Homoclinic Twist Bifurcations with $\mathbb{Z}_2$ Symmetry

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Summary. We analyze bifurcations occurring in the vicinity of a homoclinic twist point for a generic two-parameter family of $\mathbb{Z}_2$ equivariant ODEs in four dimensions. The results are compared with numerical results for a system of two coupled Josephson junctions with pure capacitive load.

Key words. bifurcation, $\mathbb{Z}_2$ symmetry, homoclinic orbits, periodic orbits

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1. Introduction

In a recent article, Aronson, Golubitsky, and Krupa [AGK91] have studied a class of differential equations describing the dynamics of large-series arrays of Josephson junctions coupled through an external load. An important feature of the equations is $\mathbb{S}_n$ symmetry realized through permutation of the junctions. The problem addressed in the article is the existence, stability, and bifurcations of periodic solutions invariant under the $\mathbb{S}_n$ symmetry. A numerical study indicates that such solutions exist in a region of the two-parameter space bounded on one side by a curve of homoclinic solutions. This curve contains an interesting codimension 2 point which can be roughly characterized in the following way. Let F be the vector field defined by the Josephson junctions equations. [AGK91] show that for certain values of the parameters of the system, F has an $\mathbb{S}_n$ invariant homoclinic orbit Γ . Let s be the saddle to which Γ is asymptotic. Consider the differential equation obtained by linearizing F around Γ . The combination of the $\mathbb{S}_n$ equivariance and some special properties of F imply that this linearized system has n mutually orthogonal invariant subspaces, each spanned by

two eigenvectors of $dF(s)$. Both of these eigenvectors correspond to real eigenvalues, one of them stable and one unstable. The linearized equation around Γ restricted to one of the invariant subspaces has a unique, up to a constant multiple, solution whose direction limits in the backward time on the direction of the unstable eigenvector. In the forward time limit this solution generically approaches precisely the same direction or exactly the opposite direction. [AGK91] show numerically that as parameters vary along the curve of homoclinic solutions, a transition occurs between these two cases. They call a point of such transition a *homoclinic twist point* and show that in every neighborhood of this point there must be points at which the symmetric periodic orbit has a Floquet multiplier $+1$ and a Floquet multiplier -1 .

In this article we analyze bifurcations occurring in the vicinity of a homoclinic twist point for a generic two-parameter family of $\mathbb{Z}_2$ equivariant ODEs in four dimensions. The following is the overview of our results. We show that, generically, in the two-parameter space there exists a region of stability of the symmetric periodic solution bounded by a curve of period-doubling bifurcations (multiplier -1) and a curve of pitchfork bifurcations (multiplier $+1$). These bifurcation curves terminate at the twist point. The period doubling leads to creation of a single doubly winding periodic solution whereas the pitchfork bifurcation produces two symmetry-related periodic orbits. We prove also that there exists a curve in the parameter space, also terminating at the twist point, where there exists a pair of symmetry-related homoclinic loops. The periodic orbits born in the period doubling and pitchfork bifurcations continue in the parameter space to the line of the two homoclinic loops and terminate there in an infinite period bifurcation. We show also that there exists a branch of doubly winding homoclinic loops. Parallel with the theoretical analysis we carry out path-following experiments with AUTO and the Josephson junction system of [AGK91]. We find qualitative agreement with our theoretical results.

An important research topic in symmetric bifurcation theory has been spontaneous symmetry breaking. This term is applied to the situation when a symmetric invariant set undergoes a bifurcation producing invariant sets with less symmetry. There exists an extensive theory of symmetry breaking from invariant equilibria and periodic orbits (see [GSS88] and references therein). Our results also fit into the context of spontaneous symmetry breaking; the period doubling and pitchfork bifurcations produce solutions with less symmetry, and the pair of homoclinic loops bifurcating from the twist point also have less symmetry than the original homoclinic orbits. The situation we describe is a prototype of symmetry breaking from an invariant homoclinic orbit. In our opinion the class of Josephson junction equations studied by [AGK91] provides an example of such symmetry breaking with symmetry group $\mathbb{S}_n$ for arbitrary choice of n .

Twist-point bifurcations for generic vector fields (no symmetry) have received recently a fair amount of attention. They were studied in particular by Yanagida [Yan87], who asserted that they provide a route to homoclinic doubling, that is, that the dynamics near a twist point contain a homoclinic orbit making two loops around the original homoclinic solution. In a recent article, Kisaka et al. [KK092] provide a rigorous proof of the result of Yanagida. We prove that homoclinic doubling also occurs in the $\mathbb{Z}_2$ -symmetric context (Theorem 6.1). Deng [Den92] has recently described a scenario by which twist-point bifurcations can lead to cusp horseshoes, hence com-

plex dynamical behavior. We believe that similar phenomena are also present in the symmetric context.

This article is organized as follows. In Sec. 2 we describe families of vector fields near a twist point and state our main results. In Sec. 3 we discuss basic properties of the return map derived by following the trajectories of the flow along the homoclinic orbit. This return map is our main tool in analyzing the dynamics near a twist point. In Sec. 4 we analyze existence and stability of symmetric fixed points of the return map. Translated in the context of vector fields this corresponds to existence and stability of symmetric periodic orbits. In Sec. 5 we restate and then prove the main results in terms of the return map. In Sec. 6 we prove the existence of homoclinic doubling. Finally, in Sec. 7 we report on the AUTO computations and compare them with the theoretical results.

2. Formulation of the Twist Bifurcation Problem with $\mathbb{Z}_2$ Symmetry

Consider a family of ordinary differential equations

$$\dot{x} = F(x; \alpha, \lambda), \quad (2.1)$$

where $x \in \mathbb{R}^4$ and $F : \mathbb{R}^4 \times \mathbb{R}^2 \rightarrow \mathbb{R}^4$ is a C^∞ smooth function. We assume that the family F has the following symmetry properties:

H1. F commutes with the action of $\mathbb{Z}_2$ given by $\kappa(x_1, y_1, x_2, y_2) = (x_1, y_1, -x_2, -y_2)$.

Let $S = \mathbb{R}^2 \times \{0\}$, and $R = \{0\} \times \mathbb{R}^2$. In S , κ acts trivially and in R , κ acts as $-Id$. Equivariance of F implies that S is invariant under the flow of F . We consider families with the following additional property:

H2. For $\alpha = 0$ (2.1) has a smooth family of homoclinic orbits $\gamma_\lambda(t)$ to a saddle s_λ and $\gamma_\lambda(t) \in S$.

Let

$$\Gamma_\lambda = \{\gamma_\lambda(t) : t \in \mathbb{R}\}. \quad (2.2)$$

Symmetry implies that $dF(s_\lambda; 0, \lambda)$ has the form

$$dF(s_\lambda; 0, \lambda) = \begin{pmatrix} A_\lambda & 0 \\ 0 & B_\lambda \end{pmatrix}, \quad (2.3)$$

where A_λ, B_λ are 2×2 blocks. Similarly,

$$dF(\gamma_\lambda(t); 0, \lambda) = \begin{pmatrix} A_\lambda(t) & 0 \\ 0 & B_\lambda(t) \end{pmatrix}. \quad (2.4)$$

Clearly, A_λ must have one negative and one positive eigenvalue. We make the following additional hypothesis.

H3. B_λ has one positive and one negative eigenvalue.

Suppose F satisfies H1–H3. Let v_+ (v_-) be the eigenvector corresponding to positive (negative) eigenvalue of B_λ . Consider the linearized equation

$$\dot{y} = dF(\gamma_\lambda(t); 0, \lambda)y. \quad (2.5)$$

For $|t|$ large the behavior of solutions of (2.5) is qualitatively the same as the behavior of the solutions of

$$\dot{y} = dF(s_\lambda; 0, \lambda)y \quad (2.6)$$

(see [CL55], Chapter 3). It follows that there exists a unique solution of (2.5), $y_\lambda(t)$ such that $\lim_{t \rightarrow -\infty} [y_\lambda(t)] / |y_\lambda(t)| = v_+$ and $|y_\lambda(0)| = 1$. Let

$$y_\infty(\lambda) = \lim_{t \rightarrow \infty} \frac{y_\lambda(t)}{|y_\lambda(t)|}.$$

Generically $y_\infty(\lambda) = \pm v_+$. We say that the point $(\alpha, \lambda) = (0, \lambda_0)$ is a *homoclinic twist point* if $y_\infty(\lambda)$ points in opposite directions for $\lambda > \lambda_0$ and $\lambda < \lambda_0$.

Note that there exists also a unique solution of (2.5), $z_\lambda(t)$ such that $\lim_{t \rightarrow \infty} \frac{z_\lambda(t)}{|z_\lambda(t)|} = v_-$ and $|z_\lambda(0)| = 1$. Let

$$z_{-\infty}(\lambda) = \lim_{t \rightarrow -\infty} \frac{z_\lambda(t)}{|z_\lambda(t)|}.$$

Generically $z_{-\infty}(\lambda) = \pm v_-$. The invariance of S and the fact that a flow has to preserve orientation imply that $z_{-\infty}(\lambda)$ and $y_\infty(\lambda)$ point to the same side of v_- and v_+ , respectively. It follows that if $(0, \lambda_0)$ is a twist point then $z_{-\infty}(\lambda)$ must point to the opposite directions of v_- for $\lambda > \lambda_0$ and $\lambda < \lambda_0$.

It is well known that under mild genericity conditions (2.1) has for each (α, λ) on one side of the line $\alpha = 0$ a unique periodic orbit $P_{(\alpha, \lambda)} \subset S$. The orbits $P_{(\alpha, \lambda)}$ limit on Γ_λ as $\alpha \rightarrow 0$. In our analysis we will also encounter periodic orbits lying outside of S . Such orbits can many times enter and leave a neighborhood of the saddle s_λ , before coming back to the initial condition. Let V_λ be a small neighborhood of s_λ . We say that a periodic orbit $p(t)$ with minimal period T is a *1-periodic orbit* if the set

$$PV = \{p(t) : t \in [0, T]\} \cap V_\lambda$$

is connected. We say that $p(t)$ is a *2-periodic orbit* if PV has precisely two connected components. We now state the two main theorems of the article.

Theorem 2.1. *Suppose F satisfies H1–H3 and $(\alpha, \lambda) = (0, 0)$ is a twist point. Then, generically, the following statements hold.*

1. *There exists a continuous curve $\tau_1 : [0, 1] \rightarrow \mathbb{R}^2$, with $\tau_1(0) = 0$, such that as (α, λ) cross τ_1 , $P_{(\alpha, \lambda)}$ has a symmetry-breaking pitchfork bifurcation.*

2. There exists a continuous curve $\tau_1 : [0, 1) \rightarrow \mathbb{R}^2$, with $\tau_1(0) = 0$, such that as (α, λ) cross τ_1 , $P_{(\alpha, \lambda)}$ has a period-doubling bifurcation. The resulting 2-periodic solution is invariant under κ as a set but not pointwise.
3. (2.1) has a unique pair of nonsymmetric homoclinic orbits (not contained in S) along a smooth curve $\tau_3 : [0, 1) \rightarrow \mathbb{R}^2$, with $\tau_3(0) = 0$. This orbit limits on Γ_0 as (α, λ) approaches 0 along τ_3 .

Theorem 2.2. For a generic F satisfying the assumptions of Theorem 2.1 there exist regions $D_i, i \in \{1, 2\}$ whose boundaries are given by $\tau_i, i \in \{1, 2\}$ and τ_3 with the following properties:

1. For each (α, λ) in D_1 there exists a unique pair of symmetry related 1-periodic orbits not contained in S . For each (α, λ) in D_2 there exists a unique 2-periodic orbit P_2 with the property $\kappa P_2 = P_2$.
2. The periods of the periodic orbits in the regions D_1 and D_2 depend continuously on (α, λ) and tend to ∞ as $(\alpha, \lambda) \rightarrow \tau_3$.

Figure 1 shows the four possible relative positions of the curves τ_i . The regions D_1 and D_2 correspond to the following:

(a) $D_1 = \text{IV}, D_2 = \text{III},$	(b) $D_1 = \text{III} \cup \text{IV}, D_2 = \text{III},$
(c) $D_1 = \text{IV} \cup \text{V}, D_2 = \text{I} \cup \text{II},$	(d) $D_1 = \text{I} \cup \text{II} \cup \text{III}, D_2 = \text{I} \cup \text{II}.$

Remark 2.3. If $\text{tr } A_0 < 0$ then $P_{(\alpha, \lambda)}$ is stable in the region bounded by the curves τ_1 and τ_2 and not containing the line $\alpha = 0$ (see Corollary 5.4). We have not been able to carry out the stability of the solutions whose existence is asserted in Theorem 2.2.

Theorems 2.1 and 2.2 will be proved in Sec. 5 by analyzing a return map defined on a hyperplane transverse to Γ_λ near the saddle s_λ . The genericity conditions required in Theorems 2.1 and 2.2 will be stated precisely for the return map.

3. Return Map Near the Homoclinic Orbit Γ

Let $e_i, i = 1, \dots, 4$ be the standard basis vectors in $\mathbb{R}^4$. We assume that $s_\lambda \equiv 0$ and that $e_i, i = 1, \dots, 4$ are the eigendirections for $dF(0; 0, \lambda)$. Let (x_1, y_1) be the coordinates in S and (x_2, y_2) the coordinates in R . We assume that x_1, x_2 correspond to the stable directions and y_1, y_2 to the unstable directions. In the subsequent analysis we will assume that the vector field is linear in a neighborhood of 0. This assumption is justified by the theorem of Sternberg [Ste58] stating that every C^k smooth vector field can be linearized in a neighborhood of a hyperbolic equilibrium provided that the eigenvalues of the linearization at the equilibrium are not at resonance up to order k . The linearizing transformation is C^l , with $l \rightarrow \infty$ as $k \rightarrow \infty$. We now state and sketch the proof of a straightforward generalization of the result of Sternberg to the symmetric case.

	$(P_{(\alpha,\lambda)}, 0, 0)$	1-per	2-per
I	0	0	0
II	1	0	0
III	1	0	1
IV	1	1	0
V	1	0	0

	$(P_{(\alpha,\lambda)}, 0, 0)$	1-per	2-per
I	0	0	0
II	1	0	0
III	1	1	1
IV	1	1	0
V	1	0	0

	$(P_{(\alpha,\lambda)}, 0, 0)$	1-per	2-per
I	0	0	1
II	1	0	1
III	1	0	0
IV	1	1	0
V	0	1	0

	$(P_{(\alpha,\lambda)}, 0, 0)$	1-per	2-per
I	0	1	1
II	1	1	1
III	1	1	0
IV	1	0	0
V	0	0	0

Fig. 1. The four possible relative positions of the curves τ_1 , τ_2 , and τ_3 .

Proposition 3.1. *Let G be a compact Lie group acting orthogonally on $\mathbb{R}^n$ and suppose F is a C^k smooth vectorfield commuting with G . Suppose also that F has a hyperbolic equilibrium s fixed by all the elements of G . Assume the eigenvalues of $dF(s)$ are nonresonant up to order k . Then, in a small neighborhood of s , F can be linearized by a C^1 smooth equivariant transformation. Moreover, $l(k) \rightarrow \infty$ as $k \rightarrow \infty$.*

Sketch of the proof. The method of proof used by Sternberg [Ste58] consists of finding a diffeomorphism conjugating the flow Φ_t of F to the flow of $dF(s)$. Let $T = \Phi_t$ for some choice of t and let $L = dF(s)$. The procedure of linearizing T consists of the following steps.

1. Poincaré–Birkhoff normal form transformation. This can be done preserving the equivariance; see [GSS88].
2. An intermediate transformation of coordinates R_0 . Let E_+ be the vector space spanned by the eigenspaces corresponding to eigenvalues with positive real parts and E_- the vector space spanned by the eigenspaces corresponding to eigenvalues with negative real parts. After application of R_0 the vector field restricted to E_+ and E_- is linear. R_0 is constructed as the limit of a sequence of transformations obtained as a repeated composition of L^{-1} , the normal form transformation, the transformation T , and the projections onto E_+ and E_- . It follows that R_0 is equivariant.
3. The transformation R achieving the linearization of coordinates. Let x_+ and x_- denote the coordinates in E_+ and E_- , respectively. To define R Sternberg considers the sets $S_+ = \{|x_+| < |x_-|\}$ and $S_- = \{|x_+| > |x_-|\}$. We consider the set S_+ ; the argument for the set S_- is similar. Let $W = \overline{S_+} \setminus LS_+$. The map R is taken to be arbitrary on W except for the set $TC \cap W$, $C = \{|x_+| = |x_-|\}$. For the further definition of R the set W is used as the fundamental domain of $L|_{S_+}$, which means that R is defined by the relation $R(L^m x) = T^s(R(x))$ for all possible choices of m and the corresponding choices of s . It is clear that the sets S^+ , W , and C are invariant under the action of G and that R can be defined to be equivariant on W . It follows that R is equivariant on S_+ and consequently on the entire neighborhood of s . $\square$

We now make the following hypothesis:

Hypothesis H4. The eigenvalues of $dF(0;0,0)$ are nonresonant to order k , with $l(k) \geq 3$.

In the sequel we assume that there exists a neighborhood V of 0 in $\mathbb{R}^4$ where the flow of (2.1) is given by the flow of (2.6). We consider the following sections of the flow.

$$\begin{aligned}\Sigma^{\text{out}} &= \{(x_1, \delta, x_2, y_2) : |(x_1, x_2, y_2)| < \delta\}, \\ \Sigma^{\text{in}} &= \{(\delta, y_1, x_2, y_2) : |(x_1, x_2, y_2)| < \delta\}.\end{aligned}$$

If we choose δ small enough, $(0, \delta, 0, 0)$ and $(\delta, 0, 0, 0)$ are in V . This implies $(0, \delta, 0, 0) \in \Gamma_\lambda$ and $(\delta, 0, 0, 0) \in \Gamma_\lambda$. By scaling the variables we can achieve that $\delta = 1$ (see Fig. 2).

Fig. 2. Poincaré map near Γ .

We assume that the stable eigenvalue in S has been scaled to -1 . Let ν_1, ν_2 denote the eigenvalues corresponding to e_2 and e_4 and $-\mu$ the eigenvalue corresponding to e_3 . The linear flow starting on Σ^{in} is given by

$$(1, y_1^{\text{in}}, x_2^{\text{in}}, y_2^{\text{in}}) \xrightarrow{\varphi^s} (e^{-s}, e^{\nu_1 s} y_1^{\text{in}}, e^{-\mu s} x_2^{\text{in}}, e^{\nu_2 s} y_2^{\text{in}}).$$

We now define the part of the return map given by the first hit map from Σ^{in} to Σ^{out} . For an initial condition $u = (1, y_1^{\text{in}}, x_2^{\text{in}}, y_2^{\text{in}}) \in \Sigma^{\text{in}}$ the condition that $\varphi^s(u) \in \Sigma^{\text{out}}$ is

$$(e^{-s}, e^{\nu_1 s} y_1^{\text{in}}, e^{-\mu s} x_2^{\text{in}}, e^{\nu_2 s} y_2^{\text{in}}) = (x_1^{\text{out}}, 1, x_2^{\text{out}}, y_2^{\text{out}}). \quad (3.1)$$

It follows that $e^{-\nu_1 s} = y_1^{\text{in}}$. Let $r = e^{-\nu_1 s}$. The first hit map from Σ^{in} to Σ^{out} is given by

$$\Pi^{\text{loc}} \begin{pmatrix} 1 \\ r \\ x_2^{\text{in}} \\ y_2^{\text{in}} \end{pmatrix} = \begin{pmatrix} r^{1/\nu_1} \\ 1 \\ r^{\mu/\nu_1} x_2^{\text{in}} \\ r^{-\nu_2/\nu_1} y_2^{\text{in}} \end{pmatrix}, \quad (3.2)$$

with domain

$$\mathcal{D}(\Pi^{\text{loc}}) = \{(1, r, x_2^{\text{in}}, y_2^{\text{in}}) \in \Sigma^{\text{in}} \mid r > 0, |y_2^{\text{in}}| \geq r^{\nu_2/\nu_1}\}.$$

The inverse of Π^{loc} on $\mathcal{R}(\Pi^{\text{loc}})$ is given by

$$(\Pi^{\text{loc}})^{-1} \begin{pmatrix} u \\ 1 \\ x_2^{\text{out}} \\ y_2^{\text{out}} \end{pmatrix} = \begin{pmatrix} 1 \\ u^{\nu_1} \\ u^{-\mu} x_2^{\text{out}} \\ u^{\nu_2} y_2^{\text{out}} \end{pmatrix}. \quad (3.3)$$

The domain of $(\Pi^{\text{loc}})^{-1}$ is equal to the range of Π^{loc} and is given by

$$\mathcal{R}((\Pi^{\text{loc}})^{-1}) = \{x \in \Sigma^{\text{out}} \mid x_1^{\text{out}} \geq 0, |x_2^{\text{out}}| < u^\mu\}.$$

We now define the part of the return map given by the first hit map from Σ^{out} to Σ^{in} . Let Π^{far} denote this map. The domain of Π^{far} is given by $\Sigma^{\text{out}} \cap V$, where V is a neighborhood of the origin containing $(0, 1, 0, 0)$ and $(1, 0, 0, 0)$. Note that Σ^{in} and Σ^{out} are invariant under κ . It follows that Π^{far} commutes with the action of κ . Also for $\alpha = 0$, $\Pi^{\text{far}}(0, 1, 0, 0) = (1, 0, 0, 0)$. Let $z = (u, x^{\text{out}}, y^{\text{out}})$ denote the coordinates in Σ^{out} and $w = (r, x^{\text{in}}, y^{\text{in}})$ the coordinates in Σ^{in} . The action of symmetry on the coordinates z and w is given by

$$\kappa \begin{pmatrix} u \\ x^{\text{out}} \\ y^{\text{out}} \end{pmatrix} = \begin{pmatrix} u \\ -x^{\text{out}} \\ -y^{\text{out}} \end{pmatrix},$$

$$\kappa \begin{pmatrix} r \\ x^{\text{in}} \\ y^{\text{in}} \end{pmatrix} = \begin{pmatrix} r \\ -x^{\text{in}} \\ -y^{\text{in}} \end{pmatrix}.$$

In these coordinates Π^{far} has the form

$$\Pi^{\text{far}} \begin{pmatrix} x_1^{\text{out}} \\ x^{\text{out}} \\ y^{\text{out}} \end{pmatrix} = \begin{pmatrix} g_1(u, x^{\text{out}}, y^{\text{out}}) \\ g_2(u, x^{\text{out}}, y^{\text{out}}) \\ g_3(u, x^{\text{out}}, y^{\text{out}}) \end{pmatrix}, \quad (3.4)$$

where $g_j, j = 1, 2, 3$ satisfy the following commutation relations.

$$\begin{aligned} g_1(u, -x^{\text{out}}, -y^{\text{out}}) &= g_1(u, x^{\text{out}}, y^{\text{out}}), \\ g_2(u, -x^{\text{out}}, -y^{\text{out}}) &= -g_2(u, x^{\text{out}}, y^{\text{out}}), \\ g_3(u, -x^{\text{out}}, -y^{\text{out}}) &= -g_3(u, x^{\text{out}}, y^{\text{out}}). \end{aligned} \quad (3.5)$$

For the moment we suppress the dependence of the functions g_j on α and λ . A straightforward argument using Taylor's formula shows that for $j = 2, 3$ the functions g_j can be written in the form

$$\begin{aligned} g_2(u, x^{\text{out}}, y^{\text{out}}) &= b_1(u, x^{\text{out}}, y^{\text{out}})x^{\text{out}} + b_2(u, x^{\text{out}}, y^{\text{out}})y^{\text{out}}, \\ g_3(u, x^{\text{out}}, y^{\text{out}}) &= b_3(u, x^{\text{out}}, y^{\text{out}})x^{\text{out}} + b_4(u, x^{\text{out}}, y^{\text{out}})y^{\text{out}}, \end{aligned}$$

where $b_j, j = 1, \dots, 4$ satisfy the relation $b_j(u, -x^{\text{out}}, -y^{\text{out}}) = b_j(u, x^{\text{out}}, y^{\text{out}})$. We write g_1 in the form $g_1(u, x^{\text{out}}, y^{\text{out}}) = \alpha + a(u, x^{\text{out}}, y^{\text{out}})$. The existence of a symmetric homoclinic for $\alpha = 0$ implies that $a(0; 0, \lambda) = 0$. We choose the parameter α in such a way that $a(0; \alpha, \lambda) = 0$.

Let $\Pi = \Pi^{\text{loc}} \circ \Pi^{\text{far}}$ and let

$$\begin{pmatrix} \hat{u} \\ \hat{x}^{\text{out}} \\ \hat{y}^{\text{out}} \end{pmatrix} = \Pi \begin{pmatrix} u \\ x^{\text{out}} \\ y^{\text{out}} \end{pmatrix}. \quad (3.6)$$

We rewrite (3.6) as

$$(\Pi^{\text{loc}})^{-1} \begin{pmatrix} \hat{u} \\ \hat{x}^{\text{out}} \\ \hat{y}^{\text{out}} \end{pmatrix} = \Pi^{\text{far}} \begin{pmatrix} u \\ x^{\text{out}} \\ y^{\text{out}} \end{pmatrix}.$$

Hence the map Π can be implicitly expressed by the equations

$$\begin{aligned} \hat{u}^{\nu_1} &= \alpha + a, \\ \hat{u}^{-\mu} \hat{x}^{\text{out}} &= (b_1 x^{\text{out}} + b_2 y^{\text{out}}), \\ \hat{u}^{\nu_2} \hat{y}^{\text{out}} &= b_3 x^{\text{out}} + b_4 y^{\text{out}}. \end{aligned} \quad (3.7)$$

The domain of Π , $\mathcal{D}(\Pi)$, is given by

$$\mathcal{D}(\Pi) = (\Pi^{\text{far}})^{-1}(\mathcal{D}\Pi^{\text{loc}}).$$

The following lemma gives a characterization of a twist point in the context of the return map Π .

Lemma 3.2. *The point $(\alpha, \lambda) = (0, 0)$ in parameter space is a twist point if $b_4(0; 0, \lambda)$ changes sign as λ passes through 0.*

Proof. Recall the definition of the vectors v_+ and v_- and the trajectory $y_\lambda(t)$ given in Sec. 2. Under the assumptions of Sec. 2, $v_- = e_3$ and $v_+ = e_4$. Let T_0, T_1 be such that $\gamma_\lambda(T_0) = (1, 0, 0, 0)$ and $\gamma_\lambda(T_1) = (0, 1, 0, 0)$. Let $v(t)$ be a solution of the variational equation 2.5 with $v(T_0) = v_0$. It is easy to check that this solution is given by $v(t + T_0) = d\Phi_t(x)v_0$. Let $t_0 = T_1 - T_0$. We will show that for $v_0 \in R$

$$d\Phi_{t_0}(\gamma_\lambda(T_0))v_0 = d\Pi^{\text{far}}(\gamma_\lambda(T_0))v_0. \quad (3.8)$$

Let $z = (u, x^{\text{out}}, y^{\text{out}})$. We use $\Phi_t(z)$ to denote $\Phi_t(1, u, x^{\text{out}}, y^{\text{out}})$. Let $h(t, z) = \Phi_t^1(z)$ (the x_1 component of $\Phi_t(z)$). h is a smooth function and the condition $h(t, z) = 1$ is equivalent to $\Phi_t(z) \in \Sigma^{\text{in}}$. Clearly, $h(t_0, 0) = 1$. Moreover, $(\partial h / \partial t)(t_0, 0) = -1 \neq 0$. Hence, by the implicit function theorem, there exists a function $t(z)$, with $t(0) = t_0$, such that $h(t(z), z) = 1$. It follows that $\Pi^{\text{far}}(z) = \Phi_{t(z)}(z)$. Hence

$$d\Pi^{\text{far}}(z)v = d\Phi_{t(z)}(z)v + \frac{d}{dt}\Phi_t(z)|_{t=t(z)} \frac{dt}{dz} \cdot v, \quad (3.9)$$

where $\cdot$ denotes the inner product. Note that $h(t, \kappa z) = h(t, z)$. Since the implicit function theorem preserves symmetry invariance we must have $t(\kappa z) = t(z)$ or, in coordinates, $t(u, -x, -y) = t(u, x, y)$. It follows that $(dt/dz)(u, 0, 0) = (*, 0, 0)$. If $v_0 \in R$ then $(dt/dz)(0, 0, 0) \cdot v_0 = 0$ and the identity (3.8) holds. Using the form of Π^{far} we obtain

$$y_\lambda(T_1) = \begin{pmatrix} 0 \\ 0 \\ b_2(0; 0, \lambda) \\ b_4(0; 0, \lambda) \end{pmatrix}.$$

Consequently, $\lim_{t \rightarrow \infty} \frac{y_\lambda(t)}{|y_\lambda(t)|} = (\text{sgn } b_4(0; 0, \lambda))v_+$. □

Note that $z_{-\infty}(\lambda) = \lim_{t \rightarrow \infty} \frac{z_\lambda(t)}{|z_\lambda(t)|} = (\text{sgn } b_4(0; 0, \lambda))v_-$. Hence the fact that $z_{-\infty}(\lambda)$ also changes direction at the twist point does not imply any further degeneracies of the map Π . We assume that

H5. $\frac{\partial}{\partial \lambda} b_4(0; 0, 0) \neq 0$.

By redefining the parameters α, λ we can write $b_4 = \lambda + f$, where $f(0; \alpha, \lambda) = 0$. We now rewrite (3.7) as

$$\begin{aligned}\hat{u}^{\nu_1} &= \alpha + a, \\ \hat{u}^{-\mu} \hat{x} &= (b_1 x + b_2 y), \\ \hat{u}^{\nu_2} \hat{y} &= b_3 x + (\lambda + f)y.\end{aligned}\tag{3.10}$$

In the sequel we will analyze the dynamical properties of the map Π as defined by (3.10) restricted to a small neighborhood of 0. Any claims regarding existence of uniqueness of solutions will be made with respect to a small neighborhood of 0.

4. Periodic Points in S

The periodic orbits $P_{(\alpha, \lambda)}$ correspond to fixed points of (3.10) with $x = y = 0$. This leads to the following equation (we suppress the dependence of α on (α, λ)):

$$u^{\nu_1} = \alpha + a(u, 0, 0).\tag{4.1}$$

We assume that

H6. $a_u = \frac{\partial}{\partial u} a(0, 0, 0) \neq 0$.

It is clear that (3.10) has a unique symmetric fixed point for each $\alpha > 0$ or $\alpha < 0$.

Let $(u^*, 0, 0)$ denote the fixed point of (4.1). Let $b_i^* = b_i(u^*, 0, 0)$, $i = 1, 2, 3$; $f^* = f(u^*, 0, 0)$; and $a_u^* = \frac{\partial}{\partial u} a(u^*, 0, 0)$. We have the following lemma:

Lemma 4.1.

$$d\Pi(u^*, 0, 0; \alpha, \lambda) = \begin{pmatrix} \frac{1}{\nu_1} (u^*)^{1-\nu_1} a_u^* & 0 & 0 \\ 0 & (u^*)^\mu b_1^* & (u^*)^\mu b_2^* \\ 0 & (u^*)^{-\nu_2} b_3^* & (u^*)^{-\nu_2} (\lambda + f^*) \end{pmatrix}.$$

Proof. The formula is obtained by implicit differentiation of (3.10) using the invariance of a, b_i , $i = 1, 2, 3$ and f under the action of κ and the fact that $(u^*, 0, 0)$ is a fixed point of Π . $\square$

Let

$$A = \begin{pmatrix} (u^*)^\mu b_1^* & (u^*)^\mu b_2^* \\ (u^*)^{-\nu_2} b_3^* & (u^*)^{-\nu_2} (\lambda + f^*) \end{pmatrix}.\tag{4.2}$$

The eigenvalues of A correspond to the eigenvalues of $d\Pi(u^*, 0, 0, \alpha, \lambda)$ in the non-symmetric directions (directions contained in R). The following proposition describes the asymptotic behavior of these eigenvalues.

Proposition 4.2. Suppose $|\lambda|$ is bounded. For α small enough A has real eigenvalues μ_+ and μ_- with $|\mu_+| > 1$ and $|\mu_-| < 1$. For fixed $\lambda \neq 0$ these eigenvalues have the following limiting behavior:

$$\begin{aligned} \lim_{u^* \rightarrow 0} \mu_- &= 0, \\ \lim_{u^* \rightarrow 0} \mu_+ &= \begin{cases} \infty & \text{if } \lambda > 0, \\ -\infty & \text{if } \lambda < 0. \end{cases} \end{aligned}$$

Proof. We have

$$\mu_{\pm} = \frac{\operatorname{tr} A \pm \sqrt{\operatorname{tr} A^2 - 4 \det A}}{2}, \quad (4.3)$$

$$\operatorname{tr} A = (u^*)^{\mu} b_1^* + (u^*)^{-\nu_2} (\lambda + f^*), \quad (4.4)$$

$$\det A = (u^*)^{\mu - \nu_2} [b_1^* (\lambda + f^*) - b_2^* b_3^*]. \quad (4.5)$$

Let $\nu_0 = \min\{1, \nu_1\}$. As $(u^*)^{\nu_1} = \alpha + a(u^*, 0, 0, \alpha, \lambda)$ it follows that $\alpha = \mathcal{O}(u^*)^{\nu_0}$ uniformly for $|\lambda|$ bounded. Hence $b_i^* = b_{i0} + \mathcal{O}(|\lambda| + (u^*)^{\nu_0})$ and $f^* = \mathcal{O}((u^*)^{\nu_0})$, where $b_{i0} = b_i(0; 0, 0, 0)$, $i = 1, 2, 3$. We conclude that

$$\operatorname{tr} A = b_{10}(u^*)^{\mu} + \lambda(u^*)^{-\nu_2} + \mathcal{O}(u^*)^{\nu_0 - \nu_2},$$

and

$$\det A = -(u^*)^{\mu - \nu_2} (b_{20} b_{30} + \mathcal{O}(|\lambda| + (u^*)^{\nu_0})).$$

Therefore, if we fix $\lambda \neq 0$ we conclude that

$$\det A = -c(\lambda)(u^*)^{\mu - \nu_2} + \mathcal{O}(u^*)^{\mu - \nu_2 + \nu_0},$$

$$\frac{\det A}{\operatorname{tr} A} = -\frac{c(\lambda)}{\lambda} (u^*)^{\mu} (1 + \mathcal{O}(u^*)^{\mu + \nu_0}),$$

where $c(\lambda) = b_{20} b_{30} + \mathcal{O}(|\lambda|)$. As

$$\mu_{\pm} = \frac{1}{2} \left\{ \operatorname{tr} A \pm \operatorname{tr} A \left(1 - 2 \frac{\det A}{(\operatorname{tr} A)^2} + \mathcal{O} \left(\left(\frac{\det A}{(\operatorname{tr} A)^2} \right)^2 \right) \right) \right\},$$

we conclude that

$$\mu_+ = \lambda(u^*)^{-\nu_2} + \mathcal{O}(u^*)^{\min\{\nu_0 - \nu_2, \mu\}},$$

$$\mu_- = -\frac{c(\lambda)}{\lambda} (u^*)^{\mu} (1 + \mathcal{O}(u^*)^{\mu + \nu_0}).$$

Hence the assertion of the proposition holds. $\square$

Remark 4.3. In [AGK91] an analogous result is proved for the system of ODEs describing large series arrays of Josephson junctions. See also Sec. 7.

5. Proofs of Theorems 2.1 and 2.2

In this section we formulate and prove Theorems 2.1 and 2.2 in the context of the map Π defined by (3.10).

Remark 5.1. Note that a fixed point of Π must be a solution of the equations

$$u^{\nu_1} = \alpha + a, \quad (5.1a)$$

$$x = u^\mu(b_1x + b_2y), \quad (5.1b)$$

$$u^{\nu_2}y = b_3x + y(\lambda + f), \quad (5.1c)$$

with $u > 0$. We must also require that $\Pi^{\text{far}}(u, x, y) \in \mathcal{D}(\Pi^{\text{loc}})$. By (3.10),

$$\Pi^{\text{far}}(u, x, y) = \begin{pmatrix} u^{\nu_1} \\ u^{-\mu}x \\ u^{\nu_2}y \end{pmatrix}. \quad (5.2)$$

Moreover, if (u, x, y) are near 0, then, by (5.1b) $u^{-\mu}x$ must be small. Hence every solution of (5.1) with $u > 0$ is a fixed point of Π corresponding to a 1-periodic orbit of the vector field F . For a solution of (5.1) to correspond to a homoclinic orbit of F we must have $u = 0$. Note that (5.1b) implies that if $u = 0$ then $x = 0$. Hence the 1-homoclinic orbits of F correspond to solutions of (5.1) of the form $(0, 0, y)$.

In the sequel we will use the notation $a_u = (\partial_a/\partial_u)(0; 0, 0)$, $f_u = (\partial_f/\partial_u)(0; 0, 0)$ and $b_{j0} = b_j(0; 0, 0)$, $j = 1, \dots, 3$. Also note that $a(0, 0, y)$ is even in y and hence is a function of y^2 . Hence we will use the notation a_{y^2} for $(\partial a/\partial y^2)(0, 0, y; 0, 0)|_{y=0}$. Analogous considerations apply to f . Note that the periodic orbit $P_{(\alpha, \lambda)}$ corresponds to the fixed point $(u^*, 0, 0)$ and a nonsymmetric homoclinic orbit of (2.1) corresponds to a fixed point of Π of the form $(0, 0, y^*)$.

Proposition 5.2. *Let Π be the map defined by (3.10). Suppose H1–H5 are satisfied. Then the following hold:*

1. *The symmetric fixed point $(u^*, 0, 0)$ has a Floquet multiplier 1 along a continuous curve τ_1 emanating from the twist point $(\alpha, \lambda) = (0, 0)$.*
2. *$(u^*, 0, 0)$ has a Floquet multiplier -1 along a continuous curve τ_2 emanating from the twist point.*
3. *Π has a fixed point of the form $(0, 0, y^*)$ along a curve τ_3 emanating from the twist point. The curve τ_3 is as smooth as the vector field F and $y^* \rightarrow 0$ as $\lambda \rightarrow 0$.*

Proposition 5.3. *Suppose H1–H5 hold. Assume in addition that $a_u f_{y^2} - a_{y^2} f_u \neq 0$, $a_{y^2} \neq 0$, $f_{y^2} \neq 0$, and $a_{y^2} b_{20} b_{30} \neq 0$. Then there exist regions D_i , $i = 1, 2$, whose boundaries are given by the curve τ_3 and the curves τ_i , $i = 1, 2$, with the following properties:*

1. *For each (α, λ) in D_1 , Π has a unique pair of fixed points of the form (u, x, y) and $(u, -x, -y)$, $u > 0$, $|(x, y)| \neq 0$. Moreover, $u \rightarrow 0$ as $(\alpha, \lambda) \rightarrow \tau_3$.*
2. *For each (α, λ) in D_2 , Π has a unique $\mathbb{Z}_2$ invariant period-2 orbit of the form $\{(u, x, y), (u, -x, -y)\}$, $u_2 > 0$, $|(x, y)| \neq 0$. Moreover, $u \rightarrow 0$ as $(\alpha, \lambda) \rightarrow \tau_3$.*

Proof of Proposition 5.2.

1. The multipliers $\mu_{\pm}$ are given in (4.3). The equation $\mu_{\pm} = 1$ is equivalent to

$$\operatorname{tr} A - 1 = \det A. \quad (5.3)$$

Using (4.4) and (4.5) we obtain

$$\lambda + f^* - (u^*)^{\nu_2} + (u^*)^{\mu} (b_2^* b_3^* - b_1^* (\lambda + f^*)) + (u^*)^{\mu + \nu_2} b_1^* = 0. \quad (5.4)$$

To obtain the curve τ_1 we solve equations (5.4) and (4.1). We define the variables $u_1 = (u^*)^{\nu_1}$ and $u_2 = (u^*)^{\mu}$ and solve (5.4) and (4.1) by the implicit function theorem for λ and α as function of u^* , u_1 , and u_2 . We then substitute the original expressions for u_1 and u_2 . This defines the curve τ_1 in the form $(\alpha, \lambda) = (\alpha(u), \lambda(u))$. At the lowest order τ_1 is given by $\alpha = \operatorname{const} \cdot (u^*)^{\min\{\nu_1\}}$, $\lambda = \operatorname{const} \cdot (u^*)^{\min\{1, \nu_2, \mu\}}$. It follows that τ_1 is, in general, only continuous.

2. Suppose $\mu_+ = -1$ or $\mu_- = 1$. Using (4.3) we arrive at the following equation.

$$\operatorname{tr} A + 1 = -\det A \quad (5.5)$$

From (5.5), (4.4), and (4.5) we obtain

$$\lambda + f^* - (u^*)^{\nu_2} - (u^*)^{\mu} (b_2^* b_3^* - b_1^* (\lambda + f^*)) + (u^*)^{\mu + \nu_2} b_1^* = 0 \quad (5.6)$$

The curve τ_2 is obtained as the solution of (5.6) and (4.1) (see the discussion of the case $\mu_{\pm} = 1$).

3. To find the curve τ_3 we look for the solutions of (5.1) of the form $(0, 0, y)$ (see Remark 5.1). This leads to the equations

$$\begin{cases} \alpha + a = 0, \\ x = 0, \\ y(\lambda + f) = 0. \end{cases} \quad (5.7)$$

We rewrite (5.7) as

$$\begin{aligned} \alpha &= -a(0, 0, y^2; \alpha, \lambda) \\ \lambda &= -f(0, 0, y^2; \alpha, \lambda). \end{aligned} \quad (5.8)$$

Equation (5.8) determines the curve τ_3 . The form of (5.8) implies that τ_3 is as smooth as the vector field F . $\square$

The proof of parts 1 and 2 of Proposition 5.2 implies the following result.

Corollary 5.4. *If $\nu_1 < 1$ then $P_{(\alpha, \lambda)}$ is stable for (α, λ) contained in the region bounded by τ_1 , τ_2 and not containing the line $\alpha = 0$.*

Proof of Proposition 5.3. We suppress the dependence of a , b_1 , b_2 , b_3 , and f on (α, λ) . This dependence is of higher order and does not affect the analysis.

We find the region D_1 by solving for fixed points of (3.10). By Remark 5.1 such fixed points must satisfy the equation (5.1). Applying the implicit function theorem in the analogous way as in the proof of Proposition 5.2 we solve (3.10) for x as a function of (u, y, α, λ) . The resulting solution $x = g(u, y)$ must be odd in y . Hence there exists a function φ such that

$$x = u^\mu y \varphi(u, y^2).$$

Substituting into (5.1a) and (5.1c) and canceling the factor y we get

$$u^{\nu_1} = \alpha + \tilde{a}, \quad (5.9a)$$

$$u^{\nu_2} = u^\mu \varphi \tilde{b}_3 + \lambda + \tilde{f}, \quad (5.9b)$$

where

$$\begin{aligned} \tilde{a}(u, y^2) &= a(u, u^\mu y \varphi(u, y^2), y), \\ \tilde{b}_3(u, y^2) &= b_3(u, u^{2\mu} y^2 \varphi^2(u, y^2), y^2), \\ \tilde{f}(u, y^2) &= f(u, u^{2\mu} y^2 \varphi^2(u, y^2), y^2). \end{aligned}$$

Using the condition $f_{y^2} \neq 0$ we solve equation (5.9b) for y^2 as a function of $(u; \alpha, \lambda)$ (the implicit function theorem is applied in the analogous way as in the proof of Proposition 5.2). Hence we obtain a function $\rho(u; \alpha, \lambda)$ such that $(u, \rho(u; \alpha, \lambda))$ solves (5.9b). This imposes a solvability condition

$$\rho(u; \alpha, \lambda) > 0. \quad (5.10)$$

Substituting $y^2 = \rho(u; \alpha, \lambda)$ into (5.9a) we obtain

$$u^{\nu_1} = \alpha + \tilde{a}(u, \rho(u; \alpha, \lambda)). \quad (5.11)$$

Let $\eta = \alpha + \tilde{a}(0, \rho(0; \alpha, \lambda))$. We replace the parameter λ by η and treat λ as a function of (α, η) . We can now write (5.11) in the form

$$\eta = u^{\nu_1} + C_1 u^{\nu_2} + C_2 u^\mu + C_3 u + o(u^{\min\{1, \mu, \nu_2\}}), \quad (5.12)$$

where, at the lowest order in u ,

$$\begin{aligned} C_1 &= -\frac{a_{y^2}}{f_{y^2}}, \\ C_2 &= \frac{a_{y^2} b_{20} b_{30}}{f_{y^2}}, \\ C_3 &= \frac{a_{y^2} f_u}{f_{y^2}} - a_u. \end{aligned}$$

It is clear that equation (5.12) has a unique solution $\eta(u; \alpha)$ provided that the coefficients C_1, C_2, C_3 are nonzero. Every solution of (5.12) for which the solvability conditions $u > 0$ and (5.10) hold corresponds to a pair of symmetry-related fixed points of $\Pi((u, x, y))$ and $(u, -x, -y)$. The two solvability conditions $u > 0$ and (5.10) define a region in the plane (α, λ) bounded by the curves $u = 0$ and $\rho(u; \alpha, \lambda) = 0$. The equation $u = 0$ is equivalent to (5.8), which, by the definition of η , is equivalent to the pair of equations $\eta = 0$ and $\alpha = -a(0, 0, y^2, \alpha, \lambda)$. For $\alpha \neq 0$ the solution is given by the curve τ_3 , which in the parameter plane (η, α) equals the halfline $\eta = 0, \alpha \cdot \text{sgn}(-a_{y^2}) \geq 0$. We now examine the other boundary of the region D_1 defined by $\rho(u; \alpha, \lambda) = 0$. When $y = 0$, (5.9a) reduces to (4.1). Also note that $\Phi(u, 0) = b_2(u, 0, 0) \left(\frac{1}{1 - u^\mu b_1(u, 0, 0)} \right)$. By substituting the explicit form of $\Phi(u, 0)$ into (5.9b) we obtain the condition (5.4). It follows that the curve defined by $\rho(u; \alpha, \lambda) = 0$ is τ_1 . Note that $u > 0$ and (5.12) imply that η must be of fixed sign. Hence the region D_1 is determined as the region bounded by τ_1 and τ_3 with the sign of η given by $u > 0$ and (5.12).

To find the region D_2 we look for points (u, x, y) such that under the transformation (3.10) $(\hat{u}, \hat{x}, \hat{y}) = (u, -x, -y)$. Substituting this relation into (3.10), we get

$$\begin{cases} u^{\nu_1} = \alpha + a, \\ x = -u^\mu(b_1x + b_2x), \\ u^{\nu_2}y = -(b_3x + y(\lambda + f)). \end{cases} \quad (5.13)$$

The same argument as used in Remark 5.1 shows that all solutions of (5.13) with $u > 0$ correspond to 2-periodic orbits of F . Through a procedure analogous to the one applied for D_1 we find the region D_2 . One of its boundaries is given by the curve τ_2 and is equal to the set where the fixed point $(u^*, 0, 0)$ has an eigenvalue of -1 . The other boundary is obtained by solving (5.8) and is given by the curve τ_3 . $\square$

The parameter η can be understood as measuring the separation between the stable and the unstable manifolds of the saddle 0 in the nonsymmetric direction. The halfline $\eta = 0, \alpha \cdot \text{sgn}(-a_{y^2}) > 0$ corresponds to the curve τ_3 . Under the assumptions of Propositions 5.2 and 5.3 there are four qualitatively different bifurcation diagrams. These bifurcation diagrams, represented in the parameter plane (η, α) , are shown in Fig. 3. Figure 1(a)–(d) can be obtained from these diagrams using the relation $\eta = \alpha + a(0, \rho(0; \alpha, \lambda))$. We assume $C_i > 0, i = 1, 2, 3$. The relative position of some of the regions depends on these coefficients, but there is no qualitative difference.

Remark 5.5. (a) Propositions 5.2 and 5.3 imply that as the parameters (α, λ) cross the line τ_1 and move into the region D_1 a unique pair of symmetry-related fixed points is born from the symmetric fixed point $(u^*, 0, 0)$. The solutions of (2.1) corresponding to these two fixed points are 1-periodic orbits lying outside of S . We refer to this bifurcation as the *pitchfork* bifurcation. Note that this bifurcation takes place as an eigenvalue of $d\Pi(u^*, 0, 0)$ passes through 1.

(b) As the parameters (α, λ) enter the region D_2 crossing the line τ_2 a unique $\mathbb{Z}_2$ -symmetric, period-two orbit bifurcates from $(u^*, 0, 0)$. The corresponding orbit of (2.1) is 2-periodic and symmetric as a set but not pointwise. We refer to this bifurcation as the *period-doubling* bifurcation.

(a)

	$(u^*, 0, 0)$	fixed point	per-2 point
I	0	0	0
II	1	0	0
III	1	0	1
IV	1	1	0
V	1	0	0
$f_{y^2} < 0, \min\{\nu_1, 1\} > \min\{\mu, \nu_2\}$

(b)

	$(u^*, 0, 0)$	fixed pt.	per 2 pt
I	0	0	0
II	1	0	0
III	1	1	1
IV	1	1	0
V	1	0	0
$f_{y^2} < 0, \min\{\nu_1, 1\} < \min\{\mu, \nu_2\}$

(c)

	$(u^*, 0, 0)$	fixed pt.	per 2 pt
I	0	0	1
II	1	0	1
III	1	0	0
IV	1	1	0
V	0	1	0
$f_{y^2} > 0, \min\{\nu_1, 1\} > \min\{\mu, \nu_2\}$

(d)

	$(u^*, 0, 0)$	fixed pt.	per 2 pt
I	0	1	1
II	1	1	1
III	1	1	0
IV	1	0	0
V	0	0	0
$f_{y^2} < 0, \min\{\nu_1, 1\} < \min\{\mu, \nu_2\}$			

Fig. 3. Bifurcation diagrams depending on the sign of f_{y^2} and the eigenvalue configuration.

(c) The curve τ_3 is the locus of existence of a pair of symmetry-related homoclinic orbits. This implies that a twist point is a point of coincidence of a curve of symmetric homoclinic orbits and a curve of nonsymmetric homoclinic orbits. As the parameters approach the twist point, the nonsymmetric homoclinic orbit approaches the symmetric one. In this sense a twist-point bifurcation can be seen as leading to homoclinic symmetry breaking. The existence of the two distinct branches of 1-homoclinic orbits does not occur in the nonsymmetric twist bifurcation problem and is a special feature of the $\mathbb{Z}_2$ -symmetric problem discussed in this article.

(d) Generically, the primary periodic orbit $P_{(\alpha,\lambda)}$ has, for parameter values arbitrarily near the twist point, multipliers 1 and -1 . This occurs regardless of the eigenvalue configuration. In the nonsymmetric twist bifurcation problem the occurrence of parameter values near the twist point where the primary periodic solution has multipliers $+1$ or -1 does depend on the eigenvalue configuration (see [KKO92]). In fact in some of the cases there are no rigorous results regarding the existence and the position of such parameter values.

6. Existence of 2-Homoclinic Orbits

In the previous sections we have focused our attention on the simplest dynamics of the map Π , namely fixed points and symmetric period-two points. A natural question is whether a twist bifurcation leads to more complicated dynamics, for example n -periodic orbits with arbitrary n . The original motivation of Yanagida [Yan87] for studying twist bifurcations (without symmetry) was to find double homoclinic orbits near the twist point. Rigorous results in this direction have been first obtained by [KK092]. They show that, given certain open conditions on the eigenvalues of the saddle, twist-point bifurcations lead to homoclinic doubling (occurrence of 2-homoclinic orbits), while no n -homoclinic orbits with $n > 2$ exist. The more recent work of Deng [Den92], Homburg, Kokubu, and Krupa [HKK93], and Sandstede [San93] show that for other eigenvalue configurations unfolding a twist point leads to the appearance of Smale horseshoes. Moreover, there exist curves in the parameter space along which such horseshoes undergo a sequence of bifurcations leading to their complete annihilation. An analogous process of horseshoe annihilation is analyzed in [Hom93].

This type of complicated behavior is also likely to take place in the symmetric case studied in this article. In fact Homburg [Hom93] proves that perturbations of pairs of symmetry-related homoclinic orbits in the case when $\nu_1 < \nu_2$ and $1 < \mu$ lead to the occurrence of a $\mathbb{Z}_2$ symmetric Smale horseshoe. As the size of the perturbation decreases the entire horseshoe limits on the pair of homoclinic orbits. The result of Homburg does not imply the existence of any n -homoclinic orbits. We have attempted to find n -homoclinic orbits near the twist point by looking for the periodic orbits of the map Π , but we have only been successful in the case $n = 2$. For $n > 2$ the computations necessary to carry out this approach seem to be very involved and a result for general n seems difficult to achieve. We have dealt with the following eigenvalue configuration:

$$\nu_1 \leq \min\{1, \mu, \nu_2\}. \quad (6.1)$$

We believe that 2-homoclinic orbits exist also for other eigenvalue configurations and our approach could still be applied. We have chosen however to deal only with the case (6.1) for which one gets the simplest computations. Also this case is the one encountered in the Josephson junction equations (7.1). The following is the main result of this section.

Theorem 6.1. *Suppose that F is a two-parameter family of four-dimensional vector fields satisfying $H1$, $H2$, (6.1) and having a twist point. Then, generically, F has a unique branch of 2-homoclinic orbits emanating from the twist point.*

The approach to proving Theorem 6.1 is as in the previous sections; we restate the theorem as a result for the return map Π . Let p_1, p_2 be a period-two orbit of Π , $p_1 = (u_1, x_1, y_1)$, $p_2 = (u_2, x_2, y_2)$. In other words,

$$\begin{aligned} p_2 &= \Pi(p_1), \\ p_1 &= \Pi(p_2). \end{aligned} \tag{6.2}$$

Using the definition of Π we write (6.2) as the following system of equations:

$$\begin{aligned} u_2^{v_1} &= \alpha + a(u_1, x_1, y_1), \\ x_2 &= u_2^\mu(b_1(u_1, x_1, y_1)x_1 + b_2(u_1, x_1, y_1)y_1), \\ u_2^{v_2}y_2 &= b_3(u_1, x_1, y_1)x_1 + (\lambda + f(u_1, x_1, y_1))y_1, \\ u_1^{v_1} &= \alpha + a(u_2, x_2, y_2), \\ x_1 &= u_1^\mu(b_1(u_2, x_2, y_2)x_2 + b_2(u_2, x_2, y_2)y_2), \\ u_1^{v_2}y_1 &= b_3(u_2, x_2, y_2)x_2 + (\lambda + f(u_2, x_2, y_2))y_2. \end{aligned} \tag{6.3}$$

We consider the solutions of (6.3) which correspond to the trajectories of the flow with initial points in Σ^{out} returning to Σ^{out} in finite time and then forward asymptotic to the saddle s . This translates to the condition

$$u_2 > 0, \quad u_1 = 0.$$

Clearly, $u_1 = 0$ implies $x_1 = 0$. Hence (6.3) reduces to

$$\begin{aligned} u_2^{v_1} &= \alpha + a(0, 0, y_1), \\ x_2 &= u_2^\mu(b_2(0, 0, y_1)y_1), \\ u_2^{v_2}y_2 &= (\lambda + f(0, 0, y_1))y_1, \\ 0 &= \alpha + a(u_2, x_2, y_2), \\ 0 &= b_3(u_2, x_2, y_2)x_2 + (\lambda + f(u_2, x_2, y_2))y_2. \end{aligned} \tag{6.4}$$

Let f_{y^2} and a_{y^2} be the lowest-order coefficients of the Taylor expansion of $f(0, 0, y)$ and $a(0, 0, y)$, that is,

$$\begin{aligned} f(0, 0, y) &= f_{y^2}y^2 + O(y^4), \\ a(0, 0, y) &= a_{y^2}y^2 + O(y^4). \end{aligned}$$

The following result holds.

Proposition 6.2. *Assume that f_{y^2} and a_{y^2} do not vanish. Then there exists a continuous curve $\Psi_2 : [0, 1) \rightarrow \mathbb{R}^2$, with $\Psi_2(0) = 0$, and a small neighborhood U of $0 \in \mathbb{R}^5$ such that for every (α, λ) on Ψ_2 there exists a unique nontrivial solution of (6.4) contained in U . This solution tends to 0 as $(\alpha, \lambda) \rightarrow 0$ along Ψ_2 . If $a_{y^2} < 0$, then Ψ_2 is tangent at 0 to τ_3 . If $a_{y^2} > 0$, then the curve Ψ_2 is tangent at 0 to the α -axis.*

Proof. Recall the system of equations (6.4). Substitute the expression for x_2 (the second equation) into the last equation. This leads to a system of four equations:

$$u_2^{\nu_1} = \alpha + a(0, 0, y_1), \quad (6.5a)$$

$$u_2^{\nu_2} y_2 = (\lambda + f(0, 0, y_1)) y_1, \quad (6.5b)$$

$$u_1^{\nu_1} = \alpha + a(u_2, u_2^\mu b_2 y_1, y_2), \quad (6.5c)$$

$$0 = b_3(u_2, u_2^\mu b_2 y_1, y_2) u_2^\mu b_2(0, 0, y_1) y_1 \\ + (\lambda + f(u_2, u_2^\mu b_2 y_1, y_2)) y_2. \quad (6.5d)$$

We now perform the following operations:

- Subtract (6.5c) from (6.5a) to eliminate α .
- Multiply (6.5b) by y_2 and subtract (6.5d) multiplied by y_1 in order to eliminate λ .

We obtain the following system of two equations.

$$u_2^{\nu_1} = a(0, 0, y_1) - a(u_2, u_2^\mu y_1, y_2), \quad (6.6) \\ 0 = u_2^{\nu_2} y_2^2 + b_2 b_3 u_2^\mu y_1^2 + (f(u_2, u_2^{\nu_2} y_2, y_2) - f(0, 0, y_1)) y_1 y_2.$$

Note that the first equation in (6.6) can be written as

$$(y_1^2 - y_2^2) = \frac{(1 + \mathcal{O}(y_1^2 + y_2^2))}{a_{y^2}} u_2^{\nu_1} + \mathcal{O}(u_2, u_2^\mu). \quad (6.7)$$

Note also that the expression $f(u_2, u_2^{\nu_2} y_2, y_2) - f(0, 0, y_1)$ in (6.6) is, at the lowest order, equal to $f_{y^2}(y_1^2 - y_2^2)$. Hence we can rewrite (6.6) in the following way:

$$0 = u_2^{\nu_2} y_2^2 + b_2 b_3 u_2^\mu y_1^2 + \left(-\frac{f_{y^2}}{a_{y^2}} + \mathcal{O}(y_1^2 + y_2^2) \right) y_1 y_2 (u_2^{\nu_1} + \mathcal{O}(u_2, u_2^\mu)). \quad (6.8)$$

The assumption (6.1) allows us to divide both sides of (6.8) by $u_2^{\nu_1}$ leading to the equation

$$0 = u_2^{\nu_2 - \nu_1} y_2^2 + b_2 b_3 u_2^{\mu - \nu_1} y_1^2 + y_1 y_2 \left(-\frac{f_{y^2}}{a_{y^2}} + g(u_2, y_1, y_2) \right), \quad (6.9)$$

where $g \in \mathcal{O}(y_1^2 + y_2^2 + u_2^{\mu - \nu_1} + u_2^{1 - \nu_1})$.

Next we show that we can rewrite the last equation in the form

$$0 = M(y_1 - Ay_2)(y_2 - By_1) \stackrel{\text{def}}{=} H(u_2, y_1, y_2), \quad (6.10)$$

with M , A , and B depending on y_1 , y_2 , and u_2 . Remark that (6.9) and (6.10) are equivalent iff

$$\begin{aligned} M(1 + AB) &= -\frac{f_{y^2}}{a_{y^2}} + g, \\ -MA &= u_2^{\nu_2 - \nu_1}, \\ -MB &= b_2 b_3 u_2^{\mu - \nu_1}. \end{aligned} \quad (6.11)$$

At $u_2 = y_1 = y_2 = 0$ this set of equations has the solution

$$M_0 = -\frac{f_{y^2}}{a_{y^2}}, \quad A_0 = 0, \quad B_0 = 0.$$

At this point the implicit function theorem applies as the determinant of the Jacobian with respect to M , A , and B equals M_0^3 which is nonzero by assumption. Hence we obtain mappings $(y_1, y_2, u_2) \mapsto M^*(y_1, y_2, u_2)$, $(y_1, y_2, u_2) \mapsto A^*(y_1, y_2, u_2)$, and $(y_1, y_2, u_2) \mapsto B^*(y_1, y_2, u_2)$ such that $M^*(0, 0, 0) - f_{y^2}/a_{y^2} = A^*(0, 0, 0) = B^*(0, 0, 0) = 0$ and (6.11) holds in a neighborhood of the origin in $\mathbb{R}^3$. It then follows that there are two solutions of equation (6.10) given by $y_1 = A^*y_2$ and $y_2 = B^*y_1$. Using again the implicit function theorem we obtain two mappings $(u_2, y_2) \mapsto y_1^*(u_2, y_2)$ and $(u_2, y_1) \mapsto y_2^*(u_2, y_1)$ such that $H(u_2, y_1^*(u_2, y_2), y_2) = H(u_2, y_1, y_2^*(u_2, y_1)) = 0$. In particular,

$$y_1^* = \frac{a_{y^2}}{f_{y^2}} u_2^{\nu_2 - \nu_1} y_2 (1 + \mathcal{O}(y_2^2 - u_2^{\mu - \nu_1} + u_2^{1 - \nu_1})), \quad (6.12)$$

$$y_2^* = \frac{b_2 b_3 a_{y^2}}{f_{y^2}} u_2^{\mu - \nu_1} y_1 (1 + \mathcal{O}(y_1^2 - u_2^{\mu - \nu_1} + u_2^{1 - \nu_1})). \quad (6.13)$$

From (6.12) we obtain $y_2^2 - (y_1^*)^2 = y_2^2 (1 - \mathcal{O}(y_2^2 + u_2^{\mu - \nu_1} + u_2^{\mu - \nu_1} + u_2^{1 - \nu_1}))$. Now, using (6.7) we derive

$$y_2^2 = -\frac{1}{a_{y^2}} u_2^{\nu_1} + o(u_2^{\nu_1}). \quad (6.14)$$

Hence this solution is valid when $a_{y^2} < 0$. We substitute the expression for y_1^* given by (6.12) into (6.5b) and cancel the factor y_2 . Now, applying (6.14) we infer that in this case $\lambda = \frac{f_{y^2}}{a_{y^2}} u_2^{\nu_1} + o(u_2^{\nu_1})$. Using (6.12), (6.14), and (6.5) we obtain $\alpha = u_2^{\nu_1} + o(u_2^{\nu_1})$. A similar procedure with (6.12) replaced by (6.13) yields

$$y_1^2 = -\frac{1}{a_{y^2}} u_2^{\nu_1} + o(u_2^{\nu_1}). \quad (6.15)$$

This gives a valid solution when $a_{y^2} > 0$. In this case it follows from (6.5) that $\alpha = o(u_2^{\nu_1})$ and $\lambda = -\frac{f_{y^2}}{a_{y^2}} u_2^{\nu_1} + o(u_2^{\nu_1})$. The proposition now follows. $\square$

7. Application to Arrays of Josephson Junctions

In [AGK91] the dynamics of current biased Josephson junctions with global coupling is considered. In the case of *pure capacitive load* the equations of motion of the phase of the k th junction φ_k are given by

$$\beta \ddot{\varphi}_k + \dot{\varphi}_k + \sin(\varphi_k) - \frac{3}{2(3 + \beta)} \sum_{j=1}^2 (\dot{\varphi}_j + \sin(\varphi_j)) = \frac{\beta}{3 + \beta} I, \quad k = 1, 2. \quad (7.1)$$

The $\mathbb{Z}_2$ symmetry is given by the permutation of junctions. The symmetric solutions of (7.1), that is, the solutions contained in the space S , are characterized by the property $\varphi_1 = \varphi_2$ and satisfy the equation

$$(3 + \beta)\ddot{\varphi} + \dot{\varphi} + \sin(\varphi) = I. \quad (7.2)$$

The space R (see Sec. 2) consists of the elements $(\varphi_1, \dot{\varphi}_1, \varphi_2, \dot{\varphi}_2)$ such that $\varphi_1 = -\varphi_2$.

Using the software package AUTO [Doe81], numerical results have been obtained for this system of two oscillators. There are two parameters in the problem: β (measuring the capacitance) and I (the applied bias current). In the case of pure capacitive load a codimension-2 bifurcation is observed where, in parameter space, a curve of period-doubling bifurcations and a curve of pitchfork bifurcations terminate on a curve of homoclinic orbits at a single point P . Here the terms *pitchfork bifurcation* and *period-doubling bifurcation* are used in the same sense as in Remark 5.5. On one side of this curve of homoclinic orbits, in-phase ($\mathbb{Z}_2$ symmetric) periodic solutions exist. In [AGK91] it is shown using a combination of analytical arguments and numerical computations that P is a twist point. The bifurcation picture computed with AUTO is shown schematically in Fig. 4; see also [AGK91]. Proposition 5.2 promises a third

Fig. 4. Homoclinic twist in the Josephson system.

curve, namely the curve τ_3 . For parameter values on this curve a pair of symmetry related homoclinic orbits Γ_+ and Γ_- exists. When the parameters approach τ_3 both the period-doubled and the asymmetric periodic solutions approach $\Gamma_+ \cup \Gamma_- \cup \{0\}$ in the set theoretic sense. We have found in Sec. 5 that there are four qualitatively different bifurcation pictures. From the numerical computations with AUTO we know the numerical values of I and β at the twist point. They are approximately $I = 0.639$, $\beta = 0.419$. The eigenvalues at these critical parameter values are given by the eigenvalues of the matrices A_λ and B_λ in (2.3). The saddle point s_λ is determined by the equation $\sin(\varphi) = 0.639$. It turns out that the symmetric eigenvalues (i.e., the eigenvalues of A_λ), are 0.350 and -0.642 . The eigenvalues in the asymmetric direction are 0.612 and -2.999 . This implies that we are in case (b) or (d) of Fig. 1. We have done numerical calculations with AUTO to find the curves τ_1 (pitchfork bifurcation), τ_2 (period-doubling bifurcation), τ_3 (asymmetric homoclinic solutions), and τ_4 (symmetric homoclinic solutions). The results are in agreement with the theory and are shown in Fig. 5. Note that the bifurcation lines are shown only in a small neighborhood of P and a large unit has been selected. Otherwise the relative positions of τ_1 , τ_2 , and τ_3 could not be seen. Unfortunately we have not yet numerically calculated the coefficient $f_{j,2}$. This would further restrict the possible bifurcation diagrams.

At the value $I = 0.8$ we have followed (as a function of β) bifurcating solutions both from the pitchfork and the period-doubling line. These solutions both disappear in a homoclinic connection when β reaches the value on the curve τ_3 (at $I = 0.8$). In Fig. 6 several solutions are depicted. We are presently looking for the 2-homoclinic orbits using AUTO.

Further results on the Josephson junction model with two junctions and capacitive load are given in [ADT92]. Numerical studies show the existence of a codimension-2 bifurcation point in the parameter region to the right of the curve τ_4 of symmetric homoclinic orbits (specifically at $\beta = .345891$, $I = .842596$). The unfolding of this point generates a wide variety of curves of homoclinic connections including the curve of asymmetric homoclinic orbits τ_3 discussed here.

Fig. 5. Bifurcations at the homoclinic twist point.

Fig. 6. Periodic solutions outside the symmetric space. In the left column coming from the pitchfork line, in the right column coming from the period doubling line. The period-doubled solutions are symmetric (as a set).

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