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switched one-way traffic lanes**

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Waiting times at periodically switched one-way traffic lanes

A periodic, two-queue polling system with random setup times

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Abstract

We study vehicle waiting times at a traffic lane that is shared by traffic from two directions. In contrast to crossovers, we focus on instances where the vehicle passing time of the shared infrastructure can be large. The motivation for this model arises from our research on underground transportation systems. We examine vehicle waiting times under *periodic* control rules, i.e. the driving direction on the infrastructure is switched between two directions according to a fixed time schedule. We analyse both symmetric and asymmetric systems (i.e., vehicle arrival rates as well as effective green and red periods may be different for both directions). In fact, we are dealing with a single server, two-queue polling system with random set-up times and periodic (non-exhaustive) service discipline. We develop approximations for the mean waiting time and we show by comparison to simulation results that the accuracy is usually in the range of 1-2% for Poisson arrivals. Also, we indicate how our approximations can be generalised to compound Poisson arrivals.

Key words: traffic control, queuing, polling, vacation

1. Introduction

When traffic infrastructure has to be shared by multiple traffic streams that cannot use this infrastructure concurrently, a control mechanism is required. Examples are crossovers and single traffic lanes that have to be shared by traffic from two directions because of road maintenance or traffic accidents on the other lane(s). We encountered another example in our research on the logistics control of automated transportation systems, where AGVs transport cargo through underground tube systems (cf. Van der Heijden et al. [2000]). Because tunneling requires high investments, some tubes may be shared by traffic from both directions. The driving time through these so-called bi-directional tubes can easily be 5-10 minutes, so an appropriate control mechanism is essential in order to reduce vehicle waiting times.

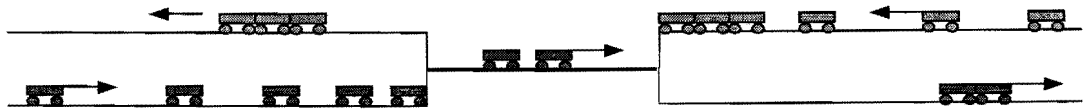


Figure 1. Bi-directional lane

The situation under consideration is depicted in Figure 1. Vehicles arrive at both sides of the bi-directional lane. A traffic light at both sides of the lane entrance shows whether vehicles may enter the lane or not. Vehicles that cannot enter the lane wait in queue until the traffic light at their side is switched to green. The traffic lights at both sides of the tube are switched between red and green according to a periodic control rule, i.e. with fixed intervals, not depending on the queue sizes. Hence, we encounter the following system states in cyclic order:

- a) Green light on the left, red light on the right: AGVs from the left may enter and AGVs from the right have to wait;
- b) Red light on both sides: AGVs from the left have to wait, but AGVs from the right are not yet able to enter the track, because it still contains AGVs driving from left to right; we shall use the term *clearing the tube* (from the left, in this case) for this system state in the sequel;
- c) Red light on the left, green light on the right: AGVs from the right may enter and AGVs from the left have to wait;
- d) Red light on both sides: AGVs from the right are stopped and AGVs from the left wait until the tube is cleared.

These four system states constitute a cycle. The time spent in system state *a* (*c*) is called the *effective green time* from the left (right). We refer to the time spent in system state *b* (*d*) as the *clearance time* from the left (right), denoted by C_L (C_R). Obviously, the clearance time is variable and depends on the moment when the last vehicle enters the lane just before the traffic light is switched to red.

The aim of a control rule is to switch between these four successive states in such a way, that the system performance is optimized. A common way to measure performance in such a system is the mean vehicle waiting time. Control rules can both be fixed and dynamic, i.e. depending on the specific

system state (such as the queue lengths at both sides of the tube). In this paper, we focus on fixed, periodic control rules. For a discussion on dynamic control rules in this context, we refer to Ebben et al. [2000].

We define a *periodic* control rule such, that the time spent in system states *b* and *c* is constant and equal to P_R , see Figure 2. Analogously, the time spent in system states *a* and *d* is constant and equal to P_L . Hence, the total cycle length equals P_R+P_L and the effective green time from the right (left) equals P_R-C_L (P_L-C_R). The key problem is to determine P_L and P_R such, that vehicle waiting times are minimized. To this end, we need to calculate the mean waiting times for vehicles arriving at both sides of the bi-directional lane as function of the control parameters P_L and P_R . If the vehicle arrival processes at both sides of the tube are the same, it is reasonable to take $P_L=P_R=P$. We refer to this specific situation as the *symmetric* case. In this paper, we both address the symmetric and the asymmetric case.

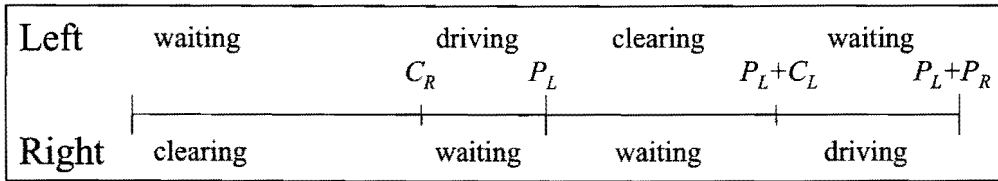


Figure 2. System states under a periodic control rule

As we shall explain in the next section, this problem is related to other models encountered in traffic management literature and has also similarities to some queuing models (polling systems, *M/D/1* vacation queue). However, we did not encounter this particular model in the literature. Therefore, we shall develop (approximate) methods to calculate the mean waiting time in traffic systems as described above for several variants. We start with the analysis of a simple case, namely a symmetric model where the *succession time*, i.e. the minimum time between successive vehicles entering the lane, is zero (and so is the minimum intermediate distance between vehicles). For this special case, we can develop an exact method to calculate the mean waiting time as well as a simple approximation. It is more realistic that minimum succession times are strictly positive, taking into account the vehicle length and a minimum safety distance to avoid collisions. For this more general case, we are only able to derive

approximate expressions. Therefore, we validate our approximation method by comparing the results to simulation results. We subsequently address the symmetric and asymmetric case. In all our models, we assume Poisson arrivals at both sides of the tube. However, extension to more general arrival processes is possible, and we shall briefly discuss how to accomplish that.

The remainder of this paper is structured as follows. First, we give an overview of literature and we state the added value of our research (Section 2). Next, we state our modelling assumptions and we give the basic notation to be used throughout the paper in Section 3. The simple case with zero minimum vehicle succession time is discussed in Section 4 and some numerical results are shown in Section 5. The analysis for strictly positive minimum succession times is the subject of Section 6, both for the symmetric and the asymmetric case. The corresponding numerical results (a.o. comparison to simulation results) are shown in Section 7. Finally, we describe how our method can be extended to compound Poisson arrivals (Section 8). Our conclusions are given in Section 9.

2. Literature

The scheduling of traffic signals at road intersections according to a fixed schedule is a rather traditional approach, cf. Bell [1992]. We see that the problem of calculating waiting times at simple traffic intersections has already been studied in the 50's and 60's. A basic paper is Webster [1958], who conducted a simulation study. Some analytical results for such models are obtained by Haight [1963]. Given the vehicle arrival distributions and the allowed driving directions (either left-right or the other way around), he derives formulas that specify queuing behaviour. Also, Miller [1968] derived an approximate method for this model. Later, many papers on this model have appeared. An interesting paper is Heidemann [1994], who derives exact formulae for the probability distributions of queue lengths and delays at traffic signals, given Poisson arrivals and a fixed-time control. Heidemann compares his results numerically to some previous approximations (Webster [1958]; Miller [1968]). Another relatively recent example is Mung, Poon and Lam [1996], who extend Haight's model to non-Poisson arrivals and derive distributions of queue lengths at fixed time traffic signals. Hu et al. [1997]

extend Heidemann's model to the multi-lane case, where multiple vehicles may enter the traffic intersection simultaneously.

In this traffic literature, it is usually assumed that the effective green period and the effective red period are *constant* and known. If this assumption is valid, we could simply use the exact method by Heidemann [1994]. As appears from our problem statement in the introduction however, this is not the case in our situation. The clearance time is a random variable and depends on the last moment at which a vehicle enters the tube before the traffic light is switched to red. Fixed effective green times is a reasonable assumption if the passing time of the lane is negligible. It is also a reasonable assumption if the system runs close to maximum capacity, so that the vehicle queue usually has not yet vanished if the traffic light is switched to red. In the latter case, the clearance time equals the (deterministic) driving time along the shared lane, so the effective green time is fixed too. Some preliminary numerical experiments using the latter approach (applying Heidemann's [1994] method to our modified model) revealed that the waiting time can be overestimated by 5-10% over the entire parameter range in this way. This is a consequence of the fact that the clearance time is less than or equal to the driving time along the lane, causing a longer effective green time. Therefore, we concluded that it is useful to construct a dedicated method to calculate mean waiting times.

Next to traffic literature, we can find similar models in queuing literature. To be specific, our model has similarities with polling systems (cf. Takagi [1990] for an overview), where a single server is handling two queues and switches between them according to some control rule. For our model, the service time should be deterministic and equal to the minimum distance between successive vehicles, expressed in time. Further, we have that both queues are attended by the server for a fixed period of time. This aspect is not common in polling models, but can be found in STDM (Service Time Division Multiplexing, cf. Kleinrock [1976]). However, STDM models do not include switchover times. In our setting, the clearance time could be modelled as a random set-up time, depending on the timing of the last service at the other queue. This aspect is not common in queueing systems.

Related to polling systems are queuing models with vacations (cf. Doshi [1990] and Takagi [1991]), i.e. the server leaves the queue regularly to perform other tasks. In our case, the vacation period can be modelled as some random variable that depends on the switching interval P , the driving time along the lane T and the vehicle arrival process. Still we would need a decomposition between the queues at both sides of the shared lane, whereas these queuing processes are actually interrelated. In Doshi's [1990] classification, our model is closest to the category with asynchronous vacations and general vacation rules. We did not find a model similar to ours in the literature on vacation queues.

From our literature survey, we conclude that similar models to the one under consideration have been analysed, but also that the combination of a fixed (periodic) service schedule, random clearance times and correlated queuing processes is new as far as we know.

3. Assumptions and basic notation

We focus on the computation of the mean waiting time at a periodically switched bi-directional traffic lane as described in the introduction. If we have an expression for the mean waiting time, we can use some standard numerical search procedure to find the optimal value of the switching period P (in the symmetric case; we have to find both P_R and P_L in the asymmetric case).

3.1. Assumptions

For clarity of understanding and to keep the formulas simple, we make the following simplifying assumptions for the analysis in the sequel:

- 1) All vehicles are identical.
- 2) When driving, the vehicles travel with a constant speed.
- 3) Queued vehicles accelerate instantaneously to their normal speed when activated and the other way around, vehicles can stop instantaneously when arriving at a queue.
- 4) Vehicles waiting in queue enter the shared lane with a fixed minimum succession time that remains constant while driving over the lane.

- 5) Vehicles arrive one-by-one according to a Poisson process (not necessarily identical at both sides of the shared lane).

In practice, the succession time between vehicles will be strictly positive. An obvious reason is the length of a vehicle, so that two vehicles cannot enter a single lane simultaneously. Also, safety margins can be included to account for a finite deceleration. This safety margins should be set such, that the collision probability is negligible. In automated transportation systems, we usually encounter the so-called *brick wall principle*. This principle states that a vehicle should be able to stop in time if the preceding vehicle driving with minimum succession time halts immediately (i.e. decreases speed from maximum to zero instantaneously).

Referring to assumption 2), we note that the vehicle speed may vary in practise, for example depending on the load carried by the vehicles. We may include the effect of fluctuations in the vehicle speed by increasing the safety margins in the succession times and by working with a slightly lower constant speed than the maximal one.

With respect to assumption 5), we note that Poisson arrivals are theoretically conflicting with the assumption on minimum succession times, as Poisson arrivals allow succession times that are almost zero. Still many papers use the assumption of Poisson arrivals for convenience. This seems not to be very harmful from a practical point of view if the probability that a vehicle arrives within the minimum succession time is negligible, which is satisfied in many practical cases. Although we shall base our derivations on Poisson arrivals, we shall discuss the generalisation to compound Poisson arrivals in Section 9.

3.2. Basic notation

Throughout our paper, we shall use the following basic notation, subdivided in model input, output and control parameter:

Model input:

T = the driving time of a single vehicle to pass the lane

$A(t)$ = the number of vehicles arriving at one side of the lane during an arbitrary time period with length t , a random variable

λ = the vehicle arrival rate at one side of the lane

δ = the minimum time between two successive vehicles entering the lane

Model output:

C = clearance time of the lane, a random variable

W = vehicle waiting time, a random variable

Control parameter:

P = Fixed period for traffic from one side to pass the lane (hence the cycle length equals $2P$)

We may give the parameters P and λ and the random variables W , C and $A()$ a subscript R or L , denoting the respective characteristics of the right and left side of the lane. In an asymmetric case with different arrival rates λ_L and λ_R , the length of the green period at the left (right) side of the lane equals $P_L - C_R$ ($P_R - C_L$). Analogously, the length of the red period at the left (right) side of the lane equals $P_R + C_R$ ($P_L + C_L$). As we consider identical vehicles, the driving time through the lane T as well as the minimum mutual distance between vehicles δ does not depend on the driving direction. Therefore, the subscripts R and L are omitted for these parameters. From a theoretical point of view, it is straightforward to include non-identical values of T and δ for the right and left side it in our analysis.

3.3. Stability conditions

Under a periodic control rule, the setting of the time interval P is restricted by the need for clearing the lane every time the lane direction is changed. As a consequence, the minimum time interval P should exceed the maximum clearance time, i.e., the driving time along the lane T . However, it is

straightforward to define necessary and sufficient conditions for the system stability. We shall give these conditions for the symmetric as well as the asymmetric case.

For the symmetric case, it is easy to derive the stability condition by intuition. First, we note that the effective green period equals $P-T$ worst case, because the clearance time C equals the driving time along the lane T in case of heavy traffic. Second, we observe that the maximum number of vehicles than can be processed during the effective green period should exceed the total number of arrivals in a cycle. Hence we find that $(P - T) / \delta > 2\lambda P$ or, equivalently,

$$(1) \quad P > \frac{T}{1 - 2\lambda\delta}$$

The latter condition is formally derived by Meissl (1963).

This stability condition can easily be extended to the asymmetric case. Then it is necessary that, at both sides of the lane, the expected number of arrivals in a cycle with length $P_L + P_R$ can be served in the expected green time. This expected green time equals $P_L - T$ and $P_R - T$ at the left and right side of the lane, respectively. Therefore, the following two stability conditions apply:

$$(2) \quad \lambda_L(P_L + P_R) < \frac{1}{\delta}(P_L - T) \Leftrightarrow P_L(1 - \lambda_L\delta) > T + \lambda_L\delta P_R$$

$$(3) \quad \lambda_R(P_L + P_R) < \frac{1}{\delta}(P_R - T) \Leftrightarrow P_R(1 - \lambda_R\delta) > T + \lambda_R\delta P_L$$

These two equations define the feasible area in \mathbb{R}^2 for which the system is stable. As can be expected, the conditions (2) and (3) reduce to the stability condition for the symmetric case (1) if $\lambda_L = \lambda_R = \lambda$.

4. Waiting times if the succession time is negligible ($\delta=0$)

In this section, we shall derive expressions for the mean waiting time $E[W]$ for the symmetric case with minimum succession time $\delta=0$. We shall omit the subscripts R and L for ease of notation.

First we note that, conditional on the clearance time C , the waiting probability equals $(P+C)/2P$.

Second, we exploit the fact that the exponential interarrival distribution corresponding to Poisson arrivals has the well-known memoryless property. So if a vehicle has to wait, the mean waiting time equals $(P+C)/2$. Hence we have

$$(4) \quad E[W | C] = \frac{(P+C)^2}{4P} = \frac{P}{4} + \frac{C}{2} + \frac{C^2}{4P} \quad \Rightarrow \quad E[W] = \frac{P}{4} + \frac{E[C]}{2} + \frac{E[C^2]}{4P}$$

So it is sufficient to derive an expression for the first two moments of the clearing time C in order to compute $E(W)$.

In the sequel, we shall show how we can compute the probability distribution of the clearance time C exactly. Therefore, we can compute the first two moments of C exactly as well.

In the case $P > 2T$, the effective green time $P-C$ certainly exceeds T , because $C \leq T$. In that case, the clearance time is fully determined by the last arrival before the driving direction is changed. Then we find that

$$\begin{aligned} \Pr\{C = 0\} &= \Pr\{\text{no arrivals in } [0, T]\} = e^{-\lambda T} \\ \Pr\{0 < t < C < T\} &= \Pr\{\geq 1 \text{ arrivals in } [0, T-t]\} = 1 - e^{-\lambda(T-t)} \end{aligned}$$

Note that in this case the distribution is independent of P .

In the case $T < P \leq 2T$, the situation is more difficult. Nevertheless, we can exploit some obvious fundamental relation that C has to satisfy. Let I_R and I_L denote indicators whether at the start of a green

traffic light interval at the right and left end of the lane, respectively, vehicles are present (indicator=1) or not (indicator=0). Let Y be the length of the interval beginning when the traffic light becomes green and ending when the last vehicle during that green traffic period enters the lane. It is clear that

$$(5) \quad \begin{aligned} C_L &= C_R + Y + T - P \quad \text{if } I_L = 1 \quad \text{and } C_R + Y + T - P > 0 \\ C_L &= 0 \quad \text{else} \end{aligned}$$

Noting that $\Pr\{I_L = 0 | C_R = \tau\} = e^{-\lambda(P+\tau)}$, we find that

$$\begin{aligned} \Pr\{0 < t < C_L < T | C_R = \tau\} &= 1 - e^{-\lambda(T-t)} \quad \text{if } \max(0, \tau - \Delta) < t < T \\ \Pr\{0 < t < C_L < T | C_R = \tau\} &= (1 - e^{-\lambda(P+\tau)}) + (1 - e^{-\lambda(P-\tau)})e^{-\lambda(P+\tau)} = 1 - e^{-2\lambda P} \quad \text{if } T > \tau > t + \Delta \end{aligned}$$

Due to the symmetry in the arrival processes, C_R and C_L are identically distributed. Let us denote their probability density as $f(t)$ and let us further introduce the shorthand notation

$$G(t) \stackrel{\text{def}}{=} \Pr\{0 < t < C < T\} = \int_0^T \Pr\{0 < t < C_L < T | C_R = \tau\} f(\tau) d\tau$$

It should be noted that $f(t)$ has a point mass at $t=0$, hence the integral should be interpreted as including this effect. It is crucial that $G(t)$ satisfies a fundamental difference equation. Using the relation with the conditional probabilities given above, we find after integration with respect to τ :

$$\begin{aligned} G(t) &= (1 - e^{-\lambda(T-t)}) \left(1 - \int_{\min(t+\Delta, T)}^T f(\tau) d\tau\right) + (1 - e^{-2\lambda P}) \int_{\min(t+\Delta, T)}^T f(\tau) d\tau = \\ &= 1 - e^{-\lambda(T-t)} + (e^{-\lambda(T-t)} - e^{-2\lambda P}) G(t + \Delta) \end{aligned}$$

where $\Delta = P - T$. Of course, $G(t + \Delta)$ has to be interpreted as 0 if $t + \Delta > T$. The structure of this equation is such that it can easily be solved recursively: first $t \in [T - \Delta, T]$, next $t \in [T - 2\Delta, T - \Delta]$, etc.

This leads us to

$$(6) \quad G(t) = \begin{cases} 1 - e^{-\lambda(T-t)} & t \in [T - \Delta, T] \\ 1 - e^{-\lambda(T-t)} + (1 - e^{-\lambda(T-t-\Delta)})(e^{-\lambda(T-t)} - e^{-2\lambda P}) & t \in [T - 2\Delta, T - \Delta] \\ (1 - e^{-\lambda(T-t)}) + \sum_{k=1}^n (1 - e^{-\lambda(T-t-k\Delta)}) \prod_{l=0}^{k-1} (e^{-\lambda(T-t-l\Delta)} - e^{-2\lambda P}) & t \in [T - (n+1)\Delta, T - n\Delta] \end{cases}$$

Equation (6) can easily be proved using induction. This analysis leads us to sums of exponential functions for $G(t)$ and $f(t)$ for $t > 0$. Finally, this computation is completed by noticing that the clearance distribution has a point mass at $t=0$ such that its total mass equals 1. In this way, all moments of C can be computed in principle. Note that in general the probability on no arrivals in a cycle will be negligible, so $e^{-2\lambda P} \approx 0$. If we substitute this in the equations above, we can derive a simple and accurate approximation for the clearance time distribution:

$$(7) \quad G(t) \approx 1 - e^{-(n+1)\lambda(T-t-\frac{1}{2}n\Delta)} \quad \text{for } t \in [T - (n+1)\Delta, T - n\Delta]$$

It is easy to derive the first two moments of the clearance time from this expression. Defining the integer N as $\lfloor T/\Delta \rfloor$, the largest number smaller than or equal to T/Δ , we find:

$$E[C] \approx T - \int_0^{T-N\Delta} e^{-(N+1)\lambda(T-t-\frac{1}{2}N\Delta)} dt - \sum_{n=0}^{N-1} \int_{T-(n+1)\Delta}^{T-n\Delta} e^{-(n+1)\lambda(T-t-\frac{1}{2}n\Delta)} dt$$

After some manipulations, this equation can be reduced to

$$(8) \quad E[C] \approx T - \frac{e^{-(N+1)(T-\frac{1}{2}N\Delta)\lambda} - e^{-\frac{1}{2}(N+1)(N+2)\lambda\Delta}}{(N+1)\lambda} - \frac{1}{\lambda} + \frac{e^{-\frac{1}{2}N(N+1)\lambda\Delta}}{N\lambda} + \sum_{n=1}^{N-1} \frac{e^{-\frac{1}{2}(n+1)n\lambda\Delta}}{n(n+1)\lambda}$$

Analogously, we can derive for the second moment

$$(9) \quad E[C^2] \approx T^2 + 2 \frac{\left([1 - (N+1)(T - N\Delta)\lambda] e^{-\frac{1}{2}N(N+1)\lambda\Delta} - e^{-(N+1)(T - \frac{1}{2}N\Delta)\lambda} \right)}{(N+1)^2 \lambda^2} + 2 \sum_{n=0}^{N-1} \frac{\left([1 - (n+1)(T - n\Delta)\lambda] e^{-\frac{1}{2}n(n+1)\lambda\Delta} - [1 - (n+1)(T - (n+1)\Delta)\lambda] e^{-\frac{1}{2}(n+1)(n+2)\lambda\Delta} \right)}{(n+1)^2 \lambda^2}$$

It is straightforward to calculate the approximate values of $E[C]$ and $E[C^2]$ from the expressions above. Hence the most convenient way to compute the mean waiting time is to combine the equations (4), (8) and (9).

5. Numerical results for $\delta=0$

We shall use our equations as obtained in the previous section for some numerical analysis in order to get some insight in the optimal value of P . We choose $T=7$ minutes for all our numerical experiments, being a representative value that we encountered during our research on automated transportation networks as mentioned in the introduction. Figure 3 shows the average waiting time as a function of the switching period P for various vehicle arrival rates λ (all time units in minutes).

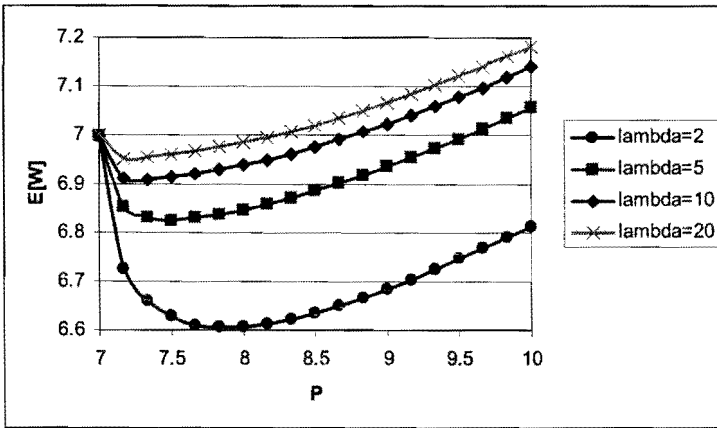


Figure 3. Expected waiting times for periodic control, $T=7$ minutes.

We see that the average waiting time only varies with a few percent around the optimum. As can be expected, the average waiting time increases with the traffic intensity. Another observation is that the optimum switching period P decreases (and so does the effective green time) as traffic intensity

increases. This is due to the assumption that a batch of vehicles can enter the lane in negligible time, irrespective of the batch size. If the driving distance between vehicles is restricted by some minimum value in order to prevent collision, we expect an opposite effect (this will be confirmed in Section 8). Therefore, the equations as derived in the previous section are *only* useful if the minimum driving distance between vehicles is negligible indeed (e.g. if a train of vehicles can be constructed by magnetic coupling of vehicles while waiting for the lane entrance).

6. Waiting times in the case of strictly positive succession time ($\delta > 0$)

For the case that mutual distances between vehicles are significant, we shall construct an approximation that is closely related to what we did in the previous case with $\delta = 0$.

6.1. Assumption and additional notation

We start from the following assumption:

The probability that a vehicle encounters more than two traffic light switches during its waiting time is negligible

Based on this assumption, we can use the following decomposition of the mean waiting time:

$$(10) \quad E[W] * E[A(cycle)] = \sum_{s=0}^2 E[K_s * W \mid S = s]$$

where

S = the number of traffic light switches during the AGV waiting time.

K_s = the number of vehicles in a cycle for which $S=s$ and $W > 0$

and where $E[A(cycle)] = \lambda(P_R + P_L)$ in the case of Poisson arrivals. Note that the assumption above is formalized as $\Pr\{S \geq 3\} = 0$.

Further we shall use the following auxiliary notation in this section:

M = the number of AGVs remaining in queue when the traffic light is switched to red

N = the number of AGVs in queue when the traffic light is switched to green

$Z(n)$ = the time until a queue with length n vanishes, if the traffic light is green and not switched to red (i.e. AGVs enter the tube at rate $1/\delta$ and additional AGVs arrive according to the process $A(t)$, being a Poisson process with rate λ)

$\lceil x \rceil$ = the smallest integer larger than or equal to x

x^+ = $\max\{x, 0\}$

For the asymmetric case, the parameter S and the random variables K_s , M , N and $Z(n)$ have a subscript R or L , referring to the side of the shared lane (right or left). We shall derive all equations for the asymmetric case and state which simplifications are possible for the symmetrical case when appropriate. The equations will be stated in terms of the waiting time at the left side, Obviously, similar expressions apply for the right side of the lane where the subscripts L and R are exchanged in all expressions.

We proceed as follows. First we derive expressions for the three components of equation (10),

$E[K_s * W \mid S = s]$ for $s=0, 1$ and 2 (subsection 6.2). We shall see that these expressions contain the first and/or second moment of the random variables M , N , $Z(n)$ and C . Therefore we shall derive expressions for these components in subsection 6.3. Next, we shall summarise our algorithm in subsection 6.4, combining the results from Sections 6.2 and 6.3.

6.2. Basic expression for the waiting time components

We first classify all vehicles arriving at the left side of the lane in a cycle with length $P_L + P_R$ as member of the sets having $S_L=0$, $S_L=1$ or $S_L=2$. In this way, we obtain approximate expressions for the numbers $K_{L,s}$ ($s=0,1,2$).

For vehicles arriving at the left side of the shared lane, a cycle with length P_L+P_R , consists of an (effective) red period with length P_R+C_R and a subsequent (effective) green period with length P_L-C_R . The mean number of vehicles arriving in this cycle equals $\lambda_L (P_L+P_R)$. The number of vehicles in queue at the start of the cycle (start of the red period) equals M_L by definition. All vehicles arriving in a green period have an even value of S_L (0 or 2, ignoring higher order terms) and all vehicles arriving in a red period have an odd value of S_L (1, ignoring higher order terms). The latter implies that

$$(11) \quad K_{L,1} \approx A_L (P_R + C_R)$$

The vehicles arriving during the green period that cannot be handled in the same green period have $S_L=2$. This number equals the number of vehicles in queue at the end of the first green period (M_L), excluding the vehicles that were already in queue at the end of the preceding red period and that could not be served in the first green period. Based on our assumption, we can ignore the latter number, as $S_L=3$ for those vehicles. Therefore, we find

$$(12) \quad K_{L,2} \approx M_L$$

The number of other vehicles that arrive in a green period, have to wait, but still can be handled in the same green period equals $K_{L,0} = \delta^{-1} * \text{Min}\{Z_L(N_L), P - C_R\} - N_L$. This stochastic variable is difficult to handle, because it requires the complete distribution of $Z_L(N_L)$. However, if the system is not close to its stability boundary, we have that $\text{Pr}\{Z_L(N_L) > P - C_R\} \approx 0$. Therefore, we use the following approximation for $K_{L,0}$:

$$(13) \quad K_{L,0} \approx \frac{Z_L(N_L)}{\delta} - N_L$$

We shall use (11) - (13) to derive expressions for the waiting time components $E[K_{L,s} * W_L | S_L = s]$ ($s=0, 1, 2$).

$S_L=1$.

The average waiting time of these vehicles consists of the following two components:

- the waiting time during the red period, being $\frac{1}{2}(P_R + C_R)$ on average;
- the time until the vehicle can enter the lane after the light has switched to red; the number of vehicles waiting at the start of a green period equals $(M_L + K_{L,1})$; as they enter the tube at rate δ , this average additional delay is $M_L \delta + \frac{1}{2}(K_{L,1} - 1)\delta$; this formula covers the fact that all $K_{L,1}$ vehicles have to wait until the first M_L vehicles enter the tube, and the fact that the n^{th} vehicle enters the tube at time $(n-1)\delta$ after the traffic light turns to green.

So we have that $E[K_{L,1} * (W_L | S_L = 1)] = E[\frac{1}{2}K_{L,1}(P_R + C_R + K_{L,1} - 1)\delta + K_{L,1}M_L \delta]$. Noting that M_L and $K_{L,1}$ are independent, but $K_{L,1}$ and C_R are not, we can write this equation as

$$(14) \quad E[K_{L,1} * (W_L | S_L = 1)] = \frac{1}{2}E[K_{L,1}C_R] + E[K_{L,1}]\{\frac{1}{2}P_R + E[M_L]\delta - \frac{1}{2}\delta\} + \frac{1}{2}\delta E[K_{L,1}^2]$$

Note that C_R refers to clearance time as part of the red period after the queue with length M_L has been created, so K_L and M_L are mutually independent. To evaluate (14), we need:

$$(15) \quad E[K_1] \approx E[A_L(P_R + E[C_R])] = \lambda_L(P_R + E[C_R])$$

$$(16) \quad Var[K_1] \approx Var[A_L(P_R + C_R)] = \lambda_L(P_R + E[C_R]) + \lambda_L^2 Var[C_R]$$

$$(17) \quad E[K_1 C_R] \approx E[C_R A_L(P_R + C_R)] = E[\lambda_L C_R(P_R + C_R)] = \lambda_L P_R E[C_R] + \lambda_L E[C_R^2]$$

Here we used that for any correlated pair of random variables X and Y the well-known conditioning formulas $E[X] = E\{E[X | Y]\}$ and $Var[X] = E\{Var[X | Y]\} + Var\{E[X | Y]\}$ apply.

$S_L=2$.

The waiting time of the $K_{L,2} \approx M_L$ vehicles remaining at the end of a green period consists of the following components:

- the length of the next red period P_R+C_R .
- the average time until the vehicles enter the lane when the light is switched to green, $\frac{1}{2}(M_L - 1)\delta$.
- the average time that the vehicles have been waiting when the light is switched to red, $\frac{1}{2}M_L / \lambda_L$

Analogously to the derivation of (14) and using that M_L and C_R are independent, we arrive at

$$(18) \quad E[K_{L,2} * (W_L | S_L = 2)] \approx E[M_L](P_R + E[C_R] - \frac{1}{2}\delta) + \frac{1}{2}E[M_L^2](\lambda_L^{-1} + \delta)$$

$S_L=0$.

These $K_{L,0}$ vehicles arrive during a green period when the queue that is present at the start of the green period (with length N_L) has not yet vanished. As noted before, this period has length $Z_L(N_L)$. The corresponding waiting time consists of the following components:

- the average time until the vehicles can enter the lane, calculated from the moment at which the light is switched to green, $N_L\delta + \frac{1}{2}(K_{L,0} - 1)\delta$.
- *minus* the average time that the vehicles arrive after the traffic light is switched to green; using that the vehicles arrive during the period $Z_L(N_L)$ and using equation (13), we find that this correction term equals $\frac{1}{2}Z_L(N_L) = \frac{1}{2}(K_{L,0} + N_L)\delta$

So we find after some algebraic manipulations:

$$(19) \quad E[K_{L,0} * (W_L | S_L = 0)] \approx \frac{1}{2}\{E[K_{L,0}N_L] - E[K_{L,0}]\}\delta$$

where

$$(20) \quad E[K_0] \approx \frac{E[Z_L(N_L)]}{\delta} - E[N_L]$$

and

$$(21) \quad E[K_0 N_L] \approx \frac{1}{\delta} E\{N_L Z_L(N_L)\} - E[N_L^2]$$

6.3. The auxiliary random variables M , N , $Z(N)$ and C

The equations (14) - (21) together with (10) give an approximation for the mean waiting time, but we observe that we need the following characteristics of the random variables M , N , $Z(N)$ and C :

- the first two moments of the number of vehicles remaining in queue at the end of a green period M_L for the equations (14) and (18),
- the first two moments of the number of vehicles in queue at the start of a green period N_L for the equations (20) and (21),
- the first two moments of the clearance time C_R for the equations (15) - (17),
- the expectation of the time until the queue at the start of a green period has vanished, $Z_L(N_L)$, for equation (20),
- the expectation of the product of $Z_L(N_L)$ and N_L for equation (21).

We shall derive formulas for these characteristics below.

6.3.1. The queue lengths M_L and N_L .

First, we observe that the following relations between M , N and C exist:

$$(22) \quad N_L = M_L + A_L(P_R + C_R)$$

$$(23) \quad M_L = [N_L + A_L(P_L - C_R) - [(P_L - C_R)/\delta]]^+ = [M_L + A_L(P_L + P_R) - [(P_L - C_R)/\delta]]^+$$

where the random variables M_L at the right and left hand side refer to two subsequent cycles.

In principle, equation (23) can be solved analytically to obtain the distribution of M_L , conditional on C_R .

Trying to obtain the unconditional distribution in this way leads to an unattractive analysis. Therefore we choose for a simpler approach, namely the moment iteration method as developed by De Kok

[1989] for the mean waiting time in the $G/G/1$ queue. This method works as follows:

1. Initialisation: assume that $M_L=0$.
2. Calculate the first two moments of the random part in the right hand side of (23),

$$X = M_L + A_L(P_L + P_R) + C_R / \delta$$
; note that this random variable has some unknown discrete probability distribution on $\{0, 1, 2, \dots\}$
3. Choose a simple discrete probability distribution function that has the same first two moments as the random variable X . For this step, we applied the procedure as developed by Adan et al [1996].
4. Use these simple probability distributions to approximate the first two moments of

$$M_L = (X - \lceil P_R / \delta \rceil)^+$$
5. Repeat the steps 2-4 until convergence is reached.

Note that we modified the approach by the De Kok [1989] to our model. Also, we used a procedure to fit *discrete* distributions on the first two moments of a discrete random variable rather than the continuous equivalent as De Kok does. If we have computed the first two moments of M_L in this way, the calculation of the first two moments of N_L is straightforward. From (22) and using (15) and (16), it is immediately clear that

$$(24) \quad E[N_L] = E[M_L] + E[A_L(P_R + C_R)] = E[M_L] + \lambda_L(P_R + C_R)$$

$$(25) \quad Var[N_L] = Var[M_L] + Var[A_L(P_R + C_R)] = Var[M_L] + \lambda_L(P_R + E[C_R]) + \lambda_L^2 Var[C_R]$$

The iterative procedure above can be followed, but only if we know the first two moments of the clearance time C_R .

6.3.2. The clearance time C_R .

In principle, equations for the distribution of C_R can be derived proceeding from equation (5). For the case $\delta > 0$, this leads to an intricate analysis. Then, we have to make several approximations to obtain an explicit solution anyway. It is interesting to observe that the distribution of C_R has a point mass in T

with strength $Pr\{C_R=T\} = Pr\{M_R>0\}$. The distribution for $0<t<T$ is considerably more complicated, however. Therefore, we choose for a much simpler approach.

We note that the probability distribution of C_R is simple for the limiting cases $P_R \rightarrow \infty$ and $P_R \downarrow P_{R,min}$, where $P_{R,min}$ represents the lower bound of P_R according to (1). If $P_R \rightarrow \infty$, the probability that the queue has vanished before the traffic light is switched to red goes to zero. Hence the limiting behaviour of clearance time is specified by $C_R = T - B_R$, where B_R refers to the timing of the last arrival before the traffic light is switched to red, so B_R is exponentially distributed with mean $1/\lambda_R$. Therefore, we find for the mean and standard deviation σ of the clearance time:

$$(26) \quad \lim_{P_R \rightarrow \infty} E[C_R] = T - \lambda_R^{-1} \quad \text{and} \quad \lim_{P_R \rightarrow \infty} \sigma[C_R] = \lambda_R^{-1}$$

On the other hand, the clearance time approached the driving time along the lane T under high utilisation, so

$$(27) \quad \lim_{P_R \downarrow P_{R,min}} E[C_R] = T \quad \text{and} \quad \lim_{P_R \downarrow P_{R,min}} \sigma[C_R] = 0$$

Now the idea is that we can use for all other values of P_R a simple weighed average of these limiting values. As weighing factor, we choose $Pr\{M_R = 0\}$, because it is clear that (27) is correct if $M_R>0$ and (26) will be a rather good approximation if the queue has vanished at the end of a green period ($M_R=0$). So we use the approximations

$$(28) \quad E[C_R] = T - \lambda_R^{-1} * Pr\{M_R = 0\} \quad \text{and} \quad \sigma[C_R] = \lambda_R^{-1} Pr\{M_R = 0\}$$

As an alternative to this approximation, we may consider to use a weighed average for the variance instead of the standard deviation. Also, we may weigh the complete probability distributions instead of

just the first and second moment. We have tested both alternatives numerically and found that the approach as stated above performs slightly better. Therefore, we choose to use the approximations (28).

In principle, we can approximate $\Pr\{M_R = 0\}$ during the iterative procedure for the calculation of the first two moments of M_R as described above. Because this moment-iteration method for $M_L (M_R)$ depends on the first two moments of the clearance time $C_R (C_L)$, we should include the approximation of $C_R (C_L)$ according to (28) in the iterative procedure. That is, after each iteration, we update $\Pr\{M_L = 0\}$ and also $E[C_R]$ and $\sigma[C_R]$ according to (28). In the asymmetric case, we have to perform the iterative procedure in parallel: During each iteration, we subsequently approximate M_L , update C_L , approximate M_R and update C_R .

6.3.3. The time $Z_L(N_L)$ until the queue vanishes.

For $Z_L(N_L)$, it is clear that the time until an initial queue with length N_L vanishes equals the time to process all vehicles in queue at the start of a green period ($N_L \delta$) plus the time required to serve all vehicles arriving while solving the queue ($A_L[Z_L(N_L)]\delta$). So we find the following equation:

$$(29) \quad Z_L(N_L) = N_L \delta + A_L[Z_L(N_L)]\delta$$

It is not straightforward to derive the distribution of $Z_L(N_L)$ from equation (29), but we can easily find the mean (and variance, if we wish to), conditionally on N_L . In case of Poisson arrivals, we find

$$(30) \quad E[Z(N_L) | N_L] = N_L \delta + \delta \lambda_L E[Z_L(N_L) | N_L] \Rightarrow E[Z(N_L) | N_L] = \frac{N_L \delta}{1 - \lambda_L \delta}$$

From this equation, we can derive the unconditional mean of $Z_L(N_L)$ as

$$(31) \quad E[Z_L(N_L)] = \frac{E[N_L]\delta}{1 - \lambda_L\delta}$$

Finally, we find for the cross-product of $Z_L(N_L)$ and N_L :

$$(32) \quad E[N_L Z_L(N_L)] = E[E[N_L Z_L(N_L) | N_L]] = \frac{E[N_L^2]\delta}{1 - \lambda_L\delta}$$

Now that we have formulas for all variables required to approximate the mean waiting time, we can summarise our algorithm.

6.4. Computation of the mean waiting time: algorithm

Now that we have distinguished between queuing behaviour at the right side and at the left side, handling both the symmetric and asymmetric case is easy. In fact, we can analyse both queues separately to a large extent. The dependencies between the queues are completely specified by the clearance times C_L and C_R . So in the symmetric case, we have a single moment iteration procedure for C and M , whereas we have a joint iteration procedure for M_L , C_L , M_R and C_R in the asymmetric case (see Subsection 6.3.2).

Now we have to take the following steps to arrive at our approximation for the mean waiting time:

1. Jointly approximate the first two moments of M_L , C_L , M_R and C_R (only M and C in the symmetric case) using the moment-iteration method as described in section 6.3.2.
2. Calculate the first component of the waiting time at the left side $E[K_{L,0} * (W_L | S_L = 0)]$ from equation (19) using (20) and (21), and based on the auxiliary expressions for:
 - the first two moments of N_L as specified by (24) and (25),
 - the mean of $Z_L(N_L)$ as specified by (31)
 - the expectation of $N_L Z_L(N_L)$ as specified by (32)

3. Calculate the second component of the waiting time at the left side $E[K_{L,1} * (W_L | S_L = 1)]$ from equation (14) using (15), (16) and (17), where the first two moments of C_R and the mean of M_L as obtained in Step 1 are substituted.
4. Calculate the third component of the waiting time at the left side $E[K_{L,2} * (W_L | S_L = 2)]$ from equation (18), where the first two moments of M_L (see Step 1) and the mean of C_R are substituted.
5. Use the three components to approximate the mean waiting time at the left side of the shared lane according to (10), i.e. add the three terms and divide the sum by $\lambda_L (P_R + P_L)$.
6. Repeat Step 2-5 for the right side, where all indices L and R are exchanged.

In the next section, we discuss the results from the numerical tests for this algorithm.

7. Numerical experiments for $\delta > 0$

In order to verify the accuracy of our approximations, we constructed a discrete-event simulation model to run a range of numerical experiments. We compare the simulation results to the approximate values. We present the results in this section, first for the symmetric case (7.1) and next for the asymmetric case (7.2). We designed our simulation experiments such, that the relative width of the 95% confidence interval for the average waiting time is less than 1% of the estimated value for $E[W]$.

7.1. Symmetric case

We use the algorithm as summarised in Section 6.4 to calculate the mean waiting time as function of the switching period P for various vehicle arrival rates λ (all time units in minutes). In all cases, the driving time along the lane equals $T=7$ minutes and the minimum driving time between vehicles equals $\delta=3\frac{1}{2}$ seconds, which are values taken from our research on automated transportation networks (see introduction). Figure 4 shows a comparison between approximation and simulation results. The figure reveals that the approximation is accurate, except when the system operates close to its stability bounds.

This is due to the approximation of the number of items remaining in queue at the end of a green period, M . There can be a significant approximation error when the system is running close to its stability bound. As waiting times are high anyway for these cases, this is not a limitation of our method from a practical point of view.

Also, we observe that the approximation quality is less for $\lambda=1$. However, we can see that this is a rather exceptional case. Close to the stability bound ($P=8$), we have that the effective red time equals $P+T=15$ and $P-T=1$, whereas only one vehicle arrives on average during the effective green time ($\lambda=1$). Hence the ratio between effective red time and effective green time is very high, whereas traffic intensity is low. Therefore, it does not seem to be a very realistic case. Nevertheless, we conclude that the location of the minimum is estimated very well in all cases, also for $\lambda=1$.

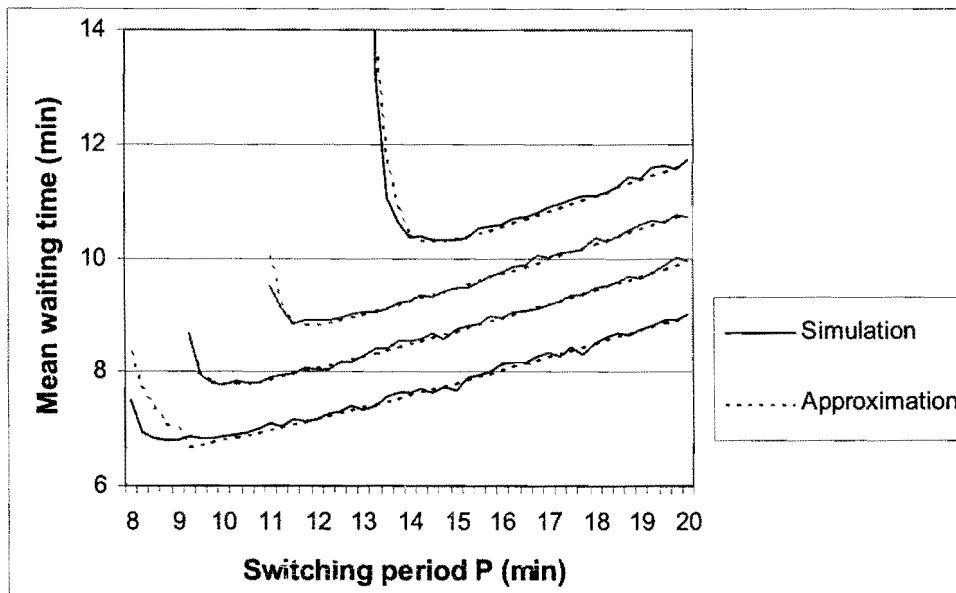


Figure 4. Mean waiting time as function of the switching period P , analytical versus simulation results ($\lambda=1,2,3,4$; waiting times increase with λ)

As can be expected, the average waiting time increases with the traffic intensity. Also we see that the optimal switching period P increases (and so does the green time) as traffic intensity increases. This is due to the fact that more green time is required to allow the vehicles waiting to enter the shared lane.

The latter observation is further clarified in Figure 5, where the optimal switching period P and the corresponding mean waiting time $E[W]$ are shown as function of the traffic intensity λ .

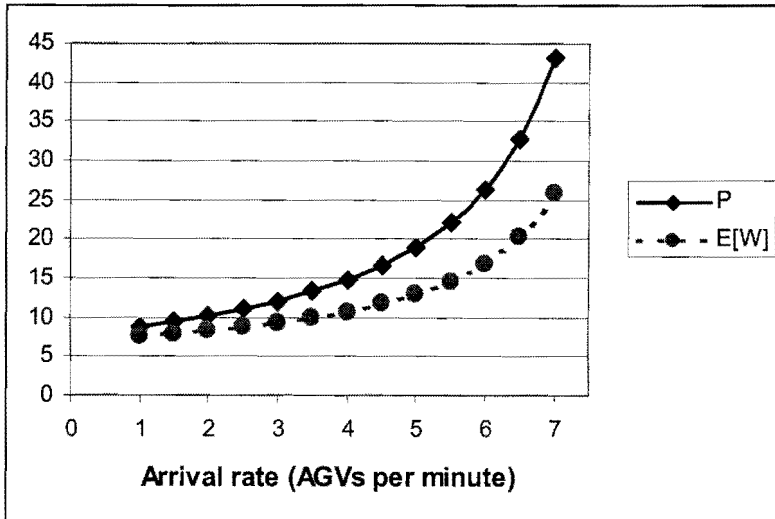


Figure 5. Optimal switching interval P and mean waiting time $E[W]$ in minutes as function of the arrival rate λ ($T=7$)

Another interesting issue is the capacity of the shared lane. From equation (1), we obtain an upper bound for λ given a fixed value of P :

$$(33) \quad \lambda < \frac{1 - T/P}{2\delta}$$

As we could expect, we note that the capacity equals $\lambda = 1/(2\delta)$ vehicles per time unit for each side, but unfortunately this level is reached if $P \rightarrow \infty$, and then also $E[W] \rightarrow \infty$. Therefore, a more realistic characteristic is the capacity of the shared lane *given an upper bound on the mean waiting time*. Using the algorithm from Section 6.4, we conduct a numerical grid search over P and λ to find the maximum value of λ satisfying the mean waiting time requirement. The results are shown in Figure 6, where the capacity of the shared lane is expressed in λ , the number of vehicles to be processed per time unit at each side of the lane. We see that the capacity of the lane, given the maximum mean waiting time, seriously decreases with increasing lane length, especially if only limited waiting time is accepted.

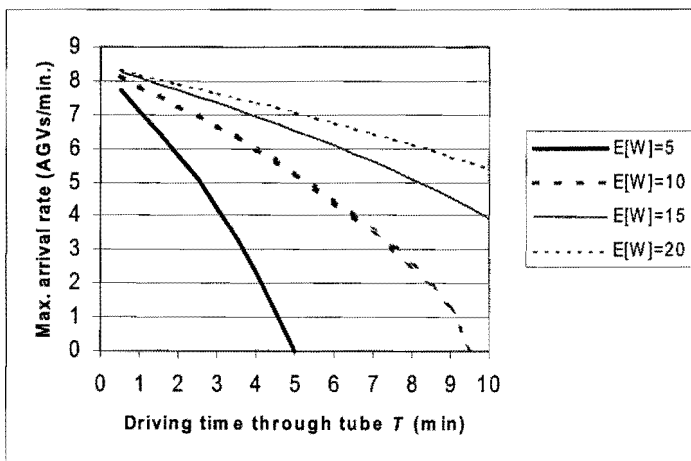


Figure 6. Tube capacity (number of vehicles per time limit per side of the lane), given a maximum mean waiting time

7.2. Asymmetric case

We can also use the algorithm from Section 6.4 to calculate the mean waiting time at each side of the shared lane as function of the two switching periods P_L and P_R . As an example, we show in Figure 7 the weighted mean waiting time $E[W] = (\lambda_R E[W_R] + \lambda_L E[W_L]) / (\lambda_R + \lambda_L)$, based on the input data $T=7$ minutes, $\delta=3\frac{1}{2}$ seconds, $\lambda_R = 2$ per minute and $\lambda_L = 3$ per minute. This figure is a so-called contour plot, i.e. it shows areas with similar values of the mean waiting time as indicated by the legend. It is clear that the weighted mean waiting time increases faster with P_R than with P_L , because of the higher arrival rate at the left side of the shared lane. The optimum is found around $(P_L, P_R) = (11.75, 10.0)$, where the weighted mean waiting time equals $E[W] \approx 8.2$

To give an indication of the accuracy of our approximation, we make a comparison to simulation results. The relative deviation between simulated and approximated values as function of P_L and P_R are shown in Figure 8. The relative error is less than 1% for most of the cases. The error is somewhat larger close to the stability bounds, but this region is less interesting from a practical point of view as pointed out in Section 7.1. The location of the minimum is slightly different for the simulated values: $(P_L, P_R) = (11.25, 10.0)$. In our opinion, this deviation is within reasonable limits, taking into account statistical fluctuations arising from the discrete event simulation.

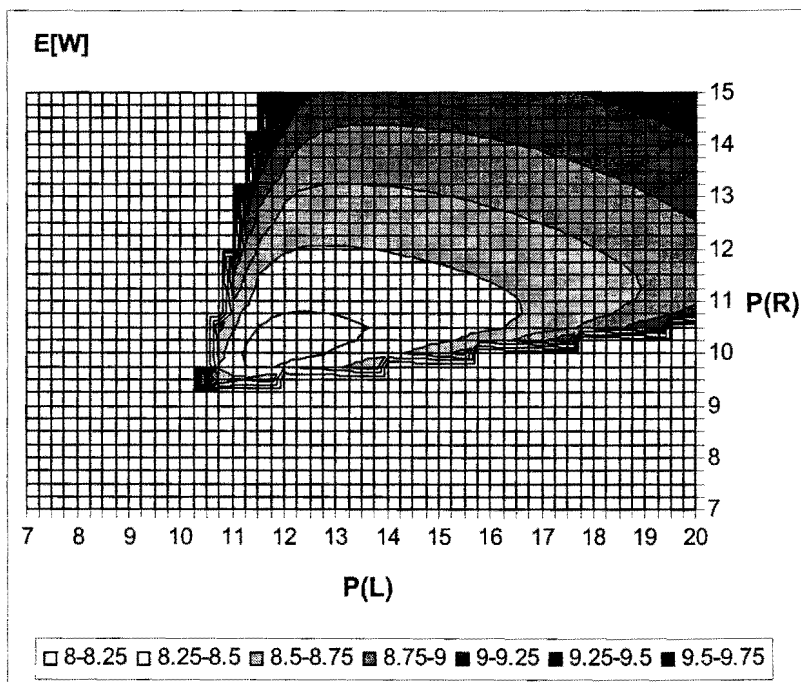


Figure 7. Approximation for the mean waiting time (weighted average of right and left side) as function of the switching periods P_R and P_L .

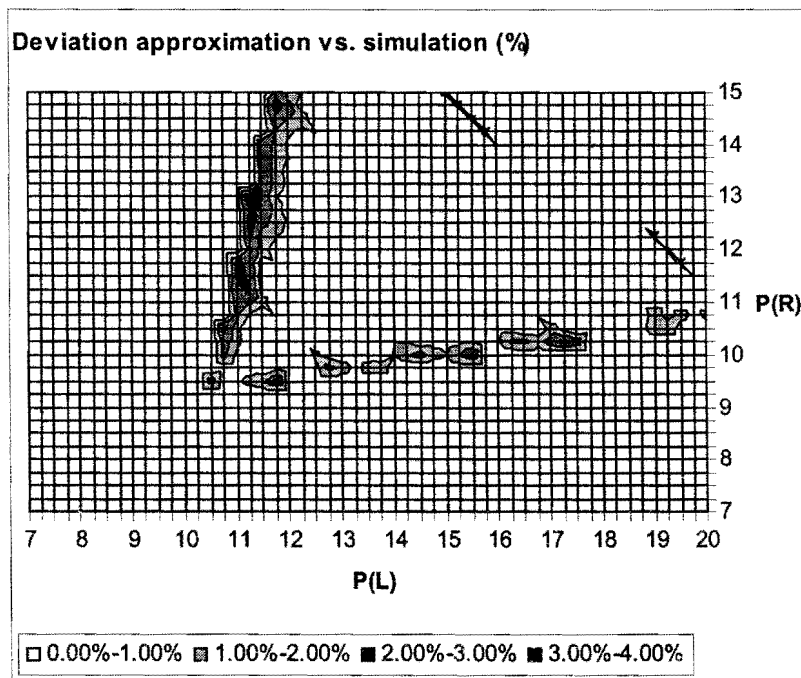


Figure 8. Relative deviation between approximation and simulation results

8. Generalisation to other arrival processes.

In principle, it is no problem to generalise our method to compound Poisson arrivals. In fact, the arrival process has impact on only a few characteristics to be discussed in this section. If we modify these

characteristics to account for other arrival processes and plug these formula in the expressions at the appropriate places, the algorithm still applies. We shall show that this can easily be accomplished for *compound* Poisson arrivals. That is, batches of vehicles arrive at the left (right) side of the shared lane according to a Poisson process with rate μ_L (μ_R), where the batch size has some discrete probability distribution function D_L (D_R) with known mean and variance. Then it is straightforward to derive that

$$(34) \quad E[A_L(P_L + P_R)] = \mu_L E[D_L](P_L + P_R)$$

$$(35) \quad E[A_L(P_R + C_R)] = \mu_L E[D_L](P_L + E[C_R])$$

$$(36) \quad Var[A_L(P_R + C_R)] = \mu_L E[D_L^2](P_L + E[C_R]) + \mu_L^2 E^2[D_L] Var[C_R]$$

$$(37) \quad E[C_R A_L(P_R + C_R)] = \mu_L E[D_L](P_L E[C_R] + E[C_R^2])$$

Further, we find from equation (29) that

$$(38) \quad E[Z_L(N_L)] = \frac{E[N_L]\delta}{1 - \mu_L E[D_L]\delta}$$

$$(39) \quad E[N_L Z_L(N_L)] = \frac{E[N_L^2]\delta}{1 - \mu_L E[D_L]\delta}$$

Using the equations (34) - (39), we can still use our algorithm. When considering further generalisation to (e.g.) compound renewal arrivals, the expressions become more complicated, because the memoryless property of the exponential interarrival distribution is lost. Still we believe that it is possible to generate approximations for the expressions as mentioned above, but the quality of the approximation has to be tested. This is a subject for further research.

9. Conclusions

In this paper we derived an algorithm for the mean waiting time in a periodically switched shared traffic lane. We showed that the algorithm works well by comparison to simulation, both for the symmetric ($\lambda_R = \lambda_L$ and $P_R = P_L$) and the asymmetric case. For the most relevant cases, the approximation error is less

than 1%. Besides, the location of the optimum (minimum mean waiting time as function of P_L and P_R) is estimated well. Although we developed our method for Poisson arrivals, we showed that extension to compound Poisson processes is straightforward. Further generalisation to compound renewal arrivals requires additional approximations and is a subject for further research.

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