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Matrix approach to the Shapley value and dual similar associated consistency*

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Abstract

Replacing associated consistency in Hamiache's axiom system by dual similar associated consistency, we axiomatize the Shapley value as the unique value verifying the inessential game property, continuity and dual similar associated consistency. Continuing the matrix analysis for Hamiache's axiomatization of the Shapley value, we construct the dual similar associated game and introduce the dual similar associated transformation matrix M_λ^{DSh} as well. In the game theoretic framework we show that the dual game of the dual similar associated game is Hamiache's associated game of the dual game. For the purpose of matrix analysis, we derive the similarity relationship $M_\lambda^{DSh} = QM_\lambda Q^{-1}$ between the dual similar associated transformation matrix M_λ^{DSh} and associated transformation matrix M_λ for Hamiache's associated game, where the transformation matrix Q represents the duality operator on games. This similarity of matrices transfers associated consistency into dual similar associated consistency, and also implies the inessential property for the limit game of the convergent sequence of repeated dual similar associated games. We conclude this paper with three tables summarizing all matrix results.

Key Words: Shapley value, Shapley standard matrix, dual game, dual matrix, dual similar associated consistency.
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1 Introduction

A *cooperative game* with transferable utility (TU) is a pair $\langle N, v\rangle$, where N is a nonempty, finite set and $v : 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. An element of N (notation: $i \in N$) and a subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) are called a *player* and *coalition* respectively, and the associated real

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number $v(S)$ is called the *worth* of coalition S. The size of coalition S is denoted by s. Particularly, n denotes the size of the player set N. We denote by $\mathcal{G}$ the universal game space consisting of all these TU-games. In this paper, a TU-game $\langle N, v\rangle$ is always denoted by its column vector of worths of all coalitions $S \subseteq N$ in the traditional order (one-person coalitions are at the top, etc.), i.e. $\vec{v} = (v(S))_{S\subseteq N, S\neq\emptyset}$. If no confusion arises, we write v instead of $\vec{v}$. A game $\langle N, v\rangle$ is said to be *inessential* if for all coalitions $S \subseteq N$, $v(S) = \sum_{i\in S} v(\{i\})$.

The solution part of cooperative game theory deals with the allocation problem of how to divide the overall earnings the amount of $v(N)$ among the players in the TU-game. There is associated a single allocation called the value of the TU-game. Formally, a *value* on $\mathcal{G}$ is a function Φ that assigns a single payoff vector $\Phi(N, v) = (\Phi_i(N, v))_{i\in N} \in \mathbb{R}^n$ to every TU-game $\langle N, v\rangle \in \mathcal{G}$. The so-called value $\Phi_i(N, v)$ of player i in the game $\langle N, v\rangle$ represents an assessment by i of his gains from participating in the game.

Without going into details, let us recall the well-known *Shapley value* $Sh(N, v)$ as follows ([4, 6]):

$$Sh_i(N, v) = \sum_{S\subseteq N, S\ni i} \frac{(s-1)!(n-s)!}{n!} \Big[v(S) - v(S \setminus \{i\})\Big] \quad \text{for all } i \in N.$$

The eldest axiomatization of the Shapley value is stated by Shapley himself ([6]) by referring to four properties called efficiency, symmetry, linearity, and dummy player property. In the framework of values for TU-games, firstly let us review several essential properties treated in former axiomatizations of the Shapley value. A value Φ on the universal game space $\mathcal{G}$ is said to be *efficient*, if $\sum_{i\in N} \Phi_i(N, v) = v(N)$ for all games $\langle N, v\rangle$; *symmetric*, if $\Phi_{\pi(i)}(N, \pi v) = \Phi_i(N, v)$ for all games $\langle N, v\rangle$, all $i \in N$, and every permutation π on N; *linear*, if $\Phi(N, \alpha \cdot v + \beta \cdot w) = \alpha \cdot \Phi(N, v) + \beta \cdot \Phi(N, w)$ for all games $\langle N, v\rangle$, $\langle N, w\rangle$, and all $\alpha, \beta \in \mathbb{R}$; *inessential*, if $\Phi_i(N, v) = v(\{i\})$ for all inessential games $\langle N, v\rangle$, all $i \in N$; *continuous*, if for all (pointwise) convergent sequences of games $\{\langle N, v_k\rangle\}_{k=0}^{\infty}$, say the limit of which is the game $\langle N, \bar{v}\rangle$, the corresponding sequences of values $\{\Phi(N, v_k)\}_{k=0}^{\infty}$ converge to the value $\Phi(N, \bar{v})$.

Hamiache's axiomatization of the Shapley value states that the Shapley value is the unique one-point solution verifying the following three axioms: inessential game property, continuity and associated consistency (see [1]). In his paper, an associated game $\langle N, v_{\lambda}^{Sh}\rangle$ is constructed. And a sequence of games is also defined, where the term of order m, in this sequence, is the associated game of the term of order $m-1$. He showed that this sequence of games converges and that the limit game is inessential. The solution is obtained using the inessential game property, the associated consistency and the continuity axioms. As a by-product, neither the linearity nor the efficiency axioms are needed.

In [7], we developed a matrix approach for Hamiache's axiomatization of the Shapley value. A new type of matrix named row (resp. column)-coalitional matrix was introduced in the framework of cooperative game theory. Particularly, both the Shapley value and Hamiache's associated game were represented algebraically by their coalitional matrices called the Shapley standard matrix M^{Sh} and the associated transformation matrix M_{λ}, respectively. The associated consistency for the Shapley value was formulated as the matrix equality $M^{Sh} = M^{Sh} \cdot M_{\lambda}$. In addition, the procedure of diagonalization of M_{λ} and the inessential property for coalitional matrices were extremely helpful to treat the continuity and inessential game property. Matrix analysis was fully adopted throughout the mathematical developments and the proofs as well. Now let us recall the notion of coalitional matrix.

Definition 1 (cf. [7]). *A matrix M is called a row (resp. column)-coalitional matrix if* *its rows (resp. columns) are indexed by coalitions $S \subseteq N$ in the traditional order (one-person coalitions are at the top, etc.). And a row-coalitional matrix $M = [\overrightarrow{m_S}]_{S\subseteq N, S\neq\emptyset}$ is row-inessential if the row-vector of M indexed by coalition S verifies $\overrightarrow{m_S} = \sum_{i\in S}\overrightarrow{m_i}$ for all $S \subseteq N$.*

And we obtained the following basic properties of row-coalitional matrix in [7].

Lemma 1.1 (cf. [7]). *Let M be a row-coalitional matrix and A be a matrix.*

1. *If M is row-inessential, then the row-coalitional matrix MA is row-inessential.*
2. *If A is invertible, then MA is row-inessential if and only if M is row-inessential.*
3. *For every game $\langle N, v\rangle \in \mathcal{G}$, if M is row-inessential, then the new game $\langle N, M\cdot v\rangle$ is inessential.*

By matrix approach we can restate the Shapley value in terminology of the Shapley standard matrix as follows.

Definition 2 (cf. [7]). *Given any game $\langle N, v\rangle$, the Shapley value $Sh(N, v)$ can be represented by the Shapley standard matrix M^{Sh} as:*

$$Sh(N, v) = M^{Sh}v,$$

where the matrix $M^{Sh} = \left[M^{Sh}\right]_{i\in N, S\subseteq N, S\neq\emptyset}$ is column-coalitional defined by

$$\left[M^{Sh}\right]_{i,S} = \begin{cases} \dfrac{(s-1)!(n-s)!}{n!}, & \text{if } i \in S; \\ -\dfrac{s!(n-s-1)!}{n!}, & \text{if } i \notin S. \end{cases}$$

And for column vectors of the Shapley standard matrix M^{Sh}, we have the following *anti-complementary property.*

Proposition 1.2. *Let $\left[M^{Sh}\right]_T$ be the column vector of M^{Sh} indexed by any coalition $T \subseteq N$. Then it holds $\left[M^{Sh}\right]_T = -\left[M^{Sh}\right]_{N\setminus T}$.*

Proof. For any coalition $T \subseteq N$, it is sufficient to show $\left[M^{Sh}\right]_T + \left[M^{Sh}\right]_{N\setminus T} = \vec{0}$. For each player $i \in N$, if $i \in T$ then $i \notin N \setminus T$, and vice versa. So only one case needs to be checked, for instance $i \in T$. By the definition of the Shapley standard matrix M^{Sh}, we know $\left[M^{Sh}\right]_{i,T} + \left[M^{Sh}\right]_{i,N\setminus T} = \frac{(t-1)!(n-t)!}{n!} - \frac{(n-t)!(t-1)!}{n!} = 0$. Therefore $\left[M^{Sh}\right]_T = -\left[M^{Sh}\right]_{N\setminus T}$ for all coalitions $T \subseteq N$. ∎

The aim of this paper is to develop the matrix approach for the Shapley value. The organization of the paper is as follows. In Section 2, continuing the matrix analysis for Hamiache's axiomatization of the Shapley value in [7], we construct the dual similar associated game and introduce the dual similar associated transformation matrix M_λ^{DSh} as well. In the game theoretic framework we show that the dual game of the dual similar associated game is Hamiache's associated game of the dual game. For the purpose of matrix analysis, we derive the similarity relationship $M_\lambda^{DSh} = QM_\lambda Q^{-1}$ between the dual similar associated transformation matrix M_λ^{DSh} and associated transformation matrix M_λ, where the transformation matrix

Q represents the duality operator on games. It yields the inessential property for the limit game of the convergent sequence of repeated dual similar associated games. In Section 3, this similarity of matrices also transfers associated consistency into dual similar associated consistency. Actually, we axiomatize the Shapley value as the unique value verifying the inessential game property, continuity and dual similar associated consistency. The concluding Section 4 provides a summary of three tables about our matrix analysis.

2 The dual similar associated game

Given any game $\langle N, v\rangle$ and $\lambda \in \mathbb{R}$, Hamiache defined its *associated game* $\langle N, v_\lambda^{Sh}\rangle$ in [1] as follows:

$$v_\lambda^{Sh}(S) := v(S) + \lambda \sum_{j\in N\setminus S} \Big[v(S\cup\{j\}) - v(S) - v(\{j\})\Big] \qquad \text{for all } \; S\subseteq N \tag{2.1}$$

or equivalently,

$$v_\lambda^{Sh}(S) = \big[1-(n-s)\lambda\big]v(S) + \lambda \sum_{j\in N\setminus S} v(S\cup\{j\}) - \lambda \sum_{j\in N\setminus S} v(\{j\}). \tag{2.2}$$

The worth $v_\lambda^{Sh}(S)$ of coalition S in the associated game differs from the initial worth $v(S)$ by taking into account the possible (weighted) net benefits $v(S\cup\{j\}) - v(S) - v(\{j\})$ arising from mutual cooperation among the coalition S itself and any of each isolated non-members $j \in N \setminus S$. In other words, for coalition S, the net benefits per non-member measures the surplus of the coalitional marginal contribution $\nabla^v(S,j) = v(S\cup\{j\}) - v(S)$ over the individual worth $\nabla^v(\emptyset, j) = v(\{j\})$ of the non-member $j \in N\setminus S$.

In [7], we introduced a coalitional matrix $M_\lambda = \big[M_\lambda\big]_{\substack{S,T\subseteq N\\ S,T\neq\emptyset}}$ named the *associated transformation matrix* for Hamiache's associated game as:

$$\big[M_\lambda\big]_{S,T} = \begin{cases} 1-(n-s)\lambda, & \text{if } T=S;\\ \lambda, & \text{if } T = S\cup\{j\} \text{ and } j\in N\setminus S;\\ -\lambda, & \text{if } T=\{j\} \text{ and } j\in N\setminus S;\\ 0, & \text{otherwise.}\end{cases}$$

Then we restated the associated game and the sequence of repeated associated games as follows.

Definition 3 (cf. [7]). *Given any game $\langle N, v\rangle$ and $\lambda\in\mathbb{R}$, the associated game $\langle N, v_\lambda^{Sh}\rangle$ can be represented as:*

$$v_\lambda^{Sh} = M_\lambda \cdot v.$$

And its sequence of repeated associated games $\{\langle N, v_\lambda^{m*Sh}\rangle\}_{m=0}^{\infty}$ is defined as:

$$v_\lambda^{m*Sh} = M_\lambda \cdot v_\lambda^{(m-1)*Sh} \quad \textit{for all } m\geq 1, \quad \textit{where } v_\lambda^{0*Sh} = v.$$

Now, we consider a new associated game by revaluing the worth of coalition S. Its new worth differs from the initial worth $v(S)$ by taking into account the possible (weighted) net benefits $\big(v(N) - v(N\setminus\{j\})\big) - \big(v(S) - v(S\setminus\{j\})\big)$. Here for coalition S, the net benefits per member measures the loss of the overall marginal contribution $\nabla^v(N,j) = v(N) - v(N\setminus\{j\})$

over the coalitional marginal contribution $\nabla^v(S,j) = v(S) - v(S\setminus\{j\})$ of the member $j \in S$. We call it dual similar associated game.

Given any game $\langle N, v\rangle$ and $\lambda \in \mathbb{R}$, define its *dual similar associated game* $\langle N, v_\lambda^{DSh}\rangle$ as follows:

$$v_\lambda^{DSh}(S) := v(S) + \lambda \sum_{j\in S} \Big[\big(v(N) - v(N\setminus\{j\})\big) - \big(v(S) - v(S\setminus\{j\})\big)\Big] \quad \text{for all} \quad S \subseteq N. \tag{2.3}$$

Notice that $v_\lambda^{DSh}(\emptyset) = 0$, $v_\lambda^{DSh}(N) = v(N)$ and moreover, $v_\lambda^{DSh} = v$ for all inessential games $\langle N, v\rangle$. We do not care about the trivial case $\lambda = 0$. Similarly, for all $S \subseteq N, S \neq \emptyset$ we can express the worth $v_\lambda^{DSh}(S)$ as:

$$v_\lambda^{DSh}(S) = \big(1 - s\lambda\big)v(S) + s\lambda v(N) - \lambda \sum_{j\in S} v(N\setminus\{j\}) + \lambda \sum_{j\in S} v(S\setminus\{j\}). \tag{2.4}$$

By matrix approach, we can define the dual similar associated game and the sequence of repeated dual similar associated games as follows.

Definition 4. *Given any game $\langle N, v\rangle$ and $\lambda \in \mathbb{R}$, its dual similar associated game $\langle N, v_\lambda^{DSh}\rangle$ can be represented as:*

$$v_\lambda^{DSh} = M_\lambda^{DSh} \cdot v,$$

where the dual similar associated transformation matrix $M_\lambda^{DSh} = \big[M_\lambda^{DSh}\big]_{\substack{S,T\subseteq N\\ S,T\neq\emptyset}}$ is both row-coalitional and column-coalitional defined by

$$\big[M_\lambda^{DSh}\big]_{S,T} = \begin{cases} 1 - s\lambda, & \text{if } T = S \text{ and } S \neq N; \\ s\lambda, & \text{if } T = N \text{ and } S \neq N; \\ \lambda, & \text{if } T = S\setminus\{j\}, j \in S \text{ and } S \neq N; \\ -\lambda, & \text{if } T = N\setminus\{j\}, j \in S \text{ and } S \neq N; \\ 1, & \text{if } T = S = N; \\ 0, & \text{otherwise.} \end{cases}$$

And its sequence of repeated dual similar associated games $\{\langle N, v_\lambda^{m*DSh}\rangle\}_{m=0}^{\infty}$ is defined as:

$$v_\lambda^{m*DSh} = M_\lambda^{DSh} \cdot v_\lambda^{(m-1)*DSh} \quad \textit{for all } m \geq 1, \quad \textit{where } v_\lambda^{0*DSh} = v.$$

Before showing the relationship between the associated game and the dual similar associated game, we should mention the concept of dual game in cooperative game theory. For a given game $\langle N, v\rangle$, its *dual game* $\langle N, v^*\rangle$ is defined as

$$v^*(S) := v(N) - v(N\setminus S) \qquad \text{for all} \quad S \subseteq N. \tag{2.6}$$

By matrix approach, it can be restated as $v^* = Q \cdot v$, where the *dual matrix* $Q = \big[Q\big]_{\substack{S,T\subseteq N\\ S,T\neq\emptyset}}$ is both row-coalitional and column-coalitional matrix defined by

$$\big[Q\big]_{S,T} = \begin{cases} -1, & \text{if } T = N\setminus S \text{ and } S \neq N; \\ 1, & \text{if } T = N; \\ 0, & \text{otherwise.} \end{cases}, \text{ or } Q = \begin{pmatrix} & & & -1 & 1 \\ & & -1 & & 1 \\ & \dots & & & \vdots \\ -1 & & & & 1 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

We have the following properties for the dual game $\langle N, v^*\rangle$ and the dual matrix Q. Let I be the identity matrix.

Proposition 2.1. *Let $\langle N, v\rangle$ be an inessential game. Then the game $\langle N, v\rangle$ is self-dual, i.e. $\langle N, v^*\rangle = \langle N, v\rangle$.*

Proof. Since $\langle N, v\rangle$ is inessential, for all coalitions $S \subseteq N$, it holds that $v(N) - v(N \setminus S) = v(S)$. By the definition of the dual game, the worth $v^*(S)$ verifies

$$v^*(S) = v(N) - v(N \setminus S) = v(S).$$

Hence an inessential game $\langle N, v\rangle$ is self-dual. ∎

Proposition 2.2. *The dual matrix verifies the duality property $Q \cdot Q = I$. That is to say, for every game $\langle N, v\rangle$, the dual game $\langle N, (v^*)^*\rangle$ of its dual game $\langle N, v^*\rangle$ is $\langle N, v\rangle$ itself.*

Proof. The duality property is easy to be checked. ∎

Proposition 2.3. *The dual matrix verifies $M^{Sh} = M^{Sh}Q$. That is to say, the Shapley values of the game $\langle N, v\rangle$ and its dual game $\langle N, v^*\rangle$ are equal.*

Proof by Matrix Approach. For any coalition $T \subseteq N$, $T \neq N$, consider the column vector $\left[M^{Sh}Q\right]_T$ indexed by T. By the definition of dual matrix Q and anti-complementary property of Shapley standard matrix M^{Sh} in Proposition 1.2, it should be

$$\left[M^{Sh}Q\right]_T = \sum_{S\subseteq N, S\neq\emptyset} \left[Q\right]_{S,T}\left[M^{Sh}\right]_S = -\left[M^{Sh}\right]_{N\setminus T} = \left[M^{Sh}\right]_T,$$

where $\left[M^{Sh}\right]_S$ is the column vector of M^{Sh} indexed by S. And if $T = N$, then

$$\left[M^{Sh}Q\right]_N = \sum_{S\subseteq N, S\neq\emptyset} \left[Q\right]_{S,N}\left[M^{Sh}\right]_S = \sum_{S\subseteq N, S\neq\emptyset} \left[M^{Sh}\right]_S = \left[M^{Sh}\right]_N.$$

This completes the algebraic proof. ∎

By far, we can conclude the following similarity relationship between the associated transformation matrix M_λ and the dual similar associated transformation matrix M_λ^{DSh}, We also conclude the corresponding relationship for the dual and two types of associated games.

Lemma 2.4. *$M_\lambda^{DSh} = QM_\lambda Q$, or equivalently $QM_\lambda^{DSh} = M_\lambda Q$. In the game theoretic context, the dual game of the dual similar associated game is the associated game of the dual game, i.e. $\langle N, (v_\lambda^{DSh})^*\rangle = \langle N, (v^*)_\lambda^{Sh}\rangle$.*

Proof. For any row-coalitional matrix M, let $\left[M\right]_{S-row}$ denote the row vector of M indexed by all coalitions $S \subseteq N$.

Since $\left[Q\right]_{N-row} = \left[M_\lambda\right]_{N-row} = \left[M_\lambda^{DSh}\right]_{N-row} = [0, \cdots, 0, 1]$, it is easy to see that $\left[QM_\lambda Q\right]_{N-row} = \left[M_\lambda^{DSh}\right]_{N-row}$. By the definitions of Q, M_λ and M_λ^{DSh}, for any coalition $S \subseteq N$, $S \neq N$, we have

$$\begin{aligned}
\left[QM_\lambda Q\right]_{S-row} &= \sum_{T\subseteq N, T\neq\emptyset} \left[Q\right]_{S,T}\left[M_\lambda Q\right]_{T-row} = \left[M_\lambda Q\right]_{N-row} - \left[M_\lambda Q\right]_{N\setminus S-row} \\
&= \sum_{T\subseteq N, T\neq\emptyset} \left[M_\lambda\right]_{N,T}\left[Q\right]_{T-row} - \sum_{T\subseteq N, T\neq\emptyset} \left[M_\lambda\right]_{N\setminus S,T}\left[Q\right]_{T-row} \\
&= \left[Q\right]_{N-row} - \left\{(1-s\lambda)\left[Q\right]_{N\setminus S-row} + \lambda\sum_{j\in S}\left[Q\right]_{(N\setminus S)\cup\{j\}-row} - \lambda\sum_{j\in S}\left[Q\right]_{\{j\}-row}\right\} \\
&\circeq \left[M_\lambda^{DSh}\right]_{S-row}.
\end{aligned}$$

The latter $S-$row vector equality needs to be checked for each entry indexed by all coalitions $T \subseteq N$. This completes the proof by matrix approach. An alternative proof based on the definitions of the dual game and the associated game can be found in the appendix. ∎

Due to the duality property $Q \cdot Q = I$, the above relationship can be recited as follows.

Corollary 2.5. *$M_\lambda = QM_\lambda^{DSh}Q$, or equivalently $QM_\lambda = M_\lambda^{DSh}Q$. In the game theoretic context, the dual game of the associated game is the dual similar associated game of the dual game, i.e. $\langle N, (v_\lambda^{Sh})^*\rangle = \langle N, (v^*)_\lambda^{DSh}\rangle$.*

The next diagram illustrates the commutative relationship between the two types of associated games in Lemma 2.4 and Corollary 2.5.

$$\begin{array}{ccc}
\langle N, v\rangle & \xrightarrow[\text{associated game}]{\text{dual similar}} & \langle N, v_\lambda^{DSh}\rangle = \langle N, \big((v^*)_\lambda^{Sh}\big)^*\rangle \\
\updownarrow \text{dual game} & & \updownarrow \text{dual game} \\
\langle N, v^*\rangle & \xrightarrow[\text{game}]{\text{associated}} & \langle N, (v^*)_\lambda^{Sh}\rangle = \langle N, \big(v_\lambda^{DSh}\big)^*\rangle
\end{array}$$

Diagram 1: The commutative relationship between the two types of associated games.

Here we recall some properties of the associated transformation matrix M_λ and the convergence property for the sequence of repeated associated games $\{\langle N, v_\lambda^{m*Sh}\rangle\}_{m=0}^{\infty}$ in [7]. Similar properties for the dual similar associated transformation matrix M_λ^{DSh} and the sequence of repeated dual similar associated games $\{\langle N, v_\lambda^{m*DSh}\rangle\}_{m=0}^{\infty}$ can be derived.

Lemma 2.6 (cf. [7]). *Let M_λ be the associated transformation matrix.*

1. *$M_\lambda = PD_\lambda P^{-1}$, where $D_\lambda = diag(\underbrace{1, \cdots, 1}_{\binom{n}{1}\ times}, \underbrace{1-2\lambda, \cdots, 1-2\lambda}_{\binom{n}{2}\ times}, \cdots, \underbrace{1-n\lambda}_{\binom{n}{n}\ times})$ and P consists of eigenvectors of M_λ corresponding to eigenvalues $1, 1-k\lambda$ $(2 \le k \le n)$.*
2. *If $0 < \lambda < \frac{2}{n}$, then $\lim\limits_{m\to\infty}(M_\lambda)^m = PDP^{-1}$, where $D = diag(\underbrace{1, \cdots, 1}_{n\ times}, \underbrace{0, \cdots, 0}_{2^n-1-n\ times})$.*
3. *The row-coalitional matrix PD equals $PD = [\overrightarrow{x^1}, \overrightarrow{x^2}, \cdots, \overrightarrow{x^n}, \vec{0}, \cdots, \vec{0}]$ and PD is row-inessential, where column vectors $\overrightarrow{x^i}$ $(i = 1, 2, \cdots n)$ are different eigenvectors of M_λ corresponding to eigenvalue 1 and $\vec{0}$ denotes a zero column vector.*

Theorem 2.7 (cf. [7]). *Let $0 < \lambda < \frac{2}{n}$, then the sequence of repeated associated games $\{\langle N, v_\lambda^{m*Sh}\rangle\}_{m=0}^{\infty}$ converges to the game $\langle N, \tilde{v}\rangle$, where $\tilde{v} = \lim\limits_{m\to\infty}(M_\lambda)^m \cdot v = PDP^{-1} \cdot v$. Furthermore, the limit game $\langle N, \tilde{v}\rangle$ is inessential.*

From Theorem 2.7 together with the relationship between the two types of associated games, we obtain the next theorem.

Theorem 2.8. *Let $0 < \lambda < \frac{2}{n}$, then the sequence of repeated dual similar associated games $\{\langle N, v_\lambda^{m*DSh}\rangle\}_{m=0}^{\infty}$ converges to the game $\langle N, \hat{v}\rangle$, where $\hat{v} = QPDP^{-1}Q \cdot v$. Furthermore, the limit game $\langle N, \hat{v}\rangle$ is inessential.*

Proof. By Lemma 2.4 and the duality property $Q \cdot Q = I$ in Proposition 2.2, we know

$$v_\lambda^{m*DSh} = (M_\lambda^{DSh})^m \cdot v = (QM_\lambda Q)^m \cdot v = Q(M_\lambda)^m Q \cdot v.$$

From the convergence property in Theorem 2.7, it follows immediately that

$$\lim_{m\to\infty} v_\lambda^{m*DSh} = Q \lim_{m\to\infty} (M_\lambda)^m Q \cdot v = QPDP^{-1}Q \cdot v = \hat{v}.$$

By Lemma 2.6, PD is a row-inessential coalitional matrix. Together with Lemma 1.1 and $P^{-1}Q$ is invertible, we derive that $PDP^{-1}Q$ is also a row-inessential coalitional matrix, that is to say $\langle N, PDP^{-1}Q \cdot v\rangle$ is an inessential game. And $\langle N, QPDP^{-1}Q \cdot v\rangle$ is just the dual game of this game. Hence by the self-duality property of the inessential game in Proposition 2.1, the limit game $\langle N, \hat{v}\rangle$ is inessential. This completes the proof. ∎

Remark 1. Notice that the limit game $\langle N, \hat{v}\rangle$ of the sequence of repeated dual similar associated games merely depends on the game $\langle N, v\rangle$ as $\hat{v} = QPDP^{-1}Q \cdot v$. The two limit games $\langle N, \tilde{v}\rangle$ and $\langle N, \hat{v}\rangle$ inherit the commutative relationship between the two types of the associated games in Diagram 1. And for any player $i \in N$, the limit worth $\hat{v}(\{i\})$ is just the inner product of the $i-$th row vector of $QPDP^{-1}Q$ and the column vector v.

3 Dual similar associated consistency and the Shapley value

In this section, by the matrix theory results from the previous sections we show that the Shapley value verifies a new consistency related to the dual similar associated game named dual similar associated consistency. Then we axiomatize the Shapley value by three axioms. Firstly, we present the system of axioms:

1. (*Dual Similar Associated Consistency*). For every game $\langle N, v\rangle$ and its dual similar associated game $\langle N, v_\lambda^{DSh}\rangle$, the solution verifies $\Phi(N, v) = \Phi(N, v_\lambda^{DSh})$.

2. (*Inessential Game Property*). For every inessential game $\langle N, v\rangle$, the solution verifies $\Phi_i(N, v) = v(\{i\})$ for all $i \in N$.

3. (*Continuity*). For every convergent sequence of games $\{\langle N, v_k\rangle\}_{k=0}^{\infty}$ the limit of which is the game $\langle N, \bar{v}\rangle$, the sequence of solutions satisfies convergence too, that is $\lim_{k\to\infty} \Phi(N, v_k)$ $= \Phi(N, \bar{v})$ (The convergence of the sequence of games is point-wise).

In Hamiache's axiomatization for the Shapley value, dual similar associated consistency is replaced by associated consistency, i.e. $\Phi(N, v) = \Phi(N, v_\lambda^{Sh})$.

Lemma 3.1 (cf. [7]). *The Shapley value satisfies the associated consistency, or equivalently,* $M^{Sh} = M^{Sh} M_\lambda$.

The axiom of dual similar associated consistency means that any player receives the same payments in the original game and in the dual similar associated game. In matrix theory, the standard matrix M^{Sh} for the Shapley value is invariant under multiplication with the dual similar associated transformation matrix M_λ^{DSh}.

Lemma 3.2. *The Shapley value satisfies the dual similar associated consistency, that is* $M^{Sh} = M^{Sh} M_\lambda^{DSh}$.

Proof. Since $Sh(N,v) = M^{Sh}v$ and $Sh(N,v_\lambda^{DSh}) = M^{Sh}(M_\lambda^{DSh} \cdot v)$, it is sufficient to check the matrix equality $M^{Sh}M_\lambda^{DSh} = M^{Sh}$. By Lemma 2.4 and Proposition 2.3, we know $M_\lambda^{DSh} = QM_\lambda Q$ and $M^{Sh} = M^{Sh}Q$. Together with Lemma 3.1, it follows

$$M^{Sh}M_\lambda^{DSh} = M^{Sh}QM_\lambda Q = M^{Sh}M_\lambda Q = M^{Sh}Q = M^{Sh}.$$

This completes the proof. ∎

Theorem 3.3. *The Shapley value is the unique solution verifying the inessential game property, continuity and dual similar associated consistency for* $0 < \lambda < \frac{2}{n}$.

Proof. Obviously, the Shapley value satisfies the inessential game and the continuity axioms, and by Lemma 3.2 we know that the Shapley value verifies the dual similar associated consistency.

So, let us now turn to the unicity proof. Consider a value Φ satisfying three listed axioms. Fix the game $\langle N, v\rangle$. We show that $\Phi(N,v) = Sh(N,v)$. By both the dual similar associated consistency and continuity, it holds

$$\Phi(N,v) = \Phi(N,\hat{v}), \quad \text{where } \hat{v} = QPDP^{-1}Q \cdot v \text{ for any game } \langle N, v\rangle.$$

Since the limit game $\langle N, \hat{v}\rangle$ is inessential by Theorem 2.8, the inessential game property for Φ yields $\Phi_i(N,\hat{v}) = \hat{v}(\{i\})$ for all $i \in N$. In summary, $\Phi(N,v) = (\hat{v}(\{i\}))_{i\in N}$.

From the proof of Theorem 3.2 in [7], we have $Sh(N,v) = Sh(N,\tilde{v})$, i.e. $M^{Sh} = M^{Sh}PDP^{-1}$. Together with Proposition 2.3, $M^{Sh} = M^{Sh}Q$. It follows that

$$M^{Sh} = M^{Sh}PDP^{-1} \iff M^{Sh}Q = M^{Sh}PDP^{-1}Q \iff M^{Sh} = M^{Sh}QPDP^{-1}Q$$

That is $Sh(N,v) = Sh(N,\hat{v})$. Since the game $\langle N,\hat{v}\rangle$ is inessential, we conclude that $Sh(N,v) = Sh(N,\hat{v}) = (\hat{v}(\{i\}))_{i\in N}$. Hence $\Phi(N,v) = Sh(N,v)$. ∎

4 Conclusions about matrix analysis

Concerning the matrix approach for the dual similar associated consistency of the Shapley value, the next three tables summarize the relevant matrices and their mutual relationships.

Matrix	Name of matrix	Value/Game	Definition
Q	dual	$v^* = Q \cdot v$	Definition
M^{Sh}	Shapley standard	$Sh(N,v) = M^{Sh}v$	Definition 2
M_λ	associated transformation	$v_\lambda^{Sh} = M_\lambda \cdot v$	Definition 3
M_λ^{DSh}	dual similar associated transformation	$v_\lambda^{DSh} = M_\lambda^{DSh} \cdot v$	Definition 4

Sequence	Limit	Matrix Representation	Property of Game	Statement
$\{\langle N, v_\lambda^{m*Sh}\rangle\}_{m=0}^{\infty}$	$\langle N,\tilde{v}\rangle$	$\tilde{v} = PDP^{-1} \cdot v$	inessential game	Theorem 2.7
$\{\langle N, v_\lambda^{m*DSh}\rangle\}_{m=0}^{\infty}$	$\langle N,\hat{v}\rangle$	$\hat{v} = QPDP^{-1}Q \cdot v$	inessential game	Theorem 2.8

Property	Hamiache	Xu-Driessen	Statement
Duality (D)		$M^{Sh} = M^{Sh}Q$	Proposition 2.3
Similarity (S)		$M^{DSh}_{\lambda} = QM_{\lambda}Q$	Lemma 2.4
		$M_{\lambda} = QM^{DSh}_{\lambda}Q$	Corollary 2.5
associated consistency	$Sh(N, v) = Sh(N, v^{Sh}_{\lambda})$	$M^{Sh} = M^{Sh}M_{\lambda}$	Lemma 3.1
DS-ass. consistency		$M^{Sh} = M^{Sh}M^{DSh}_{\lambda}$	Lemma 3.2

According to the proof of Lemma 3.2, the dual similar associated consistency of the Shapley value has been derived from Hamiache's associated consistency. We conclude the paper with the proof of the converse statement. From the duality and similarity in the third table, we obtain that our dual similar associated consistency $M^{Sh} = M^{Sh}M^{DSh}_{\lambda}$ yields Hamiache's associated consistency as:

$$M^{Sh}M_{\lambda} = M^{Sh}QM^{DSh}_{\lambda}Q = M^{Sh}M^{DSh}_{\lambda}Q = M^{Sh}Q = M^{Sh}.$$

So, the two types of associated consistency are equivalent.

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Appendix: The Alternative Proof of Lemma 2.4.

For any coalition $S \subseteq N$, by the definitions of the dual game and the associated game, we have

$$\begin{aligned}
&\Big((v^*)_\lambda^{Sh}\Big)^*(S) = (v^*)_\lambda^{Sh}(N) - (v^*)_\lambda^{Sh}(N \setminus S) \\
&= v^*(N) - \Big\{\big[1-(n-(n-s))\lambda\big]v^*(N \setminus S) + \lambda \sum_{j \in S} v^*((N \setminus S) \cup \{j\}) - \lambda \sum_{j \in S} v^*(\{j\})\Big\} \\
&= v(N) - (1-s\lambda)\big(v(N) - v(S)\big) - \lambda \sum_{j \in S} \big[v(N) - v(S \setminus \{j\})\big] + \lambda \sum_{j \in S} \big[v(N) - v(N \setminus \{j\})\big] \\
&= \big(1-s\lambda\big)v(S) + s\lambda v(N) - \lambda \sum_{j \in S} v(N \setminus \{j\}) + \lambda \sum_{j \in S} v(S \setminus \{j\}) \\
&= v_\lambda^{DSh}(S).
\end{aligned}$$

This completes the proof. ∎