CONSISTENCY OF A SYSTEM OF EQUATIONS: WHAT DOES THAT MEAN?

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ABSTRACT. The concept of (structural) consistency also called structural solvability is an important basic tool for analyzing the structure of systems of equations. Our aim is to provide a sound and practically relevant meaning to this concept. The implications of consistency are expressed in terms of explicit density and stability results. We also illustrate, by typical examples, the limitations of the concept.

Keywords: Consistency of systems of equations, structural solvability, perfect matching, density and stability results, Constraint solving.

1. INTRODUCTION

Many models in engineering lead to a Constraint Solving (CS) problem of the type: Find solutions $x \in \mathbb{R}^n$ of a system of equations and inequalities,

$$h_i(x) = 0,\ i = 1, \dots, m\ ; \quad g_l(x) \le 0,\ l = 1, \dots, k,$$

(see *e.g.*, [6]). Often in applications the system has a specific structure, *i.e.*, most of the functions only depend on a part of the variables x_j.

Remark 1. *Note, that by introducing k extra variables ξ_l, the CS-system above can be written in the equivalent form*

$$h_i(x) = 0,\ i = 1, \dots, m\ ; \quad g_l(x) + \xi_l = 0,\ l = 1, \dots, k\ ,$$

with only inequalities of the simple form $\xi_l \ge 0$.

The most crucial part of CS are the equality constraints. They, *e.g.*, determine the dimension of the solution set. Also the consistency concept is (essentially) restricted to equality constraints. Therefore, in the present article we confine our investigations to systems of equations,

$$h_i(x) = 0,\ i = 1, \dots, m\ ,$$

where all unknowns $x_j \in \mathbb{R}$ are *"continuous"* variables (no discrete variables). The concept of (structural) consistency (or structural solvability) for such systems is discussed in a number of papers (see *e.g.*, [7, 11, 12, 1, 9]). Murota ([11, 12]) has provided a mathematical foundation of this notion. All authors agree about the definition of consistency (*cf.*, Definition 1), however in these articles it does not become clear how this concept should be interpreted or what the precise implications of the concept are. Let us emphasize presently that in the context of

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this paper the term consistent system does *not* mean that the system has a solution (see also Definition 1).

The aim of the present paper is to exhibit a precise description of what is meant if we say that a system (of equations) is consistent. To do so we analyze consistency by techniques from differential geometry. In our opinion this approach leads to appropriate and satisfying interpretations and implications which are relevant from a practical viewpoint.

Note that consistency does not say something about a concrete system of equations (a concrete problem) but it reveals the (generic) properties of a class of systems of equations (problem class). Roughly speaking, if a class of problems is consistent then for almost all problems out of this class certain regularity conditions are fulfilled.

The paper is organized as follows. In Section 2 we introduce the consistency concept which is closely related to a modeling of the structure of the system of equations by a bipartite graph G. In the Sections 3 and 4 we analyze consistency by applying techniques from differential geometry, in Section 3 for linear- and in Section 4 for general nonlinear equations. The obtained results roughly speaking assert that a structured system is generically well-behaving if and only if the corresponding graph G allows a maximum matching covering all equations. The advantage of our approach is that the meaning of well-behavior can be expressed in terms of practically relevant density and stability statements. The last section presents examples which illustrate the advantage but also the limitations of the consistency concept.

The techniques used in the present paper are not new. They were developed and applied to obtain genericity and stability results for example in optimization. We refer the reader to the landmark book [8].

Let us further mention some related topics, where the bipartite graph model is used to analyze the structure of systems in order to obtain efficient solution methods. For example the so-called perfect Gaussian elimination (elimination without extra fill-in) see *e.g.*, [5]. Various studies concern the Dulmage-Mendelsohn decomposition and extensions thereof (*cf.*, [1, 12, 9]). Here, one investigates whether a system of equations allows a decomposition into smaller sub-systems. Another related subject is the problem of reducing or minimizing the bandwidth of a system (see *e.g.*, [4], [10]).

2. Consistency of structured systems of equations

According to the discussion above we consider a system of m equations in n unknowns $x_j \in \mathbb{R},\ j \in J := \{1, \ldots, n\}$,

$$h_i(x) = 0,\ i \in I := \{1, \ldots, m\}, \quad x = (x_1, \ldots, x_n)\ . \tag{1}$$

Let this system have a special structure which is defined by specifying for each i on which variables x_j the function h_i may depend. To do so, as usual (*cf., e.g.* [11, 9, 1]), we introduce a bipartite graph $G = G(I, J, E)$ with node sets I, J and a set E of edges (i, j) defined by

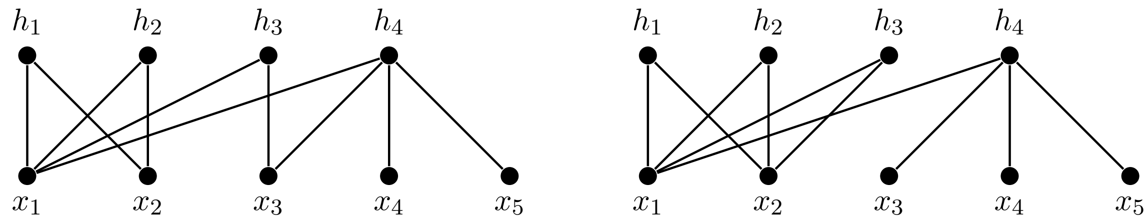
$$E = \{(i, j) \mid h_i(x) \text{ depends explicitly on } x_j\}\ .$$

Note that in this graph model the vertices in I correspond to the equations $h_i(x) = 0$ and the vertices in J are associated with the variables x_j. The following example illustrates how the bipartite graph reflects the structure of the system.

Example 1. Consider the 2 systems of 4 equations $h_i = 0$ in 5 unknowns $x = (x_1, \ldots, x_5)$,

$h_1(x) = x_1 - x_2$	$h_2(x) = x_1 + x_2 - 4$	$h_1(x) = x_1 - x_2$	$h_2(x) = x_1 + x_2 - 4$
$h_3(x) = x_1^2 - x_3$	$h_4(x) = x_1 + x_3 + x_4 + x_5$	$h_3(x) = x_1^2 - x_2$	$h_4(x) = x_1 + x_3 + x_4 + x_5$

with corresponding bipartite graphs:



For the first system we do not expect a problem. Indeed, from $h_1 = 0, h_2 = 0$ we obtain $x_1 = x_2 = 2$. The third equation yields $x_3 = 4$. Then we can chose, e.g., x_5 freely and compute x_4 by the last equation. However the second example is inconsistent in the sense that the solution $x_1 = x_2 = 2$ from the first 2 equations contradicts $h_3 = 0$.

To avoid a situation as in the second example, after a moment of reflection, one finds that the following condition should hold: For any subset $I_0 \subset I$ the number of variables appearing in the equations $h_i(x) = 0,\ i \in I_0$, should not be smaller than the cardinality $|I_0|$ of I_0. If we define the set $N(I_0)$ of neighbor nodes of I_0 in G,

$$N(I_0) = \{j \in J \mid x_j \text{ appears in at least one of the equations } h_i(x),\ i \in I_0\},$$

this condition means:

$$|N(I_0)| \geq |I_0| \quad \forall I_0 \subset I\ . \tag{2}$$

The famous theorem of Hall (see *e.g.*, [3]) says

$$(2) \quad \text{holds} \ \Leftrightarrow G \text{ has a matching covering all nodes in } I\ . \tag{3}$$

As a consequence, we call a system (1) with corresponding bipartite graph $G = G(I, J, E)$ consistent if (2) holds or equivalently if G has a matching covering all nodes in I.
Note that consistency implies $m \leq n$. Recall that a matching in G covering I defines a one to one mapping $\mu : I \to J,\ i \mapsto \mu(i),\ (i, \mu(i)) \in E$, with image $B = B_\mu := \{j = \mu(i),\ i \in I\}$. We introduce a partition $x = (x_B, x_F)$ with $x_B = (x_j,\ j \in B) \in \mathbb{R}^m$ and $x_F := (x_j,\ j \notin B) \in \mathbb{R}^{n-m}$. The $n - m$ variables of x_F are called "free variables" of the system (1). According to the consistency concept these variables can be chosen (freely) so that for any choice of x_F we are left with a (consistent) system

$$\tilde{h}_i(x_B) := h_i(x_B, x_F) = 0\ , \quad i = 1, \ldots, m\ ,$$

of m equations in the m unknowns x_B.

Example 2. The first system in Example 1 allows a matching covering $I = \{1, 2, 3, 4\}$, the second does not. For the first system such a matching is given, *e.g.*, by $\mu : I \to J$, $\mu(i) = i, i = 1, 2, 3, 4$; and x_5 is the free variable.

Remark 2. There may exist different choices for the free variables x_F. Chosing *e.g.*, in Example 2 a different matching, $\mu(I) = \{1, 2, 3, 5\}$, then x_4 becomes the free variable.

Motivated by the preceeding discussion, from now on we confine the further analysis to the case $m = n$ of a structured system of n equation in n unknowns,

$$h_i(x) = 0,\ i \in I := \{1, \dots, n\}, \quad x = (x_1, \dots, x_n) \in \mathbb{R}^n \tag{4}$$

with a structure given by a bipartite graph $G = G(I, J, E)$.

Definition 1. *Let the structure of a system (4) be given by the bipartite graph $G = G(I, J, E)$ $(I = J = \{1, \dots, n\})$. We then call the system (4)* consistent *if (2) holds or, equivalently, if G has a matching covering all nodes in I (a perfect matching).*
Often, e.g.*, in* [11]*, instead of consistency the notion* structural solvability *is used. We recall that in this paper consistency of a system does not mean its solvability. Consistency does not say something about a concrete instance of a system but it has a meaning for a whole class of systems (given by G).*

In the next sections we analyze this consistency concept, firstly for linear- and then for general nonlinear equations.

3. Structured linear equations

In this section we deal with the special case that the system (4) is linear,

$$Ax - b = 0, \qquad A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n \ ,$$

and has a structure as given by the bipartite graph $G(I, J, E)$. We then can define the corresponding structured class of linear equations.

Definition 2. *Given a bipartite graph $G = G(I, J, E)$ we introduce the corresponding class of structured matrices*

$$M_G = \{A \in \mathbb{R}^{n\times n} \mid a_{ij} = 0 \ \text{ for } (i, j) \notin E\} \cong \mathbb{R}^{|E|}.$$

We call the problem class

$$P_G : \qquad \textit{solve } Ax = b \textit{ with } A \in M_G$$

consistent *if G allows a perfect matching (*inconsistent *otherwise).*

In the following we will show that for almost all $A \in M_G$ the matrix A is non-singular (*i.e.,* $Ax = b$ has a unique solution) if and only if P_G is consistent. The basic implication of consistency is given by

Proposition 1. *The set M_G contains a non-singular matrix A_0 if and only if G allows a perfect matching.*

Proof. Let μ represent a perfect matching in $G(I, J, E)$, *i.e.*, $\mu : I \to J,\ i \mapsto \mu(i),\ (i, \mu(i)) \in E$, defines a permutation J of I. Then the matrix $A_0 \in M_G$ given by

$$A_0 = (a_{ij}) \ , \qquad a_{ij} = \begin{cases} 1 & \text{if } j = \mu(i) \\ 0 & \text{otherwise} \end{cases}$$

is obviously a permutation matrix with $\det A_0 = \pm 1$, *i.e.*, A_0 is non-singular.

Assume now that G does not allow a perfect matching. By Hall's result there must exist a subset $I_0 \subset I$ with $|N(I_0)| < |I_0|$. Defining $k := |I_0|$, we can assume $I_0 = \{1, \ldots, k\}$ and $N(I_0) = \{1, \ldots, r\}$ with $r < k$. By construction this means that the entries in the first k rows of all matrices $A \in M_G$ have value zero in the columns $j \geq k \geq r + 1$. So the first k rows of any matrix $A \in M_G$ are linearly dependent implying the singularity of A. □

To give a precise formulation of the implications of the consistency concept we use the following fundamental lemma in differential geometry.

Lemma 1. *Let $p : \mathbb{R}^K \to \mathbb{R}$ be a polynomial mapping, $p \neq 0$. Then the set $p^{-1}(0) = \{x \in \mathbb{R}^K \mid p(x) = 0\}$ has (Lebesgue) measure zero.*

Next we define a polynomial mapping on $M_G \cong \mathbb{R}^{|E|}$ by

$$p : M_G \to \mathbb{R}, \quad p(A) = \det A \ .$$

According to Proposition 1 this mapping is non-trivial ($p(A_0) \neq 0$ and thus $p \neq 0$) if and only if G allows a perfect matching. So, together with Lemma 1 we conclude.

Corollary 1. *The set $M_G^0 = \{A \in M_G \mid \det A = 0\}$ of singular matrices in M_G has (Lebesgue) measure zero if and only if G allows a perfect matching.*

By Corollary 1 the set $M_G^r := M_G \setminus M_G^0 = \{A \in M_G \mid \det A \neq 0\}$ of non-singular matrices has full Lebesgue measure. This means that for almost all $A \in M_G$ (in the Lebesgue sense) the system $Ax = b$ is uniquely solvable if and only if G allows a perfect matching. In particular M_G^r is dense in M_G. Recall the fact that $p(A) = \det(A)$ is continuous (on M_G). So if $\det(A_0) \neq 0$ holds the condition $\det(A) \neq 0$ holds in a whole neighborhood of A_0 with respect to (wrt.) some norm in $M_G \cong \mathbb{R}^{|E|}$. Altogether we have proved the following stability and density result.

Proposition 2. *The set M_G^r is dense and open in M_G if and only if G allows a perfect matching.*

We wish to mention, that in particular, with this analysis we have generalized the well-known result that for the matrix class "without any special structure" (corresponding to the complete bipartite graph $G = K_{n,n}$) the set of non-singular matrices is open and of "full measure".

4. STRUCTURED NONLINEAR EQUATIONS

We now consider general systems of (nonlinear) equations

$$h_i(x) = 0, \quad i = 1, \ldots, n \ .$$

Let the system belong to a class of problems having a special structure given by the bipartite graph $G = G(I, J, E)$. By setting $h = (h_1, \ldots, h_n)$ we define the corresponding set of functions

$$S_G = \{h : \mathbb{R}^n \to \mathbb{R}^n, h \in C^1 \mid h_i \text{ depends on } x_j \text{ only if } (i, j) \in E\} \subset C^1(\mathbb{R}^n, \mathbb{R}^n) \ .$$

To motivate our approach, recall that a standard way to solve a system $h(x) = 0$ is by Newton's method. It is well-known that for any solution $\bar{x}$ of $h(x) = 0$ the Newton iteration $x_{k+1} = x_k - [\nabla h(x_k)]^{-1} h(x_k)$ is locally quadratically convergent to $\bar{x}$ if the regularity condition holds:

$$\nabla h(\bar{x}) \qquad \text{is non-singular .} \tag{5}$$

We will show in this section, that generically (*i.e.*, for an open and dense subset in S_G) the condition (5) holds at all solutions $\bar{x}$ of $h(x) = 0$. Moreover, if G does not allow a perfect matching, then generically no solution of $h(x) = 0$ exists.

Definition 3. *We call the problem class*

$$P_G : \qquad \textit{solve } h(x) = 0 \textit{ with } h \in S_G$$

consistent *if the graph $G = G(I, J, E)$ allows a perfect matching (*inconsistent *otherwise).*

To generalize the results of Section 3 we need some preparations. Given a function $f \in C^1(\mathbb{R}^m, \mathbb{R}^r)$ the vector $0 \in \mathbb{R}^r$ is called a *regular value* of f if

$$\nabla f(y) \text{ has full rank } r \text{ for all solutions } y \text{ of } f(y) = 0 \ .$$

This condition implies that the solution set of $f(y) = 0$ is a smooth manifold of dimension $m - r$. Instead of Lemma 1, for nonlinear equations, we need

Theorem 1. *[*Parametric Sard theorem [2]*]*
Let $f(y, z),\ y \in R^m, z \in \mathbb{R}^p$ be a function in $C^k(\mathbb{R}^{m+p}, \mathbb{R}^r)$ with $k > \max\{0, m - r\}$. If 0 is a regular value of f then for almost all parameters $z \in \mathbb{R}^p$ (in the Lebesgue measure), 0 is a regular value of the function $\hat{f}_z : R^m \to \mathbb{R}^r,\ \hat{f}_z(y) = f(y, z)$.

Based on this theorem we can now prove the following basic genericity result.

Theorem 2. *Let $\hat{h} \in S_G$ be given. Then for almost all vectors $[A, d] = [a_{ij}, (i, j) \in E; d_i, i = 1, \ldots, n] \in \mathbb{R}^{|E|} \times \mathbb{R}^n$ the perturbed functions*

$$h_i(x) := \hat{h}_i(x) + \sum_{j;\ (i,j) \in E} a_{ij} x_j + d_i, \quad i = 1, \ldots, n \ , \tag{6}$$

satisfy the regularity condition (5) for all solutions $\bar{x}$ of $h(\bar{x}) = 0$.

Proof. By construction $h \in S_G$. Assume now that the statement is false. The fact, that at a solution x of $h(x) = 0$ the condition (5) fails means that after an appropriate renumbering of the h_i's, there exists a solution (x, λ) of the following system

$$\begin{array}{rcl} \nabla h_1(x) + \sum_{i=2}^n \lambda_i \nabla h_i(x) & = & 0 \\ h_i(x) & = & 0, \quad i = 1, \ldots, n. \end{array} \tag{7}$$

Some of the λ_j's might be zero. We can skip these coefficients and after a second renumbering of the indices i we arrive at a system (see (6))

$$F(x, \lambda; A, d) := \begin{array}{rcl} \nabla h_1(x) + \sum_{i=2}^k \lambda_i \nabla h_i(x) & = & 0 \\ h_i(x) & = & 0, \quad i = 1, \ldots, k. \end{array} \tag{8}$$

in (x, λ). This system depends precisely on say v variables x_j. So we can assume

$$x \in \mathbb{R}^v\ , \quad \lambda \in \mathbb{R}^{k-1}\ , \quad \lambda_i \neq 0\ \ \forall i = 2, \ldots, k\ . \tag{9}$$

We now show that for almost all parameters $[A, d]$ the system (8) doesn't allow any solution (x, λ). To do so, we apply Theorem 1 and consider the Jacobian of the system $F(x, \lambda, A, d) = 0$ in (8) with respect to (wrt.) the variables $x_j, j = 1, \ldots, v$, $\lambda_i, i = 2, \ldots, k$ and parameters

$$a_{ij}, (i, j) \in E\ , \quad d_i\ , \quad i = 1, \ldots, k\ ;\ j = 1, \ldots, v.$$

The Jacobian has the form (∂_x etc. denote the partial derivative wrt. x, $\otimes$ denote matrices of appropriate dimension)

(10)

∂_x	∂_λ	$\partial_{a_{1j}}{}'s$	$\partial_{a_{2,j}}{}'s$	$\ldots$	$\partial_{a_{kj}}{}'s$	∂_d
$\otimes$	$\otimes$	I^1	$\lambda_2 I^2$	$\ldots$	$\lambda_k I^k$	0
$\otimes$	0	$\otimes$	$\otimes$	$\ldots$	$\otimes$	I_k

with identity matrix $I_k \in \mathbb{R}^{k \times k}$ and diagonal matrices $I^i \in \mathbb{R}^{v \times v}$

$$I^i = \mathrm{diag}(d_1^i, \ldots, d_v^i), \quad d_j^i = 1 \text{ if } (i, j) \in E \text{ and } d_j^i = 0 \text{ otherwise.}$$

By construction, for any $j = 1, \ldots, v$ at least one element d_j^i, $i = 1, \ldots k$, is nonzero (each variable x_j, $j = 1, \ldots, v$, appears in (8)). Hence by recalling $\lambda_i \neq 0\ \forall i$, in each of the v first rows of the submatrix of (10) formed by the columns corresponding to the ∂a_{ij}'s at least one element is nonzero. So the first v rows are linearly independent and in view of the sub-block I_k in the last k rows of (10) this Jacobian has full row rank $k + v$. The Sard theorem implies that for almost all $[A, d]$ also the Jacobian $\partial_{x,\lambda} F$ of (8) with respect to the variables (x, λ) has full row rank $k + v$ at all solutions (x, λ) of (8). But $(x, \lambda) \in \mathbb{R}^{v+k-1}$, *i.e.*, the Jacobian $\partial_{x,\lambda} F$ only has $v + k - 1$ columns so that the (row) rank of the Jacobian cannot be equal $v + k$. Consequently for almost all $[A, d]$ there cannot exist any solution (x, λ) of (8).
By noticing that there are only finitely many choices ρ for h_1 and λ_i's equal to zero (by taking the finite intersection $\cap_\rho [A, d]^\rho$) we have proven that for almost all $[A, d]$ the system (7) doesn't have a solution. This proves the statement. □

We now describe the implications of consistency (G has a perfect matching) or inconsistency (G doesn't have a perfect matching). We begin with the latter. As a corollary of Theorem 2 we find

Corollary 2. *Suppose G does not allow a perfect matching and let $\hat{h} \in S_G$ be fixed. Then for any $x \in \mathbb{R}^n$ the Jacobian $\nabla h(x)$ is singular. Moreover, for almost all vectors $[A, d] = [a_{ij}, (i, j) \in E; d_i, i = 1, \ldots n] \in \mathbb{R}^E \times \mathbb{R}^n$ the perturbed functions*

$$h_i(x) := \hat{h}_i(x) + \sum_{j,(i,j) \in E} a_{ij} x_j + d_i, \quad i = 1, \ldots, n$$

don't allow any solution $\bar{x}$ of $h(x) = 0$.

Proof. If G doesn't posses a perfect matching then, arguing as in the second part of the proof of Proposition 1 (after an appropriate renumbering of equations and variables), with $k = |I_0|$, $r = |N(I_0)| < k$, the first k equations $h_i(x) = 0$ only depend on the r variables $x_1, \ldots, x_r$. So the first k rows of the Jacobian $\nabla h(x)$ are linearly dependent for all $x \in \mathbb{R}^n$. Consequently for all $x \in \mathbb{R}^n$ the matrix $\nabla h(x)$ is singular so that 0 can only be a regular value of h if $h(x) = 0$ does not allow any solution. The statement is now a direct consequence of Theorem 2.

□

We finally analyze the case that the problem class P_G is consistent (G has a perfect matching). For the next result we assume that the set S_G (as a sub-class of $C^1(\mathbb{R}^n, \mathbb{R}^n)$) is endowed with a special topology, the so-called strong topology. We do not go into details and refer the reader to [8]. Being interested in the set of "nice" (regular) problems we define the corresponding set of functions in S_G:

$$S_G^r = \{h \in S_G \mid (5) \text{ holds for any solution } \bar{x} \text{ of } h(x) = 0\} \ .$$

Theorem 3. *Let G allow a perfect matching. Then the following hold:*

(a) The set S_G^r contains functions h which have solutions $\bar{x}$ of $h(x) = 0$.

(b) The set S_G^r is an open and dense subset of S_G (open and dense in the strong topology).

Proof. (a) By taking the nonsingular matrix A_0 of Proposition 1, we see that for any $b \in \mathbb{R}^n$ the (linear) function $h(x) = A_0 x - b$ belongs to S_G. Moreover since $\nabla h(x) = A_0$ is nonsingular, h is a function in S_G^r and a (unique) solution of $h(x) = 0$ exists.
(b) Here we only give a sketch of the proof and refer the reader to [8] for details.
The density part is based on the perturbation result in Theorem 2 and uses the technique of partition of unity in the following way (as in the proof of [8, Th.7.1.13]). Let be given a function $\hat{h} \in S_G$. Then near each solution x_0 of $\hat{h}(x) = 0$ an (arbitrarily) small local perturbation is applied to obtain (locally defined) functions $\hat{h} \in S_G^r$. Using the partition of unity these local perturbations are "glued" together to result into a function $\widetilde{h} \in S_G^r$ close to $\hat{h}$. The proof of the openness part also uses an appropriate partition of unity to extend a local stability result into a global one.

□

In [11, Sect.5-7] and [12, Sect.4.3.1,4.3.2] Murota establishes a mathematical foundation of the consistency (structural solvability) concept. His approach relies on some assumptions ([11, GA1, p.36]) in terms of algebraic number theory. His result in [11, Theorem 7.1, 7.2] can essentially be compared with the basic statement in Theorem 3(a). Theorem 3(b) provides additional information, namely implications of consistency in terms density and stability results.

5. Interpretations and illustrative examples

In this last section we briefly comment on the interpretation of the results above from a practical perspective. We illustrate the advantage and limitations of the consistency concept. The results in Theorem 3 and Corollary 2 can be summarized as follows.

When G has a perfect matching (Theorem 3)

(i) *Openness result* (*stability*): Given a function $h \in S_G^r$ then by any (sufficiently) small perturbation $\widetilde{h}$ of h we maintain a function $\widetilde{h} \in S_G^r$ (i.e., at each solution $\bar{x}$ of $\widetilde{h}(x) = 0$

the regularity condition (5) holds). In other words, small computation errors do not destroy well-behavior.

(ii) *Density result:* Given a "bad" function $h \in S_G \backslash S_G^r$, by an arbitrarily small perturbation we can obtain a "nice" function $\widetilde{h} \in S_G^r$.

<u>When G doesn't allow a perfect matching</u>

(i) Given $h \in S_G$ and a solution $\bar{x}$ of $h(x) = 0$. Then $\nabla h(\bar{x})$ is singular. In other words, at any solution $h(\bar{x}) = 0$ of any function $h \in S_G$, the regularity condition for the Newton iteration is not satisfied.

(ii) Given a function $h \in S_G$ then by any (sufficiently) small perturbation we obtain a function $\widetilde{h} \in S_G$ such that $\widetilde{h}(x) = 0$ has no solution.

There is one essential difference between the case of a consistent system of linear and nonlinear equations.

- In the linear case (if G allows a perfect matching) then for any $A \in M_G^r$ the system $h(x) = Ax - b = 0$ has a (unique) solution.
- For nonlinear equations (with $h \in S_G^r$) the existence of a solution of $h(x) = 0$ is not guaranteed as is illustrated by the next example. We only know that if solutions exist then they are all locally unique (and regular in the sense of (5)).

Example 3. We define the systems of 2 equations in the 2 unknowns x_1, x_2, depending on the parameter $\alpha \in \mathbb{R}$:

$$(P_\alpha) \qquad h_1(x) := x_1^2 + x_2^2 - 1 = 0 \ , \quad h_2(x) := (x_1 - \alpha)^2 + x_2^2 - 1 = 0 \ ,$$

the intersection of two circles. This system is obviously consistent. For $0 < |\alpha| < 2$ the corresponding system $h = 0$ has two (regular) solutions and h is contained in S_G^r. For $\alpha = \pm 2$ we have 1 solution of $h = 0$ and for $\alpha = 0$ infinitely many (the whole circle). In both cases $h \notin S_G^r$. For $|\alpha| > 2$ there is no solution and thus trivially $h \in S_G^r$.

We finally come back to systems of n equations in more than n say $n + k$ unknowns. The result of Theorem 3 can then be interpreted as follows:
Suppose G allows a matching μ covering I and let wlog. $B_\mu = \{1, \ldots, n\}$ so that the k free variables are $x_F = (x_{n+1}, \ldots, x_{n+k})$ (*cf.*, Section 2). For any fixed $\bar{x}_F$ with $x_B = (x_1, \ldots, x_n)$ the equations

$$\hat{h}_i(x_B) := h_i(x_B, \bar{x}_F) = 0, \quad i = 1, \ldots, n \tag{11}$$

define a consistent problem (in x_B) possessing the density and stability properties above. However we cannot expect that generically (for an open and dense function set) the function $\hat{h}$ in (11) is contained in S_G^r for all x_F, unless the system (11) is linear, *i.e.*, $h(x_B) = A_1 x_B + A_2 x_F - b = 0$ with $A = [A_1, A_2] \in \mathbb{R}^{n \cdot n} \times \mathbb{R}^{n \cdot k}$. More precisely the following holds.

<u>*When G allows a matching:*</u>

- In the linear case, for any choice of the free variables x_F the system $A_1 x_B + A_2 x_F - b = 0$ has a unique solution x_B.

For systems of nonlinear equations a corresponding result is not true as illustrated by the next example.

Consider the (consistent) equation $h(x) = x_1 \cdot x_2 + x_1^2 + x_2 = 0$ in two variables. Taking x_2 as free variable we see that for the choice $\bar{x}_F = \bar{x}_2 = 0$ the system $h = 0$ does not satisfy the regularity condition at $\bar{x}_1 = 0$ since $\partial_{x_1} h(\bar{x}_1, \bar{x}_F) = 0$. Moreover, this bad situation is stable wrt. small C^1 perturbations of h. Indeed, we can show that for any small C^1 perturbation $\widetilde{h}$ of h there is a choice $\bar{x}_F \approx 0$ such that $\partial_{x_1} h(x_1, \bar{x}_F) = 0$ for a corresponding solution x_1 of $\widetilde{h} = 0$.

As a concluding remark we emphasize that in any constraint solving procedure a consistency check (check whether G allows a matching which covers I) should be done before starting to (try to) compute a solution. Such a check can be done efficiently (see *e.g.*, [13]). If the system is not consistent the solver should stop with this outcome and the user should reconsider his CS problem.

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