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Department of Applied Mathematics, University of Twente, Enschede, The Netherlands

A comparison of techniques for learning and using mathematics and a study of their relationship to logical principles

Roxana Grigoraş†, Stefan Halverscheid†, Cornelis Hoede§
†Department of Mathematics, University of Bremen, Germany
{roxana,sth}@math.uni-bremen.de
§Department of Applied Mathematics, University of Twente, The Netherlands
hoede@math.utwente.nl

Abstract. Various techniques exist for learning mathematical concepts, like *experimentation*, and *exploration*, respectively using mathematics, like *simulation* and *modelling*. For a clear application of such techniques in mathematics education, there should be a clear distinction between these techniques. A recently developed theory of fuzzy concepts can be applied to analyse the four mentioned concepts. For all four techniques one can pose the question of their relationship to *deduction*, *induction* and *abduction* as logical principles. An empirical study was conducted with 12-13 aged students, aiming at checking the three reasoning processes.

1 Introduction and Definitions

In mathematics and the natural sciences there is a strict obligation to define a concept in a precise, crisp, way. This means that all people working in the areas of research should use the same definition. In other fields, definitions are not so precise. Also for the four techniques - experimentation, exploration, simulation and modelling - definitions given are seldom identical. These concepts are *fuzzy* in the sense that elements occurring in the definitions do not occur in all definitions.

Let S denote the set of elements occurring in a given set D of definitions. Zadeh [1] introduced the concept of fuzzy set. Elements belong or do not belong to a normal set. They can be said to be member or not. A *membership functional* can be defined with value 1 for elements belonging to the set, and value 0 for elements not belonging to the set. The generalisation introduced by Zadeh is to let the functional $\mu(x)$, x an element, take values in $[0, 1]$. If μ has values 1 and 0 only, the set is called *crisp*, otherwise the set is called *fuzzy*.

There has been considerable discussion about the concept of fuzzy set, with respect to the interpretation of the value of the membership functional. It was remarked that it could be interpreted as a probability. Whether this is so in a specific case is sometimes difficult to decide. A standard example is the "set of small objects". We do not want to go into this discussion, but stress that the number of times an element of S is met in some definition in D leads to a rather natural membership functional. If an element e occurs n times, given $|D| = N$ definitions, then it can be said to have membership value

$$\mu(e) = \frac{n}{N}.$$

Hoede and Wang [2] introduced *fuzzy concepts*. The elements of S were the concepts occurring in the definitions in D **and** the pairs of concepts that came forward as related, in some way, in the definitions. Seeing the concepts as vertices of a graph and pairs of concepts as edges, a definition can be represented by a *definition graph*. There is an extensive literature on the way two concepts can be related. Several theories about *semantic networks* exist, each having its own ontology of types of relationships. We used a very simple procedure. The occurring concepts were nouns, that were investigated on synonymity. Pairs were considered linked whenever a verb or a preposition did so. Also if a noun appeared in the form of an adjective, we extracted a link, e.g. from "scientific procedure" we extracted the pair {science, procedure}. For each of the four techniques, 10 definitions were randomly chosen from the literature. It did not occur that two definitions were the same! The four are indeed fuzzy concepts. No concept occurred more than 5 times in a set of 10 definitions, so the membership value was at most 0.5. We restricted ourselves to concepts occurring at least twice.

μ	Experiment	Exploration	Modelling	Simulation
0.5	Hypothesis		System	System Model
0.4	Science Investigation Outcome		Data	Reality
0.3	Action Procedure Condition Element Strategy	Experiment Knowledge	Description	Experiment Process Feature
0.2	Fact Relationship Cause Variable	Action Understanding Circumstance	Reality Science Phenomenon	Element Strategy Object Behaviour Data Outcome Program Computer Problem Imitation Description

Table 1. Results for the four techniques

Table 1 gives the results for the four techniques. This table, essentially a frequency table, already allows some first conclusions about the interpretation that 10 different people give to these four specific fuzzy concepts. As "experiment" is used in the definitions of both exploration and simulation, it is a genus of the two other concepts. The use of "model" in the definition of simulation makes modelling more basic. The two techniques are, however, quite related. Essential differences are that modelling is **not** seen as an experiment and the explicit mentioning of the use of computer programs in simulation. For this use, the model as scientific description of reality must be given. The central concept in the definitions of experiment is "hypothesis", in particular the investigation of outcomes, so the testing, is mentioned. With the low membership values coming forward, "exploration" is the fuzziest concept and seen as "a technique for gathering knowledge".

Before choosing our definitions we give the graphs resulting from combining 10 definition graphs for each of the four concepts by identifying vertices with the same name and drawing multiple edges for occurring pairs.

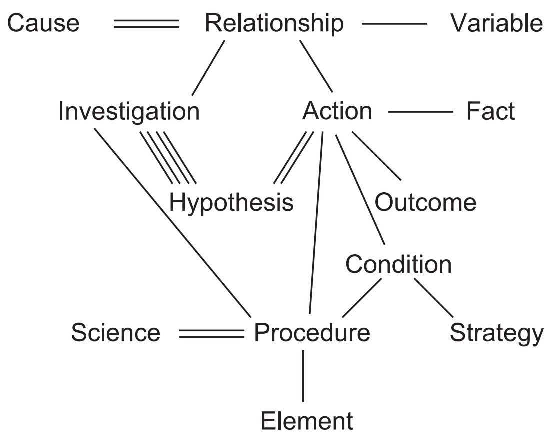

Fig. 1. Experimentation

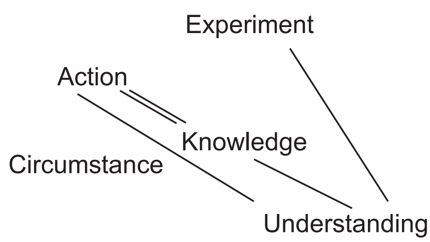

Fig. 2. Exploration

Note the difference in complexity of these two figures. It is a consequence of the natural order in which investigation takes place. ”Investigating a hypothesis” may, from a mathematical setting, also be called ”verifying a conjecture”. Before a conjecture can be verified, it has to be formulated in a process of modelling.

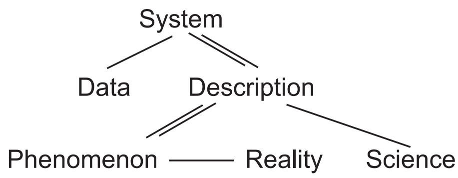

Fig. 3. Modelling

Modelling yields a conjecture in the form $P \to C$, where some premise P is set into relationship to some consequence C. Finding candidates for P and C is done in the exploration phase. So there is a natural order in the sense that exploration is needed for modelling and only after that verifying (proving) the formulated conjecture can take place.

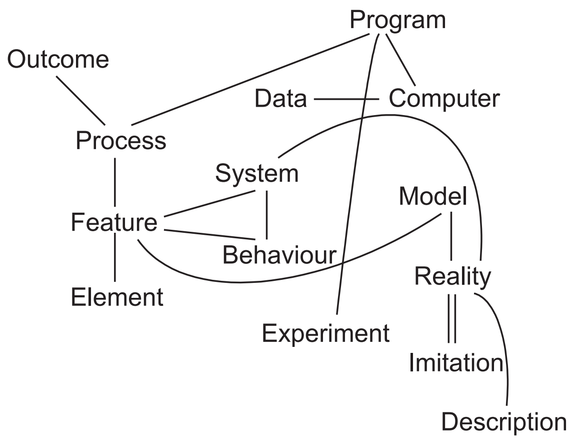

Fig. 4. Simulation

Simulation is essentially imitation of reality. The complexity of Fig. 4 is due to the various concepts that come into play when simulating, like e.g. computer, program, etc.

The graphs suggest as essential parts of the definitions:

Experimentation: scientific procedure consisting of actions to investigate a hypothesis, in particular causal relationships.
Exploration: actions taken to gather knowledge.
Modelling: description of the phenomena observable in a system.
Simulation: imitation of reality.

2 Logical principles

As an important component in mathematical education, problem solving performances are referred to, judged and explained in terms of inductive and deductive reasoning. It would be impossible not to mention, and, in fact, take as reference point, the role which Peirce [3], as semiotician, played in studying the significant role of abduction, as form of reasoning, next to the processes of induction and deduction. Abduction, as a concept, is rather often used in interdisciplinary areas as philosophy, semiotics, linguistics, or artificial intelligence, just to mention a few of them. Moreover, the term is part of our life, since its involvement in, for example, the way doctors state medical diagnosis, or jurists work out witnesses in court cases [4], is a 'sine qua non', see also Section 3. Abduction is about reasons for adopting a hypothesis. Peirce's theory of abduction of 1867 involved reasoning from data and moving towards hypothesis. All empirical sciences seem to use this type of reasoning. Peirce gave a classification of inference processes, which is not conforming to the traditional classification, but nevertheless is expressed within the Aristotelian classical logic's framework, that is, premise-conclusion [5]. The following description in terms of rules, cases and results was given by Peirce in 1878, conform Reid [6]:

Deduction
Rule: P implies C - All the beans from this bag are white.
Premise: P - These beans are from this bag.
Consequence: C - These beans are white.

Induction
Premise: P - These beans are from this bag.
Consequence: C - These beans are white.
Rule: P implies C - All the beans from this bag are white.

Abduction (Hypothesis)
Rule: P implies C - All the beans from this bag are white.
Consequence: C - These beans are white.
Premise: P - These beans are from this bag.

If going one step closer to psychology and relating the formal aspect of reasoning to cognition, we come to one of the most important principles of Peirce's theory of knowledge which has been acquired from Kant [7], namely that every mental process, that is, cognition, involves an inference. Therefore Peirce defines abduction (which he also names retroduction or presumption, or even just hypothesis) as "*an operation upon data resulting in cognition*" or "*an explanation of a phenomenon by a hypothesis*". An immediate example of natural testing hypothesis is, according to Piaget's assertion [8], the way knowledge is (mostly) subconsciously constructed from the inside by the individual. Once internally constructed, knowledge is checked against reality. Then the individual, as a learner, can disclaim, transform or rebuild knowledge (again subconsciously), according to its suitability in reality.

Peirce points out clearly that abduction suggests that *something, a premise, may be or may be not.* Deduction explains and proves that *something, a consequence, must be*, while induction evaluates and establishes that *something, a rule, is actually operative* [5]. Deduction leads to the *definitive* consequence of a rule, abduction leads to a *possible* premise and induction leads to a *possible* rule. In his early work, Peirce stresses the distinction in logical form between the three forms of reasoning. In his later work, the stress is moved to the function each reasoning form conforms to. Therefore, abduction becomes "explaining a hypothesis". Abduction corresponds to the classical logical fallacy.

However, Hoffmann [9] sees some problems regarding Peirce's claim that there is a logic of abduction, since abduction turns out to be, as Hoffmann infers from Peirce's description, "the genuine creative act of perceiving possible explanatory hypotheses". "There is a form of inference, but there are no rules for getting a hypothesis." Hoffmann [9] puts "abductive correctness" under the question mark: "...the perceptive part of abduction - indeed is conceived as the result of an inference. This *inference*, however, is not deductive but again *abductive*."

Reasoning from Peirce's concept of abduction, Hoffmann remarks that "the creativity of abduction is based in the genesis of perceptual judgements", but also that "any perceptual judgement in itself is the result of an abduction!" "...what is the second essential for Peirce's concept of abduction besides the infinite recursiveness: The continuous character of ideas and their tendency "to spread continuously and to affect certain others" offer the possibility of perceiving "unexpected" hidden ideas. In this way, there is no *creatio ex nihilo* in abduction, "new" ideas are only relatively new..."

Eco [7] distinguishes three kinds of abduction: *overcoded* abduction, for the case there is one general rule from which the premise follows, *undercoded* abduction, when they are multiple general rules, and *creative* abduction, in case no general rule from which the specific premise would follow is known to the reasoner. In this latter case, such a new general rule is invented.

3 The link of reasoning with knowledge graphs and the four techniques

In the theory of knowledge graphs, which was in principle used in analysing the definitions of the four techniques, the utilised programme can be described by the slogan "thinking is linking somethings". To illustrate this idea note that in Section 1 we only described the essential parts of the four concepts, as they came forward by being involved in multiple links. All the other concepts that were mentioned can be seen as associations linked to these essential parts, and might in principle be added.

Let us look upon the three forms of reasoning from the point of view of this slogan. For deduction, a certain structure of concepts, the premise, is assumed to be true. Deriving the truth of the consequence may be seen as step by step extension of the structure by linking other concepts, so that after each step, the structure is still true. As soon as the extended structure encompasses the structure describing the consequence, as a true statement, the consequence has been deduced.

In the case of abduction, the extension of the structure describing the consequence is now in the direction of possible premises. The two processes can be illustrated by decision support systems and expert systems. In fact knowledge graph theory was used to develop a graph theoretical medical expert system. A "description of the phenomena observable in the system that is the human body" was given in the form of a graph with mainly causal links. Note that this is precisely "modelling the human body", according to our definition of modelling! Now the huge structure can be used in two ways. Given some symptoms, by backward chaining all possible causes can be determined. This is a typical example of abductive reasoning. The structure is called an *expert system*. The other way round, so by forward chaining, the effect of a medication, change of the state of some of the vertices in the graph structure, can be calculated. This is a typical example of deductive reasoning. The same structure is now called a *decision support system*. The deductive reasoning involves here the extension of the structure by following the causal links.

As far as the graph structure is hypothetical, physicians may "use the procedure of medication to investigate its usefulness, via the hypothetical causal links". By our definitions, this reads as "by experiment".

The most difficult reasoning process seems to be induction, so finding some rule without information about the relationship between premise and consequence. Let us consider Peirce's syllogism again. "These beans are from this bag" and "These beans are white" is given. Graph theoretically, there are two structures: beans - bag and beans - white (in reduced form). The thinking about these two statements leads to the linked structure: bag - beans - white. "Bag" and "white" become associated by this structure. But trying to find some rule may yield two rules: "bag beans are white beans", so "all beans from this bag are white", as in the description given before, but also "white beans are bag beans", so "if beans are white, they are all from this bag". In fact, instead of

P
$\underline{\text{C}}$
P implies C

we may also, by symmetry, have

C
P
C implies P.

The two structures are "two pieces of knowledge that we have gathered". But this is precisely our definition of "exploration". So we have found the following associations:

Deduction - Experimentation - Forward chaining (- Decision support system)
Abduction - Modelling - Backward chaining (- Expert system)
Induction - Exploration - Potential linking rules.

Once a valid model of the human body has been developed, it can be the basis for "imitation of a real body", that is for "Simulation".

4 Translation to the math classroom setting

We now want to discuss types of reasoning and techniques of teaching in a purely mathematical setting. Given the occurrence of the concept of system in our definitions, we consider the "system" of natural numbers and therein Goldbach's conjecture: "Every even natural number greater than 2 is the sum of two primes", so "**If** $n \in \mathbb{N}$, $n \geq 4$, and N is even, **then** $N = p_1 + p_2$, p_1 and p_2 prime". This famous conjecture is so simple to state, that investigating it is well within the reach of even young school children. Our purpose is to illustrate our considerations given before.

Suppose we want to teach pupils about this conjecture and at the same time go through the various techniques, what is the appropriate order in which to proceed? The pupils have to know about natural numbers, them being even or odd, addition of them and being prime. Only the latter property will need some explanation.

4.1 Exploration

The first technique to teach would be to find out rules, so by induction. The objective would be to let pupils find that sums of two integers are even if both are even or both are odd, and odd if they are even and odd. Next to that, they should find out that, if 2 is not involved, the sum of two primes is even. The main question would be: "Can an If ... Then ... - statement be formulated?"

4.2 Modelling

In this second technique the objective would be to let pupils be given the focus on "The sum is even". They should then be asked what can be filled in: "If ... Then the sum is even". This requires abduction and may lead to various answers: "If both are odd" or "If both are even" or "If both are odd primes".

4.3 Experiment

In the third technique the objective would be to let pupils now focus on one of the possible premisses. The hypothesis may be "If n_1 and n_2 are even, then $n_1 + n_2$ is even". The experimental result is immediate from the exploration phase. So now the focus is on the **proof** of the consequence by deduction. This, as well as the proof of "If n_1 and n_2 are odd, then $n_1 + n_2$ is even" should be within the reach of even young pupils. From here the teaching may cover the following topics:

1. Seeing the addition of odd primes as special case of the addition of odd natural numbers.
2. The idea of necessary and sufficient conditions: "If a number is even, then $n = n_1 + n_2$ with n_1 and n_2 both even or both odd."
3. Goldbach's conjecture in the form "If a number n is even, $n \geq 6$, then $n = n_1 + n_2$ with n_1 and n_2 both odd primes."

2. can easily be proven, 3. turns out to be "not so easy", and we can end with letting the pupils investigate whether the conjecture is confirmed.

4.4 Simulation

The objective would be to let pupils find out on their own whether the conjecture is confirmed, by means of a hand calculator as simulation device for the "real world of natural numbers".

However we wonder whether at the age we currently focus on, using computer simulation means is appropriate as it might partially hamper developing certain cognitive abilities and skills which are essential in this phase. Therefore we intend to focus on the technique of simulation in some later stage.

We postulate that 4.1 to 4.4 gives the natural order of acquainting pupils with a mathematical subject. For the logical principles that are trained this means the order induction-abduction-deduction.

5 A set of exercises

In the previous section different subsections were allocated to each of the techniques - exploration, modelling, experiment and simulation. However, in the next sections we will describe the exercises used in our empirical study that contains tasks of exploration and/or modelling nature, as well as proofs.

Having an 'ideal' design as the one described in the previous section, we present now a series of tasks given to 7th class pupils of a Gymnasium in Bremen, Germany. We performed our experiments in two different environments.

Firstly, two children, Max and Lukas, the best belonging to a rather good collective of pupils, were brought in the university laboratory environment, where three cameras were used for recording their reactions, small talks, discussions, gestures, and so on. These tasks were given in paper-pencil form. There were four sessions, spread over three consecutive days and lasting about one and a half hour each, with a brake of fifteen minutes in between. This arrangement took place a couple of days before the new scholar year had begun. We point this out, because later results have made us consider this as a possible factor for the results-expectation balance.

The other environment used for the same (type of) experiments was the classroom itself, in the first week of the scholar year where the entire collective of 25 pupils was involved. They were free to discuss and talk to each other, in groups of 2-3 children. Nevertheless, each of them had to individually write down the results on their papers.

5.1 Goldbach's conjecture

Goldbach's conjecture was the core of one of the tasks, given to children in both environments in the following form:

Having considered the natural numbers $\mathbb{N}$, the first task consisted in finding out when the sum of two natural numbers is even.

As second task, pupils were asked to prove what they found out in the previous exercise. A hint was given here, namely the form in which odd, respectively even natural numbers can be written.

The third task was questioning which natural numbers could not be written as a product of other numbers, number 'one' being excluded. They were also asked here how many such numbers are even.

The last question was whether it is possible to write an even number as sum of two odd numbers. Here it was stated that this is a famous problem, since the 18th century, which still is not proven. The pupils were asked then to check this conjecture for the numbers between 6 and 40.

5.2 Eulerian walks

Another set of tasks to be solved by pupils starts from five given graphs as in the following figure:

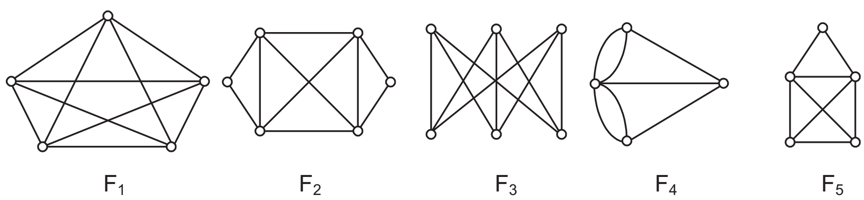

Fig. 5. Five graphs

The children had to try out for every given graph to go through all edges once, beginning from a vertex. They were told that it is not necessary to end at the starting vertex, but that they first should try to make that happen. Then came the question, which is of nature to provide some help, why is there no solution to be found for F_3 and F_4, and why it does work for F_1, F_2 and F_5. Finally, they were asked under which condition a Eulerian walk could be made, so that one ends up at the vertex it began, making it a Eulerian tour. At the end, a short story about the seven bridges of Königsberg was narrated. Herewith some historical context was provided about the beginning of graph theory in 1736.

5.3 Chinese population

Firstly, children were asked why they thought that the German population decreases, as it could be read in newspapers. Then came the question 'how many children should be born, so that the population remains constant?'. In the same context, the discussion moved now and pupils were told about the 'one-child policy' practiced by the Chinese government. They were asked what it means for the number of successors, and also what happens with the population? Then a 'mathematical' problem is formulated, namely: "Suppose that the population outside China remains constant, 5.2 billion and the 'one-child policy' is given up. The assumption was given that in one generation (that is, approximately 25 years), they reach one and a half times as many citizens as now, namely 1.3 billion. The problem was posed to calculate in how many years there will be as many Chinese people as non-Chinese people in the world."

Children were explicitly asked at this point to give their reasoning of what they calculated and to explain every step they perform.

6 Data analysis and discussion

This paper mainly focuses on the theoretical relationship between learning techniques and logical principles. The experimental part consequently is rather restricted.

After performing the previously discussed experiments, we have drawn some conclusions. We will proceed one by one, following the tasks in the order given in the previous section and we will analyse the results. Due to the fact that the chosen sample of pupils for this study was not large enough to allow some statistical analysis methods, we will be using a descriptive method of analysis.

Our main aim is to find out which logical principles were within the reach of children of this age. The tasks contained elements corresponding to each of the three principles.

6.1 Goldbach's conjecture

Most of the pupils were solving correctly the first task, meaning that induction and abduction are clearly within their reach. Nevertheless, some of them had given half-answers, concerning only the odd or the even numbers.

The interesting part comes when assessing pupils' answers for the second question. Considering the given hint, two even numbers are of the form $2n_1$ and $2n_2$, two odd numbers are of the form $2n_1 + 1$ and $2n_2 + 1$, some of them added and factorised, but ignored the indices of n_1, n_2, just making n out of both indices. This allowed them nevertheless to draw the right conclusion, that is make the correct deduction, and obtain an even number, when summing up two even, respectively two odd numbers. This is far from proving that they correctly understood what a 'proof' is. Without the given hint, probably no child would have come up with such an idea. At this point, the level of generalisation implied by handling with the two 'unknowns', when generally writing down an even, respectively an odd number, which is a deductive step, was too high for the children at this age, and for this reason, only a few of them performed adding and factorising, and even when doing so, they 'simplified' the problem, identifying the unknowns n_1 and n_2 with n, not being able to observe that different n's are involved, when writing different numbers in their general form.

When being asked how many prime numbers are even, a simple exploration, most of the pupils were tempted to state that none of the prime numbers is even. But several children did notice, with big surprise and joy, the trick of the question, and stated correctly that 2 is an exception.

When being asked to check Goldbach's conjecture for the natural numbers greater than 6 and smaller than 40, very few of them took the time to understand the stated hypothesis in the correct way, so to consider all the even numbers starting with 6 and to find correct combinations for adding and checking validity up to 40. Some pupils started to add prime numbers (1+1, 3+3, 5+5, etc), but failed to follow the direction of the conjecture itself. Interpreting the question turned out to be difficult for them. Nevertheless, some children, including Max and Lukas, had no difficulty with this task and immediately solved it. On the other hand, a number of pupils even stated that the conjecture is false, without justifying.

An interesting aspect of this first set of tasks is that Max and Lukas, as very good pupils, in the first arrangement were not performing visibly better with the first three tasks, but rather on the same level as the pupils working in the classroom environment. Our guess is that some influencing factors might have been the fact that the group of two has worked under the cameras' 'supervision', not in their familiar (classroom) environment and also the fact that things took place during their holiday. This makes us conclude that psychological aspects should be considered, when assessing results of experiments. They might have strong influence on the way children perform, and therefore to some degree cause bias.

In terms of reasoning processes which were aimed at through the described experiment, we have concluded that induction (see task 1) was the best performed by (almost all) pupils. Most of them did not succeed to write down their answer in the form of "If... then..." statements, but rather did it in a text form. Concerning the deduction process (task 2), for most of the children there is still difficult to follow such a reasoning, in a correct form, meaning that it is still too early to introduce the concept of proof at this age. Also the distinction between necessary condition and sufficient condition (task 3) is grasped only by a few, better, pupils.

6.2 Eulerian walks

Concerning this type of experiment, almost all children found out rather quickly the correct way of passing through all edges. When being asked to explain why some figures allow passing through all edges, and others do not, which is a matter of abduction, they made the remark that the number of edges involved in the figures which allow that passing, is even, whereas for the others the number is odd. That is true, but has no relevance in our problem. Only one child - Thomas (belonging to the pupils group, working in the classroom) made the connection between the number of edges meeting in one vertex, that is - odd or even - and whether for that specific graph the passing through all edges is possible or not. So, for the figures F_3 and F_4, no solution is found, essentially [1] due to the fact an odd number of edges meet in each vertex. However, the subtle meaning of this fact was not explained, namely that, in order to find a solution, the number of edges incident with a vertex should be even, but for, possibly, two vertices like in graph F_5, as the number of edges used in the walk going to one vertex has to be the same as the number going out from that vertex, making the parity property of degrees essential. Nevertheless, Thomas was the one closest to the right and complete solution.

An interesting aspect is also the fact that, when being asked to state the conditions under which the required behaviour is satisfied, most of them referred to the answer given to the previous question, where they made remarks about the 'number of lines' (meaning edges). They could not see that a general statement was to be made, but kept the connection with the five given figures. Again the step of abstraction and generalisation turned out to be the difficult one. Some of the children figured out some 'favourable' combinations, and suggested that if certain edges are erased, then passing through all edges is possible, but did not approach the problem in a 'mathematical manner', in terms of 'condition', 'statement', and so on.

It was somehow disappointing that Max and Lukas could not perform better than the class in this set of tasks. At this age even the better pupils have difficulty with abduction in a more abstract setting.

6.3 Chinese population

The answers given by children to this set of tasks were at least satisfactory. When asked to find possible reasons for the decrease of the German population, most of them provided as answer 'too few children are born, against the number of people who die', but also explanations as 'there are too many singles, who

[1] In the connected graph K_2, a single edge, both vertices also have odd degree. Yet a Eulerian walk of one step is possible.

want to remain independent and concentrate on career, so that they have no time for children' or 'military service, bad medicin, drugs, poverty' and so on, were found in their answers, which proves that they are quite informed from the mass-media about the situation in the world/country.

A majority of them saw as solution for keeping the population constant, that every person gets a child, that means two children for each pair, while others formulated in the way of 'there should be born exactly as many children as there are people dying'.

Regarding the Chinese population, with very few exceptions, they correctly understood that the 'one-child policy' will cause the decrease of population, some of them even predicted the halving of it. Their answers so far have shown that they are at the age when abductive thinking is in progress for less abstract settings.

The way they solved the 'real mathematical problem' were not very easy to follow, since most of them performed calculations for which they did not provide any explanation. Their final calculations gave answers like 60, 75, 100 and 150 years. A frequent modelling children made while solving was to consider the population existing after one generation as initial data for calculating it for the next generation.

The way problems are formulated clearly can cause more than one abduction, here the way the growth takes place.

6.4 Discussion

We mention that during this empirical study, we encouraged and explicitly asked the children to provide explanations and detailed argumentation of all the steps they undertake while solving the problems. It turned out they are not used to do so, they are rather lapidary in the problem-solving process, they simply write down mathematical calculations, but do not explain what they do and how they think. Such a 'good' example is Andreas, who, though he was the only one giving the expected answer to the Chinese population problem, one cannot get any clue about the path he followed while getting to his solution. However, it should not be forgotten that the teacher plays the main role in ascertaining situations which practice justifications and explanations of problems. As Yackel [10] points out "it is the teacher's responsibility to help students learn how to describe and talk about their mathematical thinking, to help them learn what constitutes an acceptable explanation... the teacher can ask children whether they understand and encourage them to ask questions and request clarifications."

In order to know children's opinion about the given set of tasks, we have addressed this question at the end of last session to Lukas and Max. As answers, we have got 'It was really OK', respectively 'I find it really interesting and worthwhile doing this kind of tests'. Knowing that Lukas is a rather phlegmatic type, his feedback was a positive one, while Max proved once again his special interest for mathematics and challenges of this nature.

After having finished the experiment with the 7th class children, we had the idea of testing how the 13th class pupils perform on the same set of tasks, in order to find out in which way the intermediary development stages are to be recognized in their problem-solving procedures. Eulerian walk and Chinese population were given to a number of 19 pupils, belonging to the 13th class, in the same Gymnasium to which the younger belonged. Some of them worked in groups of 2-3 colleagues, and the results were rather surprising, in the sense that the older pupils did not perform better than the younger ones in any respect. Concerning the first set of tasks, Lisa was the only 13th class pupil who observed the connection between the parity of edges meeting in a vertex and the possibility to go through all edges and come back to the same point. The Chinese population problem was solved in a similar manner as the young children did, even the same ambiguity of modelling occurred, that is next to the assumption of linear growth the assumption that after each generation, the actual population becomes initial data for calculations for the next generation. Several pupils tried some other method of solving, namely trying to use knowledge of exponential functions. The reason for the latter could be the fact that the previous mathematics classes, they worked with exponential functions, so that they made the connection with it and used the most recent acquaintance.

Since our focus was on certain techniques for learning mathematical concepts and their relationship to the logical principles, we give in the following our concluding findings, in form of a table, where the correspondent reasoning process for each learning technique is presented, as well as the presumed proper ages to be introduced to pupils.

Learning technique	Reasoning process	Age
Exploration	Induction	12-14 y.o.
Modelling	Abduction	14-16 y.o.
Experimentation	Deduction	16-18 y.o.

Table 2. Learning techniques, reasoning processes and corresponding ages

We note here that our suggestion concerning ages we think as properly corresponding to each learning process, respectively reasoning process, is based on rough empirical observations and psychology reading, and validation requires further research.

7 Summary and conclusions

We have used fuzzy concepts and a graph theoretical method for agreeing on definitions for *exploration*, *modelling*, *experimentation* and *simulation*, in order to check the accuracy and suitability of the terms used when defining these learning techniques. We also considered the logical principles *induction*, *abduction* and *deduction* as being involved in the mentioned techniques. We have chosen ten definitions for each of the techniques and created graphs, having as vertices the concepts which occur at most twice in all definitions. From the drawn graphs, using the essential parts of all definitions, we decided and agreed on certain definitions.

Originating from Peirce's theory of abduction, we describe the three reasoning processes in terms of *premise*, *consequence* and *rule*. It turned out that the logical principles could constitute the core of the four learning procedures in a certain order. A translation to the mathematical classroom setting is then made, by means of three sets of tasks, that each involve different learning methods and logical principles. After having performed the empirical study in classroom (respectively in the laboratory environment), we analysed the result in a descriptive manner.

When assessing the obtained results, we esteemed as appropriate to also consider psychological components. According to Piaget [2], children's thinking does not follow a smooth development process, but there are certain stages, where some 'take off' takes place, and pupils pass on to an entirely different and new field of abilities. It seems 12-13 years is such an age, and since the points where these transitions occur are not fixed, some children belonging to our sample performed better than others with respect to the reasoning processes that we have focused upon.

As a recommendation we advice to use Table 2 for guiding us in designing the teaching of mathematics. The table suggests a clear separation of teaching the different aspects. This is, however, NOT what we would prescribe.

Clearly exploration is suggested to be focus in the lower classes, as is modelling in the following classes. Proving theorems should be the main focus in the highest classes, but this does not mean that we recommend to avoid discussing the concept of proof in the lower classes. On the contrary, getting acquainted with the three logical principles and with abstraction should take place in all classes but with properly chosen material and to properly chosen extent.

[2] See `http://www.learningandteaching.info/learning/piaget.htm`

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