Structural Characterization of Decomposition in Rate-insensitive Stochastic Petri Nets

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Abstract

This paper focuses on stochastic Petri nets that have an equilibrium distribution that is a product form over the number of tokens at the places. We formulate a decomposition result for the class of nets that have a product form solution irrespective of the values of the transition rates. These nets where algebraically characterized by Haddad et al. as $\mathcal{S}\Pi^2$ nets. By providing an intuitive interpretation of this algebraical characterization, and associating state machines to sets of T-invariants, we obtain a one-to-one correspondence between the marking of the original places and the places of the added state machines. This enables us to show that the subclass of stochastic Petri nets under study can be decomposed into subnets that are identified by sets of its T-invariants.

Keywords: Stochastic Petri net, Product form, Decomposition, T-invariant, P-invariant.

1 Introduction

Competition over resources is an important issue in many practical systems. Examples of such systems are computer systems, telecommunication networks, flexible manufacturing systems and hospitals. Typical questions arising are identification of bottlenecks, achievable throughput and maximization of resource utilization. Therefore, performance analysis is an important issue in the design and implementation of such real life systems. In this paper, we focus on analytical performance analysis with the formalism of Stochastic Petri nets.

Composition and decomposition of closed form results contribute to less computational effort requirements and greater understanding of network behavior and performance. It allows studying a system by analyzing the characteristics of separate components. In this paper, we study closed form results for the equilibrium distribution of the number of tokens at the places of a stochastic Petri net and the decomposition of this equilibrium distribution into several components corresponding to subnets of the stochastic Petri net. Exact analytical results for the distribution of the number of items at places in performance models are in general very difficult to obtain. One of the most important analytical results for the equilibrium distribution describing the number of items at places in a performance model is the so-called *product form* equilibrium distribution found for a fairly wide class of theoretical queueing models. However, practical performance models seldom satisfy the product form conditions. Still, results obtained via the theoretical product form distributions are used for practical problems since these results are found to be robust, that is models which violate the product form conditions are often found to behave in a way very similar to a product form counterpart. The obvious advantages of these product form distributions are their simplicity, since the network behavior is captured in closed form in only a limited set of parameters. This makes product form solutions easy and powerful to use for computational issues as well as for theoretical reflections for performance models involving congestion. Another important advantage of product form solutions is that it enables to break-down the analysis of a network in the analysis of separate components of the network.

The current paper is a follow-up of [22]. The contribution of that paper was threefold. First, we surveyed the various structural results that are known for stochastic Petri nets with a product form equilibrium distribution over the number of tokens at the places and rephrases all these results in terms of T-invariants. Second, we unified and extended the product form results for stochastic Petri nets by showing that *group-local-balance* can be identified as the concept underlying all these structural results and we provide additional structural implications and an intuitive explanation of the known and new results, all based on T-invariants only. Based on [3, 4, 10, 12, 15, 16], we showed that group-local-balance requires the stochastic Petri net to be an $\mathcal{S}\Pi$-net, a stochastic Petri net in which each transition is covered by a minimal support T-invariant. Third, we described a structural decomposition result for $\mathcal{S}\Pi$-nets formulated exclusively in terms of P- and T-invariants using so-called conflict places (places that are shared by different minimal closed support T-invariants) and surplus places (places that can be omitted in characterizing the marking of the Petri net). Using the P-invariants to assign conflict places as surplus places, an algorithmic procedure was formulated to decompose a product form stochastic Petri net into subnets. The subnets corresponded to one or more common input bag classes, the equivalence classes of T-invariants of the stochastic Petri nets that share an input bag.

In the current paper, we take the results from [22] as starting point to formulate an additional decomposition result. We focus on the subclass of $\mathcal{S}\Pi$-nets that have a product form equilibrium distribution irrespective of the transition rates. These nets where algebraically characterized by Haddad et al. as $\mathcal{S}\Pi^2$-nets (see Definition 7 in [15]). In [22] we showed that these are the Petri nets in which each minimal support T-invariant is a closed support T-invariant. We will present a decomposition theorem by which all $\mathcal{S}\Pi^2$-nets can be separated in all their common input bag classes.

We build on the characterization of $\mathcal{S}\Pi^2$-nets provided by Haddad et al. [15], by establishing an interpretation of the vectors $\boldsymbol{a_r}$ that can be calculated for each bag $\boldsymbol{r} \in \mathcal{R}(T)$ according to Definition 7 of [15]. Starting from an arbitrary $\mathcal{S}\Pi^2$-net, and introducing 'bag count places', we introduce the Bag-Count-Place-Extended Petri net of an $\mathcal{S}\Pi^2$-net ($\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net). The Petri net that is formed by exclusively the bag count places consists of a set of state machine, one state machine per common input bag class. Along the concept of bag count places we show that the $\boldsymbol{a_r}$-vectors provide the explicit relation between a marking difference $\boldsymbol{m} - \boldsymbol{m}'$ and the number of times each bag $\boldsymbol{r}$ is used in a firing sequence that is associated with this marking difference. This relation induces a one-to-one correspondence between the marking of the original places and the additionally constructed bag-count places.

The one-to-one correspondence implies that the bag count places of a $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$ form a sufficient place set. From [22] we then know that the equilibrium distribution of the bag count places provides an equilibrium distribution of the original places. In addition, by construction the bag count places a $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net are non-conflict places. This enables us to apply decomposition Theorem 6.10 from [22] to the $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net. We obtain that the invariant measure of any $\mathcal{S}\Pi^2$-net factorizes in the invariant measures of the separate state machines that are associated with each of the common input bag classes.

The paper is organized as follows. For self-containedness, Section 2 provides an introduction into the (stochastic) Petri net formalism and a summary of previous results. Section 3 defines the bag count places, introduces $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-nets, and discusses the interpretation of the $\boldsymbol{a_r}$-vectors. Section 4 formulates the decomposition result, and Section 5 provides several examples.

2 Preliminaries

This section provides a general introduction into the formal Petri net language, the Petri net concepts focusing on product form and decomposition results, and previous results on stochastic Petri nets. For additional definitions, properties and results see e.g. [22, 29, 31].

2.1 Model and definitions

Definition 2.1 (Marked stochastic Petri net)**.** A marked stochastic Petri net is a 6-tuple, $\mathcal{SPN} = (P, T, I, O, R, \boldsymbol{m}_0)$, where $P = \{p_1, \dots, p_N\}$ is a finite set of places; $T = \{t_1, \dots, t_M\}$ is a finite set of transitions; $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$; $I, O : P \times T \to \mathbb{N}_0$ are the input and output functions identifying the relation between the places and the transitions; $R = (r_{t_1}, \dots, r_{t_M})$ is a set of firing rates drawn from exponential distributions; and $\boldsymbol{m}_0$ is the initial marking.

A *marking* $\boldsymbol{m} = (\boldsymbol{m}(n),\ n = 1, \dots, N)$ of a Petri net is a vector in $\mathbb{N}_0^N$, where $\boldsymbol{m}(n)$ represents the number of tokens at place p_n, $n = 1, \dots, N$. Distributions associated with different transitions are independent, and each transition of the Petri net is due to exactly one transition $t \in T$ that fires. The *execution policy* of the stochastic Petri net is the *race model with age memory* (cf. [27]).

The vectors $\boldsymbol{I}(t) = (I_1(t), \dots, I_N(t))$, and $\boldsymbol{O}(t) = (O_1(t), \dots, O_N(t))$ are called *input, and output bags* of transition $t \in T$, representing the number of tokens consumed at the places when transition t fires, and the number of tokens released to the places after firing of transition t. If transition t is *enabled* in marking $\boldsymbol{m}$ and fires, then the next state of the Petri net is $\boldsymbol{m'} = \boldsymbol{m} - \boldsymbol{I}(t) + \boldsymbol{O}(t)$, denoted as $\boldsymbol{m}[t > \boldsymbol{m'}$. A necessary and sufficient condition for t to be enabled is that $m(n) \geq I_n(t)$, $n = 1, \dots, N$.

A finite sequence of transitions $\sigma = t_{\sigma_1} t_{\sigma_2} \cdots t_{\sigma_k}$ is a finite *firing sequence* of the Petri net if there exists a sequence of markings $\boldsymbol{m}_{\sigma_1}, \dots, \boldsymbol{m}_{\sigma_k}$ for which $\boldsymbol{m}_{\sigma_i}[t_{\sigma_i} > \boldsymbol{m}_{\sigma_{i+1}}$, $i = 1, \dots, k-1$. In this case marking $\boldsymbol{m}_{\sigma_k}$ is *reachable* from marking $\boldsymbol{m}_{\sigma_1}$ by firing σ, denoted as $\boldsymbol{m}_{\sigma_1}[\sigma > \boldsymbol{m}_{\sigma_k}$. The *reachability set* $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ is a subset of $\mathbb{N}_0^N$ and gives all possible markings of Petri net $\mathcal{SPN}$ with initial marking $\boldsymbol{m}_0$.

A transition $t \in T$ is *live* if, no matter what marking has been reached from $\boldsymbol{m}_0$, it is possible to ultimately fire transition t by progressing through some further firing sequence [29]. A Petri net is *live* if all its transitions are live. A Petri net is *structurally live* if there exists an initial marking $\boldsymbol{m}_0$ for which the Petri net is live. A Petri net is *bounded* if the number of tokens in each place does not exceed a finite number k for any marking in the reachability set. It is *structurally bounded* if it is bounded for all initial markings.

The *incidence matrix* is the $N \times M$ matrix $\boldsymbol{A}$ with entries $A(i,t) = O_i(t) - I_i(t)$ describing the change in the number of tokens in place p_i if transition t fires, $i = 1, \ldots, N$, $t \in T$. A vector $\boldsymbol{\sigma}$ is the *firing count vector* of the firing sequence σ if $\boldsymbol{\sigma}$ equals the number of times transition t occurs in the firing sequence σ. If $\boldsymbol{m}_0[\sigma > \boldsymbol{m}$, then $\boldsymbol{m} = \boldsymbol{m}_0 + A\boldsymbol{\sigma}$, an equation referred to as the *state equation* of the Petri net.

A vector $\boldsymbol{x} \in \mathbb{N}_0^M$ is a *T-invariant* if $\boldsymbol{x} \neq 0$, and $\boldsymbol{A}\boldsymbol{x} = 0$. From the state equation we obtain that a T-invariant corresponds to a firing sequence that brings a marking back to itself. The *support* of a T-invariant $\boldsymbol{x}$ is the set of transitions corresponding to non-zero entries of $\boldsymbol{x}$, and is denoted by $\|\boldsymbol{x}\|$, i.e., $\|\boldsymbol{x}\| = \{t \in T \mid \boldsymbol{x}(t) > 0\}$. A T-invariant $\boldsymbol{x}$ is a *minimal T-invariant* if there is no other T-invariant $\boldsymbol{x}'$ such that $x'(t) \leq x(t)$ for all t. A support is minimal if no proper nonempty subset of the support is also a support of a T-invariant. From [28] we obtain that there is a unique minimal T-invariant corresponding to a minimal support (*minimal support T-invariant*), and any T-invariant can be written as a linear combination of minimal support T-invariants. A vector $\boldsymbol{y} \in \mathbb{N}_0^N$ is a *P-invariant* if $\boldsymbol{y} \neq 0$, and $\boldsymbol{y}\boldsymbol{A} = 0$. A P-invariant identifies a set of places such that the weighted sum of the number of tokens distributed over these places remains constant for all markings in the reachability set. Definitions of and results for minimal support etc. are analogous to those for T-invariants.

A particular type of T-invariant is essential for the analysis presented in this paper: the *minimal closed support T-invariant* [4]. For $\mathcal{T} \subseteq T$ define $\mathcal{R}(\mathcal{T})$, the set of input and output bags for the transitions in $\mathcal{T}$, as $\mathcal{R}(\mathcal{T}) = \bigcup_{t\in\mathcal{T}}\{\boldsymbol{I}(t) \cup \boldsymbol{O}(t)\}$. $\mathcal{R}(\mathcal{T})$ is a *closed set* if for all $\boldsymbol{r} \in \mathcal{R}(\mathcal{T})$ there exist $t, t' \in \mathcal{T}$ such that $\boldsymbol{r} = \boldsymbol{I}(t)$, as well as $\boldsymbol{r} = \boldsymbol{O}(t')$, that is if each output bag is also an input bag, and each input bag is also an output bag for some transition in $\mathcal{T}$. A T-invariant is *closed* if the set of input and output bags for the transitions in its support, $\mathcal{R}(\|\boldsymbol{x}\|)$, is a closed set. A T-invariant is a *minimal closed support T-invariant* if it is closed and has minimal support. From [4] we obtain that a T-invariant $\boldsymbol{x}$ is a minimal closed support T-invariant if the firing sequence of $\boldsymbol{x}$ is *linear*, that is for each $t \in \|\boldsymbol{x}\|$ there is a unique $t' \in \|\boldsymbol{x}\|$ such that $\boldsymbol{O}(t) = \boldsymbol{I}(t')$. As a consequence $x_i \leq 1$, $i = 1, \ldots, M$. Conversely, if the firing sequence of a T-invariant $\boldsymbol{x}$ is linear, then $\boldsymbol{x}$ is a closed support T-invariant. A Petri net is a *state machine* if and only if $\sum_p I_p(t) = 1$ and $\sum_p O_p(t) = 1$ for all transitions. A Petri net consisting of minimal closed support T-invariants is the natural extension of a state machine.

Let $\boldsymbol{x}, \boldsymbol{x}' \in ClT$. We say that $\boldsymbol{x}, \boldsymbol{x}'$ are in *common input bag relation* (notation: $\boldsymbol{x}\ CI\ \boldsymbol{x}'$) if there exist $t \in ||\boldsymbol{x}||, t' \in ||\boldsymbol{x}'||$ such that $I(t) = I(t')$. The relation CI^* is the transitive closure of CI (see [14]). The *common input bag class* $CI(\boldsymbol{x})$ is the equivalence class of $\boldsymbol{x} \in ClT$, that is $CI(\boldsymbol{x}) = \{\boldsymbol{x}' | \boldsymbol{x}\ CI^*\ \boldsymbol{x}'\}$. Let $\mathcal{C} = \{CI^1, \ldots, CI^K\}$ be the set of all common input bag classes. The transition set $\mathcal{T}(CI^i)$ of common input bag class CI^i is the set of all transitions belonging to common input bag class CI^i, i.e., $\mathcal{T}(CI^i) = \{t \in T | \exists \boldsymbol{x} \in CI^i : t \in ||\boldsymbol{x}||\}$. The place set $\mathcal{P}(CI^i)$ of common input bag class CI^i is the set of all places belonging to common input bag class CI^i, i.e., $\mathcal{P}(CI^i) = \{p \in P | \exists t \in \mathcal{T}(CI^i) : I(p,t) > 0\}$. We say that common input bag classes CI^i and CI^j are *connected* if $\mathcal{P}(CI^i) \bigcap \mathcal{P}(CI^j) \neq \emptyset$.

The stochastic process describing the evolution of the Petri net is a continuous-time Markov chain $\mathbf{X}$ with state space $\mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$. A transition t in marking $\boldsymbol{m}$ can be enabled only if $\boldsymbol{m} - \boldsymbol{I}(t) \in \mathbb{N}_0^N$, so that the rate for this transition is $q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{m} - \boldsymbol{I}(t)) = r_t \mathbb{I}(\boldsymbol{m} - \boldsymbol{I}(t) \in \mathbb{N}_0^N)$, bringing $\boldsymbol{m}$ to $\boldsymbol{m}' = \boldsymbol{m} - \boldsymbol{I}(t) + \boldsymbol{O}(t)$. Note that a transition from marking $\boldsymbol{m}$ to marking $\boldsymbol{m} - \boldsymbol{I}(t) + \boldsymbol{O}(t)$ may occur due to other transitions too. Denote the transition rates of $\mathbf{X}$ by $Q = (q(\boldsymbol{m}, \boldsymbol{m}'), \boldsymbol{m}, \boldsymbol{m}' \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0))$, with

$$q(\boldsymbol{m}, \boldsymbol{m}') = \sum_{\{\boldsymbol{n} \in \mathbb{N}_0^N,\, t \in T:\, \boldsymbol{n}+\boldsymbol{I}(t)=\boldsymbol{m},\, \boldsymbol{n}+\boldsymbol{O}(t)=\boldsymbol{m}'\}} q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{n}). \tag{1}$$

When analyzing the Markov chain $\mathbf{X}$ describing the behavior of a stochastic Petri net, it is convenient to aggregate transitions with identical input bag into one transition with a probabilistic output bag, the so-called probabilistic output bag transformation. Let $t_{i_1}, \dots, t_{i_k}$ have input bag $\boldsymbol{I}(t)$. Then, transition t with input bag $\boldsymbol{I}(t)$ fires with rate $\mu(t) = \sum_{j=1}^{k} r(t_{i_j})$, and the output bag is $\boldsymbol{O}(t_{i_j})$ with probability $p(\boldsymbol{I}(t), \boldsymbol{O}(t_{i_j})) = r(t_{i_j})/\mu(t)$, i.e.,

$$q(\boldsymbol{I}(t), \boldsymbol{O}(t); \boldsymbol{m} - \boldsymbol{I}(t)) = \mu(t) p(\boldsymbol{I}(t), \boldsymbol{O}(t)) \mathbb{I}(\boldsymbol{m} - \boldsymbol{I}(t) \in \mathbb{N}_0^N). \tag{2}$$

We will restrict ourselves to Petri nets that are structurally live and structurally bounded, which implies that Markov chain $\mathbf{X}$ irreducible and positive recurrent [22]. A structurally bounded and structurally live Petri net is covered by both P-invariants and T-invariants [29]. Then, a unique collection of positive numbers $\pi = (\pi(\boldsymbol{m}), \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0))$ summing to unity exists satisfying the *global balance equations*:

$$\sum_{\boldsymbol{m}' \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)} \left\{ \pi(\boldsymbol{m}) q(\boldsymbol{m}, \boldsymbol{m}') - \pi(\boldsymbol{m}') q(\boldsymbol{m}', \boldsymbol{m}) \right\} = 0\ , \qquad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0). \tag{3}$$

This $\pi = (\pi(\boldsymbol{m}), \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0))$ is called the *equilibrium distribution.*

2.2 Product form

Various authors focused on the characterization of classes of stochastic Petri nets that have a product form equilibrium distribution for the number of tokens at the places, of which an extensive survey is provided in [22]. The first results were based on behavioral properties (properties that are dependent on the initial marking $\boldsymbol{m}_0$), which as a consequence required analyzing the potentially very large reachability set. Lazar and Robertazzi [24] identified the class of stochastic Petri nets consisting of ‘linear task sequences’.

Deriving the first structural product form results, Henderson et al. [16, 17, 18] translated and extended product form results for batch routing queueing networks to stochastic Petri nets, which are equivalent to batch routing queueing networks at the level of the underlying stochastic process. The starting point for structural analysis of product form stochastic Petri nets is the assumption that a positive solution exists for the *routing chain*, the Markov chain $\mathbf{Y} = (Y(t), t \geq 0)$ defined on finite state space $S = \{\boldsymbol{I}(t), t \in T\}$ with transition rates $q_{\mathbf{Y}}(\boldsymbol{I}(t), \boldsymbol{I}(t')) = \mu(t) p(\boldsymbol{I}(t), \boldsymbol{I}(t'))$. The global balance equations for routing chain $\mathbf{Y}$ are, for $t \in T$,

$$\sum_{t' \in T} \{ y(\boldsymbol{I}(t)) \mu(t) p(\boldsymbol{I}(t), \boldsymbol{I}(t')) - y(\boldsymbol{I}(t')) \mu(t') p(\boldsymbol{I}(t'), \boldsymbol{I}(t)) \} = 0. \tag{4}$$

A characterization of the structure of stochastic Petri nets necessary and sufficient for the existence of a positive solution for the routing chain was obtained in [3, 12]: ‘all transitions of the Petri net should be covered by *minimal closed support T-invariants*’. This type of T-invariant was introduced in [3, 12] and it closely resembles the ‘task sequences’ used by Lazar and Robertazzi [24].

Definition 2.2 (**$\mathcal{S}\Pi$-net**)**.** A Π-net is a Petri net in which all transitions $t \in T$ are covered by minimal closed support T-invariants $\boldsymbol{x}^i, i = 1, \ldots, k$, that is for all $t \in T$ there exists an $i \in \{1, \ldots, k\}$ such that $t \in \|\boldsymbol{x}^i\|$ and $\|\boldsymbol{x}^i\|$ is a closed set. An $\mathcal{S}\Pi$-net is a stochastic Π-net.

For an $\mathcal{S}\Pi$-net, the structure of the minimal closed support T-invariants implies that the routing chain partitions into $|\mathcal{C}| = K$ irreducible sets: $\mathcal{R}(\mathcal{T}(CI^i)), i = 1, \ldots, K$. This yields that the global balance equations for the routing chain partition into K independent systems of equations, which all have a unique solution up to a multiplicative constant. Thus, for the stochastic Petri net $\mathcal{SPN}$ a positive solution for the routing chain (4) exists if and only if $\mathcal{SPN}$ is an $\mathcal{S}\Pi$-net [4]. The existence of a positive solution for the routing chain is the first requirement for product form. Product form also requires a numerical condition on the transition rates [11].

Haddad et al. [15] and Mairesse et al. [26] established characterizations of $\mathcal{S}\Pi$-nets possessing a product form solution irrespective of the values of the transition rates. Haddad et al. [15] achieved this via the concept of $\mathcal{S}\Pi^2$-nets and Mairesse et al. [26] via the concept of 'zero-deficiency' $\mathcal{S}\Pi$-nets. In this paper, we will build upon the characterization provided in [15].

Definition 2.3 ($\mathcal{S}\Pi^2$-net)**.** A Π^2-net is a Π-net such that for every $\boldsymbol{r} \in \mathcal{R}(T)$, there is an $\boldsymbol{a_r} \in \mathbb{Q}^N$ such that

$$\boldsymbol{a_r A} = \boldsymbol{b}_r$$

in which for $t = 1, \ldots, N$,

$$\boldsymbol{b_r}(t) = \begin{cases} -1 & \text{if } \boldsymbol{r} = \boldsymbol{I}(t), \\ 1 & \text{if } \boldsymbol{r} = \boldsymbol{O}(t), \\ 0 & \text{otherwise.} \end{cases}$$

An $\mathcal{S}\Pi^2$-net is a stochastic Π^2-net.

The equilibrium distribution of an $\mathcal{S}\Pi^2$-net with transition rates of the form (2) is given by [11, 15]:

$$\pi(\boldsymbol{m}) = B \prod_{\boldsymbol{r} \in \mathcal{R}(T)} (y(\boldsymbol{r}))^{\boldsymbol{a_r} \cdot \boldsymbol{m}}, \qquad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0).$$

The conditions for an $\mathcal{S}\Pi$-net to satisfy Definition 2.3 are algebraic and lack intuition in terms of Petri net structure. Its explanation in terms of T-invariants is provided in [22]. Kortbeek and Boucherie [22] show that an $\mathcal{S}\Pi$-net is an $\mathcal{S}\Pi^2$-net if and only if all minimal support T-invariants $\boldsymbol{x}$ are minimal closed support T-invariants. The interpretation of the vectors $\boldsymbol{a_r}, \boldsymbol{r} \in \mathcal{R}(T)$, will be considered in Section 3. These vectors provide the explicit relation between a marking difference $\boldsymbol{m} - \boldsymbol{m'}$ and the number of times each bag $\boldsymbol{r}$ is used in a firing sequence that is associated with this marking difference.

2.3 Decomposition

A network can be decomposed if its stationary distribution factorizes into the stationary distributions of the nodes of which the network is comprised. Apart from the theoretical interest, decomposition results are also of substantial practical importance: finding the stationary distribution of an entire network usually requires an enormous computational effort, whereas the stationary distribution of a single node can often be found relatively easily. The first, and perhaps most famous, decomposition results for queueing networks is the classical Jackson product form result [21]. Decomposition of networks into subnetworks have been a topic of research for queueing networks. Two streams of literature developed in parallel: results based on partial balance (e.g. [5, 7, 8, 19, 23]) and results based on quasi-reversibility (e.g. [1, 6, 32, 33]). Recently, in a setting of general stochastic processes, these results have been unified and extended in [9, 20].

For stochastic Petri nets decomposition results were initialized by Lazar and Robertazzi [25] for connected subnets of task sequences and extended by Boucherie [2] in the framework of competing Markov chains. Frosch and Natarajan [13, 14] derived product form results for so-called closed synchronized systems of stochastic sequential processes, a class of Petri nets in which state machines are synchronized via buffer places. The results in these references may also be interpreted as composition results since the networks are essentially obtained by composing subnets in to a larger net, similar to the composition structure of stochastic process algebras.

In this paper, we will build upon the decomposition result of Kortbeek and Boucherie [22], who present a decomposition result for $\mathcal{S}\Pi$-nets based on the structure of the Petri net formulated that is exclusively in terms of P- and T-invariants using so-called conflict places (places that are shared by different minimal closed support T-invariants) and surplus places (places that can be omitted in characterizing the marking of the Petri net). Using the P-invariants to assign conflict places as surplus places, an algorithmic procedure is provided to verify whether product form holds and to decompose the stochastic Petri net into subnets. The subnets correspond to one or more common input bag classes, the equivalence classes of T-invariants of the stochastic Petri nets that share an input bag.

Definition 2.4 (**Conflict place - Conflict place set**)**.** Let $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ be minimal closed support T-invariants such that $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$ are not in common input bag relation, i.e., $CI(\boldsymbol{x}^1) \neq CI(\boldsymbol{x}^2)$. Let p be a place that is an element of both $\boldsymbol{x}^1$ and $\boldsymbol{x}^2$, i.e., $p \in \big(P(CI(\boldsymbol{x}^1)) \cap P(CI(\boldsymbol{x}^2))\big)$. Then p is called a *conflict place* of $CI(\boldsymbol{x}^1)$ and $CI(\boldsymbol{x}^2)$. The *conflict place set* is the subset $\mathcal{P}^{con} \subseteq P$, of places that are a conflict place between any two common input bag classes:

$$\mathcal{P}^{con} = \left\{p \in P \mid \exists i, j \text{ with } CI(\boldsymbol{x}^i) \neq CI(\boldsymbol{x}^j) \text{ and } p \in \big(P(CI(\boldsymbol{x}^i)) \cap P(CI(\boldsymbol{x}^j))\big)\right\}.$$

Definition 2.5 (**Sufficient place set and Surplus place set**)**.** A subset of places $\mathcal{P}^{suf} \subseteq P$ is a sufficient place set if for each initial marking $\mathbf{m}_0$, the marking of the places $p \in \mathcal{P}^{suf}$ combined with $\boldsymbol{m}_0$ provides sufficient information to uniquely define the marking of all places. A subset of places $\mathcal{P}^{sur} \subseteq P$ is a surplus place set if the subset of places $P \backslash \mathcal{P}^{sur}$ is a sufficient place set.

The following result obtained by Kortbeek and Boucherie [22] the decomposes an $\mathcal{S}\Pi$-net in several subnets such that a subnet is formed by one or more common input bag classes. This result is the starting point to derive a decomposition result for $\mathcal{S}\Pi^2$-nets in Section 3, which decomposes an $\mathcal{SPN}$ in all its common input bag classes.

Theorem 2.6 ([22])**.** Consider a product form $\mathcal{SPN}$ with transition rates (2), and a surplus place set $\mathcal{P}^{sur}$ with corresponding sufficient place set $\mathcal{P}^{suf}$. If $\nexists t \in T$ for which $\{p \in P \mid I_p(t) > 0\} \subseteq \mathcal{P}^{int} = \{p \in P \mid p \in (\mathcal{P}^{con} \cap \mathcal{P}^{sur})\}$, then

- removing all places $p \in \mathcal{P}^{int}$ and all arcs incident to the places $p \in \mathcal{P}^{int}$ yields s product form $\mathcal{S}\Pi$-nets: $\mathcal{SPN}^1, \ldots, \mathcal{SPN}^s$; each $\mathcal{SPN}^i$ corresponding of one or more connected common input bag classes,

- the equilibrium distribution π of $\mathcal{SPN}$ is a product over the invariant measures of the subnets:

$$\pi(\boldsymbol{m}) = B \prod_{i=1}^{s} \pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i), \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0),$$

where $\boldsymbol{m}^i$ is the submarking in places that belong to subnet $\mathcal{SPN}^i$, $\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i)$ is the invariant measure of subnet $\mathcal{SPN}^i$ with

$$\pi_y^{\mathcal{SPN}^i}(\boldsymbol{m}^i) = \prod_{\{p \in \cup_{j \in I^i} P(CI^j) \setminus \mathcal{P}^{int}\}} f_p^{m_p},$$

where $CI^j, j \in I^i$, denote the common input bag classes contained in subnet $\mathcal{SPN}^i$, and B is a normalizing constant.

3 Bag count places

This section introduces the Bag Count Place Extended Petri-net of a bounded $\mathcal{S}\Pi^2$-net ($\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net). For every input/output bag of an $\mathcal{S}\Pi^2$-net a 'bag count' place is added to the original net. By connecting the bag count places to the existing transitions, the marking of these places will track the marking of the original places by counting the net number of times each bag $\boldsymbol{r} \in \mathcal{R}(T)$ is consumed and deposited. It will be shown that the $\boldsymbol{a}_{\boldsymbol{r}}$-vectors from Definition 2.3 induce a one-to-one correspondence between the marking of the original places and the bag count places.

Definition 3.1 ($\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net)**.** Let $\mathcal{SPN} = (P, T, I, O, Q)$ be a structurally bounded $\mathcal{S}\Pi^2$-net. For each $\boldsymbol{r} \in \mathcal{R}(T)$, add bag-count place $\underline{p}_{\boldsymbol{r}}$ to P. The Bag-Count-Place-Extended $\mathcal{S}\Pi^2$-net ($\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net) of $\mathcal{SPN}$ is $\mathcal{SPN}^* = (\bar{P}, \bar{T}, \bar{I}, \bar{O}, Q)$, where

- $\bar{P} = P \cup \mathcal{P}^*$, with $\mathcal{P}^* = \bigcup_{\boldsymbol{r} \in \mathcal{R}(T)} p^*_{\boldsymbol{r}}$,
- $\bar{I}, \bar{O} : \bar{P} \times T \to \mathbb{N}$ with

$$\bar{I}(p,t) = \begin{cases} I(p,t) & , \text{if } p \in P, \\ 1 & , \text{if } p = p^*_{\boldsymbol{r}}, \boldsymbol{r} = \boldsymbol{I}(t), \\ 0 & , \text{otherwise}, \end{cases}$$

and

$$\bar{O}(p,t) = \begin{cases} O(p,t) & , \text{if } p \in P. \\ 1 & , \text{if } p = p^*_{\boldsymbol{r}}, \boldsymbol{r} = \boldsymbol{O}(t), \\ 0 & , \text{otherwise}. \end{cases}$$

Note that the marking of a bag count place $\underline{p}_{\boldsymbol{r}}$ changes if and only if a transition fires that either uses $\boldsymbol{r}$ as its input bag (in this case the marking of $\underline{p}_{\boldsymbol{r}}$ decreases by one), or creates $\boldsymbol{r}$ as its output bag (in this case the marking of $\underline{p}_{\boldsymbol{r}}$ increases by one). So the marking of $\underline{p}_{\boldsymbol{r}}$ indicates the number of times bag $\boldsymbol{r}$ is created minus the number of times bag $\boldsymbol{r}$ is used. This insight is the starting point to obtain the marking of the original places from the marking of the bag count places. To this end, first, in Lemma 3.2, we show that a $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net is an $\mathcal{S}\Pi^2$-net. Later, we will show in Lemma 3.5 that the initial marking on the bag count places can be chosen such that the marking of these places always remains positive, so that a $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net is an $\mathcal{SPN}$.

Definition is 3.1 a structural characterization. Lemmas 3.3 and 3.5 will show that for certain initial markings the behavior of a $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net is equivalent to its defining $\mathcal{S}\Pi^2$-net. Lemma 3.3 provides two conditions on the initial marking of the $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net which guarantee that a firing sequence σ can be fired in the original net if and only if σ can be fired in the $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net. Lemma 3.5 shows that for each structurally bounded $\mathcal{S}\Pi^2$-net, an initial marking satisfying the conditions of Lemma 3.3 can indeed be found. In Theorem 3.4 it is shown that there exists a one-to-one correspondence between the marking of the original places and the marking of the bag count places. Lemma 3.6 shows that a $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net is structurally bounded, a property that is a prerequisite for the decomposition result presented in Section 4. The decomposition result uses the result of Lemma 3.7 which gives the physical interpretation of $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-nets and therefore the $\boldsymbol{a_r}$-vectors in terms of state machines.

Lemma 3.2. The $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net $\mathcal{SPN}^*$ of an $\mathcal{S}\Pi^2$-net $\mathcal{SPN}$ is an $\mathcal{S}\Pi^2$-net.

Proof. Consider a minimal closed support T-invariant $\boldsymbol{x}$ of $\mathcal{SPN}$. For any transition $t \in \|\boldsymbol{x}\|$ there is a unique $t' \in \|\boldsymbol{x}\|$ such that $\boldsymbol{O}(t) = \boldsymbol{I}(t')$. By the construction of the $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net this yields $\boldsymbol{\alpha}(p^*, \boldsymbol{I}(t))\,\boldsymbol{x} = 0$, where $\boldsymbol{\alpha}(p^*, \boldsymbol{I}(t))$ denotes the row of the incidence matrix $\bar{\boldsymbol{A}}$ corresponding to place $p^*_{\boldsymbol{r}}$ with $\boldsymbol{r} = \boldsymbol{I}(t)$. Thus, $\boldsymbol{x}$ is also a T-invariant of $\mathcal{SPN}^*$. In addition, to see that $\boldsymbol{x}$ is a minimal closed support T-invariant of $\mathcal{SPN}^*$, observe that by construction if $\boldsymbol{I}(t) = \boldsymbol{O}(t')$ then $\bar{\boldsymbol{I}}(t) = \bar{\boldsymbol{O}}(t')$ also.

Next, every T-invariant of $\mathcal{SPN}^*$ is a T-invariant of $\mathcal{SPN}$, because the rows of $\bar{\boldsymbol{A}}$ for $p \in P$ are equal to the corresponding rows of $\boldsymbol{A}$, and thus, $\bar{\boldsymbol{A}}\boldsymbol{x} = 0 \Rightarrow \boldsymbol{A}\boldsymbol{x} = 0$. So, every minimal support T-invariant of $\mathcal{SPN}^*$ is a minimal closed support T-invariant.

Finally, since $\mathcal{SPN}$ and $\mathcal{SPN}^*$ have the same transition set T, it follows that in $\mathcal{SPN}^*$ every transition is covered by a minimal closed support T-invariant. □

Lemma 3.3. If the initial marking, $\bar{\boldsymbol{m}}_0$, of a $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net $\mathcal{SPN}^*$ corresponding to the marked $\mathcal{S}\Pi^2$-net $(\mathcal{SPN}, \boldsymbol{m}_0)$, is chosen such that $(\mathcal{SPN}^*, \bar{\boldsymbol{m}}_0)$ satisfies:

1. $\bar{m}_0(p) = m_0(p)$, for $p \in P$, and
2. for all $\bar{\boldsymbol{m}} \in \mathcal{M}(\mathcal{SPN}^*, \bar{m}_0)$, $\bar{m}(p) \geq 1$, for $p \in \mathcal{P}^*$,

then any firing sequence σ can be fired in $\mathcal{SPN}$ from $\boldsymbol{m}_0$ if and only if σ can be fired in $\mathcal{SPN}^*$ from $\bar{\boldsymbol{m}}_0$.

Proof. First, we show that every firing sequence σ that can be fired from $\boldsymbol{m}_0$ in $\mathcal{SPN}$ can be fired from $\bar{\boldsymbol{m}}_0$ in $\mathcal{SPN}^*$. Since $\bar{I}(p,t) = I(p,t)$ and $\bar{O}(p,t) = O(p,t)$ for places $p \in P$, these places will never disable a transition that is enabled in $\mathcal{SPN}$. Because $\bar{I}(p,t) \leq 1$ for $p \in \mathcal{P}^*$, condition 2 ensures that the same holds for these places.

Conversely, every firing sequence σ that can be fired from $\bar{\boldsymbol{m}}_0$ in $\mathcal{SPN}^*$ can be fired from $\boldsymbol{m}_0$ in $\mathcal{SPN}$, because $\bar{I}(p,t) = I(p,t)$ and $\bar{O}(p,t) = O(p,t)$ for places $p \in P$, and any transition $t \in T$ consumes and deposits the same number of tokens from the same places $p \in P$ in both nets. □

Theorem 3.4. Let $(\mathcal{SPN}^*, \bar{\boldsymbol{m}}_0)$ be a marked $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net corresponding to the marked $\mathcal{S}\Pi^2$-net $(\mathcal{SPN}, \boldsymbol{m}_0)$, and consider the markings $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$ and $\bar{\boldsymbol{m}} \in \mathcal{M}(\mathcal{SPN}^*, \bar{\boldsymbol{m}}_0)$.

1. The marking of the places $p \in \mathcal{P}^*$ in the $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net can be expressed in terms of the marking of the places $p \in P$ as follows:

$$\bar{m}(\underline{p}_{\boldsymbol{r}}) = \bar{m}_0(\underline{p}_{\boldsymbol{r}}) + \boldsymbol{a_r}(\boldsymbol{m} - \boldsymbol{m}_0), \tag{5}$$

where $\boldsymbol{a_r}$ is a vector as given in Definition 2.3.

2. The marking of the places $p \in P$ can be expressed in the marking of the places $p \in \mathcal{P}^*$ as follows:

$$\boldsymbol{m} = \boldsymbol{m}_0 + \sum_{\boldsymbol{r} \in \mathcal{R}(T)} (\bar{m}(p^*_{\boldsymbol{r}}) - \bar{m}_0(p^*_{\boldsymbol{r}}))\boldsymbol{r}.$$

As a consequence, there is a unique relation between the marking $\boldsymbol{m}$ of $\mathcal{SPN}$ and $\bar{\boldsymbol{m}}$ of $\mathcal{SPN}^*$.

Proof.

1. For every reachable marking $\bar{\boldsymbol{m}}$ there is a firing sequence σ such that $\bar{\boldsymbol{m}}_0[\sigma > \bar{\boldsymbol{m}}$, i.e., $\bar{\boldsymbol{m}} - \bar{\boldsymbol{m}}_0 = \bar{\boldsymbol{A}}\bar{\boldsymbol{\sigma}}$. Combining Definition 3.1 with Definition 2.3 it follows that $\boldsymbol{\alpha}_{p^*_r} = \boldsymbol{b_r} = \boldsymbol{a_r}\,\boldsymbol{A}$. Combining these results for $p \in \mathcal{P}^*$ gives:

$$\bar{m}(p^*_{\boldsymbol{r}}) - \bar{m}_0(p^*_{\boldsymbol{r}}) = \boldsymbol{\alpha}_{p^*_r}\bar{\boldsymbol{\sigma}} = \boldsymbol{a_r}\boldsymbol{A}\bar{\boldsymbol{\sigma}} = \boldsymbol{a_r}(\boldsymbol{m} - \boldsymbol{m}_0).$$

   It should be noted that neither $\boldsymbol{a}_r$ nor σ is uniquely defined. However, for all $\boldsymbol{a}^1_{\boldsymbol{r}}, \boldsymbol{a}^2_{\boldsymbol{r}}$ satisfying the conditions in Definition 2.3 and all σ_i such that $\boldsymbol{m}_0[\sigma_i > \boldsymbol{m}, i \in \{1, 2\}$, we have

$$\boldsymbol{a}^1_{\boldsymbol{r}}\boldsymbol{A}\bar{\boldsymbol{\sigma}}_1 = \boldsymbol{b}_r\bar{\boldsymbol{\sigma}}_1 = \boldsymbol{a}^2_{\boldsymbol{r}}\boldsymbol{A}\bar{\boldsymbol{\sigma}}_1 = \boldsymbol{a}^2_{\boldsymbol{r}}(\boldsymbol{m} - \boldsymbol{m}_0) = \boldsymbol{a}^2_{\boldsymbol{r}}\boldsymbol{A}\bar{\boldsymbol{\sigma}}_2,$$

   so that the marking of the places $p \in \mathcal{P}^*$ is uniquely determined from the marking of the places $p \in P$, independent of the choice of $\boldsymbol{a_r}$ and firing sequence σ.

2. By construction of the bag count places, for every firing sequence σ from $\boldsymbol{m}_0$ to $\boldsymbol{m}$, for every bag $\boldsymbol{r}$, $\bar{m}(\underline{p}_{\boldsymbol{r}}) - \bar{m}_0(p^*_{\boldsymbol{r}})$ indicates exactly how many times bag $\boldsymbol{r}$ is deposited minus the number of times bag $\boldsymbol{r}$ is consumed. Part 1 of the proof indicates that there is a unique difference $\bar{m}(\underline{p}_{\boldsymbol{r}}) - \bar{m}_0(p^*_{\boldsymbol{r}})$ corresponding to $\boldsymbol{m} - \boldsymbol{m}_0$. As a consequence, $\sum_{\boldsymbol{r} \in \mathcal{R}(T)}(\bar{m}(p^*_{\boldsymbol{r}}) - \bar{m}_0(p^*_{\boldsymbol{r}}))\,\boldsymbol{r}$ is independent of σ and thus $\boldsymbol{m}$ can be found by adding $\bar{m}(p^*_{\boldsymbol{r}}) - \bar{m}_0(p^*_{\boldsymbol{r}})$ times bag $\boldsymbol{r}$ for every bag $\boldsymbol{r} \in \mathcal{R}(T)$ to the initial marking $\boldsymbol{m}_0$. □

Lemma 3.5. Let $\mathcal{SPN}$ be a structurally bounded $\mathcal{S}\Pi^2$-net and $\mathcal{SPN}^*$ its corresponding $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net. For every initial marking $\boldsymbol{m}_0$ of $\mathcal{SPN}$, an initial marking $\bar{\boldsymbol{m}}_0$ of $\mathcal{SPN}^*$ can be chosen such that $\bar{m}(p^*_{\boldsymbol{r}}) \geq 1$, $\boldsymbol{r} \in \mathcal{R}(T)$, for all $\bar{\boldsymbol{m}} \in \mathcal{M}(\mathcal{SPN}^*, \bar{\boldsymbol{m}}_0)$.

Proof. Theorem 3.4 provides $\bar{m}(p^*_{\boldsymbol{r}}) - \bar{m}_0(p^*_{\boldsymbol{r}}) = \boldsymbol{a_r}(\boldsymbol{m} - \boldsymbol{m}_0)$ and since $(\mathcal{SPN}, \boldsymbol{m}_0)$ is bounded there is a constant C_p such that $0 \leq m(p) < C_p$ for all $p \in P$. Therefore

$$C_1 = \sum_{p \in P} \min(0, a_{\boldsymbol{r}}(p)C_p) \leq \boldsymbol{a_r}\boldsymbol{m} \leq \sum_{p \in P} \max(0, a_{\boldsymbol{r}}(p)C_p) = C_2,$$

so taking initial marking $\bar{m}_0(p^*_{\boldsymbol{r}}) = 1 - C_1 + \boldsymbol{a_r}\boldsymbol{m}_0$, we get

$$\bar{m}(p^*_{\boldsymbol{r}}) = \bar{m}_0(p^*_{\boldsymbol{r}}) + \boldsymbol{a_r}(\boldsymbol{m} - \boldsymbol{m}_0) = 1 - C_1 + \boldsymbol{a_r}\boldsymbol{m} \geq 1.$$ □

Lemma 3.6. The $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net $\mathcal{SPN}^*$ corresponding to a structurally bounded $\mathcal{S}\Pi^2$-net $\mathcal{SPN}$ is structurally bounded.

Proof. By Theorem 3.4, in $\mathcal{SPN}^*$ there is a one-to-one correspondence between the marking of the places $p \in P$ and the marking of the places $p \in \mathcal{P}^*$. Since $\mathcal{SPN}$ is bounded for every initial marking $\boldsymbol{m}_0$ and the marking of places $p \in \mathcal{P}^*$ is given by the linear equations (5), $\mathcal{SPN}^*$ is also bounded for every initial marking $\bar{\boldsymbol{m}}_0$. □

Lemma 3.7. Consider the $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net $\mathcal{SPN}^* = (\bar{P}, T, \bar{I}, \bar{O}, Q)$ of an $\mathcal{S}\Pi^2$-net $\mathcal{SPN}$. Removing all original places $p \in P$ from $\mathcal{SPN}^*$ and all arcs incident to the places $p \in P$ yields ℓ state machines: $\mathcal{SM}^1, \ldots, \mathcal{SM}^\ell$. Each $\mathcal{SM}^i$ corresponds to a common input bag class: $\mathcal{SM}^i = (\mathcal{P}^i, \mathcal{T}^i, I^i, O^i, Q^i)$, with $\mathcal{P}^i = \mathcal{P}(CI^i) \bigcap \mathcal{P}^*$, $\mathcal{T}^i = \mathcal{T}(CI^i)$, and where I^i, O^i, Q^i are $\bar{I}, \bar{O}, Q$ restricted to $\mathcal{P}^i$ and $\mathcal{T}^i$.

Proof. The proof follows by construction of the $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net. Every transition has exactly one bag count place in its input bag and exactly one bag count place in its output bag. Therefore, removing all original places from the net will yield a state machine. This state machine consists of ℓ separate components, because two bag count places p_1^* and p_2^* are connected in this state machine if and only if there is a CI-class CI^i such that $p_1^*, p_2^* \in \mathcal{P}(CI^i)$. □

Observe that marking $\boldsymbol{m}$ of $\mathcal{SPN}$ is characterized by the marking of the places $p \in \mathcal{P}^*$ in $\mathcal{SPN}^*$. Lemma 3.7 expresses that $\mathcal{SPN}^*$ without the original places yields ℓ state machines, one for each CI-class. We have the following interpretation of $\mathcal{S}\Pi^2$-nets: the marking $\boldsymbol{m}$ of an $\mathcal{S}\Pi^2$-net is characterized by the combination of the 'states' of each of its CI-classes, where the state of each CI-class is tracked by the marking of its state machine in the corresponding $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net.

Theorem 3.4 provides the interpretation of the $\boldsymbol{a_r}$-vectors. Every firing sequence in $\mathcal{SPN}$ which brings $\boldsymbol{m}_0$ to $\boldsymbol{m}$ is associated with a unique value for the difference in the number of times each bag $\boldsymbol{r}$ is deposited and consumed in the firing sequence. The vector $\boldsymbol{a_r}$ gives the transformation to calculate this number: $\boldsymbol{a_r}(\boldsymbol{m} - \boldsymbol{m}_0)$, that turns out to be independent of the firing sequence. Thus, the $\boldsymbol{a_r}$-vectors are used to track the 'state' of each of the CI -classes.

4 Decomposition

Building on the insights of the previous section, in this section we will decompose the equilibrium distribution of an $\mathcal{S}\Pi^2$ into a product of the invariant measures of the state machines corresponding to these CI-classes. In Theorem 2.6, the decomposition of an $\mathcal{S}\Pi$-net can be such that a subnet is formed by multiple connected common input bag classes. Here, we take Theorem 2.6 as a starting point to derive a decomposition result for $\mathcal{S}\Pi^2$-nets, which decomposes an $\mathcal{SPN}$ in all its common input bag classes.

Recall that in decomposition Theorem 2.6 two types of place sets play a key-role: the conflict place set and the surplus place set. Decomposition is established if the places in the intersection of those two sets can be removed from the net so that live components remain. Since in a $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$ the bag count places form a sufficient place set, the direct consequence is that the set of all original places forms a surplus place set, which implies that all conflict places can be assigned to be surplus places. This leads to the application of Theorem 2.6 in Theorem 4.1.

Note that a state machine Petri net is equivalent to a Jackson network, see also [30]. So, the routing chain of a state machine is equivalent to the well-known traffic equations from queueing theory. And since the structure of a state machine induces that each T-invariant has a closed support, with $\bar{\boldsymbol{m}}^i$ its marking, the equilibrium distribution of a state machine $\mathcal{SM}^i$ as introduced in Lemma 3.7 is as follows:

$$\pi^{SM}(\bar{\boldsymbol{m}}^i) = C \prod_{\boldsymbol{r}\in\mathcal{R}(\mathcal{T}^i)} y^i(\boldsymbol{r})^{\bar{m}^i(p_r^*)} \qquad , \bar{\boldsymbol{m}}^i \in \{\bar{\boldsymbol{m}}^i : \sum_{\boldsymbol{r}\in\mathcal{R}(\mathcal{T}^i)} \bar{\boldsymbol{m}}^i(p_{\boldsymbol{r}}^*)\},$$

where $y^i(\cdot)$ is the solution of the routing chain (4) of state machine $\mathcal{SM}^i$, and C is a normalizing constant.

Theorem 4.1. Consider an $\mathcal{S}\Pi^2$-net $\mathcal{SPN} = (P, T, I, O, Q)$ with its $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net $\mathcal{SPN}^*$, a set of vectors $\boldsymbol{a}_{\boldsymbol{r}}, \boldsymbol{r} \in \mathcal{R}(T)$ satisfying the conditions of Definition 2.3, and an initial marking $\bar{\boldsymbol{m}}_0$ satisfying the conditions of Lemma 3.3. Then, the equilibrium distribution π of $\mathcal{SPN}$ is equal to the equilibrium distribution $\bar{\pi}$ of $\mathcal{SPN}^*$, of which the invariant measure is a product over the invariant measures of the state machines:

$$\pi(\boldsymbol{m}) = \bar{\pi}(\bar{\boldsymbol{m}}) = B \prod_{i=1}^{\ell} \pi^{\mathcal{SM}^i}(\bar{\boldsymbol{m}}^i) \quad , \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0), \tag{6}$$

Proof. By Lemma 3.7, removing all original places $p \in P$ from $\mathcal{SPN}^*$ yields ℓ state machines: $\mathcal{SM}^1, \ldots, \mathcal{SM}^\ell$; each $\mathcal{SM}^i$ corresponding to exactly one common input bag class. Next, we obtain from Theorem 3.4 that $\mathcal{P}^*$ is a sufficient place set. Therefore, the set of original places P is a surplus place set. By construction, all conflict places of a $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net are original places, i.e., $\mathcal{P}^{con} \subseteq P$. Since every transition is connected to a bag count place, no complete input bag is contained in the conflict place set, i.,e $\nexists t \in \bar{P}$ for which $\{p \in \bar{P} \mid \bar{I}_p(t) > 0\} \subset (\mathcal{P}^{con} \bigcap P)$. Theorem 2.6 and Lemma 3.6 complete the proof. □

5 Examples

This section illustrates the similarities and differences between Theorem 2.6 and Theorem 4.1 via three examples. The first example is an $\mathcal{S}\Pi^2$-net consisting of two CI-classes linked by a single conflict place. This conflict place will form a surplus place set by itself which means that both Theorem 2.6 and Theorem 4.1 give us the means to decompose it into two separate CI-classes. This example shows that both methods result in the same decomposition, however they follow a different path to obtain this decomposition. The second example is an $\mathcal{S}\Pi^2$-net, with three CI-classes, that can be decomposed in two ways into two parts using Theorem 2.6. Theorem 4.1, enables us to decompose it into three parts, one for each CI-class. The third example is an $\mathcal{S}\Pi^2$-net that has three CI-classes, where all places are conflict places. Obviously, Theorem 2.6 will not lead to a decomposition, whereas Theorem 4.1 again allows complete decomposition into CI-classes. This example shows that even if the CI-classes are strongly intertwined and the product form over the places does not seem to be able to be decomposed, it is still possible to separate the different CI-classes and identify their behavior separately. Finally, Example 5.4 is obtained from [15], and provides an illustration of Theorem 4.1 when a probabilistic output bag is involved.

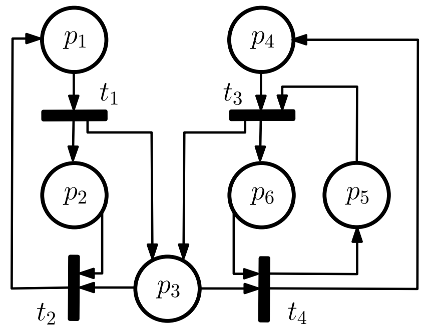


Figure 1: $\mathcal{SPN}$ of Example 5.1.

Example 5.1. Consider the stochastic Petri net $\mathcal{SPN}$ displayed in Figure 1. From the incidence matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix},$$

we obtain two minimal support T-invariants $\boldsymbol{x}^1 = (1100)$ and $\boldsymbol{x}^2 = (0011)$, and five minimal support P-invariants $\boldsymbol{y}^1 = (110000)$, $\boldsymbol{y}^2 = (101100)$, $\boldsymbol{y}^3 = (101010)$, $\boldsymbol{y}^4 = (000101)$ and $\boldsymbol{y}^5 = (000011)$ of which the first four are linearly independent. The two T-invariants are both closed and cover all transitions, so $\mathcal{SPN}$ is an $\mathcal{S}\Pi^2$-net. The T-invariants are not in common input bag relation, therefore $\mathcal{SPN}$ has two common input bag classes $CI^1 = \{\boldsymbol{x}^1\}$ and $CI^2 = \{\boldsymbol{x}^2\}$. This gives us one conflict place set $\{p_3\}$. Using the P-invariants we find that $\mathcal{P}^1 = \{p_2, p_3, p_5, p_6\}$ and $\mathcal{P}^2 = \{p_1, p_3, p_4, p_6\}$ are surplus place sets. Both these sets give $\mathcal{P}^{sur} \bigcap \mathcal{P}^{con} = \{p_3\}$, so in both cases Theorem 2.6 provides a decomposition into SPN^1 consisting of places $\{p_1, p_2\}$ and transitions $\{t_1, t_2\}$ and SPN^2 consisting of places $\{p_4, p_5, p_6\}$ and transitions $\{t_3, t_4\}$ (see Figure 2a). The equilibrium distribution of $\mathcal{SPN}$ is given by:

$$\begin{aligned} \pi(\boldsymbol{m}) &= B\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1)\pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2) \\ &= B\left(\frac{\mu_1}{\mu_2}\right)^{m(p_2)}\left(\frac{\mu_4}{\mu_3}\right)^{m(p_5)} \qquad (7) \\ &= B\left(\frac{\mu_2}{\mu_1}\right)^{m(p_1)}\left(\frac{\mu_4}{\mu_3}\right)^{m(p_4)}, \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0), \qquad (8) \end{aligned}$$

where the form (7) is obtained when surplus place set $\mathcal{P}^1$ is used, and (8) when surplus place set $\mathcal{P}^2$ is used.

Now, let us apply Theorem 4.1. First we construct the $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net of $\mathcal{SPN}$ by adding four bag count places, $p_1^*, \ldots, p_4^*$. Now, removing the original places $p_1, \ldots, p_6$, gives the net shown in Figure 2b. This leads to the following equilibrium distribution, for

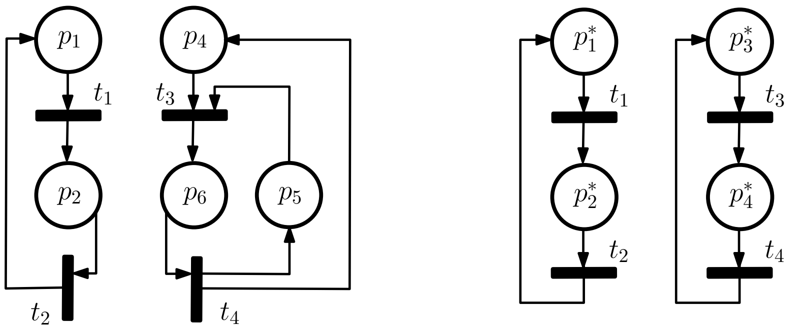


Figure 2: (a) Decomposition Ex. 5.1 via Thm. 2.6 (b) Decomposition via Thm. 4.1.

$\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$:

$$\begin{aligned}
\pi(\boldsymbol{m}) &= B\pi^{\mathcal{SM}^1}(\bar{\boldsymbol{m}}^1)\pi^{\mathcal{SM}^2}(\bar{\boldsymbol{m}}^2) \\
&= B\left(\frac{1}{\mu_1}\right)^{\bar{m}(p_1^*)}\left(\frac{1}{\mu_2}\right)^{\bar{m}(p_2^*)}\left(\frac{1}{\mu_3}\right)^{\bar{m}(p_3^*)}\left(\frac{1}{\mu_4}\right)^{\bar{m}(p_4^*)} \\
&= B\left(\frac{1}{\mu_1}\right)^{\boldsymbol{a}_{\boldsymbol{I}(t_1)}\boldsymbol{m}}\left(\frac{1}{\mu_2}\right)^{\boldsymbol{a}_{\boldsymbol{I}(t_2)}\boldsymbol{m}}\left(\frac{1}{\mu_3}\right)^{\boldsymbol{a}_{\boldsymbol{I}(t_3)}\boldsymbol{m}}\left(\frac{1}{\mu_4}\right)^{\boldsymbol{a}_{\boldsymbol{I}(t_4)}\boldsymbol{m}}.
\end{aligned}$$

One of the possible choices for the vectors $\boldsymbol{a_r}$ is $\boldsymbol{a}_{\boldsymbol{I}(t_1)} = (1,0,0,0,0,0)$ and $\boldsymbol{a}_{\boldsymbol{I}(t_3)} = (0,0,0,1,0,0)$. This choice corresponds to (7), so to choosing $\mathcal{P}^{sur} = \mathcal{P}^1$ in Theorem 2.6. A second possible choice is $\boldsymbol{a}_{\boldsymbol{I}(t_1)} = (0,\text{-}1,0,0,0,0)$ and $\boldsymbol{a}_{\boldsymbol{I}(t_3)} = (0,0,0,0,1,0)$, which corresponds to (8), so to choosing $\mathcal{P}^{sur} = \mathcal{P}^1$ in Theorem 2.6.

The first observation is that in this example Theorem 2.6 and Theorem 4.1 both lead to decomposition into two subnets and that the subnets correspond to the same parts of $\mathcal{SPN}$. However, the structure of the pieces is not necessarily the same. The subnet corresponding to CI^1 is the same in both cases, however the part corresponding to CI^2 has a different structure. The second observation is that a zero entries in all $\boldsymbol{a_r}$-vectors for a specific place $p \in P$, corresponds to assigning p as a surplus place. □

Example 5.2. Consider the $\mathcal{SPN}$ depicted in Figure 3a. From the incidence matrix:

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix},$$

we obtain the three minimal support T-invariants $\boldsymbol{x}^1 = (110000)$, $\boldsymbol{x}^2 = (01100)$ and $\boldsymbol{x}^3 = (000011)$ and four minimal support P-invariants $\boldsymbol{y}^1 = (1101000)$, $\boldsymbol{y}^2 = (1010010)$, $\boldsymbol{y}^3 = (0001100)$ and $\boldsymbol{y}^4 = (0000011)$, which are linearly independent. As the minimal support T-invariants are all closed and they cover all transitions, $\mathcal{SPN}$ is an $\mathcal{S}\Pi^2$-net. Furthermore,

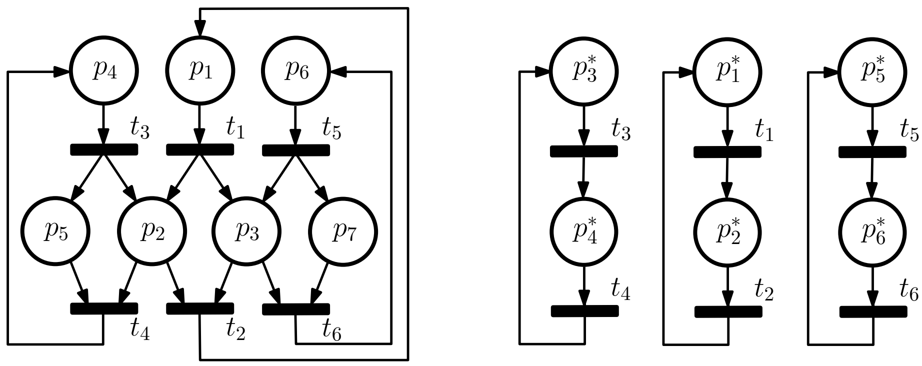


Figure 3: (a) $\mathcal{SPN}$ of Example 5.2 (b) Decomposition via Theorem 4.1.

$\boldsymbol{x}^1$, $\boldsymbol{x}^2$ and $\boldsymbol{x}^3$ are not in common input bag relation so they result in three CI-classes, $CI^1 = \{\boldsymbol{x}^1\}$, $CI^2 = \{\boldsymbol{x}^2\}$ and $CI^3 = \{\boldsymbol{x}^3\}$. This results in the following conflict place set: $\mathcal{P}^{con} = \{p_2, p_3\}$.

Since the complete input bag of transition t_1 is contained in $\mathcal{P}^{con}$, Theorem 2.6 is not able to separate all CI-classes. However, both $\mathcal{P}^1 = \{p_2\}$ and $\mathcal{P}^2 = \{p_3\}$ are surplus place sets. Both lead to a decomposition of the equilibrium distribution:

$$\pi(\boldsymbol{m}) = B\pi_y^{\mathcal{SPN}^1}(\boldsymbol{m}^1)\pi_y^{\mathcal{SPN}^2}(\boldsymbol{m}^2),$$

where in the case of decomposition via $\mathcal{P}^1$ the two subnetworks are $\mathcal{SPN}^1 = \{CI(\boldsymbol{x}^1), CI(\boldsymbol{x}^3)\}$ and $\mathcal{SPN}^2 = \{CI(\boldsymbol{x}^2)\}$, while via $\mathcal{P}^2$ the two subnetworks are $\mathcal{SPN}^1 = \{CI(\boldsymbol{x}^1), CI(\boldsymbol{x}^2)\}$ and $\mathcal{SPN}^2 = \{CI(\boldsymbol{x}^3)\}$.

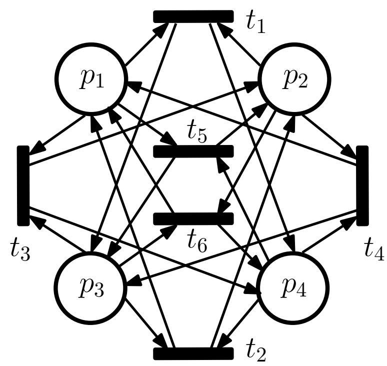
To illustrate the power of Theorem 4.1 over Theorem 2.6, we construct the $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net of $\mathcal{SPN}$. By adding the six bag count places, $p_1^*, \ldots, p_6^*$, to the net and then removing all original places, $p_1, \ldots, p_7$, we obtain the net shown in Figure 3b. A simple choice of the $\boldsymbol{a_r}$-vectors is allowed, similar to the previous example: $\boldsymbol{a}_{\boldsymbol{I}(t_1)} = (1,0,0,0,0,0)$, $\boldsymbol{a}_{\boldsymbol{I}(t_2)} = (\text{-}1,0,0,0,0,0)$, $\boldsymbol{a}_{\boldsymbol{I}(t_3)} = (0,0,0,1,0,0,0)$, $\boldsymbol{a}_{\boldsymbol{I}(t_4)} = (0,0,0,\text{-}1,0,0,0)$, $\boldsymbol{a}_{\boldsymbol{I}(t_5)} = (0,0,0,0,0,1,0)$ and $\boldsymbol{a}_{\boldsymbol{I}(t_6)} = (0,0,0,0,0,\text{-}1,0)$. This yields the following equilibrium distribution:

$$\begin{aligned}
\pi(\boldsymbol{m}) &= B\pi^{\mathcal{SM}^1}(\bar{\boldsymbol{m}}^1)\pi^{\mathcal{SM}}(\bar{\boldsymbol{m}}^2)\pi^{\mathcal{SM}}(\bar{\boldsymbol{m}}^3) \\
&= B\left(\frac{\mu_2}{\mu_1}\right)^{\bar{m}(p_1^*)}\left(\frac{\mu_4}{\mu_3}\right)^{\bar{m}(p_3^*)}\left(\frac{\mu_6}{\mu_5}\right)^{\bar{m}(p_5^*)} \\
&= B\left(\frac{\mu_2}{\mu_1}\right)^{m(p_1)}\left(\frac{\mu_4}{\mu_3}\right)^{m(p_4)}\left(\frac{\mu_6}{\mu_5}\right)^{m(p_6)}, \quad \boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0).
\end{aligned}$$

So Theorem 4.1 enables a decomposition in the three cyclic state machines corresponding to the three CI-classes. □

Example 5.3. Consider the $\mathcal{SPN}$ of Figure 4a, with the following incidence matrix

$$\boldsymbol{A} = \begin{pmatrix} -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \end{pmatrix}$$



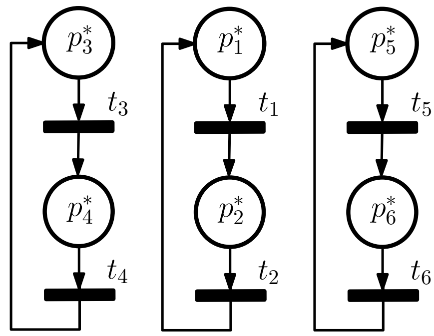


Figure 4: (a) $\mathcal{SPN}$ of Example 5.3 (b) Decomposition via Theorem 4.1.

There are three minimal support T-invariants $\boldsymbol{x}^1 = (110000)$, $\boldsymbol{x}^2 = (001100)$ and $\boldsymbol{x}^3 = (000011)$ and one minimal support P-invariant $\boldsymbol{y}^1 = (1111)$. All the T-invariants are closed so it is an $\mathcal{S}\Pi^2$-net and none of the T-invariants are in common input bag relation, so there are three CI-classes, $CI^1 = \{\boldsymbol{x}^1\}$, $CI^2 = \{\boldsymbol{x}^2\}$ and $CI^3 = \{\boldsymbol{x}^3\}$. All places belong to each of the three CI-classes so the set of conflict places is $\{p_1, p_2, p_3, p_4\}$. Clearly, Theorem 2.6 does not lead to a decomposition. For Theorem 4.1, add the six bag count places to obtain the $\mathcal{BCPE}$-$\mathcal{S}\Pi^2$-net with incidence matrix:

$$\bar{\boldsymbol{A}} = \begin{pmatrix} -1 & 1 & -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

A possible choice is for a set of vectors $\boldsymbol{a}_r, \boldsymbol{r} \in \mathcal{M}(T)$ is: $\boldsymbol{a}_{\boldsymbol{I}(t_1)} = (1/2, 1/2, 0, 0)$, $\boldsymbol{a}_{\boldsymbol{I}(t_2)} = (-1/2, -1/2, 0, 0)$, $\boldsymbol{a}_{\boldsymbol{I}(t_3)} = (1/2, 0, 1/2, 0)$, $\boldsymbol{a}_{\boldsymbol{I}(t_4)} = (-1/2, 0, -1/2, 0)$, $\boldsymbol{a}_{\boldsymbol{I}(t_5)} = (0, -1/2, -1/2, 0)$, and $\boldsymbol{a}_{\boldsymbol{I}(t_6)} = (0, 1/2, 1/2, 0)$.

By removing the original places from the net we obtain the net shown in Figure 4b. Note that this net is the same as the reduced net we obtained in Example 5.2. Thus, we obtain the following equilibrium distribution, for $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$:

$$\begin{aligned} \pi(\boldsymbol{m}) &= B\pi^{\mathcal{SM}^1}(\bar{\boldsymbol{m}}^1)\pi^{\mathcal{SM}^2}(\bar{\boldsymbol{m}}^2)\pi^{\mathcal{SM}^3}(\bar{\boldsymbol{m}}^3) \\ &= B\left(\frac{\mu_2}{\mu_1}\right)^{\bar{m}(p_1^*)}\left(\frac{\mu_4}{\mu_3}\right)^{\bar{m}(p_3^*)}\left(\frac{\mu_6}{\mu_5}\right)^{\bar{m}(p_5^*)} \end{aligned}$$

$$= B\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{2}(m(p_1)+m(p_2))}\left(\frac{\mu_4}{\mu_3}\right)^{\frac{1}{2}(m(p_1)+m(p_3))}\left(\frac{\mu_5}{\mu_6}\right)^{\frac{1}{2}(m(p_2)+m(p_3))} \qquad \square$$

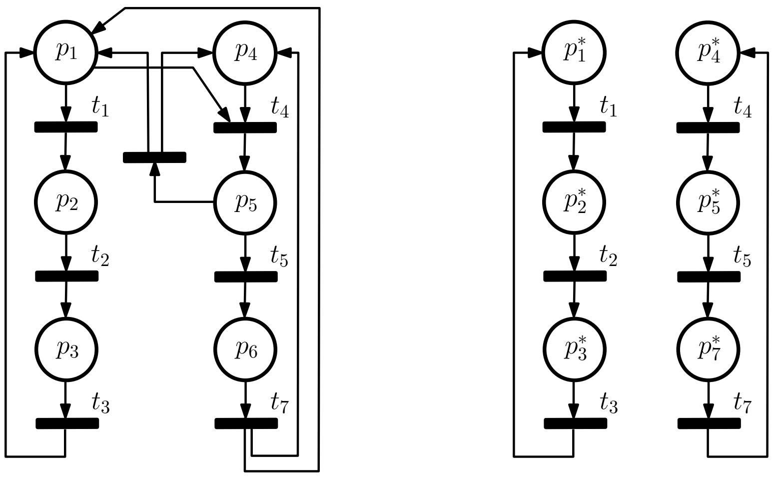


Figure 5: (a) $\mathcal{SPN}$ of Example 5.4 (b) Decomposition via Theorem 4.1.

Example 5.4. Consider the $\mathcal{S}\Pi^2$-net of Figure 5a, taken from [15], which has minimal minimal closed support T-invariants $\boldsymbol{x}^1 = (111000), \boldsymbol{x}^2 = (0001100)$ and $\boldsymbol{x}^3 = (0001011)$. The CI-classes are: $CI^1 = \{\boldsymbol{x}^1\}$ and $CI^2 = \{\boldsymbol{x}^2, \boldsymbol{x}^3\}$. Theorem 2.6 does not provide a decomposition, since it would require the removal of the complete input bag of transition t_1. Since t_5 are t_6 have the same input bag, the probabilistic output bag transformation is applied, and Theorem 4.1 requires the creation of only six bag count places. The decomposed net is shown in Figure 5b. A possible choice for the $\boldsymbol{a_r}$-vectors is (also see [15]): $\boldsymbol{a}_{\boldsymbol{I}(t_1)} = (0,\text{-}1,\text{-}1,0,0,0), \boldsymbol{a}_{\boldsymbol{I}(t_2)} = (0,1,0,0,0,0), \boldsymbol{a}_{\boldsymbol{I}(t_3)} = (0,0,1,0,0,0), \boldsymbol{a}_{\boldsymbol{I}(t_4)} = (0,0,0,1,0,0), \boldsymbol{a}_{\boldsymbol{I}(t_5)} = (0,0,0,0,1,0)$, and $\boldsymbol{a}_{\boldsymbol{I}(t_7)} = (0,0,0,0,0,1)$, which leads to the following equilibrium distribution, for $\boldsymbol{m} \in \mathcal{M}(\mathcal{SPN}, \boldsymbol{m}_0)$:

$$\begin{aligned}
\pi(\boldsymbol{m}) &= B\pi^{\mathcal{SM}^1}(\bar{\boldsymbol{m}}^1)\pi^{\mathcal{SM}^2}(\bar{\boldsymbol{m}}^2) \\
&= B\left(\frac{\mu_1}{\mu_2}\right)^{m(p_2)}\left(\frac{\mu_1}{\mu_3}\right)^{m(p_3)}\left(\frac{1}{\mu_4}\right)^{m(p_4)}\left(\frac{1}{\mu_5}\right)^{m(p_5)}\left(\frac{\mu_6}{(\mu_5+\mu_6)\mu_7}\right)^{m(p_6)}
\end{aligned}$$

□

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