Different Approaches on Stochastic Reachability as an Optimal Stopping Problem

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Abstract. Reachability analysis is the core of model checking of time systems. For stochastic hybrid systems, this safety verification method is very little supported mainly because of complexity and difficulty of the associated mathematical problems. In this paper, we develop two main directions of studying stochastic reachability as an optimal stopping problem. The first approach studies the hypotheses for the dynamic programming corresponding with the optimal stopping problem for stochastic hybrid systems. In the second approach, we investigate the reachability problem considering approximations of stochastic hybrid systems. The main difficulty arises when we have to prove the convergence of the value functions of the approximating processes to the value function of the initial process. An original proof is provided.
Keywords: Stochastic hybrid systems, Markov processes, reachability problem, optimal stopping.

1 Introduction

Stochastic hybrid systems are a class of non-linear stochastic continuous time/space hybrid dynamical systems. For these systems different models have been developed by many researchers in the field of hybrid systems. These models can be used to analyse and design complex embedded systems that operate in the presence of variability and uncertainty, and incorporate complex (hybrid/stochastic) dynamics, randomness, multiple modes of operations. Under some natural assumptions on their parameters, their behaviour can be described by stochastic processes having good properties. A very important verification problem for such systems consists mainly in reachability analysis. The aim of reachability analysis is to determine the probability that the system will reach a set of desirable/unsafe states, and the difficulty of this problem comes from the interaction between discrete/continuous dynamics and the active boundaries.

The paper addresses the reachability problem for stochastic hybrid systems. Starting with the characterization of the reachability problem as an optimal stopping problem for the Markov processes that describe the semantics of a stochastic hybrid systems, we further develop different foundational approaches that hopefully will conduct, in final, to computational methods. The main difficulty comes from the fact that these Markov processes belong to a relatively restrictive class of stochastic processes. They have "good" mathematical properties (like strong Markov property, some continuity properties of traces, etc), in fact, they are included in the large class of Borel right (Markov) processes[8]. But, because of the influence of the boundaries on the dynamics, these processes cover only a specific subclass of Borel processes, with a very little intersection with other subclasses of processes more popular in the theory of stochastic control (Feller-Markov processes [22], standard processes [40], jump-diffusion processes [38]). In consequence, for these processes, the optimal control problems have been studied mostly at a theoretical level. This situation compels us to study the optimal stopping problem for Borel right processes, in a separate section of the paper, and to give an overview for the different methods that can be employed in dealing with this problem.

For a stochastic hybrid system, the reach set probabilities coincide with the value functions of some particular optimal stopping problems corresponding to the indicator functions of the target sets [6]. These optimal stopping problems are formulated in the language of the Markov process that describes the realizations of the given hybrid system. We make a guiding inventory of the possible methods that could be used for solving the optimal stopping problems. Considering the complexity of the right (Markov) processes that appear in the context of stochastic hybrid systems, we further develop two such methods.

One approach is based on the characterization of the value functions as viscosity solutions of some variational inequalities associated to an integro-differential operator (that is represented by the infinitesimal generator of the underlying Markov process). The main drawbacks and difficulties when they are applied to the optimal stopping of stochastic hybrid systems come from the fact that we need some continuity assumptions for the reward function or for the transition probabilities of the Markov processes involved. But, for the reachability problem, the reward function is discontinuous, and the continuity of the transition probabilities imply no boundary activity.

Another method is to approximate the stochastic hybrid systems realization by simpler Markovian processes (like purely jump processes or Markov chains) and to derive convergence results for the sequences of value functions associated to the optimal stopping problem corresponding to the approximating processes. The corner stone of this approach is, in fact, proving the convergence results under such general hypotheses (the reward function is not continuous and the limiting process is only a Borel right process). We use an original argument based on the correspondence between reach set probabilities and the so-called Choquet capacities developed by us in [9].

Furthermore, for a particular class of stochastic hybrid systems, namely for Piecewise deterministic Markov processes different other methods to deal with optimal stopping problem, so with stochastic reachabil.ity, are available. These are explained briefly at the end of the paper.

2 Stochastic Hybrid Systems

Stochastic Hybrid Systems can be described as an interleaving between a finite or countable family of diffusion processes (or, sometimes, only dynamical systems) and a Markov chain. Modelling and analysis of these systems have been proved to be a very difficult task from a mathematical point of view. The stochastic analysis apparatus, employed to study their probabilistic properties is complex and rather difficult to manage. This study involves the ability to combine tools available for diffusion processes and jump processes, in order to characterise the semantics of these systems. The switching mechanism (governed by a Markov chain in most cases) between the continuous dynamic of the modes, together with the interaction between paths and boundaries, make studying the stochastic processes that arise in this way very difficult and challenging.

2.1 General Stochastic Hybrid Systems

We adopt the General Stochastic Hybrid System model presented in [8, 11]. This subsection describes the model and establishes the notation.

Let Q be a set of discrete states. For each $q \in Q$, we consider the Euclidean space $\mathbb{R}^{d(q)}$ with dimension $d(q)$ and we define an *invariant* as an open subset X^q of $\mathbb{R}^{d(q)}$. The hybrid state space is the set $X(Q, d, \mathcal{X}) = \bigcup_{i \in Q} \{i\} \times X^i$ and $x = (i, z^i) \in X(Q, d, \mathcal{X})$ is the hybrid state. The closure of the hybrid state space will be $\overline{X} = X \cup \partial X$, where $\partial X = \bigcup_{i\in Q}\{i\} \times \partial X^i$. It is known that X can be endowed with a metric ρ whose restriction to any component X^i is equivalent to the usual component metric [16]. Then $(X, \mathcal{B}(X))$ is a Borel space (homeomorphic to a Borel subset of a complete separable metric space), where $\mathcal{B}(X)$ is the Borel σ-algebra of X. Let $\mathbf{B}(X)$ be the Banach space of bounded positive measurable functions on X with the norm given by the supremum.

Definition 1. *A (General) Stochastic Hybrid System (SHS) is a collection* $H = ((Q, d, \mathcal{X}), b, \sigma, Init, \lambda, R)$ *where*

- Q *is a countable set of discrete states (modes);*
- $d : Q \to \mathbb{N}$ *is a map giving the dimensions of the continuous state spaces;*
- $\mathcal{X} : Q \to \mathbb{R}^{d(\cdot)}$ *maps each* $q \in Q$ *into an open subset* X^q *of* $\mathbb{R}^{d(q)}$*;*
- $b : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)}$ *is a vector field;*
- $\sigma : X(Q, d, \mathcal{X}) \to \mathbb{R}^{d(\cdot)\times m}$ *is a* $X^{(\cdot)}$*-valued matrix,* $m \in \mathbb{N}$*,*
- $Init : \mathcal{B}(X) \to [0, 1]$ *is an initial probability measure on* $(X, \mathcal{B}(X))$*;*
- $\lambda : \overline{X}(Q, d, \mathcal{X}) \to \mathbb{R}^+$ *is a transition rate function;*
- $R : \overline{X} \times \mathcal{B}(\overline{X}) \to [0, 1]$ *is a transition measure.*

The realization of an SHS is built as a *Markov string H* [8] obtained by the concatenation of some diffusion processes (z^i_t), $i \in Q$ together with a jumping mechanism given by a family of stopping times (S^i). Let ω_i be a diffusion trajectory, which starts in $(i, z^i) \in X$. Let $t_*(\omega_i)$ be the first hitting time of ∂X^i of the process (x^i_t). Define the function

$$F(t, \omega_i) = I_{(t<t_*(\omega_i))} \exp(-\int_0^t \lambda(i, z^i_s(\omega_i)))ds. \tag{1}$$

This function will be the survivor function for the stopping time S^i associated to the diffusions (z^i_t).

Definition 2 (SHS Execution). *A stochastic process* $x_t = (q(t), z(t))$ *is called an SHS execution if there exists a sequence of stopping times* $T_0 = 0 < T_1 < T_2 \leq \ldots$ *such that for each* $k \in \mathbb{N}$*,*

- $x_0 = (q_0, z_0^{q_0})$ *is a* $Q\times X$*-valued random variable extracted according to the probability measure Init;*
- *For* $t \in [T_k, T_{k+1})$*,* $q_t = q_{T_k}$ *is constant and* $z(t)$ *is a solution of the stochastic differential equation (SDE):*

$$dz(t) = b(q_{T_k}, z(t))dt + \sigma(q_{T_k}, z(t))dW_t \tag{2}$$

where W_t *is a the* m*-dimensional standard Wiener process;*

- $T_{k+1} = T_k + S^{i_k}$ *where* S^{i_k} *is chosen according with the survivor function (1).*
- *The probability distribution of* $x(T_{k+1})$ *is governed by the law* $R\left((q_{T_k}, z(T^-_{k+1})), \cdot\right)$.

It is known, from [8], that the realization of any SHS, H, under standard assumptions (about the diffusion coefficients, non-Zeno executions, transition measure, etc, see [8] for a detailed presentation) is a strong Markov process (see the definition, for example, in [19]). Let $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ be the Markov process associated to H, where $(\Omega, \mathcal{F})$, $\{x_t\}$ is a collection of X-valued random variables, $\{\mathcal{F}_t\}$ is the natural filtration of the process (the 'history' of the process). The meaning of the elements of M can be found in any source treating continuous-parameter Markov processes (for e.g. [5, 19, 16]). We adjoin an extra point Δ (the cemetery) to X as an isolated point, $X_\Delta = X \cup \{\Delta\}$. The existence of Δ is assumed in order to have a

probabilistic interpretation of $P_x(x_t \in X) < 1$, i.e. Δ is the state where the process lies when it 'dies'. Then, the 'termination time' $\zeta(\omega)$ is the random time when the process M escapes to and is trapped at Δ.
Let $\mathcal{P} = (P_t)_{t>0}$ denote the *semigroup of operators* associated to M, which maps $\mathbf{B}(X)$ into itself given by

$$P_t f(x) = E_x f(x_t), \forall x \in X \tag{3}$$

where E_x is the expectation w.r.t. P_x. The semigroup $\mathcal{P} = (P_t)_{t>0}$ can be thought of as an *abstraction* of M, since from $\mathcal{P}$ one can recuperate the initial process [5]. Recall that a nonnegative function $f \in \mathbf{B}(X)$ is called α-*excessive* ($\alpha \geq 0$) if $e^{-\alpha t}P_t f \leq f$ for all $t \geq 0$ and $e^{-\alpha t}P_t f \nearrow f$ as $t \searrow 0$. If $\alpha = 0$, a 0-excessive function is simply called *excessive function*. Let us denote the *cone of excessive functions* by $\mathcal{E}_M$. In the theory of Markov processes, the excessive functions play the role of the superharmonic functions from the theory of partial differential equations (for e.g. a function $f \geq 0$ is superharmonic w.r.t. the Laplace operator if $\Delta f \leq 0$). Note, that the definition of excessive function can be given in terms of the *operator resolvent* $\mathcal{U}$, which is the Laplace transform of $\mathcal{P}$. The *operator resolvent* $\mathcal{U} = (V_r)_{r\geq 0}$ associated with $\mathcal{P}$ is

$$V_r f(x) = \int_0^\infty e^{-rt} P_t f(x) dt, f \in \mathbf{B}(X),\ x \in X. \tag{4}$$

The infinitesimal generator $\mathcal{L}$ is the derivative of P_t at $t = 0$. Let $D(\mathcal{L}) \subset \mathcal{B}_b(X)$ be the set of functions f for which the following limit exists (denoted by $\mathcal{L}f$)

$$\lim_{t\searrow 0} \frac{1}{t}(P_t f - f) \tag{5}$$

The following SHS property, proved in [8], has a major influence over the coming results.

Proposition 1. *Under the standard assumptions the realization M of an SHS is a Borel right process with cadlag property.*

Recall that a Borel right process is defined by the following properties: (i) its sample paths $t \to x_t$ are right-continuous almost sure. (ii) X is a separable metric space homeomorphic to a Borel subset of some compact metric space, equipped with Borel σ-algebra $\mathcal{B}(X)$ or shortly $\mathcal{B}$ (i.e. X is a Lusin state space). (iii) The operator semigroup of M, given by (3), maps $\mathbf{B}(X)$ into itself. (iv) If f is an α-excessive function for $\mathcal{P}$, then the sample path $t \to f(x_t(\omega))$ is a.s. right continuous (this property is equivalent with the fact M is a strong Markov process).

The sample paths of M are right continuous with left limit, i.e. are cadlags [8]. Moreover, the cadlag property added to the fact that the state space is a Lusin space, which insures a 'tightness' property of this right process, that it is concentrated on compacts.
The infinitesimal generator of an SHS is an integro-differential operator (according with the terminology of [22]). We have proved in [11] that the extended generator of an SHS has the following expression:

$$\mathcal{L}f(x) = \mathcal{L}_{cont} f(x) + \lambda(x) \int_{\overline{X}} (f(y) - f(x)) R(x, dy) \tag{6}$$

where $\mathcal{L}_{cont} f(x)$ has the standard form of the diffusion infinitesimal operator. What makes this generator different from the generator of a Feller Markov process (see [22]) is its domain that contains at least the set of second order differentiable functions that satisfy the boundary condition, as follows:

$$f(x) = \int_{\mathbb{X}} f(y) R(x, dy),\ x \in \partial X. \tag{7}$$

In the presence of forced jumps, the generator of an SHS is an operator that is difficult to deal with, since its domain does not even contain the set of all compactly supported C^∞ functions.

2.2 Piecewise Deterministic Markov Processes

Piecewise deterministic Markov processes (PDMP) represent a very general class of non-diffusion processes that can be considered a particular class of SHS. The standard monograph presenting the theory of PDMP is considered to be [16]. However, the PDMP applications are presented in very rich series of papers including [14, 15, 33, 20]. SHS models from the previous subsection have been tailored after the model of PDMP. This means that if in the SHS models the SDEs, which govern the continuous evolutions between jumps, degenerate in ordinary differential equations (ODE), one can obtain the PDMP model. The generator of a PDMP is also with the generator of an SHS, the only difference is that the 'continuous operator' $\mathcal{L}_{cont}$ from (6) is replaced by the Lie derivative.

Many results available for PDMP have been proved also for SHS [8], but, of course, there is still place for many generalizations from PDMP to SHS. These generalizations are not straightforward; the main difficulty is given by the continuous evolution in the SHS modes described by diffusion processes. In fact, many times, in this process of adapting results from PDMP to SHS, new research issues appear and some PDMP mathematical objects have to be completely redesigned for SHS.

The executions of a PDMP depend on three local characteristics, namely the flow $\varphi(\cdot, x)$, the jump rate $\lambda(x)$ and the reset map (stochastic kernel) $R(x, \cdot)$. All the mathematical objects associated to a PDMP are defined in an analogous way as those for SHS, with the natural differences given by the missing of diffusions. For a detailed presentation of PDMP, consult [16].

2.3 Stochastic Reachability

Let us consider $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ being a (strong right) Markov process, the realization of a stochastic hybrid system. For this strong Markov process we address a verification problem consisting of the following *stochastic reachability problem*.

Given a target set, the objective of the reachability problem is to compute the probability that the system trajectories from an arbitrary initial state will reach the target set.
Formally, given a set $A \in \mathcal{B}(X)$ and a time horizon $T > 0$, let us define :

$$\begin{aligned} Reach_T(A) &= \{\omega \in \Omega \mid \exists t \in [0, T] : x_t(\omega) \in A\} \\ Reach_\infty(A) &= \{\omega \in \Omega \mid \exists t \geq 0 : x_t(\omega) \in A\}. \end{aligned} \tag{8}$$

These two sets are the sets of trajectories of M, which reach the set A (the flow that enters A) in the interval of time $[0, T]$ or $[0, \infty)$.
The reachability problem consists of determining the probabilities of such sets. It can be shown that under our assumptions, since the process M is Borel right process and has the cadlag property, the reachability problem is well-defined, i.e. $Reach_T(A)$, $Reach_\infty(A)$ are indeed measurable sets [12]. Then the probabilities of reach events are

$$P(T_A < T) \text{ or } P(T_A < \zeta) \tag{9}$$

where ζ is the life time of M and T_A is the first hitting time of A

$$T_A = \inf\{t > 0 | x_t \in A\} \tag{10}$$

and P is a probability on the measurable space $(\Omega, \mathcal{F})$ of the elementary events associated to M. P can be chosen to be P_x (if we want to consider the trajectories that start in x) or P_μ (if we want to consider the trajectories that start in with an initial condition given by the distribution μ). Recall that

$$P_\mu(A) = \int P_x(A) d\mu, A \in \mathcal{F}.$$

3 Stochastic Reachability as an Optimal Stopping Problem

In this section, in the framework of SHS, we explain how the stochastic reachability problem can be transformed in an equivalent optimal stopping problem.

3.1 Optimal Stopping Problem

In the following, the optimal stopping problem for a (strong right) Markov process $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$ taking values in a Lusin space is briefly reviewed.
Let Σ denote the set of stopping times (finite or not) with respect to the filtration $\{\mathcal{F}_t\}$ (i.e. $\tau \in \Sigma \Leftrightarrow \forall t,\ \{\tau \leq t\} \in \mathcal{F}_t$). Consider $g : X \to \mathbb{R}$ a bounded measurable function called the *reward function* (the interpretation being that if we stop the process at a point $x \in X$ we obtain a reward $g(x)$). Obviously, the definition of OSP requires some integrability conditions over the paths of M (see, for example [18], for more details). Let $(y_t)_{t \geq 0}$ be the *reward process* defined by

$$y_t = g(x_t),\ t \geq 0.$$

The *maximal payoff function* (or the *value function*, in the terminology of [16]) is

$$v(x) := \sup\{E_x y_\tau | \tau \in \Sigma\}. \tag{11}$$

The value function has been characterised in terms of the minimal excessive function lying above the reward function for standard Markov processes [40], or more general for right Markov processes [18].

3.2 Stochastic reachability as an optimal stopping problem

Let us introduce the *reachability function* $w_A : X \to [0, 1]$ associated to A, defined as

$$w_A(x) := P_x[Reach_\infty(A)]. \tag{12}$$

Taking the reward function g to be equal with the indicator function of A, i.e. $g := 1_A$ according to the characterization of the reach set probability derived in the previous subsection we obtain the following result:

Proposition 2. *[6] If $A \in \mathcal{B}(X)$ then the reachability function w_A coincides with the value function of the reward process $y_t = 1_A(x_t)$, i.e.*

$$w_A(x) = \sup\{P_x(x_\tau \in A) | \tau \in \Sigma\},\ \forall x \in X.$$

4 Optimal Stopping Problem for Borel Right Processes

The realizations of SHS are (Borel) right processes, and therefore the general theory of optimal stopping developed for right processes [3, 36] can be applied. This theory is foundational since it provides mathematical characterizations of the value function using different tools available for right processes:

• The approach presented in [3] relies on a well-known connection between excessivity and a special type of functional concavity.

• The main result of [36] shows that the value function of an optimal stopping problem coincides with *the Snell's enevelope* of the reward process. The *Snell's envelop* is the smallest supermartingale that dominates the reward process.

Markov processes, which appear in the SHS semantics are (Borel) right processes, but they may or may not be

• standard Markov processes (whose theory is well-developed in [5]) because the quasi-left continuity might fail, due to the existence of the active boundaries when the process jumps in a new mode;

• or, Feller processes (processes with continuous transition probabilities) since they have predictable jumps (i.e. forced transitions). See [16], for discussion of the Feller property for piecewise deterministic Markov processes.

Therefore, the optimal stopping times need not exist, and the treatment of the OSP requires some additional hypotheses. We recall the following inclusions (which are classical in the literature of Markov processes [23]) among the various classes of processes:

$$\text{(Feller)}\subset\text{(Hunt)}\subset\text{(special standard)}\subset\text{(right)}$$

These different types of processes were introduced at various stages during of the recent theory of Markov processes. This remark leads to the fact that the well developed OSP methods available for standard Markov processes [40], or those available for Feller Markov processes (and their elliptic integro-differential operators) [22] are certainly not directly applicable for the Borel right processes that arise in the SHS context.

For right Markov processes, the value function has been characterised as the minimal excessive function lying above the reward function [18]. Therefore, for Borel right processes, the computation of the value function corresponding to the OSP has to be based on specific features of these processes. Then, we distinguish:

• *analytic methods*, when the characterization of the value function for the OSP has to consider:
(i) different representation way of excessive functions (as integrals of the Green kernel of the process or the Riesz decomposition [24]);
(ii) variational inequalities associated to some energy functional constructed using the hitting/balayage operator [25];
(iii) proving that the value function is the solution of some variational inequality [21, 33], then solving numerically this inequality;

• *probabilistic methods* that consist in
(i) approximations of the underlying Markov process by a Markov chain and compute the value function corresponding to the chain by some specific algorithms [30, 39];
(ii) martingale methods based on Snell's envelope;
(iii) Monte Carlo Methods [28].

4.1 Variational inequalities

Let X be a bounded open set in $\mathbb{R}^N$ with *smooth boundary*. $\mathbb{R}^N$ can be thought of as the Euclidean space where the state space of a stochastic hybrid system can be embedded.

According to [1, 33], for the existence of the viscosity solutions some assumptions are necessary. For the *Dirichlet problem* given by (14) and (15), these assumptions can be formulated as follows:
(A.1) $F \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}_N \times \mathbb{R})$,
(A.2) F satisfies the local and non-local *degenerate ellipticity condition(s)*: for any $x \in \mathbb{R}^N$, $u \in \mathbb{R}$, $p \in \mathbb{R}^N$, $A, B \in \mathcal{S}^N$, $l_1, l_2 \in \mathbb{R}$

$$F(x, u, p, A, l_1) \leq F(x, u, p, B, l_2) \text{ if } A \geq B,\ l_1 \geq l_2$$

(A.3) $R(x, \cdot)$ is a probability measure on X for $x \in \partial X$ such that the linear operator

$$Rv(x) = \int_X v(y)R(x, dy) \tag{13}$$

satisfies

$$|Rv(x)| \leq C||v||_{L^1(X)}, \text{ for all } v \in L^1(X)$$

where C does not depend on v.
(A.4) The function $x \longmapsto Rv(x)$ is continuous w.r.t. $x \in \overline{X}$, uniformly for $v \in L^\infty(X)$.

Motivated by the expression of the generator associated to an SHS, let us consider the linear integro-differential equations of the following form:

$$F(x, u, D_x u, D_x^2 u, \int_X u(y)R(x, dy)) = 0, \tag{14}$$

where $D_x u$ denotes the space gradient, $D_x^2 u$ the matrix of second derivatives and $R(x, \cdot)$ is a probability kernel. Here, $\mathcal{S}^N$ denotes the space of symmetric $N \times N$ real valued matrices. The applications for (14) are dynamic programming equations associated with the control of the right Markov processes that appear as SHS realizations.

In the case when the state space X is a bounded domain of a Euclidean space, the process jumps back into X upon hitting the boundary, which leads to the following boundary condition to be coupled with the equation (14),

$$u(x) = \int_X u(y)R(x, dy),\ x \in \partial X. \tag{15}$$

For a bounded function $u : X \to \mathbb{R}$, its upper/lower semicontinuous envelopes can be defined in a standard way [33, 1]. Furthermore, the definitions of the viscosity (sub/super) solutions for second-order elliptic integro-differential equations are well established now in the literature [1].

Let u be a bounded function.
(i) u^* is a *viscosity subsolution* of (14) if

$$F(x, u^*, D_x\phi, D_x^2\phi, \int_X u^*(y)R(x, dy) \leq 0$$

for any $\phi \in C^2(X)$ and any local maximum x for $u^* - \phi$.
(ii) u_* is a *viscosity supersolution* of (14) if

$$F(x, u_*, D_x\phi, D_x^2\phi, \int_X u_*(y)R(x, dy) \geq 0$$

for any $\phi \in C^2(X)$ and any local minimum x for $u_* - \phi$.
(iii) u is a *viscosity solution* if u is a viscosity sub- and supersolution.

A bounded function $u : \overline{X} \to \mathbb{R}$ is a *viscosity* subsolution (resp. supersolution) of the *Dirichlet problem* given by (14) and (15), if it is a subsolution (resp. supersolution) of (14) in X and, any $\phi \in C^2(X)$ and any local maximum (resp. minimum) $x \in \partial X$ for $u^* - \phi$ (resp. $u_* - \phi$) $\min\{u^*(x) - k(x), F(x, u^*, D_x\phi, D_x^2\phi, \int_X u^*(y)R(x, dy)\} \leq 0$ (resp. $\max\{u_*(x) - k(x), F(x, u_*, D_x\phi, D_x^2\phi, \int_X u_*(y)R(x, dy)\} \geq 0)$ where $k(x) := \int_X u(y)R(x, dy)$, $x \in \partial X$.

In general, the existence of the solutions is proved by *Perron's method*, introduced in the viscosity setting in [31]. That is, one proves that the supremum of a suitable set of subsolutions is the solution. In order to do this, one needs the help of a *comparison principle*.

In particular, for an appropriate choice of F, this Dirichlet problem becomes

$$\min(-\mathcal{L}u, u - g) = 0 \text{ in } X, \tag{16}$$

$$u(x) = \int_X u(y)R(x, dy) \text{ on } \partial X \tag{17}$$

where, $\mathcal{L}$ is the generator associated to an SHS, given by (6). Equation (16) with the boundary condition (17) is *the dynamic programming equation associated with the optimal stopping problem for SHS* [10]. In this case, the assumption A.2 involves that the diffusion term is non-degenerate. This is also in force in [29]. The assumption A.3 hints at the stochastic kernel R (the SHS reset map) that should provide a bounded linear operator and the assumption A.4 involves the Feller property of the SHS realization [16]. For the case of Feller processes, the reward function is allowed to be semicontinuous, and the value function will be also semicontinuous [2].

The main problem, in this context, is that an SHS is not a Feller process unless there are no active boundaries. Then, these results can be applied only in some particular cases. For PDMP, dynamic programming equations have been developed in a series of papers (see [14][16][20], and the references therein), but, in all these papers, the reward function has some continuity properties (on the whole space, or only on the trajectories of the process). Therefore, again these results can not be applied to reachability analysis of PDMP since the reward functions for OSP associated do not have such continuities.

5 Approximation Methods for Stochastic Reachability

In Section 3, we have seen that computation of the reach set probabilities could be reduced to the computation of the *value function* of an optimal stopping problem with the reward function given by the indicator function of the target set involved.

The OSP methods discussed in the Section 4 can be adapted to SHS realizations considering the special features of SHS (mixture of deterministic/stochastic continuous motion with random jumps), in order to obtain specific optimal stopping methods where the randomness and hybridicity of SHS are clearly illustrated. On the other hand, these features can be employed in order to obtain direct approximations of value function of the OSP. We consider that for the stochastic processes that arise in the SHS semantics, numerical computation of the reach set probabilities as value functions for some optimal stopping problems could be supported by the following methods:

1. approximate the underlying Markov process by a Markov chain [30] and then compute the value function corresponding to the chain.
2. prove that the value function of the indicator function of a measurable set is the fixed point of some "jump operators" [14, 15]

5.1 Approximations of Stochastic Hybrid Systems

In this section, we summarize briefly the exponentially timestepping approximation scheme (ETAS) for strong Markov processes with cadlag property developed in [7].

Let us consider a strong Markov process $M = (\Omega, \mathcal{F}, \mathcal{F}_t, x_t, P_x)$. Suppose that M has the cadlag property and the state space $(X, \mathcal{B})$. M is thought of as the realization of a stochastic hybrid system H. Let d be a compatible metric on X. Let $(P_t)_{t>0}$ (resp. $(V_r)_{r\geq 0}$) be its operator semigroup 3 (resp. operator resolvent 4).

Fix $x \in X$; in the following discussion, P_x is the law of M under the initial condition $x_0 = x$. In order to construct the sequence of jump processes that approximate M, we need the following ingredients:

1. A sequence of Markov chains (α^n). Each $\alpha^n = (\alpha^n_k)_{k=0,1,2,...}$ is a Markov chain on X_Δ with some initial distribution ν and the (homogeneous) transition function, K_n (i.e. a time-homogeneous Markov chain), given by

$$K_n(x, dy) := nV_n(x, dy) \tag{18}$$

where V_n is the stochastic kernel computed from formula (4), i.e. is the Laplace transform of the transition probability function of M for $r = n$.

2. A sequence of Poisson processes (θ^n). Each $\theta^n = (\theta^n_t)_{t\geq 0}$ is a Poisson process[1] with the parameter n, independent of α^n.

Using these ingredients, we then define, for each $n \geq 1$, a *continuous-time (regular) Markov step or Markov jump* process on X_Δ by

$$\rho^n_t := \alpha^n_{\theta^n_t},\ t \geq 0. \tag{19}$$

whose embedded marked point process has the intensity equal to n and state space X_Δ. This means that the jump times of the process (ρ^n_t) are given by the arrival times of the Poisson process (θ^n_t) and its values between jumps are provided by the Markov chain (α^n_k).

Note that $K_n(x, \cdot)$, given by (18), can be thought of as the P_x-distribution of x_T, where T is a random time independent of M and exponentially distributed with rate n [27]. The kernel V_n can be computed using the generator L of the process M by formula

$$V_n := (nI - L)^{-1}, n \geq 1. \tag{20}$$

where I is the identity operator [19]. Moreover, V_n is the *potential kernel* of the process M killed with the exponential rate n [27].

The above sequence of step processes converges in the Skorokhod topology and consequently it converges weakly (in distribution)[2] to the initial Markov process.

Theorem 1. *[7] If $\alpha^n_0 = x$, then the sequence $\{\rho^n\}_{n\geq 1}$ of step processes converges weakly to M (under P_x) as $n \to \infty$.*

We explain how the hybrid structure of an SHS dynamics is considered in ETAS. For each $\omega \in \Omega$, a hybrid trajectory $x_t(\omega) = (q_t(\omega), z_t(\omega))$ of an SHS, H, can be thought of as the union of 'diffusion components' $\{z_t(\omega) | T_k(\omega) \leq t < T_{k+1}(\omega),\ k = 1, 2, ...\}$ where $T_1 < T_2 < ...$ represent the jump times of H. Each component is provided with the label $q_{T_k(\omega)}(\omega)$ since

[1] i.e. $P(\theta^n_t = k) = \exp(-nt)\frac{(nt)^k}{k!}$

[2] A sequence of r.v. $(x_n)_{n=1,2,...}$ X-valued, defined on $(\Omega, \mathcal{F}, \mathbf{P})$ *converges weakly* (or *in distribution*) to a r.v. x_0 if $\mathbf{E}f(x_n) \to \mathbf{E}f(x_0)$ as $n \to \infty$ $\forall f$ bounded continuous on X. Here, $\mathbf{E}f(x_n) = \int_X f(x)P_n(dx)$ for $n \geq 0$, where $P_n = \mathbf{P} \circ x_n^{-1}$ for $n \geq 0$.

$q_t(\omega)$ is constant in the random time interval $[T_k(\omega), T_{k+1}(\omega))$. Then, a cadlag trajectory of H is implicitly carrying the hybrid dynamics structure. In the ETAS, we do *not* interpolate the Poisson times of step processes considered there with the jumping times of H. The reason for not doing this is that the latter jumping times can not be explicitly computed since a jumping time might be the first boundary hitting time of some diffusion process or some random time exponentially distributed with a rate depending on the piece of diffusion trajectory covered until that moment.

In the ETAS, the trajectories of the system are considered 'first class citizens' and the methodology is heavily based on the use of a metric defined on the space of all possible trajectories. Moreover, in this approximation scheme, the approximating processes are step processes that can be thought of as the realizations of some simple particular SHSs. Since, the forced transitions are removed, step processes belong to a nicer class of Markov processes, namely standard Markov processes [5]. Therefore, computational methods for the optimal stopping problem of step processes are well developed [40]. Moreover, we will see that because of the weak convergence of the approximating processes in ETAS, we can derive convergence results for the optimal stopping value functions. Then, the reach set probabilities for an SHS can be approximated by the reach set probabilities of the step processes constructed in the ETAS.

5.2 Approximation of the reach set probabilities

Let us consider the reachability problem defined in Subsection 2.3 for an SHS, H. Suppose that the target set A is a measurable set of X. If A is open (closed) then its indicator function $g := 1_A$ is a lower (upper) semicontinuous function. We can define also the reward processes associated to the step processes ρ^n that are defined in ETAS

$$y_t^n := 1_A(\rho_t^n).$$

Even in the case when the reward function is semicontinuous, the reward processes (y_t), (y_t^n) are not longer cadlag processes (as the realization of H). Therefore, the results about convergence of values in optimal stopping nicely developed in [13] are not directly applicable.

Practically, when we are studying the convergence of value functions (v^n) (w.r.t. (y_t^n)) to the desired value function v (w.r.t. (y_t)), we need to consider different aspects related to:

- the convergence of (ρ^n) to M (in the Skorokhod topology);
- the semicontinuity of the reward function g;
- the convergence of (y_t^n) to (y_t).

Since the realization M of H is a Borel right process that may or may not have the property of quasi left continuity (i.e. whenever T_n is an increasing sequence of stopping times with limit T, then almost surely $x_{T_n} \to x_T$), we can not work under the hypotheses of the papers [13], where this property is a datum from the beginning. Moreover, the results about the convergence of the sequence of value functions have been also studied in a particular case in [32]. In the above cited paper, the Markov processes considered are Feller, and the authors make some assumptions about the density of $C_0(X)$ (the space of continuous real functions on X vanishing at infinity) in the intersections of generator domains corresponding to approximating processes (ρ_t^n) and initial process (x_t). For the approximation scheme described in the previous subsection, these assumptions are not longer in forced due to the peculiarity of the generator domain of an SHS (see the boundary condition (15)).

In the above mentioned papers, the convergence results are based on the martingale problem associated to the Markov processes involved. The type of functions that belong to the domain of an SHS generator constitutes a corner stone of using its associated martingale in studying the OSP defined in section 3. Since the indicator function 1_A does not belong to $D(\mathcal{L})$ (the domain of the SHS generator) we can not reason about the OSP (that appears in related to stochastic reachability) using the martingale problem.

The above discussion expresses, in fact, a very difficult situation and a major contribution needs to be done. In the following, a propose a solution for proving the convergence of the value functions based on the correspondence between the reach set probabilities (9) and the so-called Choquet capacities [9].

In the context of stochastic reachability, one can define a *random set*

$$
\begin{aligned}
S \ : \ \Omega \rightarrow \mathcal{B} \\
\omega \mapsto \{x_t | 0 \leq t \leq T\}
\end{aligned}
$$

Then

$$
Reach_T(A) = \{\omega | S(\omega) \cap A \neq \emptyset\}
$$

and the reach set probability gives rise to a subadditive set function (called *capacity*)

$$
\begin{aligned}
cap_T : \mathcal{B} \rightarrow [0, 1] \\
cap_T(A) := P[Reach_T(A)]
\end{aligned}
$$

that, w.r.t. the random set S, plays the same role as a distribution for a random variable. In the same way, cap_∞ can be defined. Analogously, we may define the capacities cap_T^n, cap_∞^n corresponding to the step processes (ρ_t^n). Then, studying the convergence of the reach set probabilities corresponding to the approximating processes means studying the convergence $cap_T^n \rightarrow cap_T$.

Theorem 2. *If the sequence $\{\rho_t^n\}_{n \geq 1}$ of strong Markov processes converges weakly to (x_t) (under P) as $n \rightarrow \infty$, then*

$$
cap_T^n \rightarrow cap_T.
$$

Proof. The proof is lengthy and very technical. Due to the inherent room limitations, we describe the main steps:

1. Weak convergence can be characterized in terms of extended generators of the processes.
2. The convergence of generators can be characterised in terms of convergence of the operator semigroups and resolvents (Trotter-Kato theorem [19]).
3. Convergence of resolvents involves the convergence of the associated Dirichlet forms [37].
4. Convergence of the Dirichlet forms implies the convergence of their capacities [41].
5. Convergence of the Dirichlet form capacities conducts to the convergence the capacities associated to the corresponding Markov processes.

The key of the proof is provided by the characterization of Markov processes by Dirichlet forms. A Dirichlet form is a quadratic form that can be naturally associated to the generator of a Markov process [35]. ■

Remark 1. We formulated the above convergence result in a very general case (not only for step processes), such that it may be used for different kinds of approximations.

5.3 Stochastic Reachability for PDMP

In this subsection, furthermore, we investigate the stochastic reachability, as an optimal stopping problem, for PDMP. The motivation for doing this is the fact, for PDMP, characterizations of the OSP abound in the literature [14, 15, 26, 34].

The approach of [34], extended then in [33], is inspired by the theory of viscosity solutions associated to first order integro-differential operators. In the above cited papers, the results are based on some "continuity" assumption on the reset map R (associated to a PDMP, see Subsection 2.2). According to [16], this assumption makes the PDMP a Feller-Markov process, and involves no boundary activity. So, practically, in this case, the PDMP is not a hybrid system in the traditional sense. However, the optimal control problems for Feller-Markov processes are well understood now [22], and many other results can be derived in this particular setting.

We are more interested in the approach developed in [26], and generalized in [15], and then in [14]. Mainly, in these papers the value function of the optimal stopping problem is characterized as the unique fixed point of the first jump operator. Moreover, since the stochastic reachability is equivalent with an appropriate optimal stopping problem with a discontinuous reward function, the results for OSP from [15] can be adapted for our problem in a fruitful way.

Let us recall that for the OSP problem studied in [15] optimal stopping is defined for a function g that is a real valued bounded lower semianalytic function on X as $\inf_{\tau\in\Sigma} E_x(g(x_\tau))$. Dually, one can take the reward function g as a bounded upper semianalytic function on X and study the OSP defined as in Subsection 3.1.

However, since the indicator functions are measurable functions (so, both upper/lower semianalytic), for the reachability problem we adapt the results from [15] for the value function defined using the supremum. Denote by $\mathbf{B}^*(X)$ the set of bounded upper semianalytic functions. Let $t^*(x)$ the hitting time of the active boundary for the flow started in x. Define $\Lambda := \int_0^t \lambda(\varphi(s,x))ds$ where $0 \le t \le t^*(x)$. For $x \in X$ and $0 \le t \le t^*(x)$, for a PDMP the following standard operators can be defined [16, 15]:

$$\begin{aligned}
\text{(a) } J(v_1,v_2)(t,x) \; &: = E_x[v_1(\varphi(t,x))1_{[T_1>t]} + v_2(x_{T_1})1_{[T_1\le t]}] \\
&= v_1(\varphi(t,x)e^{-\Lambda(t,x)} + \int_0^t Rv_2(\varphi(s,x))\lambda(\varphi(s,x))e^{-\Lambda(s,x)}ds \\
\text{(b) } Kv_2(x) \; &: = E_x[v_2(x_{T_1})] \\
\text{(c) } L(v_1,v_2)(x) \; &: = \{\inf_{0\le t<t^*(x)} J(v_1,v_2)(t,x)\} \wedge Kv_2(x)
\end{aligned}$$

L is called the *first jump operator*. Define the sequence of functions $(\rho_n)_{n\ge 0}$ by $\rho_0 := g$, $\rho_{n+1} := L(g,\rho_n)$. Clearly, ρ_n is increasing and denote by ρ its limit.

Theorem 3. *Let ρ_n and ρ be defined as above. Then*
(a) $\rho_{n+1} = L(\rho_n,\rho_n)$
(b) ρ is the smallest solution of $v = L(g,v)$, $v \in \mathbf{B}^*(X)$;
(c) ρ is the smallest solution of $v = L(v,v)$, $v \ge g$, $v \in \mathbf{B}^*(X)$
(d) ρ coincides with the value function defined by (11).

Proof. See Prop.1, Prop.2 and Cor.1 from [15].

Proposition 3. *If $A \in \mathcal{B}$, the reachability function w_A is the smallest bounded upper semianalytic function of $v = L(1_A,v)$.*

Proof. Take in Th. 3, $g = 1_A$, and use the characterization of w_A as a value function for an OSP.

6 Conclusions

In this paper, we have used the characterisation of the reachability problem of stochastic hybrid systems as an optimal stopping problem with a discontinuous reward function developed in [6], we have investigated further developments. Because of the fact that the semantics of stochastic hybrid systems cover only a particular subset of the class of (right) Markov processes, solving the optimal stopping problem for such processes is difficult and challenging. For these processes, characterizing the reachability problem as a viscosity solution for some variational inequalities corresponding to their stochastic generators needs additional assumptions regarding the continuity properties of their internal structure (transition probabilities, stopping times). For stochastic hybrid systems, due to the interaction between continuous dynamics and boundary, these assumptions may not be fulfilled. Therefore, to deal with the stochastic reachability, we need to consider new approaches..

One of the major contribution of this paper is to provide rigorous evidence about reliability of verification of SHS by optimal control. The behaviour of a complex SHS can be constructively approximated by simpler Markov processes, for which the optimal stopping problem is well understood. The key for achieving this is the mathematical result that provides the approximation of the reach set probabilities.

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