Generating All Circular Shifts by Context-Free Grammars in Greibach Normal Form

Peter R.J. Asveld

Department of Computer Science, Twente University of Technology
P.O. Box 217, 7500 AE Enschede, the Netherlands
e-mail: `infprja@cs.utwente.nl`

Abstract

For each alphabet $\Sigma_n = \{a_1, a_2, \dots, a_n\}$, linearly ordered by $a_1 < a_2 < \dots < a_n$, let C_n be the language of circular or cyclic shifts over Σ_n , i.e., $C_n = \{a_1 a_2 \dots a_{n-1} a_n, a_2 a_3 \dots a_n a_1, \dots, a_n a_1 \dots a_{n-2} a_{n-1}\}$. We study a few families of context-free grammars G_n ($n \geq 1$) in Greibach normal form such that G_n generates C_n . The members of these grammar families are investigated with respect to the following descriptonal complexity measures: the number of nonterminals $\nu(n)$, the number of rules $\pi(n)$ and the number of leftmost derivations $\delta(n)$ of G_n . As in the case of Chomsky normal form, these ν , π and δ are functions bounded by low-degree polynomials. However, the question whether there exists a family of grammars that is minimal with respect to all these measures remains open.

Keywords: context-free grammar, Greibach normal form, permutation, circular shift, cyclic shift, descriptonal complexity, unambiguous grammar.

1 Introduction

Let $\Sigma_n = \{a_1, a_2, \dots, a_n\}$ be an alphabet, linearly ordered by $a_1 < a_2 < \dots < a_n$, and let L_n be the language over Σ_n of the $n!$ permutations of $a_1 a_2 \dots a_n$. In 2002 G. Satta [15] conjectured that “any context-free grammar G_n in Chomsky normal form (CNF) that generates L_n must have a number of nonterminal symbols that is not bounded by any polynomial function in n ”. This statement has been proved in [9], but without showing how to generate $\{L_n\}_{n \geq 1}$ by context-free grammars $\{G_n\}_{n \geq 1}$ in CNF. In [2] we provided several grammar families for $\{L_n\}_{n \geq 1}$ together with the usual descriptonal complexity measures as the number of nonterminals $\nu(n)$ and the number of rules $\pi(n)$; cf. [11, 13, 14, 7, 5, 1, 6] for

these measures. The relative descriptive complexity of these grammar families is anything but straightforward and the quest for a family of minimal grammars (with respect to these complexity measures) remains a challenging problem. Then in [3] we investigated some specific permutations over Σ_n , viz. the circular or cyclic shifts, defined by

$$C_n = \{a_1a_2 \cdots a_{n-1}a_n, a_2a_3 \cdots a_na_1, a_3a_4 \cdots a_1a_2, \dots, a_na_1 \cdots a_{n-2}a_{n-1}\}.$$

An alternative definition of C_n in terms of the so-called *circular closure operator* c on languages L , which is defined by $c(L) = \{vu \mid uv \in L\}$ [8], is: $C_n = c(\{a_1a_2 \cdots a_n\})$.

The fact that the n elements of C_n is much less than the $n!$ elements of L_n is also reflected by the complexity measures of the corresponding grammar families: for $\{L_n\}_{n \geq 1}$, the functions $\nu(n)$ and $\pi(n)$ are exponential functions [15, 9, 2], whereas for $\{C_n\}_{n \geq 1}$, they are bounded by low-degree polynomial functions; cf. [3].

In this paper we investigate a few ways of generating the family $\{C_n\}_{n \geq 1}$ by context-free grammars in Greibach normal form (GNF) and for these families of grammars we determine the complexity measures ν , π and δ as functions of n .

The results we obtain are rather similar to those in [3], as is the organization of this paper. §2 is devoted to preliminaries and in §3 we consider elementary properties of grammars G_n in GNF for C_n . As using the arbitrary GNF for C_n is trivial (§3), we focus in §§4–7 on the Greibach k -form ($k = 1, 2$). An approach based on the set of circularly ordered strings results in a grammar family in Greibach 2-form (§4). Modifying this family into Greibach 1-form in §5 results in a family with less rules. Unambiguous grammars for C_n are studied in §6: then $\nu(n)$ and $\pi(n)$ are related in a simple way. In §7 we discuss minimality for unambiguous grammars in GNF but the existence of a family of grammars for which these complexity measures are minimal remains open. Finally, some concluding remarks are in §8.

2 Preliminaries

For rudiments of discrete mathematics, particularly of combinatorics, and of formal language theory, we refer to standard texts like [10] and [12], respectively.

We denote the empty word by λ and the length of the word x by $|x|$. For each word w over Σ , $\mathcal{A}(w)$ is defined as the set of all symbols from Σ that do occur in w . Formally, $\mathcal{A}(\lambda) = \emptyset$, and $\mathcal{A}(ax) = \{a\} \cup \mathcal{A}(x)$ for each $a \in \Sigma$ and $x \in \Sigma^*$. This mapping is extended to languages L over Σ by $\mathcal{A}(L) = \bigcup \{\mathcal{A}(w) \mid w \in L\}$.

Remember that a λ -free context-free grammar $G = (V, \Sigma, P, S)$ is in *Chomsky normal form* (CNF) if $P \subseteq N \times (N - \{S\})^2 \cup N \times \Sigma$ where $N = V - \Sigma$. And G is in *Greibach normal form* (GNF) if $P \subseteq N \times \Sigma(N - \{S\})^*$. Particularly, G is in *Greibach k -form* if $P \subseteq N \times \Sigma(\bigcup_{i=0}^k (N - \{S\})^i)$.

For a context-free grammar $G = (V, \Sigma, P, S)$ with $\alpha \in V$, $L(G, \alpha)$ denotes the language defined by $L(G, \alpha) = \{w \in \Sigma^* \mid \alpha \Rightarrow^* w\}$. Thus for the language $L(G)$ generated by G , we have $L(G) = L(G, S)$. Notice that, if G is in CNF or GNF, then G has no useless symbols, $L(G, \alpha)$ is a nonempty language for each symbol α in V , and $L(G, \alpha) = \{\alpha\}$ for each α in Σ .

In studying C_n we need, as in [3], subwords of $a_1a_2 \cdots a_na_1 \cdots a_{n-1}$; so we consider the set F_k^n all subwords of length k ($1 \leq k \leq n$) that obey the *circular succession relation* $\prec$ on Σ_n defined by: $a_i \prec a_j$ if and only if either (i) $i < n$ and $j = i + 1$ or (ii) $i = n$ and $j = 1$; cf. §6.1 in [10]. Clearly, $\prec$ is not a transitive relation: it is a kind of successor relation. Then the formal definition of F_k^n reads

$$F_k^n = \{x \in \Sigma_n^* \mid \exists u, v \in \Sigma_n^* : uxv = a_1a_2 \cdots a_na_1a_2 \cdots a_{k-1}; |x| = k\}$$

with $1 \leq k \leq n$; their partial unions $\mathcal{Q}_m^n$ are defined by $\mathcal{Q}_m^n = \bigcup_{k=1}^m F_k^n$ ($1 \leq m \leq n$).

For a finite set X , we denote the cardinality of X by $\#X$. Then we obviously have $C_n = F_n^n = \mathcal{Q}_n^n - \mathcal{Q}_{n-1}^n$, $\#F_k^n = n$ ($1 \leq k \leq n$), $\#\mathcal{Q}_m^n = mn$ and $\#C_n = \#F_n^n = n$.

3 Elementary Properties

We first recall some simple properties of grammars in GNF that generate L_n (the language of all permutations over Σ_n). From [4] we quote the following results.

Proposition 3.1. *For $n \geq 1$, let $G_n = (V_n, \Sigma_n, P_n, S_n)$ be a context-free grammar in GNF that generates L_n , and let $A, B \in N_n = V_n - \Sigma_n$.*

(1) *The language $L(G_n, A)$ is a nonempty subset of an isomorphic copy M_k of the language L_k for some k ($1 \leq k \leq n$). Consequently, each string z in $L(G_n, A)$ has length k , z consists of k different symbols, and $\mathcal{A}(z) = \mathcal{A}(L(G_n, A))$.*

(2) *If $L(G_n, A) \cap L(G_n, B) \neq \emptyset$, then $\mathcal{A}(L(G_n, A)) = \mathcal{A}(L(G_n, B))$.*

(3) *If $A \rightarrow aA_1A_2 \cdots A_m$ is a rule in G_n , then for each (i, j) with $1 \leq i < j \leq m$, $\mathcal{A}(L(G_n, A_i)) \cap \mathcal{A}(L(G_n, A_j)) = \emptyset$, $a \notin \mathcal{A}(L(G_n, A_k))$ with $1 \leq k \leq m$, and*

$$\mathcal{A}(L(G_n, A)) = \{a\} \cup \mathcal{A}(L(G_n, A_1)) \cup \mathcal{A}(L(G_n, A_2)) \cup \cdots \cup \mathcal{A}(L(G_n, A_m)). \quad \square$$

This result gives rise to the following equivalence relation on N_n : A and B in N_n are called *equivalent* if $|x| = |y|$ for some $x \in L(G_n, A)$ and some $y \in L(G_n, B)$. The equivalence classes are denoted by $\{E_{n,k}\}_{k=1}^n$. The number of elements $\#E_{n,k}$ of the equivalence class $E_{n,k}$ will be denoted by $D(n, k)$ ($1 \leq k \leq n$).

Example 3.2. Consider $G_3^P = (V_3, \Sigma_3, P_3, A_{123})$ with $N_3 = \{A_{123}, A_{12}, A_{13}, A_{23}, A_1, A_2, A_3\}$ and P_3 consists of $A_{123} \rightarrow a_1A_{23} \mid a_2A_{13} \mid a_3A_{12}$, $A_{12} \rightarrow a_1A_2 \mid a_2A_1$, $A_{13} \rightarrow a_1A_3 \mid a_3A_1$, $A_{23} \rightarrow a_2A_3 \mid a_3A_2$, $A_1 \rightarrow a_1$, $A_2 \rightarrow a_2$ and $A_3 \rightarrow a_3$. Clearly, G_3^P is in GNF. Then $L(G_3^P) = L_3$, $E_{3,3} = \{A_{123}\}$, $E_{3,2} = \{A_{12}, A_{13}, A_{23}\}$, $E_{3,1} = \{A_1, A_2, A_3\}$, and hence $D(3, 3) = 1$ and $D(3, 2) = D(3, 1) = 3$. $\square$

Proposition 3.1 relies on the fact that each word in $L(G_n)$ is a permutation. As circular shifts are special permutations, Proposition 3.1 still applies; but what is particular about generating C_n rather than L_n is expressed in Proposition 3.4, the proof of which depends on the following result from [3].

Lemma 3.3. *If X is a nonempty proper subalphabet of Σ_n , then there exists at most one word x with $\mathcal{A}(x) = X$ such that x satisfies the circular succession relation. And if*

$X = \{b_1, b_2, \dots, b_l\}$, then $x = b_{p(1)}b_{p(2)} \cdots b_{p(l)}$ provided there exists a permutation p of $\{1, 2, \dots, l\}$ such that $b_{p(1)} \prec b_{p(2)} \prec \cdots \prec b_{p(l)}$. $\square$

For each $w \in \Sigma_n^*$, let $\underline{\alpha}(w)$ be the first and $\underline{\omega}(w)$ be the last symbol of w . Thus if $w = \sigma_1\sigma_2 \cdots \sigma_m$ with $\sigma_i \in \Sigma$ ($1 \leq i \leq m$), then $\underline{\alpha}(w) = \sigma_1$ and $\underline{\omega}(w) = \sigma_m$.

Proposition 3.4. *Let $G_n = (V_n, \Sigma_n, P_n, S_n)$ be a context-free grammar in GNF that generates C_n , and let $\alpha, \beta \in V_n - \{S_n\}$.*

- (1) *For each α , the language $L(G_n, \alpha)$ is a singleton.*
- (2) *If $L(G_n, \alpha) \cap L(G_n, \beta) \neq \emptyset$, then $L(G_n, \alpha) = L(G_n, \beta)$.*
- (3) *If $A \rightarrow aA_1A_2 \cdots A_m$ is in P_n with $L(G_n, A_i) = \{x_i\}$ ($1 \leq i \leq m$), then $a \prec \underline{\alpha}(x_1)$ and for each i ($1 \leq i < m$), $\underline{\omega}(x_i) \prec \underline{\alpha}(x_{i+1})$. Consequently, if $L(G_n, A_i) = \{x_i\} \subseteq F_{\Lambda(i)}^n$, then $L(G_n, A) = \{ax_1x_2 \cdots x_m\} \subseteq F_k^n$ with $k = 1 + \sum_{i=1}^m \Lambda(i)$.*

Proof. (1) G_n is in GNF; so each symbol α in $V_n - \{S_n\}$ is useful: there is a derivation $S_n \Rightarrow^+ \varphi\alpha\psi \Rightarrow^+ \varphi x_\alpha \psi \Rightarrow^+ x$ where x is a circular shift, $\varphi\psi \neq \lambda$, $\mathcal{A}(x) = \Sigma_n$, and $1 \leq \#\mathcal{A}(x_\alpha) < n$. Now, by Lemma 3.3, $L(G_n, \alpha)$ contains at most one word over Σ_n , and since $L(G_n, \alpha)$ is nonempty, $L(G_n, \alpha)$ is a singleton.

(2) As $L(G_n, \alpha)$ and $L(G_n, \beta)$ are singletons by (1), $L(G_n, \alpha) \cap L(G_n, \beta) \neq \emptyset$ implies that they are equal.

Finally, (3) is a direct consequence of the fact that $L(G_n, A) \subseteq \mathcal{Q}_n^n$. $\square$

Henceforth, in examples we will always assume tacitly that $E_{n,1} = \{A_1, \dots, A_n\}$ and we will use $R_n = \{A_i \rightarrow a_i \mid A_i \in E_{n,1}\}$.

Example 3.5. Consider $G_4^U = (V_4, \Sigma_4, P_4, S_4)$ in GNF with $P_4 = \{S_4 \rightarrow a_1A_{23}A_4 \mid a_2A_{34}A_1 \mid a_3A_{41}A_2 \mid a_4A_{12}A_3, A_{12} \rightarrow a_1A_2, A_{23} \rightarrow a_2A_3, A_{34} \rightarrow a_3A_4, A_{41} \rightarrow a_4A_1\} \cup R_4$. Then $L(G_4^U) = C_4$, $E_{4,4} = \{S_4\}$, $E_{4,3} = \emptyset$, $E_{4,2} = \{A_{12}, A_{23}, A_{34}, A_{41}\}$, $E_{4,1} = \{A_1, A_2, A_3, A_4\}$, $D(4, 4) = 1$, $D(4, 3) = 0$ and $D(4, 2) = D(4, 1) = 4$. Since $S_4 \rightarrow a_3A_{41}A_2$ is in P_4 , we have $L(G_4, A_{41}) = \{a_4a_1\}$, $L(G_4, A_2) = \{a_2\}$, $a_3 \prec a_4 = \underline{\alpha}(a_4a_1)$ and $\underline{\omega}(a_4a_1) = a_1 \prec a_2 = \underline{\alpha}(a_2)$. $\square$

As measures for the descriptonal complexity of G_n from $\{G_n\}_{n \geq 1}$, we use $\nu(n) = \#N_n$ and $\pi(n) = \#P_n$; cf. [11, 13, 14, 7, 5, 1, 6]. A less-known measure has been introduced in [2, 3]; viz. the number of left-most derivations $\delta(n)$ of G_n . Remember that in a leftmost derivation the leftmost nonterminal is always expanded. Thus $\delta(n) = \#\{S_n \Rightarrow_L^* x \mid x \in L(G_n)\}$, where $\Rightarrow_L$ denotes the leftmost derivation relation. Clearly, this measure makes sense when we generate a finite language by a λ -free grammar with bounded ambiguity.

Notice that these descriptonal complexity measures depend on n as well as on the family under consideration; so we use $\nu_\alpha(n)$, $\pi_\alpha(n)$ and $\delta_\alpha(n)$ in the context of a family $\{G_n^\alpha\}_{n \geq 1}$ of which the individual members are labeled by α .

Example 3.6. For G_3^P of Example 3.2, we have $\nu_P(3) = 7$, $\pi_P(3) = 12$ and $\delta_P(3) = 3! = 6$, since G_3^P is unambiguous [2]. Similarly, for the unambiguous G_4^U of Example 3.5, we have $\nu_U(4) = 9$, $\pi_U(4) = 12$ and $\delta_U(4) = 4$. $\square$

For each family $\{G_n^\alpha\}_{n \geq 1} = \{(V_n, \Sigma_n, P_n, S_n)\}_{n \geq 1}$ for $\{C_n\}_{n \geq 1}$ to be considered in the sequel, we assume that the first two (unspecified) elements G_1^α and G_2^α satisfy

- $N_1 = \{S_1\}$ and $P_1 = \{S_1 \rightarrow a_1\}$ for G_1^α , and
- $N_2 = \{S_2, A_1, A_2\}$, $P_2 = \{S_2 \rightarrow a_1 A_2 \mid a_2 A_1, A_1 \rightarrow a_1, A_2 \rightarrow a_2\}$ for G_2^α .

Then $\nu_\alpha(1) = \pi_\alpha(1) = \delta_\alpha(1) = 1$, $\nu_\alpha(2) = 3$, $\pi_\alpha(2) = 4$, $\delta_\alpha(2) = 2$, whereas for $n \geq 3$, $\nu_\alpha(n) = \sum_{k=1}^n D(n, k) \geq n + 1$, $\pi_\alpha(n) \geq n + 2$ and $\delta_\alpha(n) \geq n$. This implies that specifying a family $\{G_n^\alpha\}_{n \geq 1}$ reduces to defining the family $\{G_n^\alpha\}_{n \geq 3}$.

As an illustration we consider a simple family of grammars in GNF for $\{C_n\}_{n \geq 1}$. It is based on a single nonterminal S_n and the trivial set of rules $\{S_n \rightarrow w \mid w \in C_n\}$. To obtain grammars in GNF we need isomorphisms $\varphi_n : \Sigma_n \rightarrow \{A_1, A_2, \dots, A_n\}$ defined by $\varphi_n(a_i) = A_i$ ($1 \leq i \leq n$), that are extended to words by $\varphi_n(\sigma_1 \sigma_2 \cdots \sigma_k) = \varphi_n(\sigma_1) \varphi_n(\sigma_2) \cdots \varphi_n(\sigma_k)$ ($\sigma_i \in \Sigma_n$, $1 \leq i \leq k$).

Definition 3.7. $\{G_n^T\}_{n \geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n \geq 1}$ with for $n \geq 3$,

- $N_n = V_n - \Sigma_n = \{S_n\} \cup \{A_i \mid 1 \leq i \leq n\}$,
- $P_n = \{S_n \rightarrow \sigma_1 \varphi_n(\sigma_2 \cdots \sigma_n) \mid \sigma_1 \sigma_2 \cdots \sigma_n \in C_n\} \cup \{A_i \rightarrow a_i \mid 1 \leq i \leq n\}$. □

In the sequel we will slightly change our notation in order to reduce the number of subscript levels: if $x = a_j \cdots a_k$, we will write $A_{j \cdots k}$ for A_x instead of $A_{a_j \cdots a_k}$; cf. Examples 3.2 and 3.5 above. In this way the set of indices $\{1, 2, \dots, n\}$ inherits the linear order of Σ_n ; a similar remark applies with respect to the $\prec$ -relation.

Example 3.8. For $n = 3$, we have $G_3^T = (V_3, \Sigma_3, P_3, S_3)$ with $P_3^T = \{S_3 \rightarrow a_1 A_2 A_3 \mid a_2 A_3 A_1 \mid a_3 A_1 A_2\} \cup R_3$. Now $E_{3,3} = \{S_3\}$, $E_{3,2} = \emptyset$, $E_{3,1} = \{A_1, A_2, A_3\}$, $D(3, 3) = 1$, $D(3, 2) = 0$, $D(3, 1) = 3$, $\nu_T(3) = 4$, $\pi_T(3) = 6$ and $\delta_T(3) = 3$. □

The following result easily follows from Definition 3.7.

Proposition 3.9. For the family $\{G_n^T\}_{n \geq 1}$ of Definition 3.7 we have for $n \geq 3$,

- (1) $D(n, n) = 1$, $D(n, k) = 0$ ($1 < k < n$), and $D(n, 1) = n$,
- (2) $\nu_T(n) = n + 1$,
- (3) $\pi_T(n) = 2n$,
- (4) $\delta_T(n) = n$. □

4 Greibach 2-form — A Straightforward Approach

The trivial family $\{G_n^T\}_{n \geq 1}$ of grammars in (unrestricted) GNF of Definition 3.7 gives rise to simple results that are not very interesting. Therefore we restrict ourselves in the sequel to grammars in Greibach k -form with $k=1, 2$. It turns out that in those cases the corresponding descriptonal complexity measures are less trivial.

The idea on which our next family of grammars is based stems from Proposition 3.4: we have nonterminals A_x for all strings x in $\mathcal{Q}_{n-1}^n$ with $|x| = k < n$ such that $L(G_n, A_x) = \{x\}$

and $A_x \in E_{n,k}$. For the words in $C_n = \mathcal{Q}_n^n - \mathcal{Q}_{n-1}^n$ we have rules $S_n \rightarrow aA_xA_y$ for all nonempty words x and y with $axy \in C_n$ and $E_{n,n} = \{S_n\}$.

Definition 4.1. $\{G_n^0\}_{n \geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n \geq 1}$ with for $n \geq 3$,

- $N_n = V_n - \Sigma_n = \{S_n\} \cup \{A_x \mid x \in \mathcal{Q}_{n-2}^n\}$,
- $P_n = \{S_n \rightarrow aA_xA_y \mid axy \in C_n; a \in \Sigma_n; x, y \in \Sigma_n^+\} \cup \{A_a \rightarrow a \mid a \in \Sigma_n\} \cup \{A_{axy} \rightarrow aA_xA_y \mid a \in \Sigma_n; axy \in \mathcal{Q}_{n-1}^n; x, y \in \Sigma_n^+\} \cup \{A_{ab} \rightarrow aA_b \mid a, b \in \Sigma_n; a \prec b\}$. □

Example 4.2. Consider $G_5^0 = (V_5, \Sigma_5, P_5, S_5)$ with $P_5 = \{S_5 \rightarrow a_1A_2A_{345} \mid a_1A_{23}A_{45} \mid a_1A_{234}A_5 \mid a_2A_3A_{451} \mid a_2A_{34}A_{51} \mid a_2A_{345}A_1 \mid a_3A_4A_{512} \mid a_3A_{45}A_{12} \mid a_3A_{451}A_2 \mid a_4A_5A_{123} \mid a_4A_{51}A_{23} \mid a_4A_{512}A_3 \mid a_5A_1A_{234} \mid a_5A_{12}A_{34} \mid a_5A_{123}A_4, A_{123} \rightarrow a_1A_2A_3, A_{234} \rightarrow a_2A_3A_4, A_{345} \rightarrow a_3A_4A_5, A_{451} \rightarrow a_4A_5A_1, A_{512} \rightarrow a_5A_1A_2, A_{12} \rightarrow a_1A_2, A_{23} \rightarrow a_2A_3, A_{34} \rightarrow a_3A_4, A_{45} \rightarrow a_4A_5, A_{51} \rightarrow a_5A_1\} \cup R_5$.

We have $E_{5,5} = \{S_5\}$, $E_{5,4} = \emptyset$, $E_{5,3} = \{A_{123}, A_{234}, A_{345}, A_{451}, A_{512}\}$, $E_{5,2} = \{A_{12}, A_{23}, A_{34}, A_{45}, A_{51}\}$, $E_{5,1} = \{A_1, A_2, A_3, A_4, A_5\}$, $D(5,5) = 1$, $D(5,4) = 0$, $D(5,3) = D(5,2) = D(5,1) = 5$. Consequently, $\nu_0(5) = 16$, $\pi_0(5) = 30$ and $\delta_0(5) = 15$; hence G_5^0 is ambiguous. □

In general, if S is a statement that can be true or false, then $[S]$ is equal to 1 if S is true, and to 0 otherwise; cf. [10].

Proposition 4.3. For the family $\{G_n^0\}_{n \geq 1}$ of Definition 4.1 we have for $n \geq 3$,

- (1) $D(n, n) = 1$, $D(n, n-1) = 0$, and $D(n, k) = n$ with $1 \leq k < n-1$,
- (2) $\nu_0(n) = n^2 - 2n + 1$,
- (3) $\pi_0(n) = [n \geq 4] \cdot (\frac{1}{2}n^3 - \frac{7}{2}n^2 + 6n) + n^2$,
- (4) $\delta_0(n) = n^2 - 2n$.

Proof. From Definition 4.1 it follows that for $n \geq 3$, $\nu_0(n) = \#N_n = 1 + \#\mathcal{Q}_{n-2}^n = 1 + (n-2)n = n^2 - 2n + 1$, while $\pi_0(n) = h_0(n) + h_1(n) + h_2(n) + h_3(n)$ with

$$h_0(n) = \#\{S_n \rightarrow aA_xA_y \mid axy \in C_n; a \in \Sigma_n; x, y \in \Sigma_n^+\},$$

$$h_1(n) = \#\{A_{axy} \rightarrow aA_xA_y \mid a \in \Sigma_n; axy \in \mathcal{Q}_{n-1}^n; x, y \in \Sigma_n^+\},$$

$$h_2(n) = \#\{A_{ab} \rightarrow aA_b \mid a, b \in \Sigma_n; a \prec b\},$$

$$h_3(n) = \#\{A_a \rightarrow a \mid a \in \Sigma_n\}.$$

Clearly, $h_0(n) = n(n-2)$ and $h_2(n) = h_3(n) = n$. For h_1 we observe that $h_1(3) = 0$, and for $n \geq 4$, we have $h_1(n) = \sum_{k=3}^{n-2} (k-2) \cdot n = \frac{1}{2}n(n-3)(n-4) = \frac{1}{2}n^3 - \frac{7}{2}n^2 + 6n$. So $\pi_0(n) = n(n-2) + [n \geq 4] \cdot (\frac{1}{2}n^3 - \frac{7}{2}n^2 + 6n) + n + n = [n \geq 4] \cdot (\frac{1}{2}n^3 - \frac{7}{2}n^2 + 6n) + n^2$.

The grammar G_n^0 generates n strings, each of which can be obtained by a left-most derivation in $n-2$ ways ($n \geq 3$); consequently, we have $\delta_0(n) = n(n-2)$. □

5 Greibach 1-form — An Improvement

The next grammar family $\{G_n^1\}_{n \geq 1}$ to generate $\{C_n\}_{n \geq 1}$ consists of context-free grammars in Greibach 1-form; this family is closely related to Definition 5.1 in [3] which in turn has been inspired by generating $\{C_n\}_{n \geq 1}$ with regular grammars.

Definition 5.1. $\{G_n^1\}_{n \geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n \geq 1}$ with for $n \geq 3$,

- $N_n = V_n - \Sigma_n = \{S_n\} \cup \{A_x \mid x \in \mathcal{Q}_{n-1}^n\}$,
- $P_n = \{S_n \rightarrow aA_x \mid ax \in C_n; a \in \Sigma_n\} \cup \{A_a \rightarrow a \mid a \in \Sigma_n\} \cup \{A_{ax} \rightarrow aA_x \mid ax \in \mathcal{Q}_{n-1}^n; a \in \Sigma_n, x \in \Sigma_n^+\}$. □

Example 5.2. For $n = 3$, we obtain $G_3^1 = (V_3, \Sigma_3, P_3, S_3)$ with $P_3 = \{S_3 \rightarrow a_1A_{23} \mid a_2A_{31} \mid a_3A_{12}, A_{12} \rightarrow a_1A_2, A_{23} \rightarrow a_2A_3, A_{31} \rightarrow a_3A_1\} \cup R_3$

Then we have $E_{3,3} = \{S_3\}$, $E_{3,2} = \{A_{12}, A_{23}, A_{31}\}$, $E_{3,1} = \{A_1, A_2, A_3\}$, $D(3,3) = 1$, $D(3,2) = D(3,1) = 3$, $\nu_1(3) = 7$, $\pi_1(3) = 9$ and $\delta_1(3) = 3$. □

The proof of the following result is similar to the one of Proposition 5.3 in [3].

Proposition 5.3. For the family $\{G_n^1\}_{n \geq 1}$ of Definition 5.1 we have for $n \geq 3$,

- (1) $D(n, n) = 1$ and $D(n, k) = n$ with $1 \leq k < n$,
- (2) $\nu_1(n) = n^2 - n + 1$,
- (3) $\pi_1(n) = n^2$,
- (4) $\delta_1(n) = n$. □

Comparing Propositions 4.3 and 5.3 yields for $n \geq 4$, $\nu_0(n) < \nu_1(n)$, $\pi_0(n) > \pi_1(n)$ and $\delta_0(n) > \delta_1(n)$. The latter two inequalities may be considered as an improvement; the price we have to pay is n additional nonterminal symbols.

6 Families of Unambiguous Grammars

In [3] we argued that a first step towards minimal grammars in CNF is to avoid ambiguity. The situation for the GNF is very similar: the following crucial result and its proof are almost identical to the one for CNF in [3].

Proposition 6.1. Let $\{G_n\}_{n \geq 1}$ be a family of grammars in GNF that generates $\{C_n\}_{n \geq 1}$. Then $\delta(n) = n$ if and only if $\pi(n) = \nu(n) + n - 1$. □

The proof tells us that in an unambiguous grammar for C_n , there are n rules for S_n and a single rule for each $A \in N_n - \{S_n\}$. So we try to minimize $\nu(n)$, and as a consequence $\pi(n)$ will reach its minimum value as well. Clearly, Proposition 6.1 applies to $\{G_n^1\}_{n \geq 1}$ but not to $\{G_n^0\}_{n \geq 1}$. However, $\{G_n^1\}_{n \geq 1}$ is not the only family satisfying Proposition 6.1; another one will be introduced now.

Definition 6.2. $\{G_n^2\}_{n \geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n \geq 1}$ with for $n \geq 3$,

- $N_n = V_n - \Sigma_n = \{S_n\} \cup \{A_a \mid a \in \Sigma_n\} \cup M_n$ with for $m \geq 2$,

$M_{2m} = \{A_x \mid x \in F_{2m-2}^{2m} \cup F_{2m-4}^{2m} \cup \dots \cup F_2^{2m}\}$, and

$M_{2m-1} = \{A_x \mid x \in F_{2m-3}^{2m-1} \cup F_{2m-5}^{2m-1} \cup \dots \cup F_3^{2m-1}\}$,

- $P_n = \{S_n \rightarrow aA_bA_x \mid a, b \in \Sigma_n; x \in \Sigma_n^+; abx \in C_n; x \in F_{n-2}^n\} \cup Q_n \cup$
 $\cup \{A_{abx} \rightarrow aA_bA_x \mid a, b \in \Sigma_n; A_{abx} \in M_n; x \in \Sigma_n^+\} \cup \{A_a \rightarrow a \mid a \in \Sigma_n\}$

with for $m \geq 2$, $Q_{2m} = \{A_{ab} \rightarrow aA_b \mid a, b \in \Sigma_n; a \prec b\}$, and $Q_{2m-1} = \emptyset$. $\square$

Example 6.3. Let $G_6^2 = (V_6, \Sigma_6, P_6, S_6)$ with $P_6 = \{S_6 \rightarrow a_1A_2A_{3456} \mid a_2A_3A_{4561} \mid a_3A_4A_{5612} \mid a_4A_5A_{6123} \mid a_5A_6A_{1234} \mid a_6A_1A_{2345}, A_{1234} \rightarrow a_1A_2A_{34}, A_{2345} \rightarrow a_2A_3A_{45}, A_{3456} \rightarrow a_3A_4A_{56}, A_{4561} \rightarrow a_4A_5A_{61}, A_{5612} \rightarrow a_5A_6A_{12}, A_{6123} \rightarrow a_6A_1A_{23}, A_{12} \rightarrow a_1A_2, A_{23} \rightarrow a_2A_3, A_{34} \rightarrow a_3A_4, A_{45} \rightarrow a_4A_5, A_{56} \rightarrow a_5A_6, A_{61} \rightarrow a_6A_1\} \cup R_6$.

Now $E_{6,6} = \{S_n\}$, $E_{6,5} = \emptyset$, $E_{6,4} = \{A_{1234}, A_{2345}, A_{3456}, A_{4561}, A_{5612}, A_{6123}\}$, $E_{6,3} = \emptyset$, $E_{6,2} = \{A_{12}, A_{23}, A_{34}, A_{45}, A_{56}, A_{61}\}$ and $E_{6,1} = \{A_1, A_2, A_3, A_4, A_5, A_6\}$. Then we obtain $\nu_2(6) = 19 < 31 = \nu_1(6)$ and $\pi_2(6) = 24 < 36 = \pi_1(6)$. $\square$

Proposition 6.4. For the family $\{G_n^2\}_{n \geq 1}$ of Definition 6.2, we have

- (1) $D(n, n) = 1$, $D(n, n-1) = 0$, $D(n, 1) = n$, and for $2 \leq k < n-1$,

$$D(n, k) = \text{if } k \equiv n \pmod{2} \text{ then } n \text{ else } 0,$$

- (2) for $n \geq 3$, $\nu_2(n) = \frac{1}{2}n^2 + \frac{1}{2}n \cdot [n \text{ is odd}] + 1$,

- (3) for $n \geq 3$, $\pi_2(n) = \frac{1}{2}n^2 + \frac{1}{2}n \cdot ([n \text{ is odd}] + 2)$,

- (4) $\delta_2(n) = n$.

Proof. From Definition 6.2, Proposition 6.4(1) and (4) easily follow. Then for even n with $n \geq 4$, we have $\nu_2(n) = 1 + 2n + \sum_{k=4}^{n-2} D(n, k) = 1 + 2n + \frac{1}{2}(n-4)n = \frac{1}{2}n^2 + 1$. For odd n with $n \geq 3$, we obtain $\nu_2(n) = 1 + 2n + \sum_{k=3}^{n-2} D(n, k) = 1 + 2n + \frac{1}{2}(n-3)n = \frac{1}{2}n^2 + \frac{1}{2}n + 1$. Combining these two cases results in $\nu_2(n) = \frac{1}{2}n^2 + \frac{1}{2}n \cdot [n \text{ is odd}] + 1$ for $n \geq 3$. Finally, Proposition 6.1 implies Proposition 6.4(3). $\square$

From Propositions 5.3 and 6.4 it follows that for $n \geq 4$, we have $\nu_2(n) < \nu_1(n)$ and, consequently, $\pi_2(n) < \pi_1(n)$.

It is possible to continue in this way by introducing families $\{G_n^k\}_{n \geq 1}$ ($k \geq 3$) such that for each rule $A \rightarrow aBC$, $L(G_n^k, B)$ consists of a single word of length $k-1$. As in [3] the objections are twofold: the definitions become more complicated as k increases, and we are probably left with $\nu_k(n)$ and $\pi_k(n)$ being functions in $\Theta(n^2)$.

7 Towards a Family of Minimal Grammars

For the CNF we defined in [3] a family of grammars $\{G_n^k\}_{n \geq 1}$ to be minimal if each G_k is unambiguous and $\nu(n) \in \Theta(n)$; cf. Proposition 6.1. As we will see, this latter condition is likely to be too ambitious for the GNF. In [3] we also established the existence of a minimal family for the CNF; it turns out that the corresponding problem for the GNF remains open. But let us first have a look at a GNF-family as simple as the minimal CNF-family of [3].

Definition 7.1. $\{G_n^\diamond\}_{n \geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n \geq 1}$ with for even $n \geq 4$,

- $N_n = V_n - \Sigma_n = \{S_n, A_1, A_2, \dots, A_n\} \cup \{A_{i \dots k} \mid i = 1, 3, 5, \dots, n-1;$
 $a_i \dots a_k \in F_{n-2} \cup F_{n-4} \cup F_{n-6} \cup \dots \cup F_2\},$
- $P_n = \{S_n \rightarrow a_i A_j A_{k \dots m} \mid a_j A_{k \dots m} A_i \mid i = 1, 3, \dots, n-1; a_i a_j a_k \dots a_m \in C_n\} \cup$
 $\{A_k \rightarrow a_k \mid 1 \leq k \leq n\} \cup \{A_{i \dots m} \rightarrow a_i A_j A_{k \dots m} \mid i = 1, 3, \dots, n-1;$
 $a_i a_j a_k \dots a_m \in F_{n-2} \cup F_{n-4} \cup F_{n-6} \cup \dots \cup F_2\},$
 where $A_{k \dots m}$ is taken equal to λ whenever $a_k \dots a_m$ equals λ ;

and for odd $n \geq 3$,

- $N_n = V_n - \Sigma_n = \{S_n, A_1, A_2, \dots, A_n\} \cup \{A_{i \dots k} \mid i = 1, 3, 5, \dots, n;$
 $a_i \dots a_k \in F_{n-2} \cup F_{n-4} \cup F_{n-6} \cup \dots \cup F_3\},$
- $P_n = \{S_n \rightarrow a_i A_j A_{k \dots m} \mid a_j A_{k \dots m} A_i \mid a_n A_{12 \dots (n-2)} A_{n-1} \mid i = 1, 3, \dots, n-2;$
 $a_i a_j a_k \dots a_m \in C_n\} \cup \{A_k \rightarrow a_k \mid 1 \leq k \leq n\} \cup \{A_{i \dots m} \rightarrow a_i A_j A_{k \dots m} \mid$
 $i = 1, 3, \dots, n-2; a_i a_j a_k \dots a_m \in F_{n-2} \cup F_{n-4} \cup F_{n-6} \cup \dots \cup F_3\} \cup$
 $\{A_{n12 \dots k} \rightarrow a_n A_{12 \dots (k-1)} A_k \mid k = 2, 4, \dots, n-3; a_n a_1 \dots a_k \in F_{n-2} \cup$
 $F_{n-4} \cup F_{n-6} \cup \dots \cup F_3\}.$ $\square$

Example 7.2. Let $G_7^\diamond = (V_7, \Sigma_7, P_7, S_7)$ with $P_7 = \{S_7 \rightarrow a_1 A_2 A_{34567} \mid a_2 A_{34567} A_1 \mid$
 $a_3 A_4 A_{56712} \mid a_4 A_{56712} A_3 \mid a_5 A_6 A_{71234} \mid a_6 A_{71234} A_5 \mid a_7 A_{12345} A_6, A_{12345} \rightarrow a_1 A_2 A_{345},$
 $A_{34567} \rightarrow a_3 A_4 A_{567}, A_{56712} \rightarrow a_5 A_6 A_{712}, A_{71234} \rightarrow a_7 A_{123} A_4, A_{123} \rightarrow a_1 A_2 A_3, A_{345} \rightarrow$
 $a_3 A_4 A_5, A_{567} \rightarrow a_5 A_6 A_7, A_{712} \rightarrow a_7 A_1 A_2\} \cup R_7.$

Then $\nu_\diamond(7) = 16 < 29 = \nu_2(7)$, $\pi_\diamond(7) = 22 < 35 = \pi_2(7)$ and $\delta_\diamond(7) = 7$. $\square$

Proposition 7.3. For the family $\{G_n^\diamond\}_{n \geq 1}$ of Definition 7.1 we have

- (1) $D(n, n) = 1$, $D(n, 1) = n$, and
 for even n and $k = 2, 4, \dots, n-2$, $D(n, k) = \frac{1}{2}n$,
 for odd n and $k = 3, 5, \dots, n-2$, $D(n, k) = \lceil \frac{1}{2}n \rceil$,
- (2) $\nu_\diamond(n) = \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4} + \frac{3}{4} \cdot [n \text{ is even}]$,
- (3) $\pi_\diamond(n) = \frac{1}{4}n^2 + \frac{3}{2}n - \frac{3}{4} \cdot [n \text{ is odd}]$,
- (4) $\delta_\diamond(n) = n$.

Proof. It is easy to establish (1) and (4); then for even n we have

$$\nu_\diamond(n) = 1 + n + (\frac{1}{2}n - 1)\frac{1}{2}n = \frac{1}{4}n^2 + \frac{1}{2}n + 1,$$

and for odd n ,

$$\nu_\diamond(n) = 1 + n + \lfloor \frac{1}{2}n - 1 \rfloor \lceil \frac{1}{2}n \rceil = \frac{1}{4}n^2 + \frac{1}{2}n + \frac{1}{4}.$$

Finally, (3) follows from (2), (4) and Proposition 6.1. $\square$

Although this is an improvement with respect to Propositions 4.3, 5.3 and 6.4, $\{G_n^\diamond\}_{n \geq 1}$ is by no means a minimal family as we will see from the following “divide-and-conquer” family; cf. §8 in [2].

```

 $P_n := \{A_i \rightarrow a_i \mid a_i \in \Sigma_n\} \cup \{S_n \rightarrow a_i A_x A_y \mid a_i x y \in C_n; |x| = \lceil \frac{1}{2}(n-1) \rceil\};$ 
 $N_n := \{S_n\} \cup \{A_i \mid a_i \in \Sigma_n\}; \quad M := \{x, y \mid S_n \rightarrow a_i A_x A_y \in P_n\};$ 
while  $M - \Sigma_n \neq \emptyset$  [i.e.  $\exists x \in M : |x| \geq 2$ ] do
  begin  $N_n := N_n \cup \{A_x\}; M := M - \{x\};$ 
    case  $|x|$  of
      = 2:  $P_n := P_n \cup \{A_x \rightarrow a_i A_j \mid a_i a_j = x\};$ 
      = 3:  $P_n := P_n \cup \{A_x \rightarrow a_i A_j A_k \mid a_i a_j a_k = x\};$ 
       $\geq 4$ : begin
         $P_n := P_n \cup \{A_x \rightarrow a_i A_y A_z \mid a_i y z = x; |y| = \lceil \frac{1}{2}(|x| - 1) \rceil\};$ 
         $M := M \cup \{y, z \mid a_i y z = x; |y| = \lceil \frac{1}{2}(|x| - 1) \rceil\};$ 
      end
    endcase
  end

```

Figure 1: Algorithm to determine N_n and P_n of $G_n^\bullet$.

Definition 7.4. $\{G_n^\bullet\}_{n \geq 1}$ is given by $\{(V_n, \Sigma_n, P_n, S_n)\}_{n \geq 1}$ where the sets N_n and P_n are determined by the algorithm in Figure 1. $\square$

Example 7.5. $G_7^\bullet = (V_7, \Sigma_7, P_7, S_7)$ with $P_7 = \{S_7 \rightarrow a_1 A_{234} A_{567} \mid a_2 A_{345} A_{671} \mid a_3 A_{456} A_{712} \mid a_4 A_{567} A_{123} \mid a_5 A_{671} A_{234} \mid a_6 A_{712} A_{345} \mid a_7 A_{123} A_{456}, A_{123} \rightarrow a_1 A_2 A_3, A_{234} \rightarrow a_2 A_3 A_4, A_{345} \rightarrow a_3 A_4 A_5, A_{456} \rightarrow a_4 A_5 A_6, A_{567} \rightarrow a_5 A_6 A_7, A_{671} \rightarrow a_6 A_7 A_1, A_{712} \rightarrow a_7 A_1 A_2\} \cup R_7$. Now $\nu_\bullet(7) = 15 < 16 = \nu_\diamond(7)$, $\pi_\bullet(7) = 21 < 22 = \nu_\diamond(7)$ and $\delta_\bullet(7) = 7$. $\square$

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\nu_\bullet(n)$	4	9	11	19	15	33	28	41	34	61	53	71	46	96
$\pi_\bullet(n)$	6	12	15	24	21	40	36	50	44	72	65	84	60	111

Table 1: $\nu_\bullet(n)$ and $\pi_\bullet(n)$ for $3 \leq n \leq 16$.

As usual in analyzing such a divide-and-conquer approach, a closed form for $\nu_\bullet(n)$ and $\pi_\bullet(n)$ is very hard or even impossible to obtain; for small values we refer to Table 1. Only for special values of n we can infer some manageable expressions.

Proposition 7.6. For the family $\{G_n^\bullet\}_{n \geq 1}$ we have in case $n = 2^k - 1$ ($k \geq 2$),

- (1) $D(n, n) = 1$, $D(n, 2^i - 1) = n$ ($i = 1, 2, \dots, k-1$), and $D(n, i) = 0$ otherwise,
- (2) $\nu_\bullet(n) = n \cdot \log_2(n+1) - n + 1$,
- (3) $\pi_\bullet(n) = n \cdot \log_2(n+1)$,
- (4) $\delta_\bullet(n) = n$. $\square$

So using this divide-and-conquer approach we end up with $\nu_{\bullet}(n)$ and $\pi_{\bullet}(n)$ in $\Theta(n \cdot \log_2 n)$, rather than in $\Theta(n^2)$ as for the previous families.

8 Concluding Remarks

We discussed a few ways of generating the languages $\{C_n\}_{n \geq 1}$ of circular shifts by context-free grammars $\{G_n\}_{n \geq 1}$ in GNF, and we compared these families with respect to the measures ν , π and δ . Our results give rise to the following observation.

Conjecture 8.1. *Any family of context-free grammars in GNF $\{G_n\}_{n \geq 1}$ that generates $\{C_n\}_{n \geq 1}$ must have measures $\nu(n)$ and $\pi(n)$ that are not bounded by any linear function in n .* $\square$

The situation in the GNF-case differs considerably from the CNF-case: in [3] we established the existence of a minimal family in CNF for which ν and π are linear functions in n (even with small coefficients). For the GNF the definition of minimality remains a problem; viz. setting “ $\{G_n\}_{n \geq 1}$ in GNF is *minimal* for $\{C_n\}_{n \geq 1}$ if (i) each G_n is unambiguous, and (ii) $\nu(n) \in \Theta(f(n))$ ” leaves us with the question of an adequate choice for $f(n)$. Conjecture 8.1 implies $f(n) \in \omega(n)$.

Taking $f(n)$ equal to $n \cdot \log_2 n$ results in the minimality of $\{G_n^{\bullet}\}_{n \geq 1}$ (Proposition 7.6), but the question whether this family is also minimal in an absolute sense (i.e., does there exist no family with $\nu(n) \in \Theta(n \cdot \log_2 n)$ and $\nu(n) < n \cdot \log_2(n+1) - n + 1$ for n large enough with $n = 2^k - 1$ and $k \geq 2$?) remains an open problem as well.

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