Tree, web and average web value for cycle-free directed graph games*

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Abstract

On the class of cycle-free directed graph games with transferable utility solution concepts, called web values, are introduced axiomatically, each one with respect to some specific choice of a management team of the graph. We provide their explicit formula representation and simple recursive algorithms to calculate them. Additionally the efficiency and stability of web values are studied. Web values may be considered as natural extensions of the tree and sink values as has been defined correspondingly for rooted and sink forest graph games. In case the management team consists of all sources (sinks) in the graph a kind of tree (sink) value is obtained. In general, at a web value each player receives the worth of this player together with his subordinates minus the total worths of these subordinates. It implies that every coalition of players consisting of a player with all his subordinates receives precisely its worth. We also define the average web value as the average of web values over all management teams in the graph. As application the water distribution problem of a river with multiple sources, a delta and possibly islands is considered.

Keywords: TU game, cooperation structure, Myerson value, efficiency, deletion link property, stability

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1 Introduction

In standard cooperative game theory it is assumed that any coalition of players may form. However, in many practical situations the collection of coalitions that can be formed is restricted by some social, economical, hierarchical, communication, or technical structure. The study of games with transferable utility and limited cooperation introduced by means of communication graphs was initiated by Myerson

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[6]. In this paper we restrict our consideration to the class of cycle-free digraph games in which the players are partially ordered and the communication via bilateral agreements between players is represented by a directed graph without directed cycles. A cycle-free digraph cooperation structure allows modeling of various flow situations when several links may merge at a node, while other links split at a node into several separate ones.

It is assumed that a directed link represents a one-way communication situation. This restricts the set of coalitions that can be formed. There are different scenarios possible for controlling cooperation in case of directed communication. It is possible that players are controlled only by their predecessors. Another scenario assumes that players are controlled only by their successors. But it is also possible that the management team is located neither at the top nor at the bottom of the given directed communication structure but somewhere in between and each manager keeps control over all of his successors and predecessors.

We introduce web values for cycle-free digraph games axiomatically, each one with respect to a chosen management team, and provide their explicit formula representation. On the class of cycle-free digraph games with a fixed management team the web value is completely characterized by web efficiency (WE), web successor equivalence (WSE) and web predecessor equivalence (WPE), where a value is web efficient, if for every manager of the given management team it holds that the payoff for this manager together with all his successors and all his predecessors is equal to the total worth they can get by their own. A value satisfies WSE if when a link towards a player from one of the managers or one of the successors of the given management team is deleted, this player and all his successors will get the same payoff, and a value satisfies WPE if when a link from a player being a predecessor of the management team is deleted, this player and all his predecessors will get the same payoff. It implies that the web value assigns to every player what he contributes when he joins his subordinates in the graph and that the total payoff for any player together with all his subordinates is equal to the worth they can get all together by their own. It is worth to emphasize that the web value should not be considered as personal payment by one player to another one (the boss to his subordinate) but as distribution of the total worth according to the proposed scheme. We also provide simple recursive computational methods for computing web values and study their efficiency and when possible stability.

The values are introduced for arbitrary cycle-free digraph games and can be considered as natural extensions of the tree and sink values defined for rooted and sink forest digraph games, respectively (cf. [2], [5]). Besides, we define the average web value by taking the average of web values over all management teams of the graph. This value depends only on a given TU game and a given cycle-free directed communication graph and does not depend on the choice among different options for controlling cooperation. Furthermore, we extend the Ambec and Sprumont ([1]) line-graph river game model of sharing a river to the case of a river with multiple sources, a delta and possibly islands by applying the results obtained to this more general setting of sharing a river among different agents located at different levels along the river bed restated in terms of a cycle-free digraph game.

The paper has a following structure. Basic definitions and notation are introduced in Section 2. In Section 3 we discuss different scenarios possible for controlling the situation defined by a digraph communication structure with respect to the chosen management team (anti-chain in the digraph). Section 4 investigates a particular case when the control is going from the top to the bottom, which provides the so-called tree value. In Section 5 the general case of web values is studied. The average web value is introduced in Section 6. In Section 7 the application to the water distribution problem of a river with multiple sources, a delta and possibly islands is considered.

2 Preliminaries

A *cooperative game with transferable utility (TU game)* is a pair $\langle N, v\rangle$, where $N = \{1, \ldots, n\}$ is a finite set of n, $n \geq 2$, players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a *coalition* and the associated real number $v(S)$ represents the *worth* of coalition S. The set of TU games with fixed player set N we denote $\mathcal{G}_N$. For simplicity of notation and if no ambiguity appears, we write v when we refer to a TU game $\langle N, v\rangle$. A game $v \in \mathcal{G}_N$ is *superadditive* if $v(S \cup T) \geq v(S) + v(T)$ for all $S, T \subseteq N$, such that $S \cap T = \emptyset$, and $v \in \mathcal{G}_N$ is *convex* if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$, for all $S, T \subseteq N$. A *value* on a subset $\mathcal{G}$ of $\mathcal{G}_N$ is a function $\xi\colon \mathcal{G} \to \mathbb{R}^N$ that assigns to every game $v \in \mathcal{G}$ a vector $\xi(v) \in \mathbb{R}^N$; the number $\xi_i(v)$ represents the *payoff* to player i, $i \in N$, in the game v. In the sequel we use standard notation $x(S)=\sum_{i\in S} x_i$, $x_S=(x_i)_{i\in S}$ for any $x\in\mathbb{R}^N$ and $S \subseteq N$, $|A|$ for the cardinality of a finite set A, and omit brackets when writing one-player coalitions such as i instead of $\{i\}$, $i \in N$.

For a game $v \in \mathcal{G}_N$, a payoff vector $x \in \mathbb{R}^N$ is *efficient* if $x(N) = v(N)$ and is *feasible* if $x(N) \leq v(N)$.

The *core* [3] of a game $v \in \mathcal{G}_N$ is defined as

$$C(v) = \{x \in \mathbb{R}^N \mid x(N) = v(N),\ x(S) \geq v(S), \text{ for all } S \subseteq N\}.$$

For a game $v \in \mathcal{G}_N$ we may also consider the *weak core* defined as

$$\tilde{C}(v) = \{x \in \mathbb{R}^N \mid x(N) \leq v(N),\ x(S) \geq v(S), \text{ for all } S \subsetneq N\}.$$

A value ξ on a subset $\mathcal{G}$ of $\mathcal{G}_N$ is *stable* if for any game $v \in \mathcal{G}$ it holds that $\xi(v) \in C(v)$, and a value ξ on $\mathcal{G}$ is *weakly stable* if for any game $v \in \mathcal{G}$ it holds that $\xi(v) \in \tilde{C}(v)$.

The *cooperation structure* on the player set N is specified by a graph, directed or undirected, on N. An *undirected graph* on N consists of a set of nodes, being the elements of N, and a collection of unordered pairs of nodes $\Gamma \subseteq \Gamma_N^c$, where $\Gamma_N^c = \{\, \{i,j\} \,|\, i,j \in N,\ i \neq j\}$ is the complete undirected graph without loops on N and an unordered pair $\{i,j\} \in \Gamma$ is a *link* between $i, j \in N$. A *directed graph*, or *digraph*, on N is given by a collection of ordered pairs of nodes $\Gamma \subseteq \bar{\Gamma}_N^c$, where $\bar{\Gamma}_N^c = \{(i,j) \,|\, i,j \in N,\ i \neq j\}$ is the complete directed graph without loops on N and an ordered pair $(i,j) \in \Gamma$ is a *directed link* between $i, j \in N$. In this paper we study cooperation structures represented by directed graphs. A subset Γ' of a (directed or undirected) graph Γ on N is a *subgraph* of Γ. For a subgraph Γ' of a digraph Γ on N, $N(\Gamma') \subseteq N$ is the set of nodes in Γ', i.e., $N(\Gamma') = \{i \in N \mid \exists j \in N\colon \{(i,j),(j,i)\} \cap \Gamma' \neq \emptyset\}$. For a digraph Γ on N and a coalition $S \subseteq N$, the *subgraph of Γ on S* is the digraph $\Gamma|_S = \{(i,j)\in\Gamma \mid i,j\in S\}$ on S.

In a graph Γ on N a sequence of different nodes $p = (i_1, \ldots, i_r)$, $r \geq 2$, is a *path* in Γ from node i_1 to node i_r if for $h=1,\ldots,r-1$ it holds that $\{i_h, i_{h+1}\} \in \Gamma$ when Γ is undirected and $\{(i_h, i_{h+1}), (i_{h+1}, i_h)\} \cap \Gamma \neq \emptyset$ when Γ is directed. In a digraph Γ a path $\vec{p} = (i_1, \ldots, i_r)$ is a *directed path* from node i_1 to node i_r if for all $h=1,\ldots,r-1$ it holds that $(i_h, i_{h+1}) \in \Gamma$. For a digraph Γ on N and any $i, j \in N$ we denote by $\vec{P}_\Gamma(i,j)$ the set of all directed paths from i to j in Γ. Any node i of a (directed) path p we denote as an element of p, i.e., $i \in p$. Moreover, when for a directed path $\vec{p}$ in a digraph Γ we write $(i,j) \in \vec{p}$, we assume that i and j are consecutive nodes in $\vec{p}$. For any set P of (directed) paths, by $N(P) = \{i \in p \mid p \in P\}$ we denote the set of nodes determining the paths in P. In a digraph Γ a directed link $(i,j) \in \Gamma$ for which there exists a directed path $\vec{p}$ in Γ from i to j such that $\vec{p} \neq (i,j)$ is *inessential*, otherwise (i,j) is an *essential* link. In a digraph Γ a directed path $\vec{p}$ is a *proper path* if it contains only essential links.

Given a graph Γ on N, two nodes i and j in N are *connected* in Γ if there exists a path in Γ from node i to node j. Γ is *connected* if any two nodes in N are connected. A coalition $S \subseteq N$ is *connected* in Γ if the subgraph $\Gamma|_S$ on S is connected. For a coalition $S \subseteq N$, $C_\Gamma(S)$ is the set of all connected subcoalitions of S in Γ, S/Γ is the set of maximally connected subcoalitions of S in Γ, called the *components of* S in Γ, and $(S/\Gamma)_i$ is the component of S in Γ containing player $i \in S$.

For a digraph Γ on N and any $i, j \in N$, j is a *(proper) successor* of i and i is a *(proper) predecessor* of j if there is a directed (proper) path from i to j. For a directed (essential) link $(i,j) \in \Gamma$, i is the *origin* and j is the *terminus*, i is a *(proper) immediate predecessor* of j and j is a *(proper) immediate successor* or *(proper) follower* of i. Node $j \in N$ is a brother of node $i \in N$ if both have a same predecessor in Γ. For $i \in N$, we denote by $P_\Gamma(i)$ the set of predecessors of i in Γ, by $O_\Gamma(i)$ the set of immediate predecessors of i in Γ, by $O^*_\Gamma(i)$ the set of proper immediate predecessors of i, by $F_\Gamma(i)$ the set of immediate successors of i in Γ, by $F^*_\Gamma(i)$ the set of proper immediate successors of i, by $S_\Gamma(i)$ the set of successors of i in Γ, and by $B_\Gamma(i)$ the set of brothers of i. Moreover, for $i \in N$, we define $\bar{P}_\Gamma(i) = P_\Gamma(i) \cup i$, $\bar{S}_\Gamma(i) = S_\Gamma(i) \cup i$, and $\bar{B}_\Gamma(i) = B_\Gamma(i) \cup i$.

For a digraph Γ on N and a node $i \in N$, the set $W_\Gamma(i) = S_\Gamma(i) \cup P_\Gamma(i) \cup i$ defines the *web* of i in Γ with i being its *hub*, and all $j \in W_\Gamma(i) \backslash \{i\}$ are called *subordinates* of i. A coalition $S \subseteq N$ is a *full successors set* in Γ, if $S = \bar{S}_\Gamma(i)$ for some $i \in N$, and is a *full predecessors set* in Γ, if $S = \bar{P}_\Gamma(i)$ for some $i \in N$. A node $i \in N$ having no predecessor in Γ, i.e., $P_\Gamma(i) = \emptyset$, is a *source* in Γ. A node $i \in N$ having no successor in Γ, i.e., $S_\Gamma(i) = \emptyset$, is a *sink* in Γ. For any $S \subseteq N$ we denote by $R_\Gamma(S)$ the set of sources in $\Gamma|_S$ and by $L_\Gamma(S)$ the set of sinks in $\Gamma|_S$. For simplicity of notation, for a digraph Γ on N and $i \in N$, by Γ^i we denote the subgraph $\Gamma|_{\bar{S}_\Gamma(i)}$ and by Γ_i the subgraph $\Gamma|_{\bar{P}_\Gamma(i)}$.

Given a digraph Γ on N and a node $i \in N$, the *in-degree* of i is given by $d_\Gamma(i) = |O^*_\Gamma(i)|$ and the *out-degree* of i by $\tilde{d}_\Gamma(i) = |F^*_\Gamma(i)|$, and for $j \in S_\Gamma(i)$ the *in-degree of* j *with respect to* i is given by $d^i(j) = |O^*_{\Gamma^i}(j)|$ and for any $j \in P_\Gamma(i)$ the *out-degree of* j *with respect to* i is given by $d_i(j) = |F^*_{\Gamma_i}(j)|$. Given a digraph Γ on N and a set of paths $\vec{P} \subseteq \vec{P}_\Gamma(i,j)$, $i \in N$, $j \in S_\Gamma(i)$, a node $h \in N(\vec{P})$ such that $d^i(h) \cdot d_j(h) > 1$ is called a *proper intersection point* in $N(\vec{P})$. The subset of $N(\vec{P})$ composed by i, j, all proper immediate successors $h \in F^*_\Gamma(i) \cap N(\vec{P})$ of

i and all proper intersection points in $N(\vec{P})$ defines the *upper covering set* $C(\vec{P})$ for $\vec{P}$, and the subset of $N(\vec{P})$ composed by i, j, all proper immediate predecessors $h \in O^*_\Gamma(j) \cap N(\vec{P})$ of j and all proper intersection points in $N(\vec{P})$ defines the *lower covering set* $\tilde{C}(\vec{P})$ for $\vec{P}$.

For any digraph Γ on N, $i \in N$ and $j \in S_\Gamma(i)$, the set of paths $\vec{P}_\Gamma(i,j)$ can be partitioned into a number of separate subsets of two types, possibly only one subset of one or another type, or a subset containing only one path, such that paths from different subsets do not intersect between i and j, in subsets of the first type all paths belonging to the same subset have at least one common node different from i and j, and in each subset of the second type paths do intersect but have no other nodes in common than i and j. More exactly, given a digraph Γ on N, for every $i \in N$ and $j \in S_\Gamma(i)$ there exist two integers $q_{ij} \geq 1$ and $0 \leq q'_{ij} \leq q_{ij}$, and a partition

$$\vec{P}_\Gamma(i,j) = \bigcup_{h=1}^{q_{ij}} \vec{P}_h \tag{1}$$

such that (*i*) $\vec{p}_1 \cap \vec{p}_2 = \{i,j\}$, for all $\vec{p}_1 \in \vec{P}_h$, $\vec{p}_2 \in \vec{P}_l$, $h, l = 1, ..., q_{ij}$, $h \neq l$;
(*ii*) $\big(\bigcap_{\vec{p} \in \vec{P}_h} \vec{p}\big) \setminus \{i,j\} \neq \emptyset$ for all $h = 1, ..., q'_{ij}$;
(*iii*) $\bigcap_{\vec{p} \in \vec{P}_h} \vec{p} = \{i,j\}$, for all $h = q'_{ij}+1, ..., q_{ij}$.

In a digraph Γ a path $(i_1, \ldots, i_r)$, $r \geq 3$, is a *cycle* in Γ if $\{(i_r, i_1), (i_1, i_r)\} \cap \Gamma \neq \emptyset$. In a digraph Γ a directed path $(i_1, \ldots, i_r)$, $r \geq 2$, is a *directed cycle* in Γ if $(i_r, i_1) \in \Gamma$.[1] A digraph Γ on N is *cycle-free* if it contains no directed cycles, i.e., no node is a successor of itself. A digraph Γ on N is *strongly cycle-free* if it is cycle-free and contains no cycles. Remark that in a strongly cycle-free digraph all links are essential.

A cycle-free directed graph Γ on N is a *(rooted) tree* if it has only one source, called the *root* and denoted $r(\Gamma)$, and for any other node in N there is a unique directed path in Γ from the root to this node. A directed graph Γ on N is a *sink tree* if it has only one sink and for any other node in N there is a unique directed path in Γ from this node to the sink. A directed graph Γ is a (rooted or sink) *forest* if it is composed by a number of disjoint (rooted or sink) trees. A *line-graph* is a forest in which each node has at most one immediate successor and at most one immediate predecessor. Both a rooted tree and a sink tree, and in particular a line-graph, are strongly cycle-free. A subgraph T of a digraph Γ is a *subtree* of Γ if T is a tree on $N(T)$. A subtree T of Γ is a *full subtree* if its node set consists of the root $r(T)$ all successors of $r(T)$, i.e., $N(T) = \bar{S}_\Gamma(r(T))$. A full subtree T of Γ is a *maximal subtree* if the root $r(T)$ is a source of Γ.

In what follows it is assumed that the cooperation structure on the player set N is specified by a cycle-free directed graph, not necessarily being strongly cycle-free. A pair $\langle v, \Gamma \rangle$ of a TU-game $v \in \mathcal{G}_N$ and a cycle-free directed communication graph Γ on N constitutes a game with cycle-free digraph communication structure and is called a *directed cycle-free graph game* or *cycle-free digraph game*. The set of all cycle-free digraph games on a fixed player set N is denoted $\mathcal{G}_N^\Gamma$. A *value* on a subset $\mathcal{G}$ of $\mathcal{G}_N^\Gamma$ is a function $\xi\colon \mathcal{G} \to \mathbb{R}^N$ that assigns to every cycle-free digraph game

[1] Notice that in a digraph a cycle of length 2 is not well defined.

$\langle v, \Gamma \rangle \in \mathcal{G}$ a vector of payoffs $\xi(v, \Gamma) \in \mathbb{R}^N$. For any graph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$, a payoff vector $x \in \mathbb{R}^N$ is *component efficient* if for every component $C \in N/\Gamma$ it holds that $x(C) = v(C)$, and x is *component feasible* if for every component $C \in N/\Gamma$ it holds that $x(C) \leq v(C)$.

3 Web connectedness and management teams

For a directed link in an arbitrary digraph there are two different interpretations possible. One interpretation is that a link is directed to indicate which player has initiated the communication, but at the same time it represents a fully developed communication link. In such a case, following Myerson [6], it is assumed that cooperation is possible among any set of connected players, i.e., the coalitions in which players are able to cooperate, the *productive coalitions*, are all the connected coalitions. In this case the focus is on component efficient values. Another interpretation of a directed link assumes that a directed link represents the only one-way communication situation. In that case not every connected coalition might be productive. In this paper we abide by the second interpretation of a directed link and consider different scenarios possible for controlling cooperation and creation of productive coalitions under the assumption of one-directional communication.

In a directed graph every player is able to communicate only with his successors and his predecessors with whom he is connected via directed paths and no communication is possible with other players. In general any player can be chosen as a manager for controlling the situation and he keeps control over his full web set that in this case can be interpreted as the set of his subordinates. For a coalition of players to create a management team the necessary conditions are, first, that they are independent from each other, and second, that they all together keep control over the entire society represented by N.

Given a digraph Γ on N, a coalition $M \subset N$ is a *management team* in Γ if

(*i*) $W_\Gamma(M) = N$,

(*ii*) $\bar{S}_\Gamma(i) \cap \bar{P}_\Gamma(j) = \emptyset \;\; \forall\, i, j \in M,\, i \neq j$.

Given a digraph Γ the set of all possible management teams we denote by $\mathcal{M}(\Gamma)$. We write $M(\Gamma)$ instead of M when we need to emphasize that management team M depends on graph Γ. Remark that a management team is an antichain in terms of graph theory.

Observe that we prescribe the subordination of players in a given digraph Γ when we choose a management team. It is easy to see that for every $i \in N$ there exists at least one management team $\mathcal{M}(\Gamma)$ containing i. Whence, in particular, it follows that some managers might be simply sources or sinks in Γ. Moreover, there exist two particular management teams – one composed by all sources in Γ and another one composed by all sinks in Γ. Furthermore, as a consequence of condition (*ii*), we obtain that each management team is minimal since $W_\Gamma(M \backslash \{j\}) \neq N$ for all $j \in M$. It is important to notice that the *set of successors* of M in Γ given by $S_\Gamma(M) = \bigcup_{i \in M} S_\Gamma(i)$ and the *set of predecessors* of M in Γ given by $P_\Gamma(M) = \bigcup_{i \in M} P_\Gamma(i)$ are well defined in the sense that $S_\Gamma(M) \cap P_\Gamma(M) = \emptyset$. More precisely, $\{P_\Gamma(M), M, S_\Gamma(M)\}$ forms a partition of the player set N. Later on we also consider the sets $\bar{S}_\Gamma(M) = S_\Gamma(M) \cup M$ and $\bar{P}_\Gamma(M) = P_\Gamma(M) \cup M$.

Given a digraph Γ on N and management team $M \in \mathcal{M}(\Gamma)$, to keep the subordination prescribed by M we define the *management team* $M(S)$ *of a coalition* $S \subseteq N$ *induced by* M as a subcoalition of S composed by

(i) all managers in M that belong to S,

(ii) all predecessors of M in Γ belonging to S that are not covered by the web $W_{\Gamma|_S}(M \cap S)$ and all whose immediate successors belong to $S_\Gamma(M)$,

(iii) all successors of M in Γ belonging to S that are not covered by the web $W_{\Gamma|_S}(M \cap S)$ and all whose immediate predecessors belong to $P_\Gamma(M)$ except those that are already covered by (ii),

(iv) all predecessors of M in Γ that are sinks in S,

(v) all successors of M in Γ that are sources in S,

i.e.,

$$M(S) = M^1(S) \cup M^2(S) \cup M^3(S) \cup M^4(S) \cup M^5(S),$$

where $M^1(S) = M \cap S$,

$$M^2(S) = \{i \in P_\Gamma(M) \cap S \mid i \notin W_{\Gamma|_S}(M \cap S) \text{ and } F_{\Gamma|_S}(i) \subseteq S_\Gamma(M)\},$$

$$M^3(S) = \{i \in S_\Gamma(M) \cap S \mid i \notin W_{\Gamma|_S}(M \cap S) \text{ and } O_{\Gamma|_S}(i) \subseteq P_\Gamma(M) \backslash M^2(S)\},$$

$$M^4(S) = P_\Gamma(M) \cap S \cap L_\Gamma(S),$$

$$M^5(S) = S_\Gamma(M) \cap S \cap R_\Gamma(S).$$

It is not difficult to check that this procedure uniquely defines $M(S)$ and that $M(S)$ is a management team in the subgraph $\Gamma|_S$. Moreover, $M(S)$ inherits the subordination in M in the sense that if $i \in P_\Gamma(M) \cap S$ then $i \in P_{\Gamma|_S}(M(S))$ and if $i \in S_\Gamma(M) \cap S$ then $i \in S_{\Gamma|_S}(M(S))$.

In case when a directed link binding a manager is broken we admit the following rule.

Management team development rule (MTDR): Given digraph Γ on N and management team M in Γ, for any immediate successor $j \in F_\Gamma(i)$ of some manager $i \in M$, $M \cup \{j\}$ becomes a management team in $\Gamma \backslash \{(i,j)\}$ if $j \notin F_\Gamma(h)$ for all $h \in M$, $h \neq i$, and similar, for an immediate predecessor $k \in O_\Gamma(i)$ of some $i \in M$, $M \cup \{k\}$ becomes a management team in $\Gamma \backslash \{(k,i)\}$ if $k \notin O_\Gamma(h)$ for all $h \in M$, $h \neq i$.

Observe that in the first case it is not necessarily the case that the adjunct manager j is a source in $\Gamma \backslash \{(i,j)\}$ because j may have predecessors among players in $P_\Gamma(M)$, in particular, j might be a sink in $\Gamma \backslash \{(i,j)\}$ (see Example 1 below). A similar remark concerns the second case when the adjunct manager k is not a sink in $\Gamma \backslash \{(k,i)\}$ when k has successors among players in $S_\Gamma(M)$.

Example 1 Consider the cycle-free cycle-free digraph Γ depicted in Figure 1. Then the set of management teams in Γ equals

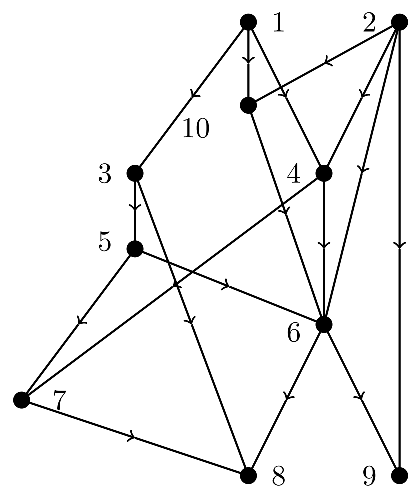

Figure 1

$$\mathcal{M}(\Gamma) = \big\{\{1,2\},\{2,3\},\{2,5\},\{3,4,10\},\{4,5,10\},\{6,7\},\{7,9\},\{8,9\}\big\}.$$

For management team $M = \{4,5,10\}$ the deletion of link $(5,6)$ does not lead to the change of the management team while in case of management team $M = \{7,9\}$ the deletion of link $(7,8)$ is accompanied by the creation of a new management team $M(\Gamma\backslash\{(7,8)\}) = (\{7,8,9\}$. In the latter case the adjunct manager 8 is a sink in the digraph $\Gamma\backslash\{(7,8)\}$.

In real-life situations usually no one accepts that one of his subordinates becomes his equal partner if a coalition forms. So, given a digraph Γ on N and a management team $M \in \mathcal{M}(\Gamma)$, we assume that the only productive coalitions are the so-called *M-web connected* coalitions, for a digraph Γ being the connected coalitions $S \in C_\Gamma(N)$ that meet the condition that for every manager $i \in M(S)$ it holds that $i \notin W_\Gamma(j)$ for any other manager $j \in M(S)$. It is not difficult to see that the latter condition guarantees that every M-web connected coalition inherits the subordination of players prescribed by M in Γ. Obviously, every component $C \in N/\Gamma$ is M-web connected. Moreover, any full web set in Γ with its hub being a manager in M is M-web connected. A M-web connected coalition is *full M-web connected* if it together with its management team contains also all their subordinates. Observe that a full M-web connected coalition is the union of several full webs sets. For a given cycle-free digraph Γ on N, management team $M \in \mathcal{M}(\Gamma)$ and coalition $S \subseteq N$ let $C^M_\Gamma(S)$ denote the set of all M-web connected subsets of S, by $[S/\Gamma]^M$ the set of maximally M-web connected subsets of S, called the *M-web components of S*, and by $[S/\Gamma]^M_i$ the M-web component of S containing player $i \in S$.

In what follows we assume that for every cycle-free digraph Γ on N some management team $M \in \mathcal{M}(\Gamma)$ is a priori fixed. The set of cycle-free digraph games $\langle v, \Gamma, M\rangle$ on N with management team M, $M \in \mathcal{M}(\Gamma)$, we denote by $\mathcal{G}^{\Gamma,M}_N$.

For efficiency of a value we require that every M-web connected coalition composed by one of the managers together with all subordinates of this manager fully realizes its worth. This gives the first axiom a value must satisfy, called web efficiency.

A value ξ on $\mathcal{G}_N^{\Gamma,M}$ is *web efficient* (WE) if for every cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}_N^{\Gamma,M}$ it holds that

$$\sum_{j\in W_\Gamma(i)} \xi_j(v,\Gamma,M) = v(W_\Gamma(i)), \qquad \forall\, i \in M.$$

WE generalizes the usual definition of efficiency for a (rooted/sink) tree. Indeed, in a (rooted) tree when it is assumed that there is only one manager - its root, the web efficiency just says that the total payoff should be equal to the worth of the grand coalition N. A similar remark holds true for a sink tree with only one sink-manager as well. Still, WE is not the productive component efficiency condition. Different from the Myerson [6] case with undirected communication graph we assume that not every productive component is able to realize its exact capacity but only those with a web structure. For example, if one worker works in two different divisions, the two managers of these firms and the worker create a productive coalition. Yet, it is impossible to guarantee the efficiency of this coalition because there is no communication link between the managers of the two divisions.

The next two axioms reflect the desirable property of stability of the management system – any changes on the upper levels of the management hierarchy should not destroy the stable performance at the lower levels. The first axiom, called web successor equivalence, says that if a link with the terminus being a successor of a given management team is deleted, the terminus of this link and all his successors still receive the same payoff.

A value ξ on $\mathcal{G}_N^{\Gamma,M}$ is *web successor equivalent* (WSE) if for every cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}_N^{\Gamma,M}$ it holds that for all $(i,j) \in \Gamma$ such that $i, j \in \bar{S}_\Gamma(M)$,

$$\xi_k(v, \Gamma\backslash(i,j), M) = \xi_k(v,\Gamma,M), \qquad \forall\, k \in \bar{S}_\Gamma(j).$$

WSE means that the payoff to any member in the full successors set of any player being a successor of the given management team does not change if any of the immediate predecessors of that player breaks his link to that player. It implies that for every successors set of a successor or member of the given management team the payoff distribution is completely determined by the players of this set.

The second axiom, called web predecessor equivalence, says that if a link with the origin being a predecessor of a given management team is deleted, the origin of this link and all his predecessors still receive the same payoff.

A value ξ on $\mathcal{G}_N^{\Gamma,M}$ is *web predecessor equivalent* (WPE) if for every cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}_N^{\Gamma,M}$ it holds that for all $(i,j) \in \Gamma$ such that $i, j \in \bar{P}_\Gamma(M)$,

$$\xi_k(v, \Gamma\backslash(i,j), M) = \xi_k(v,\Gamma,M), \qquad \forall\, k \in \bar{P}_\Gamma(i).$$

WPE means that the payoff to any member in the full predecessors set of any player being a predecessor of the given management team does not change if any of the immediate successors of that player breaks his link to that player. It implies that for every predecessors set of a predecessor or member of the given management team the payoff distribution is completely determined by the players of this set.

Along with WE we consider also two stronger efficiency properties requiring that the full sets of subordinates of any player are able to realize their full capacity. Web full-tree efficiency and web full-sink efficiency require correspondingly that every full successors set within the set of successors of a given management team and every full predecessors set within the set of predecessors of a given management team realize their worths.

A value ξ on $\mathcal{G}_N^{\Gamma,M}$ is *web full-tree efficient* (WFTE) if for every cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}_N^{\Gamma,M}$ it holds that

$$\sum_{j\in\bar{S}_\Gamma(i)} \xi_j(v,\Gamma,M) = v(\bar{S}_\Gamma(i)), \qquad \forall\, i \in S_\Gamma(M).$$

A value ξ on $\mathcal{G}_N^{\Gamma,M}$ is *web full-sink efficient* (WFSE) if for every cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}_N^{\Gamma,M}$ it holds that

$$\sum_{j\in\bar{P}_\Gamma(i)} \xi_j(v,\Gamma,M) = v(\bar{P}_\Gamma(i)), \qquad \forall\, i \in P_\Gamma(M).$$

4 The tree value

Consider first the situation when a management team is composed by the set of all sources of a given graph.

4.1 Axiomatic definition

In this case web connectedness can be restated in terms of tree connectedness. For a digraph Γ a connected coalition $S \in C_\Gamma(N)$ is *tree connected*, or simply *t-connected*, if it meets the condition that for every source $i \in R_\Gamma(S)$ it holds that $i \notin S_\Gamma(j)$ for any other source $j \in R_\Gamma(S)$. A t-connected coalition is *full t-connected*, if it together with its sources contains all successors of these sources. Observe that a full t-connected coalition is the union of one or more full successors sets.

In what follows for a cycle-free digraph Γ on N and a coalition $S \subseteq N$, let $C^t_\Gamma(S)$ denote the set of all t-connected subsets of S, $[S/\Gamma]^t$ the set of maximally t-connected subsets of S, called the *t-connected components of* S, and $[S/\Gamma]^t_i$ the t-connected component of S containing player $i \in S$.

In the considered case web efficiency reduces to maximal-tree efficiency, web successor equivalence to successor equivalence and web full-tree efficiency to full-tree efficiency, while the axioms of web predecessor equivalence and web full-sink efficiency become redundant.

A value ξ on $\mathcal{G}_N^{\Gamma}$ is *maximal-tree efficient* (MTE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ it holds that

$$\sum_{j\in\bar{S}_\Gamma(i)} \xi_j(v,\Gamma) = v(\bar{S}_\Gamma(i)), \qquad \text{for all } \ i \in R_\Gamma(N).$$

A value ξ on $\mathcal{G}_N^{\Gamma}$ is *successor equivalent* (SE) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^{\Gamma}$ it holds that for all $(i,j) \in \Gamma$

$$\xi_k(v,\Gamma\backslash(i,j)) = \xi_k(v,\Gamma), \qquad \text{for all } \ k \in \bar{S}_\Gamma(j).$$

A value ξ on $\mathcal{G}_N^\Gamma$ is *full-tree efficient* (FTE) if for every cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ it holds that

$$\sum_{j \in \bar{S}_\Gamma(i)} \xi_j(v, \Gamma) = v(\bar{S}_\Gamma(i)), \qquad \text{for all } i \in N. \tag{2}$$

Proposition 1 *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ MTE and SE together imply FTE.*

Proof. Let ξ be a value on $\mathcal{G}_N^\Gamma$ that meets MTE and SE, and let a cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ be arbitrarily chosen. For every given $i \in N$, the subgraph Γ^i is a maximal tree in the subgraph $\Gamma' = \Gamma \backslash \bigcup_{j \in O_\Gamma(i)} \{(j, i)\}$. Since $\bar{S}_{\Gamma'}(i) = \bar{S}_\Gamma(i)$, $i \in R_{\Gamma'}(N)$ and due to MTE,

$$\sum_{j \in \bar{S}_\Gamma(i)} \xi_j(v, \Gamma \backslash \bigcup_{k \in O_\Gamma(i)} \{(k, i)\}) \overset{\text{MTE}}{=} v(\bar{S}_\Gamma(i)).$$

By successive application of SE,

$$\xi_j(v, \Gamma \backslash \bigcup_{k \in O_\Gamma(i)} \{(k, i)\}) \overset{\text{SE}}{=} \xi_j(v, \Gamma), \qquad \text{for all } j \in \bar{S}_\Gamma(i).$$

Whence,

$$\sum_{j \in \bar{S}_\Gamma(i)} \xi_j(v, \Gamma) = v(\bar{S}_\Gamma(i)), \qquad \text{for all } i \in N,$$

i.e., the value ξ meets FTE. ∎

Given a digraph Γ on N, for all $i \in N$ and $j \in S_\Gamma(i)$ we define

$$\kappa_{ij} = \sum_{r=0}^{n-2} (-1)^r \kappa_{ij}^r, \tag{3}$$

where, for $r = 0, 1, \ldots, n-2$, κ_{ij}^r is the number of tuples $(i_0, \ldots, i_{r+1})$ such that $i_0 = i$, $i_{r+1} = j$, $i_h \in S_\Gamma(i_{h-1})$, $h = 1, \ldots, r+1$.

It turns out that MTE and SE uniquely define a value on the class of cycle-free digraph games.

Theorem 1 *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ there is a unique value t that satisfies MTE and SE. For every cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$, the value $t(v, \Gamma)$ satisfies the following conditions:*

(i) it obeys the recursive equality

$$t_i(v, \Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} t_j(v, \Gamma), \qquad \textit{for all } i \in N; \tag{4}$$

(ii) it admits the explicit representation in the form

$$t_i(v, \Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j \in S_\Gamma(i)} \kappa_{ij}\, v(\bar{S}_\Gamma(j)), \qquad \textit{for all } i \in N. \tag{5}$$

Proof. Due to Proposition 1 the value t on $\mathcal{G}_N^\Gamma$ that satisfies MTE and SE meets FTE as well, wherefrom the recursive equality (4) follows straightforwardly. Next, we show that the representation in the form (4) is equivalent to the representation in the form (5). According to (4) it holds for the value t that every player receives what this player together with his successors can get on their own, their worth, minus what all his successors will receive by themselves. Since the same property holds for these successors as well, it is not difficult to see that (5) follows directly from (4) by successive substitution. Indeed, for any $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ and $i \in N$ it holds that

$$t_i(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j\in S_\Gamma(i)} t_j(v,\Gamma) \overset{(4)}{=}$$

$$v(\bar{S}_\Gamma(i)) - \sum_{j\in S_\Gamma(i)} v(\bar{S}_\Gamma(j)) + \sum_{j\in S_\Gamma(i)}\sum_{k\in S_\Gamma(j)} t_k(v,\Gamma) \overset{(4)}{=}$$

$$v(\bar{S}_\Gamma(i)) - \sum_{j\in S_\Gamma(i)} v(\bar{S}_\Gamma(j)) + \sum_{j\in S_\Gamma(i)}\sum_{k\in S_\Gamma(j)} v(\bar{S}_\Gamma(k)) - \sum_{j\in S_\Gamma(i)}\sum_{k\in S_\Gamma(j)}\sum_{h\in S_\Gamma(k)} t_h(v,\Gamma) \overset{(4)}{=}$$

$$\ldots = v(\bar{S}_\Gamma(i)) - \sum_{j\in S_\Gamma(i)}\sum_{r=0}^{n-2}(-1)^r\kappa_{ij}^r\, v(\bar{S}_\Gamma(j)) = v(\bar{S}_\Gamma(i)) - \sum_{j\in S_\Gamma(i)} \kappa_{ij}\, v(\bar{S}_\Gamma(j)).$$

From (5), we obtain immediately that the value t meets SE, because in any digraph Γ for all $(i,j) \in \Gamma$ and $k \in \bar{S}_\Gamma(j)$ the full subtrees Γ^k and $(\Gamma\backslash(i,j))^k$ coincide. This completes the proof, since MTE follows from FTE automatically. ∎

According to (4) the value t assigns to every player the worth of his full successors set minus the total payoff to his successors.

Corollary 1 There exists a simple recursive algorithm for computing the value t going upstream from the sinks of the given digraph.

The computation of the coefficients κ_{ij}, $i \in N$, $j \in S_\Gamma(i)$, defined by (3) in the explicit formula representation (5) requires, in general, the enumeration of quite a lot of possibilities. We show below that in many cases the coefficients κ_{ij} can be easily computed and the value t can be presented in a computationally more transparent and simpler form. To do that observe first that for a given digraph Γ on N, for any $i \in N$ and $j \in S_\Gamma(i)$, all nodes forming a tuple $(i_0, ..., i_{r+1})$ in which $i_0 = i$, $i_{r+1} = j$, $i_h \in S_\Gamma(i_{h-1})$, $h = 1, ..., r+1$, belong to one directed path $\vec{p}$ in $\vec{P}_\Gamma(i,j)$. Wherefrom it easily follows that for all $i \in N$ and $j \in S_\Gamma(i)$, κ_{ij} given by (3) is in fact defined only via tuples of nodes from $N(\vec{P}_\Gamma(i,j))$. For $i \in N$, $j \in S_\Gamma(i)$ and $S \subseteq N(\vec{P}_\Gamma(i,j))$ containing nodes i and j, define

$$\kappa_{ij}(S) = \sum_{r=0}^{n-2}(-1)^r\kappa_{ij}^r(S), \tag{6}$$

where, for $r = 0, 1, \ldots, n-2$, $\kappa_{ij}^r(S)$ counts all tuples $(i_0, ..., i_{r+1})$ for which $i_0 = i$, $i_{r+1} = j$, and $i_h \in S_\Gamma(i_{h-1}) \cap S$, $h = 1, \ldots, r+1$. Remark that $\kappa_{ij} = \kappa_{ij}(N(\vec{P}_\Gamma(i,j)))$ for all $j \in S_\Gamma(i)$, $i \in N$.

Theorem 2 *For every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ the value t given by (5) admits the equivalent representation in the form*

$$t_i(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j\in F^*_\Gamma(i)} v(\bar{S}_\Gamma(j)) +$$

$$+ \sum_{\substack{j\in S_\Gamma(i)\\ d^i(j)>1}} \Big(q_{ij} - 1 - \sum_{h=q'_{ij}+1}^{q_{ij}} \kappa_{ij}(C(\vec{P}_h))\Big)\, v(\bar{S}_\Gamma(j)), \qquad \text{for all } i\in N, \tag{7}$$

where, for all $i \in N$ and $j \in S_\Gamma(i)$, $\vec{P}_h$, $h = 1, ..., q_{ij}$, form the partition of $\vec{P}_\Gamma(i,j)$ defined by (1).

If the consideration is restricted to only strongly cycle-free digraph games, then the above representation reduces to

$$t_i(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{j\in F_\Gamma(i)} v(\bar{S}_\Gamma(j)), \qquad \text{for all } i \in N. \tag{8}$$

For rooted-forest digraph games defined by rooted forest digraph structures that are strongly cycle-free, the value given by (8) coincides with the tree value introduced first under the name of hierarchical outcome in Demange [2], where it is also shown that under the mild condition of superadditivity it belongs to the core of the restricted game defined in Myerson [6]. More recently, the tree value for rooted-forest games was used as a basic element in the construction of the average tree solution for cycle-free undirected graph games in Herings et al. [4]. In Khmelnitskaya [5] it is shown that on the class of rooted-forest digraph games the tree value can be characterized via component efficiency and successor equivalence; moreover, it is shown that the class of rooted-forest digraph games is the maximal subclass in the class of strongly cycle-free digraph games where this axiomatization holds true. It is worth to recall that by definition for a rooted-tree digraph game every connected component is a tree. Hence, on the class of rooted-forest digraph games every connected component is productive and maximal-tree efficiency coincides with component efficiency.

From now on we refer to the value t given by (5), or equivalently by (7), as to the *root-tree value*, or simply the *tree value*, for cycle-free digraph games. The tree value assigns to every player the payoff equal to the worth of his full successors set minus the worths of all full successors sets of his proper immediate successors plus or minus the worths of all full successors sets of any other of his successors that are subtracted or added more than once. For a player $i \in N$ and his successor $j \in N$ that is not his proper immediate successor, the coefficient κ_{ij} indicates the number of overlappings of full successors sets of all proper immediate successors of i at node j. A player receives what he contributes when he joins his successors when only the full successors sets, that are the only efficient productive coalitions, are counted. Since a sink has no successors, a sink just gets his own worth. It is worth to note and not difficult to check that the right sides of both formulas (7) and (8), being considered with respect not to coalitional worths but to players in these coalitions, contain only player i when taking into account all pluses and minuses.

The validity of the first statement of Theorem 2 follows directly from Theorem 1 and Lemma 1 and Corollary 2 to it. The second statement follows easily from the first one. Indeed, in any strongly cycle-free digraph Γ all links are essential, whence $F^*_\Gamma(i) = F_\Gamma(i)$, and $d^i(j) = 1$ for all $i \in N$ and $j \in S_\Gamma(i)$.

Lemma 1 *For any digraph Γ on N, the coefficients κ_{ij}, $i \in N$, $j \in S_\Gamma(i)$, defined by (3) possess the following properties:*

(*i*) *if a link $(k,l) \in \Gamma$ is inessential, then for all $i \in N$ and $j \in S_\Gamma(i)$, κ_{ij} defined on Γ is equal to κ_{ij} defined on $\Gamma \backslash (k,l)$;*

(*ii*) $\kappa_{ij} = 1$ *for all* $i \in N$ *and* $j \in F^*_\Gamma(i)$;

(*iii*) $\kappa_{ij} = -q_{ij} + 1 + \sum\limits_{h=q'_{ij}+1}^{q_{ij}} \kappa_{ij}(C(\vec{P}_h))$ *for all* $i \in N$ *and* $j \in S_\Gamma(i) \setminus F^*_\Gamma(i) j \in$ $S_\Gamma(i) \setminus F^*_\Gamma(i)$ *with* $d^i(j) = 1$.

Proof.

(*i*). It is sufficient to prove the statement only in case when $k \in S_\Gamma(i)$ and $j \in S_\Gamma(l)$. Let $\vec{p} \in \vec{P}_\Gamma(i,j)$ be such that $\vec{p} \ni (k,l)$. By definition of an inessential link there exists $\vec{p}_0 \in \vec{P}_\Gamma(k,l)$ such that $\vec{p}_0 \neq (k,l)$. It is not difficult to see that the path $\vec{p}_1 = \vec{p} \backslash (k,l) \cup \vec{p}_0$ obtained from the path $\vec{p}$ by replacing the link (k,l) by the path $\vec{p}_0$ belongs to $\vec{P}_\Gamma(i,j)$, and moreover, all tuples $(i_0, ..., i_{r+1})$ in the definition of $\kappa_i(j)$ that belong to $\vec{p}$ also belong to $\vec{p}_1$. Whence it follows straightforwardly that deleting an inessential link does not change the value of κ_{ij}.

From now without loss of generality we may assume that $\vec{P}_\Gamma(i,j)$ is composed by only proper paths.

(*ii*). If $j \in F^*_\Gamma(i)$, then $\vec{P}_\Gamma(i,j)$ contains only the path $\vec{p} = (i,j)$. Wherefrom it follows that $\kappa_{ij} = 1$.

(*iii*). Let $j \in S_\Gamma(i) \setminus F^*_\Gamma(i)$. Since paths in $\vec{P}_\Gamma(i,j)$ are partitioned into subsets of paths $\vec{P}_h$, $h = 1, ..., q_{ij}$, such that paths from different subsets do not intersect between i and j, it holds that

$$\kappa_{ij} = \kappa_{ij}(N(\vec{P}_1)) + \left[\kappa_{ij}(N(\vec{P}_2)) - \kappa_{ij}(N(\vec{P}_1 \cap \vec{P}_2))\right] + \ldots$$

$$\ldots + \left[\kappa_{ij}(N(\vec{P}_{q_{ij}})) - \kappa_{ij}(N(\bigcap_{h=1}^{q_{ij}} \vec{P}_h))\right].$$

Since the paths from different subsets $\vec{P}_h$ do not intersect between i and j, only (i,j) belongs to all paths in $\vec{p} \in \vec{P}_\Gamma(i,j)$. Therefore, for all $k = 2, \ldots, q_{ij}$,

$$\kappa_{ij}(N(\bigcap_{h=1}^{k} \vec{P}_h)) = 1.$$

Whence it easily follows that

$$\kappa_{ij} = -q_{ij} + 1 + \sum_{h=1}^{q_{ij}} \kappa_{ij}(N(\vec{P}_h)).$$

First, let $h \in \{1, ..., q'_{ij}\}$, i.e., the subset of paths $\vec{P}_h$ is of the first type when all paths belonging to $\vec{P}_h$ have at least one common node different from i and j.

Then there exists $k \in N(\vec{P}_h)$, $k \neq i, j$, such that $k \in \vec{p}$ for all $\vec{p} \in \vec{P}_h$. By definition, $\kappa^r_{ij}(N(\vec{P}_h))$ is equal to the number of tuples $(i_0, \dots, i_{r+1})$ such that $i_0 = i$, $i_{r+1} = j$, $i_l \in S_\Gamma(i_{l-1}) \cap N(\vec{P}_h)$, $l = 1, \dots, r+1$, or equivalently, κ^r_{ij} is equal to the number of these tuples $(i_0, \dots, i_{r+1})$ that do not contain k plus the number of these tuples $(i_0, \dots, i_{r+1})$ that contain k. Since $k \in \vec{p}$ for all $\vec{p} \in \vec{P}_h$, for every $(r+2)$-tuple $(i_0, \dots, i_{r+1})$ that does not contain k there exists a uniquely defined $(r+3)$-tuple composed by the same nodes plus node k. Wherefrom together with equality (6) it follows that $\kappa_{ij}(N(\vec{P}_h)) = 0$.

Next, consider $h \in \{q'_{ij}+1, ..., q_{ij}\}$, i.e., the subset of paths $\vec{P}_h$ is of the second type when all paths belonging to $\vec{P}_h$ do intersect but have no other nodes in common than i and j. We show now that $\kappa_{ij}(N(\vec{P}_h)) = \kappa_{ij}(C(\vec{P}_h))$. Consider arbitrary $k \in N(\vec{P}_h) \setminus C(\vec{P}_h)$. We may split the computation of $\kappa_{ij}(N(\vec{P}_h))$ into two parts:

$$\kappa_{ij}(N(\vec{P}_h)) = \kappa_{ij}(N(\vec{P}_h); k) + \kappa_{ij}(N(\vec{P}_h) \setminus \{k\}),$$

where $\kappa_{ij}(N(\vec{P}_h); k)$ counts all tuples in $N(\vec{P}_h)$ containing k. By definition of upper covering set, $C(\vec{P}_h)$ contains some predecessor of k, i.e., $C(\vec{P}_h) \cap P_\Gamma(k) \neq \emptyset$. Moreover, since $k \notin C(\vec{P}_h)$, i.e., k is neither a proper immediate successor of i nor a proper intersection point in the subgraph $\Gamma|_{N(\vec{P}_h)}$, there exists $l \in C(\vec{P}_h) \cap P_\Gamma(k)$ that belongs to all paths in $\vec{P}_h$ containing k. Applying the same argument as above in the proof of $\vec{P}_h$ of the first type, now with respect to l, we obtain that $\kappa_{ij}(N(\vec{P}_h); k) = 0$. Thus $\kappa_{ij}(N(\vec{P}_h)) = \kappa_{ij}(N(\vec{P}_h) \setminus \{k\})$. Repeating the same reasoning successively with respect to all $k' \in N(\vec{P}_h) \setminus \big(C(\vec{P}_h) \cup \{k\}\big)$ we obtain $\kappa_{ij}(N(\vec{P}_h)) = \kappa_{ij}(C(\vec{P}_h))$. ∎

From (iii) of Lemma 1 we easily obtain the following.

Corollary 2 $\kappa_{ij} = 0$ for all $i \in N$ and $j \in S_\Gamma(i) \setminus F^*_\Gamma(i)$ for which $q_{ij} = q'_{ij} = 1$. In particular, $\kappa_{ij} = 0$ for all $i \in N$ and $j \in S_\Gamma(i) \setminus F^*_\Gamma(i)$ with $d^i(j) = 1$, since for all $j \in S_\Gamma(i) \setminus F^*_\Gamma(i)$ with $d^i(j) = 1$ there is a unique proper immediate predecessor of j that belongs to all paths in $\vec{P}_\Gamma(i, j)$.

A value ξ on $\mathcal{G}^\Gamma_N$ is *independent of inessential links* (IIL) if for every cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}^\Gamma_N$ and the cycle-free digraph game $\langle v, \Gamma'\rangle \in \mathcal{G}^\Gamma_N$ with Γ' being the subgraph Γ' of Γ composed by all essential links of Γ it holds that $\xi(v, \Gamma) = \xi(v, \Gamma')$.

Corollary 3 The tree value t satisfies independence of inessential links.

Example 2 The examples of digraphs depicted in Figure 2 demonstrate the situation when all paths from any $\vec{P}_\Gamma(i, j)$ constitute one subset of the second type, i.e., paths in $\vec{P}_\Gamma(i, j)$ do intersect but have no other nodes in common than i and j.

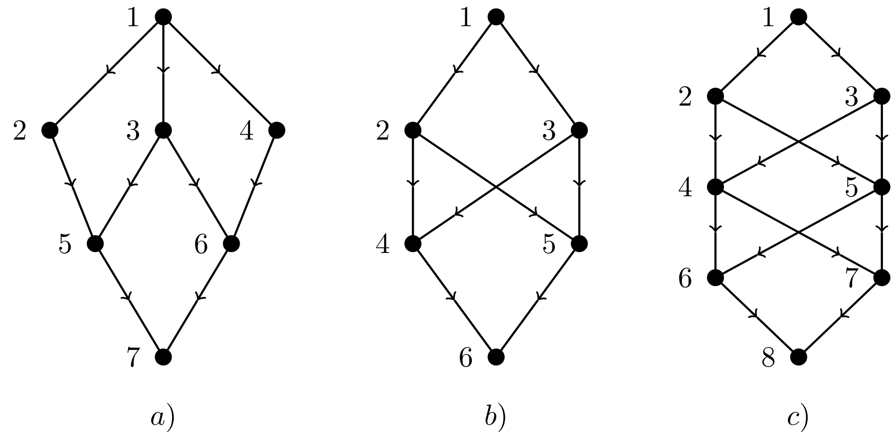

Figure 2

For the digraph depicted in Figure 2.a) we have $d^1(7) = 2$ and $\kappa_{17} = 0$, for the one in Figure 2.b) we have $d^1(6) = 2$ and $\kappa_{16} = 1$, and for the one in Figure 2.c) we have $d^1(8) = 2$ and $\kappa_{18} = -1$.

Example 3 Figure 3 provides an example of the tree value for a 10-person game with cycle-free but not strongly cycle-free digraph structure depicted in Figure 1. If there is no confusion, a set $\{i_1, ..., i_k\}$ is denoted by $i_1...i_k$.

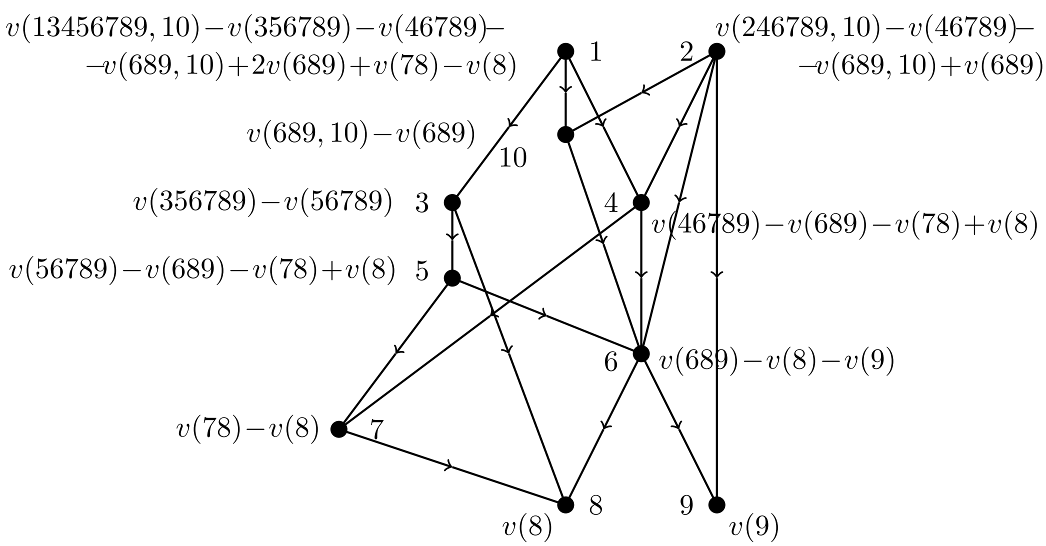

Figure 3

The tree value may be computed in two different ways, either by the recursive algorithm based on equality (4) or using the explicit formula representation (7).

We explain in detail the computation of $t_1(v, \Gamma)$ based on the explicit formula (7):
$\bar{S}_\Gamma(1) = \{1, 3, 4, 5, 6, 7, 8, 9, 10\}$.
$3, 4, 10 \in F^*_\Gamma(1) \Longrightarrow \kappa_{13} = \kappa_{14} = \kappa_{1,10} = 1$;
$S_\Gamma(1) \setminus F^*_\Gamma(1) = \{5, 6, 7, 8, 9\}$:
$d^1(5) = d^1(9) = 1 \implies \kappa_{15} = \kappa_{19} = 0$;
$\vec{P}_\Gamma(1, 6) = \{\vec{p}_1 = (1, 3, 5, 6), \vec{p}_2 = (1, 4, 6), \vec{p}_3 = (1, 10, 6)\}$, paths $\vec{p}_1$, $\vec{p}_2$ and $\vec{p}_3$ do

not intersect between 1 and 6 $\Longrightarrow$ $q_{16} = q'_{16} = 3$ $\Longrightarrow$ $\kappa_{16} = -2$;
$\vec{P}_\Gamma(1,7) = \{\vec{p}_1 = (1,3,5,7), \vec{p}_2 = (1,4,7)\}$, paths $\vec{p}_1$ and $\vec{p}_2$ do not intersect between 1 and 7 $\Longrightarrow$ $q_{17} = q'_{17} = 2$ $\Longrightarrow$ $\kappa_{17} = -1$;
$\vec{P}_\Gamma(1,8) = \{\vec{p}_1 = (1,3,5,7,8),\ \vec{p}_2 = (1,3,5,6,8),\ \vec{p}_3 = (1,10,6,8),\ \vec{p}_4 = (1,4,7,8),$ $\vec{p}_5 = (1,4,6,8),\ \vec{p}_6 = (1,3,8)\}$; we eliminate path $\vec{p}_6$ containing inessential link $(3,8)$; paths $\vec{p}_1$, $\vec{p}_2$, $\vec{p}_3$, $\vec{p}_4$ and $\vec{p}_5$ form one subset of the second type $\Longrightarrow$ $q_{18} = 1$, $q'_{18} = 0$;
$\{1,4,5,6,7,8,10\}$ is the minimal covering set $C(\vec{P}_\Gamma(1,8))$;
$\kappa_{18}(\vec{p}_1) = 0$;
$\vec{p}_2 \backslash \vec{p}_1$ contains tuples $(1,6,8)$ and $(1,5,6,8)$ $\Longrightarrow$ $\kappa_{18}(\vec{p}_2 \backslash \vec{p}_1) = 0$;
$\vec{p}_3 \backslash (\vec{p}_1 \cup \vec{p}_2)$ contains tuples $(1,10,8)$, $(1,10,6,8))$ $\Longrightarrow$ $\kappa_{18}(\vec{p}_3 \backslash (\vec{p}_1 \cup \vec{p}_2)) = 0$;
$\vec{p}_4 \backslash (\vec{p}_1 \cup \vec{p}_2 \cup \vec{p}_3)$ contains $(1,4,8)$, $(1,4,7,8))$ $\Longrightarrow$ $\kappa_{18}(\vec{p}_4 \backslash (\vec{p}_1 \cup \vec{p}_2 \cup \vec{p}_3)) = 0$;
$\vec{p}_5 \backslash (\vec{p}_1 \cup \vec{p}_2 \cup \vec{p}_3 \cup \vec{p}_4)$ contains $(1,4,6,8))$ $\Longrightarrow$ $\kappa_{18}(\vec{p}_5 \backslash (\vec{p}_1 \cup \vec{p}_2 \cup \vec{p}_3 \cup \vec{p}_4)) = 1$;
$\Longrightarrow$ $\kappa_{18} = 1$.
$t_1(v,\Gamma) = v(13456789,10) - v(356789) - v(46789) - v(689,10) + 2v(689) + v(78) - v(8)$.

Example 4 Figure 4 gives an example of the tree value for a 10-person game with strongly cycle-free digraph structure.

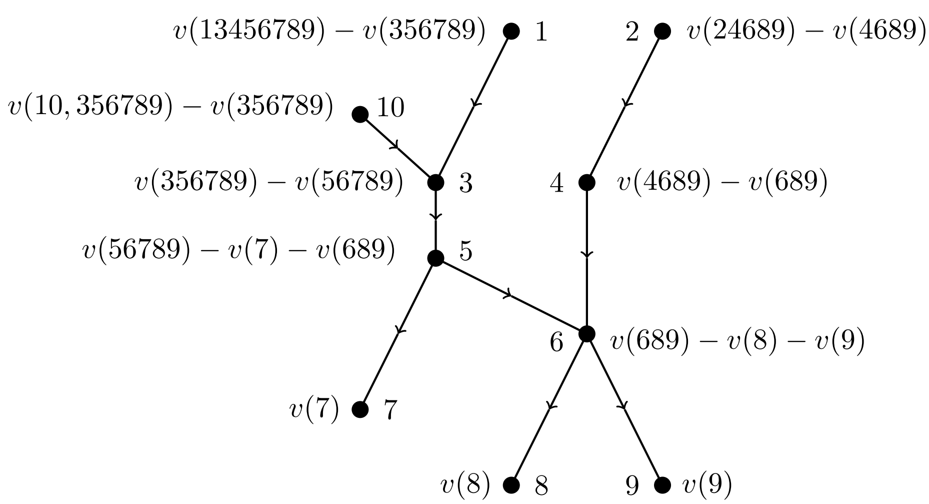

Figure 4

In Figure 2,a $7 \in S_\Gamma(1)$, $d^1(7) = 2$, and $\kappa_{17} = 0$. In Figure 2,b $6 \in S_\Gamma(1)$, $d^1(6) = 2$,and $\kappa_{16} = 1$. In Figure 2,c $8 \in S_\Gamma(1)$, $d^1(8) = 2$, and $\kappa_{18} = -1$.

It turns out that the tree value not only meets FTE but FTE alone uniquely defines the tree value on the class of cycle-free digraph games.

Theorem 3 *On the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ the tree value is the unique value that satisfies FTE.*

Proof. Since the tree value satisfies FTE, to prove the theorem it is enough to show that the tree value is the unique value that meets FTE on $\mathcal{G}_N^\Gamma$. Let a value ξ on $\mathcal{G}_N^\Gamma$ satisfy axiom FTE. Then, because of FTE, (2) holds for every $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$. Every digraph Γ under consideration is cycle-free, i.e., no player in N appears to be a successor of itself. Hence, due to the arbitrariness of game $\langle v, \Gamma \rangle$, the n equalities in (2) are independent. Therefore, we have a system of n independent linear equalities with respect to n variables $\xi_j(v, \Gamma)$ which uniquely determines the value $\xi(v, \Gamma)$ that in this case coincides with $t(v, \Gamma)$. ∎

Corollary 4 FTE on the class of cycle-free digraph games $\mathcal{G}_N^\Gamma$ implies not only MTE but SE as well.

Remark 1 Observe that the inessential links independence of the tree value can be also obtained as a corollary to Theorem 3.

4.2 Overall efficiency and stability

In this subsection we consider efficiency and stability of the tree value. First we derive for the tree value the total payoff for any t-connected coalition.

Given a digraph Γ and a t-connected coalition $S \subseteq N$, we define

$$\bar{S}_\Gamma(S) = \bigcup_{i \in R_\Gamma(S)} \bar{S}_\Gamma(i),$$

and

$$\kappa_{i,S} = \sum_{j \in \bar{P}_\Gamma(i) \cap \bar{S}_\Gamma(S)} \kappa_{ij}, \qquad \text{for all } \; i \in \bar{S}_\Gamma(S),$$

and let for every $i \in \bar{S}_\Gamma(S)$, $d_S(i)$ be the in-degree of i in the subgraph $\Gamma|_{\bar{S}_\Gamma(S)}$, i.e.,

$$d_S(i) = |O_\Gamma^*(i) \cap \bar{S}_\Gamma(S)|.$$

Remark that for all $\; i \in N$, $d_N(i) = d_\Gamma(i)$.

Theorem 4 *In a cycle-free digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$, for any t-connected coalition $S \in C_\Gamma^t(N)$ it holds that*

$$\sum_{i \in S} t_i(v, \Gamma) = \sum_{i \in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \\ - \sum_{\substack{i \in S \setminus R_\Gamma(S) \\ d_\Gamma(i) > 1}} \big(\kappa_{i,S} - 1\big)\, v(\bar{S}_\Gamma(i)) - \sum_{i \in \bar{S}_\Gamma(S) \setminus S} \kappa_{i,S}\, v(\bar{S}_\Gamma(i)). \tag{9}$$

If the consideration is restricted to only strongly cycle-free digraph games, then for any t-connected coalition $S \in C_\Gamma^t(N)$ it holds that

$$\sum_{i \in S} t_i(v, \Gamma) = \sum_{i \in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \\ - \sum_{i \in S \setminus R_\Gamma(S)} \big(d_S(i) - 1\big)\, v(\bar{S}_\Gamma(i)) - \sum_{i \in R_\Gamma(\bar{S}_\Gamma(S) \setminus S)} d_S(i)\, v(\bar{S}_\Gamma(i)). \tag{10}$$

Proof. Let $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ be a cycle-free digraph game and let S be any t-connected coalition $S \in C_\Gamma^t(N)$. Then it holds that

$$\sum_{i\in S} t_i(v,\Gamma) \overset{(5)}{=} \sum_{i\in S}\big(v(\bar{S}_\Gamma(i)) - \sum_{j\in S_\Gamma(i)} \kappa_{ij}\, v(\bar{S}_\Gamma(j))\big) =$$

$$= \sum_{i\in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \sum_{i\in S\setminus R_\Gamma(S)} \big(\sum_{j\in S_\Gamma(i)} (\kappa_{ij}-1)\, v(\bar{S}_\Gamma(i))\big) - \sum_{i\in \bar{S}_\Gamma(S)\setminus S} \big(\sum_{j\in S_\Gamma(i)} \kappa_{ij}\, v(\bar{S}_\Gamma(i))\big).$$

Since $S \in C_\Gamma^t(N)$, for all $i, j \in S$ with $j \in S_\Gamma(i)$ every path from i to j belongs to S. Then, from the last equality it follows that

$$\sum_{i\in S} t_i(v,\Gamma) = \sum_{i\in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \sum_{i\in S\setminus R_\Gamma(S)} \big(\kappa_{i,S}-1\big)\, v(\bar{S}_\Gamma(i)) - \sum_{i\in \bar{S}_\Gamma(S)\setminus S} \kappa_{i,S}\, v(\bar{S}_\Gamma(i)).$$

Next, due to Lemma 1, $\kappa_{ji} = 0$ for all $j \in \big(\bar{P}_\Gamma(i) \cap \bar{S}_\Gamma(S)\big)\setminus F_\Gamma(i)$ with $d^j(i) = 1$ and $\kappa_{ji} = 1$ for $j \in F_\Gamma(i) \cap \bar{S}_\Gamma(S)$. Whence it follows that $\kappa_{i,S} = 1$ when $d_\Gamma(i) = 1$.

In case Γ is a strongly cycle-free digraph, it holds that

$$\sum_{i\in S} t_i(v,\Gamma) \overset{(8)}{=} \sum_{i\in S}\big(v(\bar{S}_\Gamma(i)) - \sum_{j\in F_\Gamma(i)} v(\bar{S}_\Gamma(j))\big) =$$

$$= \sum_{i\in R_\Gamma(S)} v(\bar{S}_\Gamma(i)) - \sum_{i\in S\setminus R_\Gamma(S)} \big(d_S(i)-1\big)\, v(\bar{S}_\Gamma(i)) - \sum_{\substack{j\in F_\Gamma(i)\\ i\in S,\, j\notin S}} d_S(j)\, v(\bar{S}_\Gamma(j)).$$

To complete the proof of (10) it suffices to notice that, since Γ is a strongly cycle-free digraph, every immediate successor $j \in F_\Gamma(i)$ of $i \in S$ that does not belong to S is a source in $\bar{S}_\Gamma(S)\setminus S$. ∎

From Theorem 4 it follows that for any cycle-free digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ the overall efficiency is given by

$$\sum_{i\in N} t_i(v,\Gamma) = \sum_{i\in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) - \sum_{i\in N\setminus R_\Gamma(N)} \big(\kappa_{i,N}-1\big)\, v(\bar{S}_\Gamma(i)), \tag{11}$$

while if the consideration is restricted to only strongly cycle-free digraph games, (11) reduces to

$$\sum_{i\in N} t_i(v,\Gamma) = \sum_{i\in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) - \sum_{i\in N\setminus R_\Gamma(N)} \big(d_\Gamma(i)-1\big)\, v(\bar{S}_\Gamma(i)). \tag{12}$$

To support these expressions we recall the Myerson model in [6] of a game with undirected cooperation structure, in which the component efficiency entails the equality

$$\sum_{i\in N} \xi_i(v,\Gamma) = \sum_{C\in N/\Gamma} v(C). \tag{13}$$

While the right-side expression in (13) is composed by connected components that are the only efficient productive elements in the Myerson's model, the building bricks in (11) and (12) are the full successors sets which are the only efficient productive coalitions under the assumption of t-connectedness. Observe also that for strongly cycle-free rooted-forest digraph games (12) reduces to (13),

$$\sum_{i\in N} t_i(v,\Gamma) = \sum_{i\in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) = \sum_{C\in N/\Gamma} v(C).$$

For a cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$, we define the *t-core* $C^t(v,\Gamma)$ as the set of component efficient payoff vectors that are not dominated by any t-connected coalition,

$$C^t(v,\Gamma) = \{x\in \mathbb{R}^N \mid x(C)=v(C), \forall C\in N/\Gamma;\ x(S)\geq v(S), \forall S\in C^t_\Gamma(N)\}, \quad (14)$$

while the *weak t-core* $\tilde{C}^t(v,\Gamma)$ is the set of component feasible payoff vectors that are not dominated by any t-connected coalition,

$$\tilde{C}^t(v,\Gamma) = \{x\in \mathbb{R}^N \mid x(C)\leq v(C), \forall C\in N/\Gamma;\ x(S)\geq v(S), \forall S\in C^t_\Gamma(N)\}. \quad (15)$$

Theorem 5 *The tree value on the subclass of superadditive rooted-forest digraph games is t-stable.*

Proof. Let $\langle v,\Gamma\rangle \in \mathcal{G}_N^\Gamma$ be a superadditive rooted-forest digraph game arbitrarily chosen. We show that the tree value $t(v,\Gamma)$ belongs to the core $C^t(v,\Gamma)$. Consider arbitrary $C\in N/\Gamma$, then C is a tree. Let $i\in C$ be a source in Γ, then $C=\bar{S}_\Gamma(i)$ because of the rooted-forest structure of Γ. Due to the full-tree efficiency of the tree value, it holds that

$$\sum_{j\in\bar{S}_\Gamma(i)} t_j(v,\Gamma) \stackrel{FTE}{=} v(\bar{S}_\Gamma(i)),$$

wherefrom it follows that

$$\sum_{j\in C} t_j(v,\Gamma) = v(C).$$

Let now $S\in C^t_\Gamma(N)$. Because of the rooted-forest structure of Γ, it holds that $d_N(i)=1$ for all $i\in N\backslash R_\Gamma(N)$. Wherefrom it follows that $\Gamma|_S$ contains exactly one source, say, node i, $\Gamma|_S$ is a subtree, and $S\subseteq \bar{S}_\Gamma(i)$. Moreover, since Γ is strongly cycle-free, $\Gamma|_{\bar{S}_\Gamma(i)}$ is a full subtree, and because of the tree structure of $\Gamma|_S$, $\Gamma|_{\bar{S}_\Gamma(i)\backslash S}$ consists of a collection (might be empty) of disconnected full subtrees, i.e., $\Gamma|_{\bar{S}_\Gamma(i)\backslash S} = \bigcup_{k=1}^q T_k$ where $T_k\bigcap T_l=\emptyset$, $k\neq l$, and $q=|[\bar{S}_\Gamma(i)\backslash S]/\Gamma|$ is the number of components in $\bar{S}_\Gamma(i)\backslash S$. Hence,

$$\bar{S}_\Gamma(i) = S\cup\bigcup_{k=1}^q T_k.$$

Applying again the full-tree efficiency of the tree value, we obtain that

$$\sum_{j\in\bar{S}_\Gamma(i)} t_j(v,\Gamma) \stackrel{FTE}{=} v(\bar{S}_\Gamma(i)),$$

and

$$\sum_{j\in T_k} t_j(v,\Gamma) \stackrel{FTE}{=} v(T_k), \qquad \text{for all } k=1,\ldots,q.$$

From the superadditivity of v and the last three equalities, it follows that

$$\sum_{j\in S} t_j(v,\Gamma) = v(\bar{S}_\Gamma(i)) - \sum_{k=1}^{q} v(T_k) \geq v(S). \qquad \blacksquare$$

Remark 2 The statement of Theorem 5 can also be obtained as a corollary of the stability result proved in Demange [2]. Indeed, in a rooted forest every connected component has a tree structure and, therefore, is t-connected. Whence, for any rooted-forest digraph game the t-core coincides with the core of the Myerson restricted game.

The following examples show that for t-stability of a superadditive digraph game the requirement on the digraph to be a rooted forest is non-reducible. In Example 5 the tree value of a superadditive cycle-free but not strongly cycle-free digraph game violates individual rationality and, therefore, does not meet the second constraint of the weak t-core, while in Example 6 the tree value of a superadditive strongly cycle-free game in which the graph contains two sources violates feasibility.

Example 5 Consider a 4-person cycle-free superadditive digraph game $\langle v,\Gamma\rangle$ with $v(24) = v(34) = v(234) = v(N) = 1, v(S) = 0$ otherwise, and Γ depicted in Figure 5.

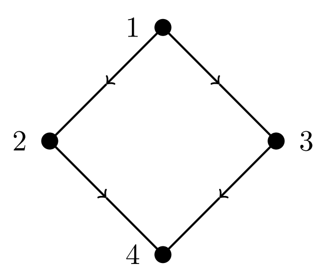

Figure 5

Then $t(v,\Gamma) = (-1,1,1,0)$, whence $t_1(v,\Gamma) = -1 < 0 = v(1)$. Remark that every singleton coalition, in particular $S = \{1\}$, is t-connected.

Example 6 Consider a 3-person cycle-free superadditive digraph game $\langle v,\Gamma\rangle$ with $v(12) = v(13) = v(N) = 1, v(S) = 0$ otherwise, and Γ depicted in Figure 6.

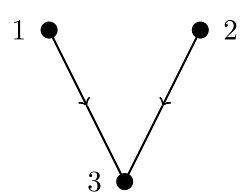

Figure 6

Then $t(v,\Gamma) = (1,1,0)$, whence $t_1(v,\Gamma) + t_2(v,\Gamma) + t_3(v,\Gamma) = 2 > 1 = v(N)$.

A cycle-free digraph game $\langle v,\Gamma\rangle$ is *t-convex*, if for all t-connected coalitions $T,Q\subset C^t_\Gamma(N)$ such that T is a full t-connected set, Q is a full successors set, and $T\cup Q\in C^t_\Gamma(N)$, it holds that

$$v(T)+v(Q)\leq v(T\cup Q)+v(T\cap Q). \tag{16}$$

Theorem 6 *The tree value on the subclass of t-convex strongly cycle-free digraph games is feasible.*

Proof. Let $\langle v,\Gamma\rangle\in\mathcal{G}^\Gamma_N$ be any t-convex strongly cycle-free digraph game. Assume that Γ is connected, otherwise we apply the same argument to any component $C\in N/\Gamma$. If there is only one source in Γ, it holds that $\sum_{i=1}^n t_i(v,\Gamma)=v(N)$ and the tree value is even efficient. So, suppose that there are q different sources $r_1,\ldots,r_q$ in Γ for some $q\geq 2$. Since Γ is connected, the sources in Γ can be ordered in such a way that

$$\Big(\bigcup_{h=1}^{j-1}\bar{S}_\Gamma(r_h)\Big)\cap\bar{S}_\Gamma(r_j)\neq\emptyset,\qquad \text{for } j=2,...,q.$$

For $j=1,...,q$ let $T_j=\bigcup_{h=1}^{j}\bar{S}_\Gamma(r_h)$. Then from the strongly cycle-freeness of Γ it follows that for $j=2,...,q$ there exists a unique $i_j\in N$ such that

$$T_{j-1}\cap\bar{S}_\Gamma(r_j)=\bar{S}_\Gamma(i_j).$$

By t-convexity of the digraph game $\langle v,\Gamma\rangle$ it holds that

$$v(T_{j-1})+v(\bar{S}_\Gamma(r_j))\leq v(T_j)+v(\bar{S}_\Gamma(i_j)),\qquad \text{for } j=2,...,q.$$

Since $T_1=\bar{S}_\Gamma(r_1)$ and $T_q=N$, then applying the last inequality successively $q-1$ times we obtain

$$\sum_{j=1}^{q}v(\bar{S}_\Gamma(r_j))\leq v(N)+\sum_{j=2}^{q}v(\bar{S}_\Gamma(i_j)).$$

Hence,

$$v(N)\geq\sum_{j=1}^{q}v(\bar{S}_\Gamma(r_j))-\sum_{j=2}^{q}v(\bar{S}_\Gamma(i_j)).$$

Since Γ is strongly cycle-free, for any $i\in N\backslash R_\Gamma(N)$, node i has $d_\Gamma(i)$ different sources as predecessors, which implies that the term $v(\bar{S}_\Gamma(i))$ appears precisely $d_\Gamma(i)-1$ times. Therefore,

$$v(N)\geq\sum_{i\in R_\Gamma(N)}v(\bar{S}_\Gamma(i))-\sum_{i\in N\backslash R_\Gamma(N)}\big(d_\Gamma(i)-1\big)\,v(\bar{S}_\Gamma(i)).\qquad\blacksquare$$

The following example of a convex strongly cycle-free digraph game shows that even under the assumption of convexity of a given digraph game, which is stronger than t-convexity, one or more constraints for not being dominated in the definition of the week t-core might be violated by the tree value, and therefore, the tree value is not weakly t-stable.

Example 7 Consider a 5-person cycle-free convex digraph game $\langle v, \Gamma \rangle$ with $v(N) = 10$, $v(123) = v(1234) = v(1235) = 3$, $v(1345) = v(2345) = 2$, $v(S) = 0$ otherwise, and the strongly cycle-free digraph Γ depicted in Figure 7.

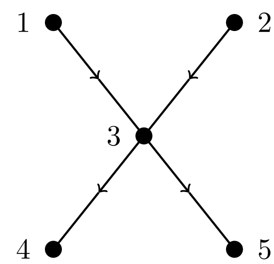

Figure 7

Then $t(v, \Gamma) = (1, 1, 0, 0, 0)$, whence, the total payoff $t_1(v, \Gamma) + t_2(v, \Gamma) + t_3(v, \Gamma)$ of t-connected coalition $S = \{1, 2, 3\}$ is equal to 2 which is smaller than $v(S)$ that is equal to 3.

From (11) it follows that for a cycle-free (for simplicity connected) digraph game $\langle v, \Gamma \rangle \in \mathcal{G}_N^\Gamma$ a necessary and sufficient condition for the feasibility of the tree value is that

$$\sum_{i \in R_\Gamma(N)} v(\bar{S}_\Gamma(i)) \leq v(N) + \sum_{i \in N \setminus R_\Gamma(N)} \left(\kappa_{i,N} - 1\right) v(\bar{S}_\Gamma(i)). \tag{17}$$

Since $N = \bigcup_{i \in R_\Gamma(N)} \bar{S}_\Gamma(i)$, the grand coalition equals the union of the successors sets of all sources in the graph Γ. In case there is only one source in Γ, condition (17) is redundant, because the left side is then equal to $v(N)$. In case there is more than one source in Γ, the different successors sets of the sources of Γ will intersect each other and for any $i \in N \setminus R_\Gamma(N)$ the number $\kappa_{i,N} - 1$ is the number of times that the successors set $\bar{S}_\Gamma(i)$ of node i equals the intersection of successors sets of the sources of Γ. Therefore, condition (17) is a kind of convexity condition for the grand coalition saying that the sum of the worths of the successors sets of all the sources of the graph should be less than or equal to the worth of the grand coalition (their union) plus the total worths of their intersections. In a firm where any full successors set of a source is a division within the firm and subdivisions that are intersections of several divisions are shared by these divisions, in (17) the left-side minus the sum in the right-side can be economically interpreted as the total worths of the divisions when they do not cooperate, while $v(N)$ is the worth of the firm when the divisions do cooperate. To have feasibility the latter value should be at least equal to the former value. Remark that $v(N)$ minus the total payoff at the tree value can be interpreted as the net profit of the firm (or the synergy effect from cooperation) that can be given to its shareholders.

5 Web values

We consider now the general case of an arbitrary management team in a given cycle-free directed communication graph.

To any digraph Γ on N and management team $M \in \mathcal{M}(\Gamma)$ we associate the digraph

$$\Gamma^M = \{(i,j) \mid (i,j) \in \Gamma,\ j \in S_\Gamma(M)\} \bigcup \{(j,i) \mid (i,j) \in \Gamma,\ i \in P_\Gamma(M)\},$$

composed by the same links as Γ but with reversed orientation of all links with origins the predecessors of M. It then holds that the set of sources in Γ^M coincides with the management team M in Γ, i.e., $R_{\Gamma^M}(N) = M$. Moreover, due to the management team development rule, the assumption of M-web connectedness with respect to M in Γ is equivalent to the assumption of tree connectedness in digraph Γ^M, and the requirements of axioms WE, WSE together with WPE, and WE together with WFTE and WFSE with respect to game $\langle v, \Gamma, M\rangle$ are equivalent to the requirements of axioms MTE, SE amd FTE with respect to game $\langle v, \Gamma^M\rangle$ correspondingly. The latter observations allow to obtain the following results relevant to the general case of M-web connectedness straightforwardly from the results proved above in Section 4 under the assumption of tree connectedness.

Proposition 2 *On the class of cycle-free digraph games $\mathcal{G}_N^{\Gamma,M}$ WE, WSE and WPE together imply WFTE and WFSE.*

WE, WSE and WPE uniquely define a value on the class of cycle-free digraph games $\mathcal{G}_N^{\Gamma,M}$.

Theorem 7 *On the class of cycle-free digraph games $\mathcal{G}_N^{\Gamma,M}$ there is a unique value w that satisfies WE, WSE and WPE. For every cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}_N^{\Gamma,M}$, the value $w(v, \Gamma, M)$ satisfies the following conditions:*

(i) it obeys the recursive equality

$$w_i(v,\Gamma,M) = \begin{cases} v(\bar{S}_\Gamma(i)) - \sum\limits_{j \in S_\Gamma(i)} w_j(v,\Gamma,M), & \forall\, i \in S_\Gamma(M), \\ v(\bar{P}_\Gamma(i)) - \sum\limits_{j \in P_\Gamma(i)} w_j(v,\Gamma,M), & \forall\, i \in P_\Gamma(M), \\ v(W_\Gamma(i)) - \sum\limits_{j \in W_\Gamma(i)\setminus\{i\}} w_j(v,\Gamma,M), & \forall\, i \in M; \end{cases} \tag{18}$$

(ii) it admits the explicit representation in the form

$$w_i(v,\Gamma,M) = \begin{cases} v(\bar{S}_\Gamma(i)) - \sum\limits_{j \in S_\Gamma(i)} \kappa_{ij} v(\bar{S}_\Gamma(j)), & \forall\, i \in S_\Gamma(M), \\ v(\bar{P}_\Gamma(i)) - \sum\limits_{j \in P_\Gamma(i)} \kappa_{ji} v(\bar{P}_\Gamma(j)), & \forall\, i \in P_\Gamma(M), \\ v(W_\Gamma(i)) - \sum\limits_{j \in S_\Gamma(i)} \kappa_{ij} v(\bar{S}_\Gamma(j)) - & \\ \qquad - \sum\limits_{j \in P_\Gamma(i)} \kappa_{ji} v(\bar{P}_\Gamma(j)), & \forall\, i \in M; \end{cases} \tag{19}$$

where for all $i \in N$ and $j \in S_\Gamma(i)$, κ_{ij} is defined by (3).

From now on we refer to the value w given by (19) as to the *M-web value* for cycle-free digraph games.

According to (18) the web value assigns to any successor of the given management team the worth of his full successors set minus the total payoff to his successors, to any predecessor of the management team the worth of his full predecessors set minus the total payoff to his predecessors, and to any member of the management team the worth of his full web minus the total payoff to his subordinates. Wherefrom we obtain a simple recursive algorithm for computing the web value by going upstream from the sinks and downstream from the sources till the chosen management team is reached.

The next theorem provides an explicit representation of the M-web value.

Theorem 8 *For any cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}_N^{\Gamma,M}$, the M-web value $w(v,\Gamma,M)$ given by (19) admits the equivalent representation in the form*

$$w_i(v,\Gamma,M)=\begin{cases} v(\bar{S}_\Gamma(i)) - \sum\limits_{j\in F^*_\Gamma(i)} v(\bar{S}_\Gamma(j))+ \\ \quad + \sum\limits_{\substack{j\in S_\Gamma(i)\\ d^{\,i}(j)>1}} \Big(q_{ij}-1-\sum\limits_{h=q'_{ij}+1}^{q_{ij}} \kappa_{ij}(C(\vec{P}_h))\Big)\, v(\bar{S}_\Gamma(j)), \ \ \forall\, i\in S_\Gamma(M), \\ \\ v(\bar{P}_\Gamma(i)) - \sum\limits_{j\in O^*_\Gamma(i)} v(\bar{P}_\Gamma(j))+ \\ \quad + \sum\limits_{\substack{j\in P_\Gamma(i)\\ d_i(j)>1}} \Big(q_{ji}-1-\sum\limits_{h=q'_{ji}+1}^{q_{ji}} \kappa_{ji}(\tilde{C}(\vec{P}_h))\Big)\, v(\bar{P}_\Gamma(j)), \ \ \forall\, i\in P_\Gamma(M), \\ \\ v(W_\Gamma(i)) - \sum\limits_{j\in F^*_\Gamma(i)} v(\bar{S}_\Gamma(j)) - \sum\limits_{j\in O^*_\Gamma(i)} v(\bar{P}_\Gamma(j))+ \\ \quad + \sum\limits_{\substack{j\in S_\Gamma(i)\\ d^{\,i}(j)>1}} \Big(q_{ij}-1-\sum\limits_{h=q'_{ij}+1}^{q_{ij}} \kappa_{ij}(C(\vec{P}_h))\Big)\, v(\bar{S}_\Gamma(j)), \\ \quad + \sum\limits_{\substack{j\in P_\Gamma(i)\\ d_i(j)>1}} \Big(q_{ji}-1-\sum\limits_{h=q'_{ji}+1}^{q_{ji}} \kappa_{ji}(\tilde{C}(\vec{P}_h))\Big)\, v(\bar{P}_\Gamma(j)), \ \ \forall\, i \in M. \end{cases} \tag{20}$$

If the consideration is restricted to only strongly cycle-free digraph games, then the above representation reduces to

$$w_i(v,\Gamma,M)=\begin{cases} v(\bar{S}_\Gamma(i)) - \sum\limits_{j\in F_\Gamma(i)} v(\bar{S}_\Gamma(j)), & \forall\, i \in S_\Gamma(M), \\ v(\bar{P}_\Gamma(i)) - \sum\limits_{j\in O_\Gamma(i)} v(\bar{P}_\Gamma(j)), & \forall\, i \in P_\Gamma(M), \\ v(W_\Gamma(i)) - \sum\limits_{j\in F_\Gamma(i)} v(\bar{S}_\Gamma(j)) - \sum\limits_{j\in O_\Gamma(i)} v(\bar{P}_\Gamma(j)), & \forall\, i \in M; \end{cases} \tag{21}$$

The M-web value assigns to every successor of a given management team the payoff equal to the worth of his full successors set minus the worths of all full successors sets of his proper immediate successors plus or minus the worths of all full successors sets of any other of his successors that are subtracted or added more than once. The M-web value assigns to every predecessor of a given management team the payoff equal to the worth of his full predecessors set minus the worths of all full predecessors sets of his proper immediate predecessors plus or minus the worths of all full predecessors sets of any other of his predecessors that are subtracted or added more than once. The M-web value assigns to every manager of a given management team the payoff equal to the worth of his full web minus the worths of all full successors sets of his proper immediate successors plus or minus the worths of all full successors sets of any other of his successors that are subtracted or added more than once and minus the worths of all full predecessors sets of his proper direct predecessors plus or minus the worths of all full predecessors sets of any other of his predecessors that are subtracted or added more than once. Moreover, for any player $i \in \bar{S}_\Gamma(M)$ and his successor $j \in S_\Gamma(i)$ that is not his proper immediate successor, the coefficient $\kappa_i(j)$ indicates the number of overlappings of full successors sets of all proper immediate successors of i at node j. While for any player $i \in \bar{P}_\Gamma(M)$ and his predecessor $j \in P_\Gamma(i)$ that is not his proper immediate predecessor, the coefficient $\kappa_i(j)$ indicates the number of overlappings of full predecessors sets of all proper immediate predecessors of i at node j. In fact each player receives what he contributes when he joins his sudordinates when we count only the efficient productive coalitions that are either full webs, full successors sets, or full predecessors sets. Besides, it is worth to note and not difficult to check that the right sides of both formulas (20) and (21) being considered with respect not to coalitional worths but to players in these coalitions contain only player i when taking into account all pluses and minuses.

Example 8 Figure 8 provides an example of the M-web value $w(v, \Gamma, M)$ for a 10-person game v with cycle-free digraph Γ given on Figure 1 and the management team $M = \{3, 4, 10\}$.

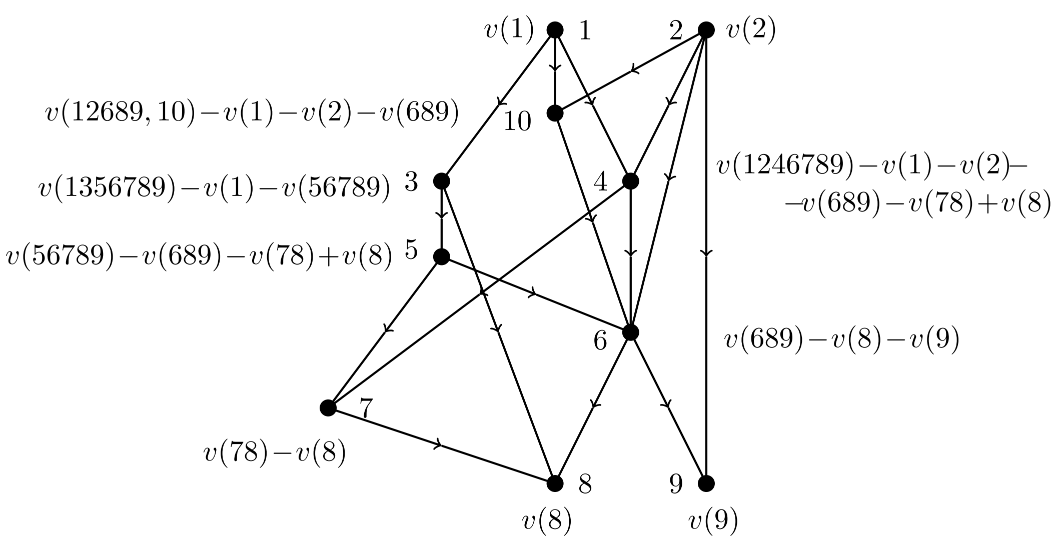

Figure 8

The M-web value not only meets WE, WFTE and WFSE but also that these three efficiency properties alone uniquely define the M-web value on the class of cycle-free digraph games $\mathcal{G}_N^{\Gamma,M}$.

Theorem 9 *On the class of cycle-free digraph games* $\mathcal{G}_N^{\Gamma,M}$ *the* M*-web value* w *is the unique value that satisfies WE, WFTE and WFSE.*

Corollary 5 WE, WFTE and WFSE together on the class of cycle-free digraph games $\mathcal{G}_N^{\Gamma,M}$ imply WSE and WPE.

Corollary 6 The M-web value w meets the independence of inessential links.

For a cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}_N^{\Gamma,M}$, we define the *$M$-web core* $C^M(v, \Gamma, M)$ as the set of component efficient payoff vectors that are not dominated by any M-web connected coalition,

$$C^M(v, \Gamma, M) = \{x \in \mathbb{R}^N \mid x(C) = v(C), \forall C \in N/\Gamma;\ x(S) \geq v(S), \forall S \in C_M^{\Gamma}(N)\}.$$

Theorem 10 *The* M*-web value on the subclass of superadditive line-graph games is* M*-web stable.*

However, for M-web stability of a superadditive digraph game the requirement on the digraph to be a line-graph is non-reducible.

A cycle-free digraph game $\langle v, \Gamma, M\rangle \in \mathcal{G}_N^{\Gamma,M}$ is *M-web-convex*, if for all M-web connected coalitions $T, Q \subset C_M^{\Gamma}(N)$ such that T is a full M-web connected set, Q is a web, and $T \cup Q \in C_M^{\Gamma}(N)$, it holds that

$$v(T) + v(Q) \leq v(T \cup Q) + v(T \cap Q). \tag{22}$$

Theorem 11 *The* M*-web value on the subclass of* M*-web-convex strongly cycle-free digraph games* $\mathcal{G}_N^{\Gamma,M}$ *is feasible.*

Remark that if the management team is composed by the set of all sinks in a given graph, web connectedness can be restated in terms of sink connectedness when for a digraph Γ a connected coalition $S \in C_\Gamma(N)$ is *sink connected*, or simply *s-connected*, if it meets the condition that for every sink $i \in L_\Gamma(S)$ it holds that $i \notin P_\Gamma(j)$ for another source $j \in L_\Gamma(S)$. In this case web efficiency reduces to maximal sink efficiency, web predecessor equivalence to predecessor equivalence, web efficiency together with web full-sink efficiency provide full sink efficiency, axioms of web successor equivalence and web full-tree efficiency become redundant, and the M-web core reduces to the *s-core* $C^s(v, \Gamma)$ defined as the set of component efficient payoff vectors that are not dominated by any s-connected coalition,

$$C^s(v, \Gamma) = \{x \in \mathbb{R}^N \mid x(C) = v(C), \forall C \in N/\Gamma;\ x(S) \geq v(S), \forall S \in C_\Gamma^s(N)\},$$

where $C_\Gamma^s(N)$ denotes the set of all s-connected subcoalitions of N. Besides, formulas (19), (20) and (21) that provide representations of M-web-value reduce correspondingly to[2]

$$s_i(v, \Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j \in P_\Gamma(i)} \kappa_{ji}\, v(\bar{P}_\Gamma(j)), \qquad \text{for all } i \in N, \tag{23}$$

[2]In the next formulas we denote the value relevant to the case of sink connectedness by s instead of w used in the general case.

$$s_i(v,\Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j\in O^*_\Gamma(i)} v(\bar{P}_\Gamma(j)) +$$

$$+\sum_{\substack{j\in P_\Gamma(i)\\ d_i(j)>1}} \Big(q_{ji}-1-\sum_{h=q'_{ji}+1}^{q_{ji}} \kappa_{ji}(\tilde{C}(\vec{P}_h))\Big)\, v(\bar{P}_\Gamma(j)), \qquad \text{for all } i\in N, \tag{24}$$

$$s_i(v,\Gamma) = v(\bar{P}_\Gamma(i)) - \sum_{j\in O_\Gamma(i)} v(\bar{P}_\Gamma(j)), \quad \text{for all } i\in N. \tag{25}$$

For sink-forest digraph games defined by sink forest digraph structures that are strongly cycle-free, the value given by (25) coincides with the sink value introduced in Khmelnitskaya [5]. By that reason from now on we refer to the value s given by (23), or equivalently by (24), as to the *sink-tree value*, or simply the *sink value*, for cycle-free digraph games.

For the sink value it holds that for any cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$ the overall efficiency is given by

$$\sum_{i\in N} s_i(v,\Gamma) = \sum_{i\in L_\Gamma(N)} v(\bar{P}_\Gamma(i)) - \sum_{i\in N\setminus L_\Gamma(N)} \big(\tilde{\kappa}_{i,N}-1\big)\, v(\bar{P}_\Gamma(i)),$$

while if the consideration is restricted to only strongly cycle-free digraph games, the last equality reduces to

$$\sum_{i\in N} s_i(v,\Gamma) = \sum_{i\in L_\Gamma(N)} v(\bar{P}_\Gamma(i)) - \sum_{i\in N\setminus L_\Gamma(N)} \big(\tilde{d}_\Gamma(i)-1\big)\, v(\bar{P}_\Gamma(i)).$$

Theorem 12 *The sink value on the subclass of superadditive sink-forest digraph games is s-stable.*

6 The average web value

In this section we introduce the average web value for cycle-free directed graphs. This value only depends on a given TU game and digraph.

For any cycle-free digraph game $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$, the *average web value* (*AW-value*) is defined as the average of M-web values over the set $\mathcal{M}(\Gamma)$ of all management teams in the digraph Γ, i.e.,

$$AW_i(v,\Gamma) = \frac{1}{|\mathcal{M}(\Gamma)|} \sum_{M\in\mathcal{M}(\Gamma)} w_i(v,\Gamma,M), \qquad \text{for all } i\in N.$$

It is not difficult to see that the AW-value inherits the independence of inessential links property from M-web values. Moreover, since convexity of a digraph game $\langle v,\Gamma\rangle \in \mathcal{G}^\Gamma_N$ is stronger then M-web-convexity with respect to any management team $M\in\mathcal{M}(\Gamma)$, we obtain from Theorem 11 the next theorem.

Theorem 13 *On the class of convex strongly cycle-free digraph games $\mathcal{G}^\Gamma_N$ the AW-value is feasible.*

The *average tree solution* (*AT solution*) for undirected cycle-free graph games, introduced in Herings et al. [4], assigns to any cycle-free graph game $\langle v, \Gamma\rangle$ to player $i \in N$ the average of his tree value payoffs in all rooted spanning trees in the subgraph $\langle (N/\Gamma)_i, \Gamma|_{(N/\Gamma)_i}\rangle$:

$$AT_i(v,\Gamma) = \frac{1}{|(N/\Gamma)_i|} \sum_{j\in(N/\Gamma)_i} t_i(v, T(j)), \qquad \text{for all } i \in N,$$

where, for $j \in (N/\Gamma)_i$, $T(j)$ is the rooted tree on $(N/\Gamma)_i$ with j as root and composed of all links of $\langle (N/\Gamma)_i, \Gamma|_{(N/\Gamma)_i}\rangle$ with orientation directed away from the root and t is the tree value given by (8).

In Herings et al. [4] it is shown that the AT solution defined on the class of superadditive cycle-free graph games is stable, and that on the entire class of cycle-free graph games the AT solution is characterized via component efficiency and component fairness.

A value ξ on the entire class of graph games is *component efficient* (CE) if, for any graph game $\langle v, \Gamma\rangle$, for all $C \in N/\Gamma$,

$$\sum_{i\in C} \xi_i(v,\Gamma) = v(C).$$

A value ξ on the entire class of graph games is *component fair* (CF) if, for any cycle-free graph game $\langle v, \Gamma\rangle$, for every link $\{i, j\} \in \Gamma$, it holds that

$$\frac{1}{|(N/\Gamma\backslash\{i,j\})_i|} \sum_{t\in(N/\Gamma\backslash\{i,j\})_i} \Big(\xi_t(v,\Gamma) - \xi_t(v,\Gamma\backslash\{i,j\}\Big) =$$

$$\frac{1}{|(N/\Gamma\backslash\{i,j\})_j|} \sum_{t\in(N/\Gamma\backslash\{i,j\})_j} \Big(\xi_t(v,\Gamma) - \xi_t(v,\Gamma\backslash\{i,j\}\Big).$$

Theorem 14 *The AW-value for a digraph game $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ coincides with the AT solution for the corresponding undirected graph game $\langle v, \tilde{\Gamma}\rangle$, i.e., $AW(v,\Gamma) = AT(v,\tilde{\Gamma})$, if and only if Γ is a line-graph.*

Proof. In a line-graph Γ on N every management team is a singleton and the web value relevant to each management team coincides with the tree value to the corresponding undirected graph $\tilde{\Gamma}$. Besides, in a line-graph Γ on N the total number of management teams in Γ is equal to n, i.e., $|\mathcal{M}(\Gamma)| = n$. Moreover, every digraph in which all management teams are singletons is a line-graph and conversely. ∎

Since on the subclass of line-graph games the requirement of WE coincides with CE, we obtain from Theorem 14 and the axiomatization of the AT solution the next theorem.

Theorem 15 *On the subclass of line-graph games $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ the AW-value is characterized by WE and CF and, moreover, on the subclass of superadditive line-graph games $\langle v, \Gamma\rangle \in \mathcal{G}_N^\Gamma$ the AW-value is stable.*

7 Sharing a river with multiple sources, a delta and possible islands

Ambec and Sprumont [1] approach the problem of optimal water allocation for a given river with certain capacity over the agents (cities, countries) located along the river from the game theoretic point of view. Their model assumes that between each pair of neighboring agents there is an additional inflow of water. Each agent, in principal, can use all the inflow between itself and its upstream neighbor, however, this allocation in general is not optimal in respect to total welfare. To obtain a more profitable allocation it is allowed to allocate more water to downstream agents which in turn can compensate the extra water obtained by side-payments to upstream ones. The problem of optimal water allocation is approached as the problem of optimal welfare distribution. Van den Brink et al. [7] show that the Ambec-Sprumont river game model can be naturally embedded into the framework of a graph game with line-graph cooperation structure. In Khmelnitskaya [5] the line-graph river model is extended to the rooted-tree and sink-tree digraph model of a river with a delta or with multiple sources, respectively. We extend the line-graph, rooted-tree or sink-tree model of a river to the cycle-free digraph model of a river with both multiple sources and a delta, and also possible islands along the river bed as well.

Let N be a set of players (users of water) located along the river from upstream to downstream. Let $e_{ki} \geq 0$, $i \in N$, $k \in O(i)$, be the inflow of water in front of the most upstream player(s) when $k = 0$, or the inflow of water entering the river between neighboring players when player k is in front of player i. Figure 9 provides a schematic representation of the model.

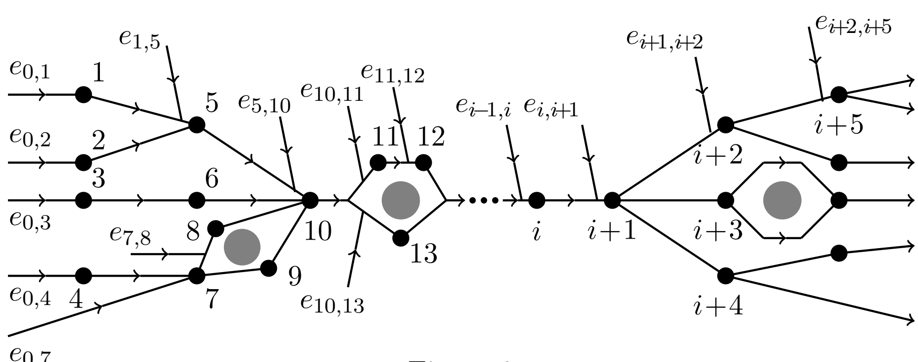

Figure 9

A river with multiple sources, a delta, and several islands along the river bed.

Following Ambec and Sprumont [1] it is assumed that each player $i \in N$ has a quasi-linear utility function given by $u^i(x_i, t_i) = b^i(x_i) + t_i$ where t_i is a monetary compensation to player i, x_i is the amount of water allocated to player i, and $b^i \colon \mathbb{R}_+ \to \mathbb{R}$ is a continuous nondecreasing function providing benefit $b^i(x_i)$ to player i when he consumes the amount x_i of water. Moreover, in case of a river with a delta it is also assumed that if a splitting of the river into branches happens to occur after a certain player, then this player takes, besides his own quota, also the responsibility to split the rest of the water flow to the branches such to guarantee the realization of the water distribution plan x^* to his successors.

The superadditive river game $v \in \mathcal{G}_N$ introduced under the same assumptions in Khmelnitskaya [5] for a river with multiple sources or a delta defined as:

for any $S \in C_\Gamma(N)$, $v(S) = \sum_{i \in S} b^i(x_i^S)$, where $x^S \in \mathbb{R}^S$ solves

$$\max_{x \in \mathbb{R}_+^S} \sum_{i \in S} b^i(x_i) \;\; s.t. \left\{ \begin{array}{rcl} \sum\limits_{j \in \bar{P}_\Gamma(i)} x_j & \leq & \sum\limits_{j \in \bar{P}_\Gamma(i)} \sum\limits_{k \in O(j)} e_{kj}, \\ \sum\limits_{j \in P_\Gamma(i) \cup \bar{B}_\Gamma(i)} x_j & \leq & \sum\limits_{j \in P_\Gamma(i) \cup \bar{B}_\Gamma(i)} \sum\limits_{k \in O(j)} e_{kj}, \end{array} \right. \quad \forall i \in S,$$

and for any other $S \subset N$, $v(S) = \sum\limits_{T \in S/\Gamma} v(T)$,

suits to the case of a river with both multiple sources and a delta, and also possible islands along the river bed as well. The tree and sink values proposed above can be applied for the solution of the river game in the general case.

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