

Linear Programming Error Bounds for Random Walks in the Quarter-plane

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Abstract

We consider approximation of the performance of random walks in the quarter-plane. The approximation is in terms of a random walk with a product-form stationary distribution, which is obtained by perturbing the transition probabilities along the boundaries of the state space. A Markov reward approach is used to bound the approximation error. The main contribution of the work is the formulation of a linear program that provides the approximation error.

1 Introduction

We consider random walks in the quarter-plane, *i.e.*, discrete-time Markov processes on state space $S = \{0, 1, \dots\}^2$. The random walks are homogeneous in the sense that within the interior of the state space, $\{1, 2, \dots\}^2$, the transition probabilities are translation invariant. In both axes and in the origin of the state space — *i.e.*, in $\{1, 2, \dots\} \times \{0\}$, $\{0\} \times \{1, 2, \dots\}$ and $\{(0, 0)\}$ — the transition probabilities are possibly distinct, but again translation invariant. Our interest is in steady-state behavior. More precisely, for a random walk with stationary distribution $\pi : S \rightarrow [0, \infty)$, our interest is in

$$F = \sum_{n \in S} f(n)\pi(n), \quad (1)$$

for some performance measure $f : S \rightarrow [0, \infty)$.

While it is possible to obtain closed form expressions for F in special cases, *e.g.*, for random walks with a product-form stationary distribution, no methods exist that provide such results for arbitrary random walks. There are some methods to find expressions for the generating functions of π , *cf.* [1,2]. However, these expressions can, in general, not be used for a straightforward calculation of F . In addition, these methods can not be straightforwardly applied. More precisely, they require a careful analysis of the the model and an adjustment of the method based on, *e.g.*, the transition probabilities.

In this work we focus on approximating F , *i.e.*, to find upper and lower bounds on F . Our method is based on the Markov reward approach to error

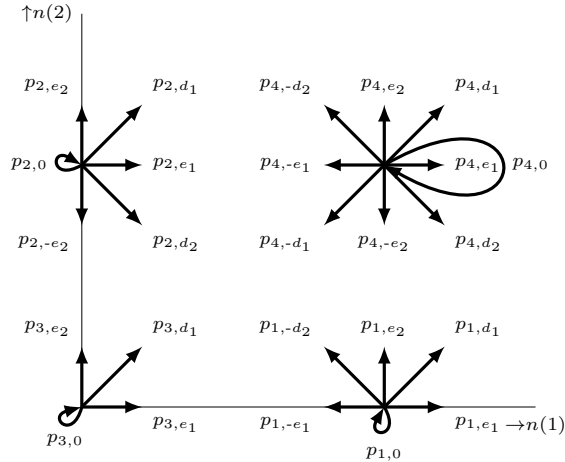


Figure 1: Random walk in the quarter-plane.

approximation as developed by van Dijk [3,4]. An introduction to this method is given in [5]. The main contribution of the current work is the formulation of linear program that applies the Markov reward approach and provides upper and lower bounds on F . The linear program accepts any random walk as an input, *i.e.*, no adjustment based on model parameters is required.

The remainder of this paper is organized as follows. In Section 2 we provide an exact statement of our model and the problem formulation. The main result is presented in Section 3. Concrete examples of random walks and an application of the results are provided in Section 4. Proofs of the results are given in Section 5.

2 Model, problem statement and notation

We consider two random walks: R and \bar{R} . Our interest is in the steady-state performance of R . However, the stationary distribution of R is unknown. Therefore, the performance of R will be approximated in terms of the stationary-distribution of \bar{R} , which is assumed to be a product-form geometric distribution.

The state space, S , of R and \bar{R} is the quarter plane, *i.e.*,

$$S = \{0, 1, \dots\} \times \{0, 1, \dots\}. \quad (2)$$

A state is represented by a pair of coordinates, *i.e.*, for $n \in S$, $n = (n(1), n(2))$. We consider a partition of S into four components: $S_1 = \{1, 2, \dots\} \times \{0\}$, $S_2 = \{0\} \times \{1, 2, \dots\}$, $S_3 = \{(0, 0)\}$ and $S_4 = \{1, 2, \dots\} \times \{1, 2, \dots\}$. We refer to these components as the horizontal axis, the vertical axis, the origin and the interior respectively. Let $k(n)$ denote the component of state $n \in S$, *i.e.*, $n \in S_{k(n)}$. We denote by N_k the nearest neighbours of a state in S_k , *i.e.*, $N_1 = \{-1, 0, 1\} \times \{0, 1\}$, $N_2 = \{0, 1\} \times \{-1, 0, 1\}$, $N_3 = \{0, 1\} \times \{0, 1\}$ and $N_4 = \{-1, 0, 1\} \times \{-1, 0, 1\}$. Also, let $N = N_4$. For notational convenience we let $e_1 = (1, 0)$, $e_2 = (0, 1)$, $d_1 = (1, 1)$ and $d_2 = (1, -1)$.

The random walks are discrete-time Markov processes, the transition probabilities of which are homogeneous in the sense that they are translation invariant

in each of the components. Transitions are to nearest neighbours only. Let $p_{k,u}$ denote the probability of R jumping from any state n in component S_k to $n+u$, where $u \in N_k$. Let $\bar{p}_{k,u}$ denote the corresponding probability for \bar{R} . For notational convenience let

$$q_{k,u} = \bar{p}_{k,u} - p_{k,u}. \quad (3)$$

We assume that the transition probabilities of \bar{R} and R are different only along the boundaries of the state space, *i.e.*, we assume that $q_{k,u} = 0$ unless

$$\begin{aligned} k = 1, u = -e_1, & & k = 1, u = e_1, \\ k = 2, u = -e_2, & & k = 2, u = e_2, \\ k = 3, u = e_1, & & k = 3, u = e_2, \end{aligned} \quad (4)$$

The stationary probability distribution of random walk \bar{R} is the distribution $\bar{\pi} : S \rightarrow [0, \infty)$ that satisfies

$$\bar{\pi}(n) = \sum_{m \in S} \sum_{\substack{u \in N_{k(m)}: \\ m+u=n}} \bar{p}_{k(m),u} \bar{\pi}(m), \quad (5)$$

for all $n \in S$. We assume that $\bar{\pi}$ is a product-form geometric distribution, *i.e.*,

$$\bar{\pi}(n) = (1 - r_1)r_1^{n_1}(1 - r_2)r_2^{n_2}, \quad (6)$$

for some $r_1, r_2 \in (0, 1)$. Our goal is to approximate steady-state performance of R in terms of in \bar{R} and $\bar{\pi}$. Let $\pi : S \rightarrow [0, \infty)$ denote the stationary distribution of R . It is assumed unknown, but used below to define the problem statement.

We will be making use of functions that are linear in each of the components of the state space. The performance measure of interest is

$$F = \sum_{n \in S} \pi(n)f(n), \quad (7)$$

where $f : S \rightarrow [0, \infty)$ is a function that is linear in each of the components of the state space, *i.e.*,

$$f(n) = \begin{cases} f_{1,0}, & \text{if } n \in S_1, \\ f_{2,0} + f_{2,1}n(1), & \text{if } n \in S_2, \\ f_{3,0} + f_{3,2}n(2), & \text{if } n \in S_3, \\ f_{4,0} + f_{4,1}n(1) + f_{4,2}n(2), & \text{if } n \in S_4, \end{cases} \quad (8)$$

where $f_{k,i}$ are the constants that define the function. We refer to functions that are linear in each of the components of the state space as componentwise linear or as S -linear. In the remainder we will use the notation

$$f(n) = f_{k(n),0} + f_{k(n),1}n(1) + f_{k(n),2}n(2). \quad (9)$$

In Section 4 we provide some examples of performance measures that can be captured by componentwise linear functions.

We introduce a final piece of notation. For a constant c , let

$$\begin{aligned} c^+ &= \max\{0, c\}, \\ c^- &= -\min\{0, c\}. \end{aligned} \quad (10)$$

3 Result

Our result builds on the Markov reward approach for error bounds as developed in, for instance [3] and [4]. An introduction to this technique is provided in [5]. The gist of the approach is to interpret f as a reward function, where $f(n)$ is the one-step reward if the random walk is in state n . We denote by $F^t(n)$ the expected cumulative reward at time t if the random walk starts from state n at time 0, *i.e.*,

$$F^t(n) = \begin{cases} 0, & \text{if } t = 0, \\ f(n) + \sum_{u \in N_{k(n)}} p_{k(n),u} F^{t-1}(n+u), & \text{if } t > 0. \end{cases} \quad (11)$$

Terms of the form $F^t(n+u) - F^t(n)$ play a crucial role in the Markov reward approach and are denoted as *bias terms*. Let $D_u^t(n) = F^t(n+u) - F^t(n)$. For the special cases $D_{e_1}^t(n)$ and $D_{e_2}^t(n)$ we introduce

$$D_1^t(n) = D_{e_1}^t(n) = F^t(n+e_1) - F^t(n), \quad (12)$$

$$D_2^t(n) = D_{e_2}^t(n) = F^t(n+e_2) - F^t(n). \quad (13)$$

The next results appears in, *e.g.*, [5], and provides a bound on the approximation error on F . In the remainder of the paper we will develop a linear programming approach to finding the approximation error.

Theorem 1 ([5]). *Let $\bar{f} : S \rightarrow [0, \infty)$ and $\Gamma : S \rightarrow [0, \infty)$ satisfy*

$$\left| \bar{f}(n) - f(n) + \sum_{u \in N_{k(n)}} q_{k(n),u} D_u^t(n) \right| \leq \Gamma(n)$$

for all $n \in S$ and $t \geq 0$. Then

$$\sum_{n \in S} [\bar{f}(n) - \Gamma(n)] \bar{\pi}(n) \leq F \leq \sum_{n \in S} [\bar{f}(n) + \Gamma(n)] \bar{\pi}(n).$$

The usual way to derive an error bound, *i.e.*, to find functions \bar{f} and Γ is by using an inductive proof over t . We will also use an inductive approach. The next result is of crucial importance.

Theorem 2. *There exist constants $g_{i,k,j,u}$, $i, j = 1, 2$, $k = 1, \dots, 4$, $u \in N_k$ that satisfy*

$$D_i^{t+1}(n) = c_{i,k(n)} + \sum_{j=1,2} \sum_{u \in N_{k(n)}} g_{i,k(n),j,u} D_j^t(n+u), \quad (14)$$

with

$$c_{i,k} = \begin{cases} f_{4,0} - f_{i,0} + f_{4,i}, & \text{if } k = i, \\ f_{i,0} - f_{3,0} + f_{i,i}, & \text{if } k = 3, \\ f_{4,i}, & \text{if } k = 4, \\ f_{i,i}, & \text{otherwise,} \end{cases} \quad (15)$$

for $i = 1, 2$, $n \in S$ and $t \geq 0$.

In Section 5 we provide a constructive proof of the above theorem. As part of the proof we give generic expressions for the constants $g_{i,j,k,u}$ that are valid for any random walk R . In the remainder of the paper we assume that constants $g_{i,k,j,u}$, $i, j = 1, 2$, $k = 1, \dots, 4$, $u \in N$, satisfying (14) are given. To illustrate notation and to demonstrate the expressions obtained in Section 5 we consider next the examples from Section 4.

The next theorem, the proof of which is given in Section 5, provides the main contribution of the current work.

Theorem 3. *Let the transition probabilities of \bar{R} and R be different only along the boundaries of the state space. Consider functions $\bar{f} : S \rightarrow \mathbb{R}$, $\Gamma : S \rightarrow \mathbb{R}$, $A_i : S \rightarrow \mathbb{R}$ and $B_i : S \rightarrow \mathbb{R}$, $i = 1, 2$. If*

$$\bar{f}(n) \geq 0, \quad A_1(n) \leq 0, \quad A_2(n) \leq 0, \quad B_1(n) \geq 0, \quad B_2(n) \geq 0, \quad (16)$$

for all $n \in S$, and

$$\begin{aligned} \bar{f}(n_i) - f(n_i) + q_{i,e_i}^+ B_i(n_i) - q_{i,e_i}^- A_i(n_i) \\ + q_{i,-e_i}^- B_i(n_i - e_i) - q_{i,-e_i}^+ A_i(n_i - e_i) \leq \Gamma(n_i), \end{aligned} \quad (17)$$

$$\begin{aligned} f(n_i) - \bar{f}(n_i) + q_{i,e_i}^- B_i(n_i) - q_{i,e_i}^+ A_i(n_i) \\ + q_{i,-e_i}^+ B_i(n_i - e_i) - q_{i,-e_i}^- A_i(n_i - e_i) \leq \Gamma(n_i), \end{aligned} \quad (18)$$

$$\begin{aligned} \bar{f}(0) - f(0) + q_{3,e_1}^+ B_1(0) - q_{3,e_1}^- A_1(0) \\ + q_{3,e_2}^+ B_2(0) - q_{3,e_2}^- B_2(0) \leq \Gamma(0), \end{aligned} \quad (19)$$

$$\begin{aligned} f(0) - \bar{f}(0) + q_{3,e_1}^- B_1(0) - q_{3,e_1}^+ A_1(0) \\ + q_{3,e_2}^- B_2(0) - q_{3,e_2}^+ B_2(0) \leq \Gamma(0), \end{aligned} \quad (20)$$

for $i = 1, 2$ and all $n_i \in S_i$, and

$$\bar{f}(n) - f(n) \leq \Gamma(n), \quad f(n) - \bar{f}(n) \leq \Gamma(n), \quad (21)$$

for all $n \in S_4$, and

$$c_{i,k(n)} + \sum_{\substack{j=1,2 \\ u \in N_{k(n)}}} [g_{i,k(n),j,u}^+ B_j(n+u) - g_{i,k(n),j,u}^- A_j(n+u)] \leq B_i(n), \quad (22)$$

$$-c_{i,k(n)} + \sum_{\substack{j=1,2 \\ u \in N_{k(n)}}} [g_{i,k(n),j,u}^- B_j(n+u) - g_{i,k(n),j,u}^+ A_j(n+u)] \leq -A_i(n), \quad (23)$$

for $i = 1, 2$, and all $n \in S$ then

$$\sum_{n \in S} [\bar{f}(n) - \Gamma(n)] \bar{\pi}(n) \leq F \leq \sum_{n \in S} [\bar{f}(n) + \Gamma(n)] \bar{\pi}(n). \quad (24)$$

In the last part of the section we will demonstrate that under the condition that the functions \bar{f} , A_1 , A_2 , B_1 , B_2 and Γ are componentwise linear, the constraints in (16)–(23) reduce to a finite number of constraints that are linear in the constants that define these functions. We will refer to componentwise linear functions as S -linear functions to express the fact that they are linear

within S_1, \dots, S_4 . Before stating our final result we introduce another partition of the state space. Let

$$\begin{aligned} T_1 &= \{(0, 0)\}, & T_4 &= \{(0, 1)\}, & T_7 &= \{0\} \times \{2, 3, \dots\}, \\ T_2 &= \{(1, 0)\}, & T_5 &= \{(1, 1)\}, & T_8 &= \{1\} \times \{2, 3, \dots\}, \\ T_3 &= \{2, 3, \dots\} \times \{0\}, & T_6 &= \{2, 3, \dots\} \times \{1\}, & T_9 &= \{2, 3, \dots\} \times \{2, 3, \dots\}. \end{aligned} \quad (25)$$

Let $t : S \rightarrow \{1, \dots, 9\}$ be defined through $n \in T_{t(n)}$. We refer to functions that are linear in each of the sets T_1, \dots, T_9 as T -linear. Similarly to a S -linear function, a T -linear function $h : S \rightarrow \mathbb{R}$ is defined through a set of coefficients $h_{t,i}$, $1 \leq t \leq 9$, $i = 0, 1, 2$, i.e.,

$$h(n) = h_{t(n),0} + h_{t(n),1}n(1) + h_{t(n),2}n(2). \quad (26)$$

The reason for introducing the new partition stems from the following result which is readily verified and stated without proof.

Lemma 1. *If \bar{f} , A_1 , A_2 , B_1 , B_2 and Γ are S -linear functions, then for each of the constraints in (16)–(23) there is a T -linear function $h(n)$ such that satisfying the constraint is equivalent to $h(n) \geq 0$, for all $n \in S$. Moreover, the coefficients of these T -linear functions are affine functions of the coefficients of \bar{f} , A_1 , A_2 , B_1 , B_2 and Γ .*

As a final technical result we give the finite number of linear constraints that are required for non-negativity of a T -linear function. The result is readily verified and stated without proof.

Lemma 2. *The T -linear function $h : S \rightarrow \mathbb{R}$ satisfies $h(n) \geq 0$ for all $n \in S$ iff*

$$\begin{aligned} h_{1,0} &\geq 0, \\ h_{2,0} + h_{2,1} &\geq 0, \\ h_{3,0} + 2h_{3,1} &\geq 0, & h_{3,1} &\geq 0, \\ h_{4,0} + h_{4,2} &\geq 0, \\ h_{5,0} + h_{5,1} + h_{5,2} &\geq 0, \\ h_{6,0} + 2h_{6,1} + h_{6,2} &\geq 0, & h_{6,1} &\geq 0, \\ h_{7,0} + 2h_{7,2} &\geq 0, & h_{7,2} &\geq 0, \\ h_{8,0} + h_{8,1} + 2h_{8,2} &\geq 0, & h_{8,2} &\geq 0, \\ h_{9,0} + 2h_{9,1} + 2h_{9,2} &\geq 0, & h_{9,1} &\geq 0, & h_{9,2} &\geq 0. \end{aligned} \quad (27)$$

Theorem 3, Lemma 1 and Lemma 2 provide the next corollary. In the corollary the exact linear expression for the upper and lower bounds on F are given. Remember, that r_1 and r_2 are the parameters of the geometric distribution of $\bar{\pi}$, as defined in (6). The corollary demonstrates that these bounds can be obtained as the solution of a linear program.

Corollary 1. *Consider S -linear functions \bar{f} , A_1 , A_2 , B_1 , B_2 and Γ . If the coefficients that define these functions satisfy the finite number of linear constraints induced by (16)–(23), then*

$$\begin{aligned} F \leq & (\bar{f}_{3,0} + \Gamma_{3,0})(1 - r_1)(1 - r_2) \\ & + r_1(1 - r_2) \left(\bar{f}_{1,0} + \Gamma_{1,0} + \frac{\bar{f}_{1,1} + \Gamma_{1,1}}{1 - r_1} \right) \\ & + (1 - r_1)r_2 \left(\bar{f}_{2,0} + \Gamma_{2,0} + \frac{\bar{f}_{2,2} + \Gamma_{2,2}}{1 - r_2} \right) \end{aligned} \quad (28)$$

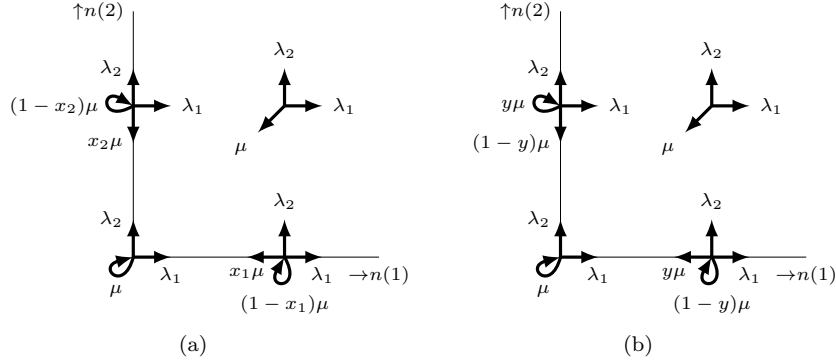


Figure 2: Random walk with joint departures.

and

$$\begin{aligned}
F \geq & (\bar{f}_{3,0} - \Gamma_{3,0})(1 - r_1)(1 - r_2) \\
& + r_1(1 - r_2) \left(\bar{f}_{1,0} - \Gamma_{1,0} + \frac{\bar{f}_{1,1} - \Gamma_{1,1}}{1 - r_1} \right) \\
& + (1 - r_1)r_2 \left(\bar{f}_{2,0} - \Gamma_{2,0} + \frac{\bar{f}_{2,2} - \Gamma_{2,2}}{1 - r_2} \right). \quad (29)
\end{aligned}$$

4 Examples

In this section we consider two examples of random walks and obtain bounds by applying Theorem 3.

4.1 Joint departures

We consider a random walk arising from an application in queueing theory. The model corresponds to two queues that are synchronized in the sense that departures from these queues are simultaneous. For efficiency reasons, if only one queue is non-empty, the other queue is serviced at a lower rate. This model arises from network coding in wireless communication networks and has recently been studied in [6].

The transition probabilities are as follows:

$$\begin{aligned}
p_{1,e_1} &= p_{2,e_1} = p_{3,e_1} = p_{4,e_1} = \lambda_1, \\
p_{1,e_2} &= p_{2,e_2} = p_{3,e_2} = p_{4,e_2} = \lambda_2, \\
p_{1,-e_1} &= x_1\mu, \\
p_{1,0} &= (1 - x_1)\mu, \\
p_{2,-e_2} &= x_2\mu, \\
p_{2,0} &= (1 - x_2)\mu, \\
p_{3,0} &= \mu, \\
p_{4,-d_1} &= \mu,
\end{aligned} \quad (30)$$

where $\lambda_1 + \lambda_2 + \mu = 1$, $0 \leq x_i \leq 1$, $i = 1, 2$. The transition diagram of the model is depicted in Figure 2, with the general case in (a) and the special case that $x_1 = y$ and $x_2 = (1 - y)$ in (b). It is known that in this special case the stationary distribution is a geometric product-form [6], *i.e.*,

$$\pi(n) = (1 - r_1)r_1^{n(1)}(1 - r_2)r_2^{n(2)}, \quad (31)$$

where r_1 and r_2 are given by the unique solution of

$$\begin{aligned} yr_1 + (1 - y)r_1r_2 &= \frac{\lambda_1}{\mu}, \\ (1 - y)r_2 + yr_1r_2 &= \frac{\lambda_2}{\mu}, \end{aligned}$$

satisfying $0 < r_1 < 1$ and $0 < r_2 < 1$.

First, we demonstrate the application of Theorem 2 by providing examples of constants $g_{i,j,k,u}$. We consider only the case that $n \in S_1$. For the first type of bias term we can write

$$\begin{aligned} D_1^{t+1}(n) &= c_{1,1} + \lambda_1 D_1^t(n + e_1) + \lambda_2 D_1^t(n + e_2) + \gamma_1 \mu D_1^t(n - e_1) \\ &\quad + (1 - \gamma_1) \mu D_1^t(n), \end{aligned} \quad (32)$$

i.e.,

$$g_{1,1,1,e_1} = \lambda_1, \quad g_{1,1,1,e_2} = \lambda_2, \quad g_{1,1,1,-e_1} = \gamma_1 \mu, \quad g_{1,1,1,0} = (1 - \gamma_1) \mu. \quad (33)$$

For the other bias term we have

$$D_2^{t+1}(n) = c_{2,1} + \lambda_1 D_2^t(n + e_1) + \lambda_2 D_2^t(n + e_2) - (1 - \gamma_1) \mu D_1^t(n - e_1), \quad (34)$$

i.e.,

$$g_{2,1,2,e_1} = \lambda_1, \quad g_{2,1,2,e_2} = \lambda_2, \quad g_{2,1,1,-e_1} = -(1 - \gamma_1) \mu. \quad (35)$$

These expressions coincide with the general forms given in the proof of Theorem 2.

Next, we provide numerical results by evaluating the bounds from Theorem 3. The performance measure that we consider is the marginal first moment in the first direction, *i.e.*, the expected number of customers in the first queue. This is achieved by taking f as

$$f_{k,i} = \begin{cases} 1, & \text{if } k = 1, i = 1, \\ 1, & \text{if } k = 4, i = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (36)$$

We restrict our attention the symmetrical case that $\lambda_1 = \lambda_2 = \lambda$ and $x_1 = x_2 = x$. The perturbed model that we use as the basis for approximating is $y = 1/2$. The upper and lower bounds on F that are obtained from Theorem 3 are depicted in Figure 3 for $\lambda = 0.1$ and various values of x . Figure 4 provides numerical results for $\lambda = 0.2$ and various values of x .

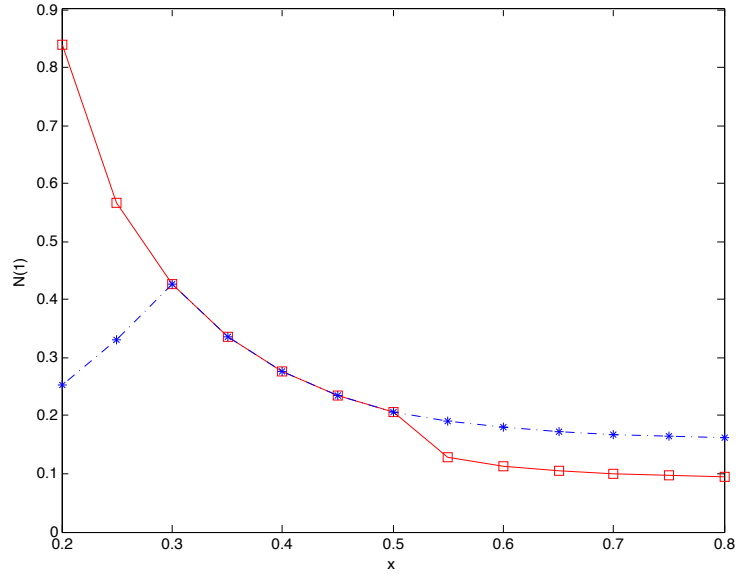


Figure 3: Joint departures, $\lambda_1 = \lambda_2 = 0.1$.

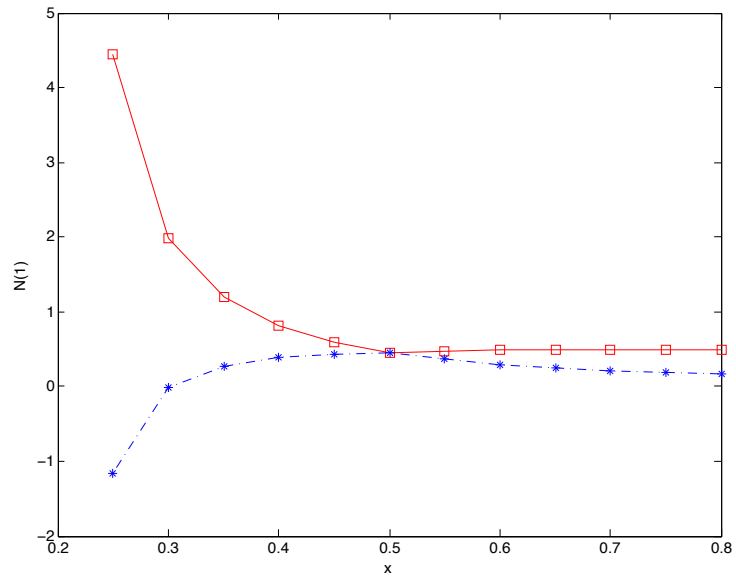


Figure 4: Joint departures, $\lambda_1 = \lambda_2 = 0.2$.

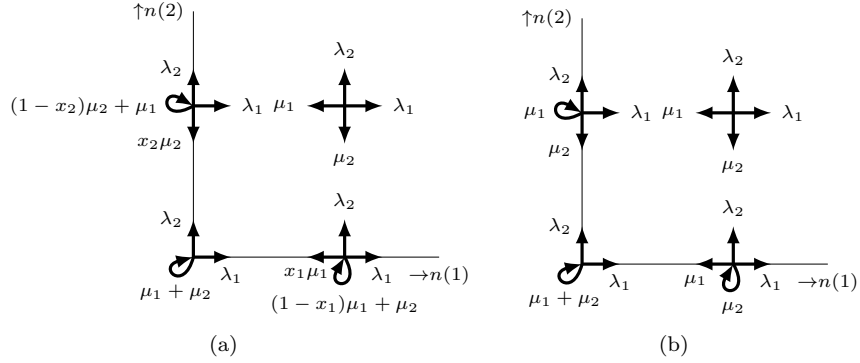


Figure 5: Random walk with coupled processors.

4.2 Coupled processors

We consider the model of coupled processors [7]. The coupling of the processors is such that in the interior of the state space the processors operate at rates μ_1 and μ_2 respectively. If one of the processors is idle, the other processor adjusts its rates. The transition probabilities are as follows:

$$\begin{aligned}
 p_{1,e_1} &= p_{2,e_1} = p_{3,e_1} = p_{4,e_1} = \lambda_1, \\
 p_{1,e_2} &= p_{2,e_2} = p_{3,e_2} = p_{4,e_2} = \lambda_2, \\
 p_{1,-e_1} &= x_1\mu_1, \\
 p_{1,0} &= (1-x_1)\mu_1 + \mu_2, \\
 p_{2,-e_2} &= x_2\mu_2, \\
 p_{2,0} &= (1-x_2)\mu_2 + \mu_1, \\
 p_{3,0} &= \mu_1 + \mu_2, \\
 p_{4,-e_1} &= \mu_1, \\
 p_{4,-e_2} &= \mu_2.
 \end{aligned} \tag{37}$$

The transition diagram is depicted in Figure 5, with the general case in (a) and the special case that $x_1 = x_2 = 1$ in (b). For the special case that $x_1 = x_2 = 1$, the model has a product-form stationary distribution with $r_1 = \lambda_1/\mu_1$ and $r_2 = \lambda_2/\mu_2$.

We consider the marginal first moment in the first direction, *i.e.*, f as in (36), for the case that $\lambda_1 = \lambda_2 = \lambda$ and $x_1 = x_2 = 2$. As a basis for approximation we use the perturbation to $x_1 = x_2 = 1$. The upper and lower bounds on F as a function of λ that are obtained from Theorem 3 are depicted in Figure 6.

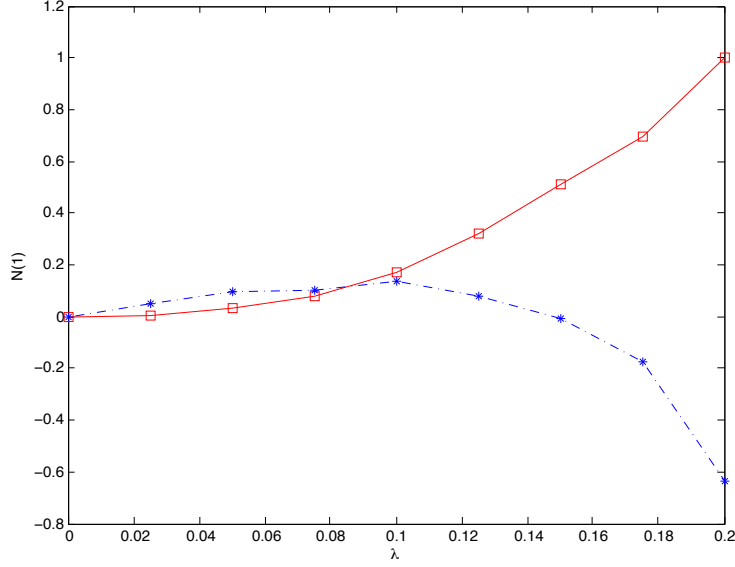


Figure 6: Coupled processors.

5 Proofs

5.1 Proof of Theorem 2

We provide a constructive proof by giving an example of such constants. For $D_1^t(n)$ we have

$$D_1^{t+1}(n) = f_{1,1} + \sum_{u \in N_1} p_{1,u} D_1^t(n+u), \quad \text{if } n \in S_1, \quad (38)$$

$$\begin{aligned} D_1^{t+1}(n) = & f_{4,0} - f_{2,0} + f_{4,1} + \sum_{u \in N_2} p_{4,u} D_1^t(n+u) + (p_{4,e_1} - p_{2,e_1}) D_1^t(n) \\ & + (p_{4,d_1} - p_{2,d_1}) D_1^t(n+e_2) + (p_{4,d_2} - p_{2,d_2}) D_1^t(n-e_2) \\ & - (p_{4,d_2} + p_{4,-e_2} + p_{4,-d_1} - p_{2,d_2} - p_{2,-e_2}) D_2^t(n-e_2) \\ & + (p_{4,d_1} + p_{4,e_2} + p_{4,-d_2} - p_{2,d_1} - p_{2,e_2}) D_2^t(n), \quad \text{if } n \in S_2, \quad (39) \end{aligned}$$

$$\begin{aligned} D_1^{t+1}(n) = & f_{1,0} - f_{3,0} + f_{1,1} + \sum_{u \in N_3} p_{1,u} D_1^t(u) \\ & + (p_{1,d_1} - p_{3,d_1}) D_1^t(e_2) + (p_{1,e_1} - p_{3,e_1}) D_1^t(0) \\ & + (p_{1,d_1} + p_{1,e_2} + p_{1,-d_2} - p_{3,d_1} - p_{3,e_2}) D_2^t(0), \quad \text{if } n \in S_3, \quad (40) \end{aligned}$$

$$D_1^{t+1}(n) = f_{4,1} + \sum_{u \in N_4} p_{4,u} D_1^t(n+u), \quad \text{if } n \in S_4. \quad (41)$$

For $D_2^t(n)$ existence of constants follows from symmetry considerations. We give example expressions for such constants for completeness.

$$\begin{aligned} D_2^{t+1}(n) = & f_{4,0} - f_{1,0} + f_{4,2} + \sum_{u \in N_1} p_{4,u} D_2^t(n+u) + (p_{4,e_2} - p_{1,e_2}) D_2^t(n) \\ & + (p_{4,-d_2} - p_{1,-d_2}) D_2^t(n-e_1) + (p_{4,d_1} - p_{1,d_1}) D_2^t(n+e_1) \end{aligned}$$

$$\begin{aligned}
& - (p_{4,-d_2} + p_{4,-e_1} + p_{4,-d_1} - p_{1,-d_2} - p_{1,-e_1}) D_1^t(n - e_1) \\
& + (p_{4,d_1} + p_{4,e_1} + p_{4,d_2} - p_{1,d_1} - p_{1,e_1}) D_1^t(n), \quad \text{if } n \in S_1, \quad (42)
\end{aligned}$$

$$D_2^{t+1}(n) = f_{2,2} + \sum_{u \in N_2} p_{2,u} D_2^t(n + u) \quad \text{if } n \in S_2, \quad (43)$$

$$\begin{aligned}
D_2^{t+1}(n) &= f_{2,0} - f_{3,0} + f_{2,2} + \sum_{u \in N_3} p_{2,u} D_2^t(u) \\
& + (p_{2,d_1} - p_{3,d_1}) D_2^t(e_1) + (p_{2,e_2} - p_{3,e_2}) D_2^t(0) \\
& + (p_{2,d_1} + p_{2,e_1} + p_{2,d_2} - p_{3,d_1} - p_{3,e_1}) D_1^t(0) \quad \text{if } n \in S_3, \quad (44)
\end{aligned}$$

$$D_2^{t+1}(n) = f_{4,2} + \sum_{u \in N_4} p_{4,u} D_2^t(n + u), \quad \text{if } n \in S_4. \quad (45)$$

5.2 Proof of Theorem 3

Using induction over t , we first prove that

$$A_i(n) \leq D_i^t(n) \leq B_i(n),$$

$i = 1, 2$. Since $A_i(n) \leq 0$ and $B_i(n) \geq 0$, from (16), and $D_i^0(n) = 0$, the bounds hold at $t = 0$. Next, assume that $A_i(n) \leq D_i^t(n) \leq B_i(n)$ for some $t > 0$. Then

$$\begin{aligned}
D_i^{t+1}(n) &= c_{i,k(n)} + \sum_{j=1,2} \sum_{u \in N} g_{i,k(n),j,u} D_j^t(n + u) \\
&= c_{i,k(n)} + \sum_{j=1,2} \sum_{u \in N} [g_{i,k(n),j,u}^+ D_j^t(n + u) - g_{i,k(n),j,u}^- D_j^t(n + u)] \\
&\leq c_{i,k(n)} + \sum_{j=1,2} \sum_{u \in N} [g_{i,k(n),j,u}^+ B_j(n + u) - g_{i,k(n),j,u}^- A_j(n + u)] \\
&\leq B_i(n), \quad (46)
\end{aligned}$$

where the first equality follows from the definition of the constants g , the first inequality from the induction hypothesis and the last inequality from (22). The lower bound $A_i(n) \leq D_i^{t+1}(n)$ follows in similar fashion from (23).

Next, we prove that

$$\left| \bar{f}(n) - f(n) + \sum_{u \in N_{k(n)}} q_{k(n),u} D_u^t(n) \right| \leq \Gamma(n). \quad (47)$$

First, since $q_{k,u} \neq 0$ only along the boundaries, we need to show that

$$|\bar{f}(n) - f(n) + q_{i,e_i} D_i^t(n) - q_{i,-e_i} D_i^t(n - e_i)| \leq \Gamma(n), \quad \text{if } n \in S_i, \quad (48)$$

for $i = 1, 2$, and that

$$|\bar{f}(0) - f(0) + q_3(e_1) D_1^t(0) + q_3(e_2) D_2^t(0)| \leq \Gamma(0), \quad \text{and} \quad (49)$$

$$|\bar{f}(n) - f(n)| \leq \Gamma(n), \quad \text{if } n \in S_4. \quad (50)$$

It is readily verified that (17) and (18) provide (48), and that (19) and (20) provide (49). Finally, (50) is included directly as condition (21). This concludes the proof of (47) and hence the proof of the theorem, which now follows directly from Theorem 1.

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