THE VORTON METHOD

FONS ALKEMADE, and F.T.M. NIEUWSTADT

Laboratory for Aero- and Hydrodynamics

Delft University of Technology

Rotterdamseweg 145

2628 AL Delft

The Netherlands

and

E. VAN GROESEN

Faculty of Applied Mathematics

University of Twente

The Netherlands

ABSTRACT. After a general introduction to the vorton method, which is a 3-D vortex particle method resembling the 2-D point-vortex method, a set of equations describing dynamics of 3-D vortex singularities (vortons) is derived, avoiding the inconsistency in the derivation of former vorton equations by Novikov and Kuwabara. Though inviscid, numerical simulations show reconnection phenomena. Current examples of vorton simulations are shown; they show at least qualitative agreement with results from other methods.

1. Introduction

The background of our research is the investigation of the dynamics of relatively simple vortex structures, like vortex rings, which may be regarded as models of some well-known and much-studied coherent structures in current turbulence research.

Application of vorticity theory to the study of turbulence is not only formally attractive. It can be claimed that the problem of turbulence is a problem of vorticity dynamics. The belief that coherent structures can be regarded as regions of concentrated vorticity embedded in essentially irrotational flow, has led to the view that turbulence should be modelled or described as the creation, evolution, interaction, and decay of these large-scale vortex structures.

It is mathematically correct and often very convenient to consider fluid dynamics in terms of *parcels* of vorticity which induce motion and deformation on each other. This is the basic idea behind the use of **vortex methods** in fluid dynamics.

An additional advantage of describing flow by means of vorticity comes from the fact that specification of the vorticity field may be far more economical than that of the velocity field, since the rotational part of the total flow field usually takes up only a small volume.

From the aim of our research given above, it may be clear that we are not thinking the vorton method is best suited. We are just investigating its value for our purposes. This means that numerical aspects like convergence, stability, computational speed, etc. are not subject of serious study or optimization.

2. Vorticity Theory

In case the fluid is perfect, the flow is incompressible, and the external forces on the fluid are conservative, vortex elements have material properties (Helmholtz's second theorem).

Vortex deformation (reorientation and stretching) is described by the Helmholtz equation:

$$\frac{Dw_i}{Dt} = \frac{\partial v_i}{\partial x_j} w_j \quad (1)$$

This equation can be shown to be equal to:

$$\frac{Dw_i}{Dt} = \frac{\partial v_j}{\partial x_i} w_j \quad (2)$$

2.1. GENERAL OUTLINE OF VORTEX METHODS

The concept of every vortex method can be summarized by the following questions:

- How do we discretize the vorticity? Which are the vortex *elements*?
- Which parts of the method are based on a local/Lagrangian description, which on a global/Eulerian description?
- Which are the equations for displacement and deformation of the discretized vortices?

The most widely used methods seem to be *molecular*, i.e. the vortex elements have a finite size or core. However, molecular vortex methods require a large amount of detailed information since entire vortex filament structures must be located and tracked, and often local curvatures of the vortex filaments have to be computed as well. Use of coreless or *atomic* vortex elements thus seems to be more attractive.

3. The Vorton Method

Referring to the three questions posed above, the atomic vorton method can shortly be described by:

- The elements are, essentially, 3-D point vortices or singular sources of vorticity.
- The method is completely local/Lagrangian: the deformation of the vortons is followed on their paths through 3-D space. Helmholtz's second theorem allows us to regard vortons as material.
- Vorton displacement is derived from the velocity field caused by the other vortons. Vorton deformation is derived from the Helmholtz equation.

A vorton labelled α has a location vector, $\mathbf{r}_\alpha$, and a vorticity intensity vector, $\boldsymbol{\gamma}_\alpha$. For the sake of simplicity, a vorton is represented by an arrow; it starts at $\mathbf{r}_\alpha$ and the arrow line itself indicates $\boldsymbol{\gamma}_\alpha$.

For every vorton we need a set of displacement and deformation equations, which can be derived once we have found the vorton velocity and vorticity field.

These fields can be found from the following vorton vector-potential field:

$$\begin{aligned}\mathbf{A}(\mathbf{x}, t) &= \frac{1}{4\pi} \sum_{\alpha} \frac{\boldsymbol{\gamma}_\alpha}{R_\alpha} \\ &\equiv \sum_{\alpha} \phi_\alpha(\mathbf{x}, t) \boldsymbol{\gamma}_\alpha(t)\end{aligned}\quad (3)$$

where $R_\alpha \equiv |\mathbf{x} - \mathbf{r}_\alpha|$ and summation is over all vortons.

From this vector-potential field we find the following velocity field:

$$\mathbf{v}(\mathbf{x}, t) = \nabla \times \mathbf{A} = \sum_{\alpha} \nabla \phi_\alpha \times \boldsymbol{\gamma}_\alpha \quad (4)$$

Finally, from (4) we get the following vorticity field:

$$\begin{aligned}\mathbf{w}(\mathbf{x}, t) &= \sum_{\alpha} \{\boldsymbol{\gamma}_\alpha \delta(\mathbf{R}_\alpha) + (\phi''_\alpha) \circ \boldsymbol{\gamma}_\alpha\} \\ &= \sum_{\alpha} \{\boldsymbol{\gamma}_\alpha \delta(\mathbf{R}_\alpha) - \nabla[\nabla \cdot (\phi_\alpha \boldsymbol{\gamma}_\alpha)]\}.\end{aligned}\quad (5)$$

Here (ϕ''_α) is a matrix whose ij -th element is given by $\partial^2 \phi_\alpha / \partial x_j \partial x_i$.

Compare this field with the non-divergencefree vorton vorticity field originally introduced by Novikov in [6] (see also [1]):

$$\mathbf{w}(\mathbf{x}, t) = \sum_{\alpha} \boldsymbol{\gamma}_\alpha(t) \delta(\mathbf{R}_\alpha(\mathbf{x}, t)) \quad (6)$$

We see that apparently the second (gradient) part of (5) makes the vorticity field divergencefree. It can be regarded as the nonlocal vorticity field surrounding the original singular vorton represented by the first part.

By applying the above vorton velocity and vorticity fields (4) and (5), we are able to derive the vorton displacement and deformation equations by inserting them into the Helmholtz equation (1) or its transposed version (2).

Since we reject self-displacement of a vorton, we have to apply the velocity field induced by all other vortons, giving for the displacement of vorton α :

$$\dot{\mathbf{r}}_\alpha = - \sum_{\beta \neq \alpha} \boldsymbol{\gamma}_\beta \times (\nabla \phi_\beta)|_{\mathbf{x}=\mathbf{r}_\alpha} \quad (7)$$

As remarked before, the deformation equation for vortons is derived from the Helmholtz equation or its transposed version. Our approach to derive the vorton deformation equation is by integrating the Helmholtz equation about a small sphere

with a small radius and centre $\mathbf{r}_\alpha$, moving accordingly to the velocity field derived above. It is assumed that the radius is so small that no other vortons are inside the sphere. Remark the resemblance of this approach to the definition of so-called weak solutions.

After elaboration we get:

$$\dot{\gamma}_\alpha = N_\alpha + S_\alpha \quad (8)$$

where

$$N_\alpha \equiv \sum_{\beta \neq \alpha} [(\phi''_{\alpha\beta}) \circ \gamma_\alpha] \times \gamma_\beta$$

and

$$S_\alpha \equiv \frac{1}{2} \sum_{\beta \neq \alpha} \gamma_\alpha \times [(\phi''_{\alpha\beta}) \circ \gamma_\beta] .$$

which latter contribution is due to the gradient part of the vorticity field (5).

In much the same way, using the transposed Helmholtz equation (2) one can derive:

$$\dot{\gamma}_\alpha = K_\alpha - S_\alpha \quad (9)$$

where

$$K_\alpha \equiv \sum_{\beta \neq \alpha} (\phi''_{\alpha\beta}) \circ (\gamma_\beta \times \gamma_\alpha) .$$

(8) and (9) are completely equivalent. The original equations used by Novikov in [6] (derived by means of vorticity field (6)) are equal to our expression N_α . The equations used by Kuwabara in [5], derived from the transposed Helmholtz equation (2) also by applying the vorticity field (6), are equal to our expression K_α . The inconsistency between Novikov's and Kuwabara's equations, i.e. $N_\alpha \neq K_\alpha$, can now be attributed to their use of the non-divergencefree vorticity field (6).

4. Requirements on the vorton method

Certain requirements should be imposed on vortex methods in general. We will treat them here for the vorton method:

- The conservation of motion-invariants should not be violated.

So far, this requirement has never been investigated seriously for the vorton method. This is partly due to difficulties in expressing invariants by means of vorton quantities.

One should realize that in case of the vorton method complete conservation of energy will never be acquired, since part of the interaction energy will be converted into "self-energy" of the vortons, and *vice versa*. This effect may appear as (positive or negative) supply of energy, which has been named "inviscid dissipation". It may account for reconnection-like phenomena, which have been observed during vorton simulations (see below).

Fig. 1. Two approaching vorton filaments (parts of vorton rings) have reconnected in applying the vorton equations as proposed by Novikov and Kuwabara (left) and the authors (right).

- The discretization by finitely many elements should converge to the exact smooth solutions as the number of elements goes to infinity.

For the relative simple case of the vortex ring, this has been shown analytically by Novikov [6].

A first mathematical proof of convergence of a 3-D vortex particle method (the vortex blob method) was given in [2]. A definitive proof for the case of a 3-D nonsmooth, grid-free method has been given in [3].

- A vortex method has to show correct reconnection phenomena.

Vortex reconnection is an important and much-studied, though still little-understood phenomenon. A remarkable feature of the inviscid vorton method is its ability to simulate this typically viscous phenomenon.

An elementary configuration showing reconnection is shown in fig. 4: two slightly bent vorton filaments (in fact parts of two large vorton rings) approach each other closely. Simulations like these may show the specific characteristics of the several vorton equations which have been proposed. However, strong criteria for "correct" reconnection behaviour do not seem to exist.

Another interesting simulation in this context is that of the behaviour of a elliptical-shaped vorton ring. In contrast to the circular vorton ring, it deforms on its own, dependently of its aspect ratio.

We have performed simulations with pseudo-elliptical vorton rings: two semi-circles connected by two straight filaments. The first results show qualitative resemblance to those of simulations performed by others [4]; see fig. 4.

- The method should not require a disproportionate amount of computational effort and computation time.

So far our numerical simulations could be performed on an ordinary minicomputer (HP9000); computational times were of order of a few hours.

At the moment, we try to study the behaviour of a vorton ring in a shear flow above a flat plate. The first attempt by Novikov [1] showed the formation of a structure resembling a horseshoe vortex.

Fig. 2. The deformation of a pseudo-elliptical vorton ring in time (from left to right and from top to bottom). For the aspect ratio used here, splitting of the ring takes place, involving reconnection.

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