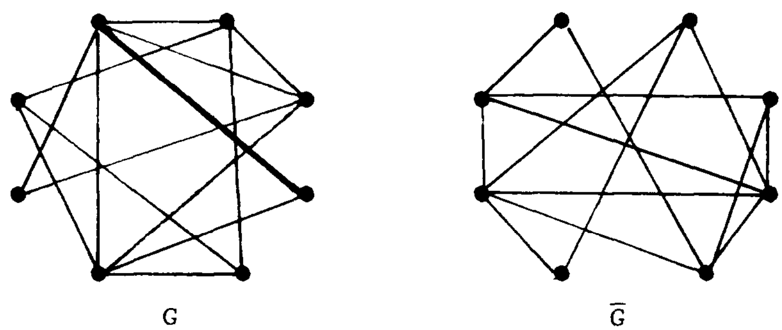

Fig. 1.

Problem 4.5. There exist independent set polynomials with only real roots as well as independent set polynomials with imaginary roots. When do these polynomials have real roots only? Hamidoune [11] has derived some results related to this problem for claw-free graphs.

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