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Memorandum No. 1796

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in cooperative game theory**

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April, 2006

ISSN 0169-2690

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Matrix analysis for associated consistency in cooperative game theory*

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Abstract

Hamiache's recent axiomatization of the well-known Shapley value for TU games states that the Shapley value is the unique solution verifying the following three axioms: the inessential game property, continuity and associated consistency. Driessen extended Hamiache's axiomatization to the enlarged class of efficient, symmetric, and linear values, of which the Shapley value is the most important representative.

In this paper, we introduce the notion of row (resp. column)-coalitional matrix in the framework of cooperative game theory. Particularly, both the Shapley value and the associated game are represented algebraically by their coalitional matrices called the Shapley standard matrix M^{Sh} and the associated transformation matrix M_λ , respectively. We develop a matrix approach for Hamiache's axiomatization of the Shapley value. The associated consistency for the Shapley value is formulated as the matrix equality $M^{Sh} = M^{Sh} \cdot M_\lambda$. The diagonalization procedure of M_λ and the inessential property for coalitional matrices are fundamental tools to prove the convergence of the sequence of repeated associated games as well as its limit game to be inessential. In addition, a similar matrix approach is applicable to study Driessen's axiomatization of a certain class of linear values. Matrix analysis is adopted throughout both the mathematical developments and the proofs. In summary, it is illustrated that matrix analysis is a new and powerful technique for research in the field of cooperative game theory.

Key Words: coalitional matrix, Shapley value, Shapley standard matrix, associated transformation matrix, associated consistency.

2000 Mathematics Subject Classifications: Primary 91A12, Secondary 15A18

1 Introduction

A *cooperative game* with transferable utility (TU) is a pair $\langle N, v \rangle$, where N is a nonempty, finite set and $v : 2^N \rightarrow \mathbb{R}$ is a *characteristic function*, defined on the power set of N , satisfying

*The research for this paper was done during a four weeks stay (October 2, 2004 till October 29, 2004) of the first author at the EEMCS, University of Twente, Enschede, The Netherlands.

$v(\emptyset) = 0$. An element of N (notation: $i \in N$) and a subset S of N (notation: $S \subseteq N$ or $S \in 2^N$ with $S \neq \emptyset$) are called a *player* and *coalition* respectively, and the associated real number $v(S)$ is called the *worth* of coalition S . The size of coalition S is denoted by s . Particularly, n denotes the size of the player set N . We denote by \mathcal{G} the universal game space consisting of all these TU-games. In this paper, a TU-game $\langle N, v \rangle$ is always denoted by its column vector of worths of all coalitions $S \subseteq N$ in the traditional order (one-person coalitions are at the top, etc.), i.e. $\vec{v} = (v(S))_{S \subseteq N, S \neq \emptyset}$. If no confusion arises, we write v instead of \vec{v} . We only consider games with at least two players. A game $\langle N, v \rangle$ is said to be *inessential* if for all coalitions $S \subseteq N$, $v(S) = \sum_{i \in S} v(\{i\})$.

The solution part of cooperative game theory deals with the allocation problem of how to divide the overall earnings the amount of $v(N)$ among the players in the TU-game. There is associated a single allocation called the value of the TU-game. Formally, a *value* on \mathcal{G} is a function Φ that assigns a single payoff vector $\Phi(N, v) = (\Phi_i(N, v))_{i \in N} \in \mathbb{R}^n$ to every TU-game $\langle N, v \rangle \in \mathcal{G}$. The so-called value $\Phi_i(N, v)$ of player i in the game $\langle N, v \rangle$ represents an assessment by i of his gains for participating in the game.

Among all the values for TU-games, the Shapley value is the best known ([1, 6, 8]). The Shapley value is also a striking example of the power of the axiomatic approach. The eldest axiomatization of the Shapley value is stated by Shapley himself ([8]) by referring to four properties called efficiency, symmetry, linearity, and dummy player property. In the framework of values for TU-games, firstly let us review several essential properties treated in former axiomatizations of the Shapley value. A value Φ on the universal game space \mathcal{G} is said to be *efficient*, if $\sum_{i \in N} \Phi_i(N, v) = v(N)$ for all games $\langle N, v \rangle$; *symmetric*, if $\Phi_{\pi(i)}(N, \pi v) = \Phi_i(N, v)$ for all games $\langle N, v \rangle$, all $i \in N$, and every permutation π on N ; *linear*, if $\Phi(N, \alpha \cdot v + \beta \cdot w) = \alpha \cdot \Phi(N, v) + \beta \cdot \Phi(N, w)$ for all games $\langle N, v \rangle$, $\langle N, w \rangle$, and all $\alpha, \beta \in \mathbb{R}$; *inessential*, if $\Phi_i(N, v) = v(\{i\})$ for all inessential games $\langle N, v \rangle$, all $i \in N$; *continuous*, if for every (pointwise) convergent sequence of games $\{\langle N, v_k \rangle\}_{k=0}^{\infty}$, say the limit of which is the game $\langle N, \bar{v} \rangle$, the corresponding sequence of values $\{\Phi(N, v_k)\}_{k=0}^{\infty}$ converges to the value $\Phi(N, \bar{v})$.

Hamiache's recent axiomatization of the Shapley value states that the Shapley value is the unique one-point solution verifying the inessential game property, continuity and associated consistency (see [3]). In his paper, an associated game $\langle N, v_{\lambda}^{Sh} \rangle$ is constructed. And a sequence of games is also defined, where the term of order m , in this sequence, is the associated game of the term of order $m - 1$. He showed that this sequence of games converges and that the limit game is inessential. The value is obtained using the inessential game property, the associated consistency and the continuity axioms. As a by-product, neither the linearity nor the efficiency axioms are needed. In [2], Driessen extended Hamiache's axiomatization to the enlarged class of efficient, symmetric, and linear values, of which the Shapley value is the most important representative. For this enlarged class of values, explicit relationships to the Shapley value are exploited in order to present a uniform approach to obtain axiomatizations of such values with reference to a slightly adapted inessential game property, continuity, and a similar associated consistency. The uniqueness proofs in Hamiache's axiomatization and Driessen's axiomatic characterization are rather tough and full of combinatorial calculations.

In cooperative game theory, linear transformations of games are widely used, for instance the dual of a game. Another well-known example is that any cooperative game can be represented as a linear combination of the unanimity games. On the other hand, there are many linear values such as the Shapley value that can be represented as a linear combination of all the worths $v(S)$, $S \subseteq N$. So algebraic representations and matrix analysis should be a justifiable technique in cooperative game theory. This motivates our present work.

In this paper, the matrix approach is adopted to develop Hamiache's axiomatization of Shapley value and Driessen's extended work. In Section 2, we introduce the notion of row (resp. column)-coalitional matrix in the framework of cooperative game theory. Particularly, both the Shapley value and the associated game are represented algebraically by their coalitional matrices called the Shapley standard matrix M^{Sh} and the associated transformation matrix M_λ , respectively. The diagonalization procedure of M_λ and the inessential property for coalitional matrices are fundamental tools to prove the convergence of the sequence of repeated associated games as well as its limit game to be inessential. In Section 3, the associated consistency for the Shapley value is formulated as the matrix equality $M^{Sh} = M^{Sh} \cdot M_\lambda$. We achieve a matrix approach for Hamiache's axiomatization of the Shapley value. In Section 4, a similar matrix approach is applicable to study Driessen's axiomatization of a certain class of linear values. To conclude with, matrix analysis is a new and powerful technique for research in the field of cooperative game theory.

2 The Shapley standard matrix and the associated transformation matrix

Firstly, let us define a new type of matrix to apply matrix theory to cooperative game theory.

Definition 1. A matrix M is called a row (resp. column)-coalitional matrix if its rows (resp. columns) are indexed by coalitions $S \subseteq N$ in the traditional order (one-person coalitions are at the top, etc.). And a row-coalitional matrix $M = [\vec{m}_S]_{S \subseteq N, S \neq \emptyset}$ is row-inessential if the row-vector of M indexed by coalition S verifies $\vec{m}_S = \sum_{i \in S} \vec{m}_i$ for all $S \subseteq N$.

Without going into details, we recall the well-known *Shapley value* $Sh(N, v)$ as follows:

$$Sh_i(N, v) = \sum_{S \subseteq N, S \ni i} \frac{(s-1)!(n-s)!}{n!} [v(S) - v(S \setminus \{i\})] \quad \text{for all } i \in N.$$

Because of its linearity property, the Shapley value can be represented by the Shapley standard matrix as follows.

Definition 2. Given any game $\langle N, v \rangle$, the Shapley value $Sh(N, v)$ can be represented by the Shapley standard matrix M^{Sh} as:

$$Sh(N, v) = M^{Sh} v,$$

where the matrix $M^{Sh} = [M^{Sh}]_{i \in N, S \subseteq N, S \neq \emptyset}$ is column-coalitional defined by

$$[M^{Sh}]_{i,S} = \begin{cases} \frac{(s-1)!(n-s)!}{n!}, & \text{if } i \in S; \\ -\frac{s!(n-s-1)!}{n!}, & \text{if } i \notin S. \end{cases}$$

Now let us recite the definition of the associated game in [3]. Given any game $\langle N, v \rangle$ and $\lambda \in \mathbb{R}$, define its *associated game* $\langle N, v_\lambda^{Sh} \rangle$ as follows:

$$v_\lambda^{Sh}(S) := v(S) + \lambda \sum_{j \in N \setminus S} [v(S \cup \{j\}) - v(S) - v(\{j\})] \quad \text{for all } S \subseteq N$$

Notice that $v_\lambda^{Sh}(\emptyset) = 0$ and moreover, $v_\lambda^{Sh} = v$ for all inessential games $\langle N, v \rangle$. We do not care about the trivial case $\lambda = 0$. The worth $v_\lambda^{Sh}(S)$ of coalition S in the associated game differs from the initial worth $v(S)$ by taking into account the possible (weighted) net benefits $v(S \cup \{j\}) - v(S) - v(\{j\})$ arising from mutual cooperation among the coalition S itself and any of each isolated non-members $j \in N \setminus S$. Obviously, the worth $v_\lambda^{Sh}(S)$ of coalition S can be expressed as

$$v_\lambda^{Sh}(S) = [1 - (n - s)\lambda]v(S) + \lambda \sum_{j \in N \setminus S} v(S \cup \{j\}) - \lambda \sum_{j \in N \setminus S} v(\{j\}).$$

In order to apply matrix theory, we introduce the associated transformation matrix to represent the associated game and the sequence of repeated associated games as follows.

Definition 3. *Given any game $\langle N, v \rangle$ and $\lambda \in \mathbb{R}$, the associated game $\langle N, v_\lambda^{Sh} \rangle$ can be represented by the associated transformation matrix M_λ as:*

$$v_\lambda^{Sh} = M_\lambda \cdot v,$$

where the matrix $M_\lambda = [M_\lambda]_{S, T \subseteq N, S, T \neq \emptyset}$ is both row-coalitional and column-coalitional defined by

$$[M_\lambda]_{S, T} = \begin{cases} 1 - (n - s)\lambda, & \text{if } T = S; \\ \lambda, & \text{if } T = S \cup \{j\} \text{ and } j \in N \setminus S; \\ -\lambda, & \text{if } T = \{j\} \text{ and } j \in N \setminus S; \\ 0, & \text{otherwise.} \end{cases}$$

And its sequence of repeated associated games $\{\langle N, v_\lambda^{m*Sh} \rangle\}_{m=0}^\infty$ is defined as:

$$v_\lambda^{m*Sh} = M_\lambda \cdot v_\lambda^{(m-1)*Sh} \text{ for all } m \geq 1, \text{ where } v_\lambda^{0*Sh} = v.$$

Now the main goal is to investigate eigenvalues and eigenvectors of the associated transformation matrix M_λ . Let I be the identity matrix.

Proposition 2.1. *1 is an eigenvalue of M_λ , and eigenvectors corresponding to eigenvalue 1 are row-inessential.*

Proof. Since $v_\lambda^{Sh}(N) = v(N)$, the last row of matrix $I - M_\lambda$ is the zero-vector. So 1 is an eigenvalue of M_λ . Let $\vec{x} = (x_S)_{S \subseteq N, S \neq \emptyset}$ be an eigenvector corresponding to eigenvalue 1. Interpret \vec{x} as a row-coalitional matrix. Since $(I - M_\lambda)\vec{x} = \vec{0}$, we have

$$(n - s)x_S - \sum_{j \in N \setminus S} x_{S \cup \{j\}} + \sum_{j \in N \setminus S} x_j = 0 \text{ for all } S \subseteq N, S \neq \emptyset.$$

By this equation, for any $N \setminus S = \{i\}$, we have $x_{N \setminus \{i\}} + x_i = x_N$. Then if $N \setminus S = \{i, j\}$, it should be

$$2x_{N \setminus \{i, j\}} + x_i + x_j = x_{N \setminus \{i\}} + x_{N \setminus \{j\}} = x_N - x_i + x_N - x_j.$$

Thus

$$x_{N \setminus \{i, j\}} + x_i + x_j = x_N \text{ for all } i, j \in N, i \neq j.$$

So, we obtain

$$x_S + \sum_{j \in N \setminus S} x_j = x_N \text{ for all } S \subseteq N, S \neq \emptyset.$$

Applying the latter equality to one-person coalitions, it holds $x_N = \sum_{j \in N} x_j$. So we conclude

$$x_S = \sum_{j \in S} x_j \text{ for all } S \subseteq N, S \neq \emptyset. \quad \blacksquare$$

From the inessential property of any eigenvector \vec{x} corresponding to eigenvalue 1, it follows immediately that the dimension of the eigenspace of eigenvalue 1 is equal to n .

Proposition 2.2. *For every k ($2 \leq k \leq n$), we have $\text{rank}[(1 - k\lambda)I - M_\lambda] \leq 2^n - 1 - \binom{n}{k}$, and hence $1 - k\lambda$ is an eigenvalue of M_λ .*

Proof. For any k ($2 \leq k \leq n$), let vector $\vec{x} = (x_S)_{S \subseteq N, S \neq \emptyset}$ be such that $[(1 - k\lambda)I - M_\lambda]\vec{x} = \vec{0}$. Then the following system of linear equations holds,

$$(n - s - k)x_S - \sum_{j \in N \setminus S} x_{S \cup \{j\}} + \sum_{j \in N \setminus S} x_j = 0 \text{ for all } S \subseteq N, S \neq \emptyset. \quad (1)$$

For the case of $S = N$, since $k \cdot x_N = 0$ and $k \neq 0$, we have $x_N = 0$. In the sequel, we show that for any k , there are $\binom{n}{k}$ identical equations in the linear system of $[(1 - k\lambda)I - M_\lambda]\vec{x} = \vec{0}$.

If $s = n - 1$ and $S = N \setminus \{j\}$, by (1) we have

$$(1 - k)x_{N \setminus \{j\}} - x_N + x_j = 0.$$

That is

$$x_{N \setminus \{j\}} = \frac{1}{k - 1} x_j \text{ for all } j \in N. \quad (2)$$

Considering $s = n - 2$ and $S = N \setminus \{i, j\}$, by (1) and (2), we conclude

$$\begin{aligned} (2 - k)x_{N \setminus \{i, j\}} - x_{N \setminus \{i\}} - x_{N \setminus \{j\}} + x_i + x_j &= 0 \\ (2 - k)x_{N \setminus \{i, j\}} &= \frac{2 - k}{k - 1}(x_i + x_j). \end{aligned}$$

If $k = 2$, these linear equations are identical equations for all coalitions S with $s = n - 2$, total $\binom{n}{2}$ equations in $[(1 - k\lambda)I - M_\lambda]\vec{x} = \vec{0}$. Otherwise, it should be

$$x_{N \setminus \{i, j\}} = \frac{1}{k - 1}(x_i + x_j). \quad (3)$$

In view of (2) and (3), for a given k , we use induction on $n - s$ to show that

$$x_S = \frac{1}{k - 1} \sum_{j \in N \setminus S} x_j \text{ for all } S \subseteq N, S \neq N, S \neq \emptyset. \quad (4)$$

Now suppose (4) is true for all $n - s \leq t - 1$, where $t \leq k$. For the case of $n - s = t$, let $S = N \setminus T$. By (1), we have

$$(t - k)x_{N \setminus T} - \sum_{i \in T} x_{(N \setminus T) \cup \{i\}} = - \sum_{j \in T} x_j.$$

By the inductive assumption and $i \in T$, we obtain

$$x_{(N \setminus T) \cup \{i\}} = \frac{1}{k - 1} \left(\sum_{j \in T} x_j - x_i \right).$$

Thus

$$(t-k)x_{N \setminus T} - \frac{t}{k-1} \sum_{j \in T} x_j + \frac{1}{k-1} \sum_{i \in T} x_i = - \sum_{j \in T} x_j.$$

$$(t-k)x_{N \setminus T} = \frac{t-k}{k-1} \sum_{j \in T} x_j. \quad (5)$$

So if $t \neq k$, then (5) implies that (4) holds for $s = n - t$.

Furthermore, by (5), if $t = k$, then $\binom{n}{k}$ linear equations in $[(1-k\lambda)I - M_\lambda]\vec{x} = \vec{0}$ are identical equations. Hence, $\text{rank}[(1-k\lambda)I - M_\lambda] \leq 2^n - 1 - \binom{n}{k}$. Consequently, $1 - k\lambda$ is an eigenvalue of M_λ for $2 \leq k \leq n$. ■

Here we recall some results in algebraic theory for getting more properties of the associated transformation matrix M_λ .

Lemma 2.3 (Algebraic results, cf. [4]). *Let A be a square matrix of order p .*

1. *The dimension d of the solution space of the linear system of equations $A\vec{x} = \vec{0}$ satisfies $d = p - \text{rank}(A)$.*
2. *For every eigenvalue of matrix A , its (algebraic) multiplicity is at least the dimension of the corresponding eigenspace.*
3. *The sum of the multiplicities of all eigenvalues of matrix A equals the order p .*
4. *The matrix A is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals p , and this happens if and only if the dimension of the eigenspace for each eigenvalue equals the multiplicity of the eigenvalue.*

Theorem 2.4. *Eigenvalues of the associated transformation matrix M_λ are $1, 1 - k\lambda$ ($k = 2, 3, \dots, n$), and multiplicities corresponding to these eigenvalues are $\binom{n}{1}, \binom{n}{k}$ ($k = 2, 3, \dots, n$).*

Proof. Let $u_1 = 1$ and $u_k = 1 - k\lambda$ ($k = 2, 3, \dots, n$). By Proposition 2.1 and 2.2, we know that u_k ($k = 1, 2, \dots, n$) are eigenvalues of M_λ . Let d_k denote the dimension of the eigenspace corresponding to $(u_k I - M_\lambda)\vec{x} = \vec{0}$. By Proposition 2.1, we obtain $d_1 = n$, whereas from Proposition 2.2 and Lemma 2.3 (1), we derive

$$d_k = 2^n - 1 - \text{rank}(u_k I - M_\lambda) \geq \binom{n}{k} \quad (k = 2, 3, \dots, n).$$

Since the multiplicity m_k of eigenvalue u_k satisfies $m_k \geq d_k$, we have

$$2^n - 1 = \sum_{k=1}^n m_k \geq n + \sum_{k=2}^n d_k \geq \binom{n}{1} + \sum_{k=2}^n \binom{n}{k} = 2^n - 1.$$

Thus $m_k = d_k = \binom{n}{k}$ for all $1 \leq k \leq n$ and so the matrix M_λ has no other eigenvalues. ■

From Theorem 2.4, we conclude that the matrix M_λ is diagonalizable. In order to prove the next theorem, we make use of the following properties of row-coalitional matrices.

Lemma 2.5. *Let M be a row-coalitional matrix and A be a matrix.*

1. If M is row-inessential, then the row-coalitional matrix MA is row-inessential.
2. If A is invertible, then MA is row-inessential if and only if M is row-inessential.
3. For every game $\langle N, v \rangle \in \mathcal{G}$, if M is row-inessential, then the new game $\langle N, M \cdot v \rangle$ is inessential.

Proof. Write $M = [\vec{m}_S]_{S \subseteq N, S \neq \emptyset}$, where \vec{m}_S is the row vector of M indexed by a coalition S .

1. Since M is row-inessential, $\vec{m}_S = \sum_{i \in S} \vec{m}_i$, so $[\vec{MA}]_S = \vec{m}_S A = (\sum_{i \in S} \vec{m}_i) A = \sum_{i \in S} (\vec{m}_i A) = \sum_{i \in S} ([MA]_i)$ for any $S \subseteq N, S \neq \emptyset$. Thus MA is row-inessential.
2. Following conclusion 1, if A is invertible, then $\vec{m}_S A = \sum_{i \in S} (\vec{m}_i A) = (\sum_{i \in S} \vec{m}_i) A$ if and only if $\vec{m}_S = \sum_{i \in S} \vec{m}_i$ for any $S \subseteq N$. That is to say, M is row-inessential.
3. If M is row-inessential, then $\vec{m}_S \cdot v = (\sum_{i \in S} \vec{m}_i) \cdot v = \sum_{i \in S} (\vec{m}_i \cdot v)$ for every game $\langle N, v \rangle \in \mathcal{G}$. Thus, $M \cdot v = [\vec{m}_S \cdot v]_{S \subseteq N}$ is inessential. ■

Now we present the following important properties of the associated transformation matrix M_λ by its diagonalization procedure and Proposition 2.1.

Lemma 2.6. Let M_λ be the associated transformation matrix.

1. $M_\lambda = PD_\lambda P^{-1}$, where $D_\lambda = \text{diag}(\underbrace{1, \dots, 1}_{\binom{n}{1} \text{ times}}, \underbrace{1 - 2\lambda, \dots, 1 - 2\lambda}_{\binom{n}{2} \text{ times}}, \dots, \underbrace{1 - n\lambda}_{\binom{n}{n} \text{ times}})$ and P consists of eigenvectors of M_λ corresponding to eigenvalues $1, 1 - k\lambda$ ($2 \leq k \leq n$).
2. If $0 < \lambda < \frac{2}{n}$, then $\lim_{m \rightarrow \infty} (M_\lambda)^m = PDP^{-1}$, where $D = \text{diag}(\underbrace{1, \dots, 1}_n, \underbrace{0, \dots, 0}_{2^n - 1 - n})$.
3. The row-coalitional matrix PD equals $PD = [\vec{x}^1, \vec{x}^2, \dots, \vec{x}^n, \vec{0}, \dots, \vec{0}]$ and PD is row-inessential, where column vectors \vec{x}^i ($i = 1, 2, \dots, n$) are different eigenvectors of M_λ corresponding to eigenvalue 1 and $\vec{0}$ denotes a zero column vector.

Using the previous results, we derive the next theorem about the convergence of the sequence of repeated associated games.

Theorem 2.7. Let $0 < \lambda < \frac{2}{n}$. The sequence of repeated associated games $\{\langle N, v_\lambda^{m*Sh} \rangle\}_{m=0}^\infty$ converges to the game $\langle N, \tilde{v} \rangle$, where $\tilde{v} = PDP^{-1} \cdot v$. Furthermore, the limit game $\langle N, \tilde{v} \rangle$ is inessential.

Proof. By the second conclusion of Lemma 2.6, $\lim_{m \rightarrow \infty} v_\lambda^{m*Sh} = \lim_{m \rightarrow \infty} (M_\lambda)^m \cdot v = PDP^{-1} \cdot v$. Due to Lemma 2.5 (3) and $\tilde{v} = PDP^{-1} \cdot v$, the game $\langle N, \tilde{v} \rangle$ is inessential whenever the matrix PDP^{-1} is row-inessential. By Lemma 2.6 (3), the matrix PD is row-inessential. Together with Lemma 2.5 (2) it follows that the matrix PDP^{-1} is row-inessential too. This completes the proof. ■

Remark 1. Notice that the limit game $\langle N, \tilde{v} \rangle$ of the sequence of repeated associated games merely depends on the game $\langle N, v \rangle$ as $\tilde{v} = PDP^{-1}v$. And for any player $i \in N$, the limit worth $\tilde{v}(\{i\})$ is just the inner product of the i -th row vector of PDP^{-1} and the column vector v .

3 Associated consistency and the Shapley value

In this section we apply the results from the previous section to develop a matrix approach for Hamiache's axiomatization of the Shapley value (see [3]). Firstly, we recall Hamiache's system of axioms:

1. (*Associated Consistency*). For every game $\langle N, v \rangle$ and its associated game $\langle N, v_\lambda^{Sh} \rangle$, the value verifies $\Phi(N, v) = \Phi(N, v_\lambda^{Sh})$.
2. (*Inessential Game Property*). For every inessential game $\langle N, v \rangle$, the value verifies $\Phi_i(N, v) = v(\{i\})$ for all $i \in N$.
3. (*Continuity*). For every convergent sequence of games $\{\langle N, v_k \rangle\}_{k=0}^\infty$ the limit of which is the game $\langle N, \bar{v} \rangle$, the sequence of values satisfies convergence too, that is $\lim_{k \rightarrow \infty} \Phi(N, v_k) = \Phi(N, \bar{v})$ (The convergence of the sequence of games is point-wise).

Here the associated consistency means that any player receives the same payments in the original game and in the associated game. In matrix theory, as the following lemma cites, the Shapley standard matrix M^{Sh} is invariant under multiplication with the associated transformation matrix M_λ .

Lemma 3.1. *The Shapley value verifies the associated consistency, that is $M^{Sh} = M^{Sh} M_\lambda$.*

Sketch of the Proof. Since $Sh(N, v) = M^{Sh}v$ and $Sh(N, v_\lambda^{Sh}) = M^{Sh}(M_\lambda \cdot v)$, it is sufficient to check the matrix equality $M^{Sh}M_\lambda = M^{Sh}$, or equivalently, $M^{Sh}(M_\lambda - I) = \mathbf{0}$, for showing that the Shapley value satisfies the associated consistency. By the definition of M^{Sh} , the entry equality $\left[M^{Sh}(M_\lambda - I) \right]_{i,T} = 0$ for all $i \in N$ and for all $T \subseteq N$, $T \neq \emptyset$, is as follows:

$$\sum_{S \subseteq N, i \in S} \frac{(s-1)!(n-s)!}{n!} [M_\lambda - I]_{S,T} - \sum_{S \subseteq N, i \notin S} \frac{s!(n-s-1)!}{n!} [M_\lambda - I]_{S,T} = 0.$$

Its proof is listed in the appendix, as well as the algebraic interpretation for the associated consistency. ■

Theorem 3.2 (cf. [3]). *For $0 < \lambda < \frac{2}{n}$, the Shapley value is the unique value verifying the associated consistency, inessential game property, and continuity.*

Proof by Matrix Approach. Obviously, the Shapley value satisfies the inessential game and the continuity axioms, and by Lemma 3.1 the Shapley value verifies the associated consistency.

So, let us now turn to the unicity proof. Consider a value Φ satisfying these three axioms. Fix the game $\langle N, v \rangle$. We show that $\Phi(N, v) = Sh(N, v)$. By both the associated consistency and continuity for Φ , it holds

$$\Phi(N, v) = \Phi(N, \tilde{v}), \quad \text{where } \tilde{v} = PDP^{-1} \cdot v.$$

Since the limit game $\langle N, \tilde{v} \rangle$ is shown to be inessential in Theorem 2.7, the inessential game property for Φ yields $\Phi_i(N, \tilde{v}) = \tilde{v}(\{i\})$ for all $i \in N$. In summary, $\Phi(N, v) = (\tilde{v}(\{i\}))_{i \in N}$. Similarly, since the Shapley value also verifies these three axioms, it follows that

$$Sh(N, v) = Sh(N, \tilde{v}) = (\tilde{v}(\{i\}))_{i \in N}.$$

From this, we conclude $\Phi(N, v) = Sh(N, v)$. This completes the proof. \blacksquare

Remark 2. Since $Sh(N, v) = M^{Sh}v$ and $\tilde{v} = PDP^{-1} \cdot v$, we deduce from $Sh(N, v) = (\tilde{v}(\{i\}))_{i \in N}$ that the Shapley standard matrix M^{Sh} is just the first part of the row-coalitional matrix PDP^{-1} indexed by one-person coalitions. In fact, PDP^{-1} is the extension of M^{Sh} by the row inessential property.

4 The \mathcal{B} –associated transformation matrix and \mathcal{B} –associated consistency

In [2], Driessen extended Hamiache’s axiomatization to the enlarged class of efficient, symmetric, and linear values, of which the Shapley value is the most important representative. The family of least square values ([7]) as well as the solidarity value ([5]) are members of this class. For this enlarged class of values, explicit relationships to the Shapley value are exploited in order to present a uniform approach to obtain axiomatizations of such values with reference to a slightly adapted inessential game property, continuity, and a similar associated consistency. Following the former matrix analysis on Hamiache’s axiomatization of the Shapley value, a similar algebraic approach is applicable to study Driessen’s work.

Throughout this section, denote by $\mathcal{B} = \{b_s^n \mid n \in \mathbb{N} \setminus \{0, 1\}, s = 1, 2, \dots, n\}$ a collection of positive scaling constants, whereas $b_n^n := 1$. Given any game $\langle N, v \rangle$ and $\lambda \in \mathbb{R}$, Driessen defined its \mathcal{B} –associated game $\langle N, v_\lambda^\mathcal{B} \rangle$ in [2] as follows: $v_\lambda^\mathcal{B}(\emptyset) = 0$ and for all $S \subseteq N$, $S \neq \emptyset$,

$$v_\lambda^\mathcal{B}(S) = v(S) + \lambda \cdot \sum_{j \in N \setminus S} \left[\frac{b_{s+1}^n}{b_s^n} \cdot v(S \cup \{j\}) - v(S) - \frac{b_1^n}{b_s^n} \cdot v(\{j\}) \right].$$

That is,

$$v_\lambda^\mathcal{B}(S) = [1 - (n - s)\lambda] \cdot v(S) + \frac{\lambda \cdot b_{s+1}^n}{b_s^n} \sum_{j \in N \setminus S} v(S \cup \{j\}) - \frac{\lambda \cdot b_1^n}{b_s^n} \sum_{j \in N \setminus S} v(\{j\}).$$

Analogical to the matrix approach for the associated game, we restate the \mathcal{B} –associated game as follows.

Definition 4. Given any game $\langle N, v \rangle$ and $\lambda \in \mathbb{R}$, the \mathcal{B} –associated game $\langle N, v_\lambda^\mathcal{B} \rangle$ of $\langle N, v \rangle$ can be represented by the \mathcal{B} –associated transformation matrix $M_\lambda^\mathcal{B}$ as:

$$v_\lambda^\mathcal{B} = M_\lambda^\mathcal{B} \cdot v,$$

where the matrix $M_\lambda^\mathcal{B}$ is both row-coalitional and column-coalitional defined by $[M_\lambda^\mathcal{B}]_{S,T} = \frac{b_t^n}{b_s^n} [M_\lambda]_{S,T}$ for all $S, T \subseteq N$ and $S, T \neq \emptyset$.

And its sequence of repeated \mathcal{B} –associated games $\{\langle N, v_\lambda^{m*\mathcal{B}} \rangle\}_{m=0}^\infty$ is defined as:

$$v_\lambda^{m*\mathcal{B}} = M_\lambda^\mathcal{B} \cdot v_\lambda^{(m-1)*\mathcal{B}} \quad \text{for all } m \geq 1, \quad \text{where } v_\lambda^{0*\mathcal{B}} = v.$$

We write M_λ instead of $M_\lambda^\mathcal{B}$ if it concerns the unit constants $b_s^n = 1$ for all $1 \leq s \leq n$, and then $\langle N, v_\lambda^\mathcal{B} \rangle$ is the associated game $\langle N, v_\lambda^{Sh} \rangle$ of $\langle N, v \rangle$. Next we show that the \mathcal{B} –associated transformation matrix $M_\lambda^\mathcal{B}$ inherits certain properties from the associated transformation matrix M_λ .

Proposition 4.1. *Let M_λ and $M_\lambda^{\mathcal{B}}$ be the associated transformation matrix and \mathcal{B} -associated transformation matrix, respectively.*

1. $M_\lambda^{\mathcal{B}} = B^{-1}M_\lambda B$, where $B = \text{diag}(b_{|S|}^n)_{S \subseteq N, S \neq \emptyset}$.
2. $M_\lambda^{\mathcal{B}}$ and M_λ have the same eigenvalues and the same (algebraic) multiplicities of eigenvalues. And \vec{y} is an eigenvector of $M_\lambda^{\mathcal{B}}$ if and only if $B\vec{y}$ is an eigenvector of M_λ .
3. If $0 < \lambda < \frac{2}{n}$, then $\lim_{m \rightarrow \infty} (M_\lambda^{\mathcal{B}})^m = B^{-1} \lim_{m \rightarrow \infty} (M_\lambda)^m B = B^{-1} P D P^{-1} B$.

Proof.

1. Since $B = \text{diag}(b_{|S|}^n)_{S \subseteq N, S \neq \emptyset}$ is diagonal and b_s^n are positive for all $1 \leq s \leq n$, its inverse matrix $B^{-1} = \text{diag}(\frac{1}{b_{|S|}^n})_{S \subseteq N, S \neq \emptyset}$ is also diagonal. For any coalition S and T , we have

$$\begin{aligned} [B^{-1}M_\lambda B]_{S,T} &= \sum_{R \subseteq N, R \neq \emptyset} [B^{-1}]_{S,R} [M_\lambda B]_{R,T} = [B^{-1}]_{S,S} [M_\lambda B]_{S,T} \\ &= \frac{1}{b_s^n} \sum_{R \subseteq N, R \neq \emptyset} [M_\lambda]_{S,R} [B]_{R,T} = \frac{1}{b_s^n} [M_\lambda]_{S,T} [B]_{T,T} = \frac{b_t^n}{b_s^n} [M_\lambda]_{S,T} \\ &= [M_\lambda^{\mathcal{B}}]_{S,T}. \end{aligned}$$

Thus, the similarity property $M_\lambda^{\mathcal{B}} = B^{-1}M_\lambda B$ holds.

2. From the similarity property of conclusion 1, it is known that $M_\lambda^{\mathcal{B}}$ and M_λ have the same eigenvalues and the same multiplicities of eigenvalues. Let \vec{y} be an eigenvector of $M_\lambda^{\mathcal{B}}$ corresponding to eigenvalue μ . Then

$$M_\lambda^{\mathcal{B}} \vec{y} = \mu \vec{y} \iff (B^{-1}M_\lambda B) \vec{y} = \mu \vec{y} \iff M_\lambda (B\vec{y}) = B(\mu \vec{y}) = \mu (B\vec{y}).$$

Clearly, μ is an eigenvalue of M_λ and $B\vec{y}$ is an eigenvector corresponding to μ .

3. It is derived immediately from conclusion 1 and Lemma 2.6 (2). ■

For any game $\langle N, v \rangle$, Driessen ([2]) defined its \mathcal{B} -scaled game $\langle N, \mathcal{B}v \rangle$ by $(\mathcal{B}v)(\emptyset) := 0$ and $(\mathcal{B}v)(S) := b_s^n \cdot v(S)$ for all $S \subseteq N$, $S \neq \emptyset$. In terms of the \mathcal{B} -scaling diagonal matrix B we can rewrite the \mathcal{B} -scaled version of the game $\langle N, v \rangle$ as

$$\mathcal{B}v = B \cdot v \quad \text{where } B = \text{diag}(b_{|S|}^n)_{S \subseteq N, S \neq \emptyset}.$$

The explicit relationship between the Shapley value and any efficient, symmetric, and linear value is listed in the following theorem and the algebraic formulation in the subsequent corollary.

Theorem 4.2 (cf. [2]). *A value ψ on the game space \mathcal{G} verifies efficiency, symmetry, and linearity if and only if there exists a collection of constants \mathcal{B} such that, for every game $\langle N, v \rangle \in \mathcal{G}$, the value $\psi(N, v) = Sh(N, \mathcal{B}v)$.*

Corollary 4.3. *A value ψ on the game space \mathcal{G} verifies efficiency, symmetry, and linearity if and only if there exists a \mathcal{B} -scaling diagonal matrix $B = \text{diag}(b_{|S|}^n)_{S \subseteq N, S \neq \emptyset}$ such that, for any game $\langle N, v \rangle \in \mathcal{G}$, the value $\psi(N, v) = M^{Sh} Bv$.*

A value Φ on the game space \mathcal{G} is said to verify \mathcal{B} -associated consistency with respect to the \mathcal{B} -associated game if $\Phi(N, v_\lambda^\mathcal{B}) = \Phi(N, v)$ for all games $\langle N, v \rangle$, and all $\lambda \in \mathbb{R}$. This property generalizes the associated consistency with respect to the associated game.

According to the next theorem, the \mathcal{B} -associated game is well chosen in order to guarantee that the corresponding efficient, symmetric, and linear value ψ satisfies the \mathcal{B} -associated consistency.

Theorem 4.4 (cf. [2]). *For a given collection of constants \mathcal{B} , let ψ be the corresponding efficient, symmetric, and linear value on \mathcal{G} . Then $\psi(N, v_\lambda^\mathcal{B}) = \psi(N, v)$ for all games $\langle N, v \rangle$, and all $\lambda \in \mathbb{R}$.*

Proof by Matrix Approach. In view of Corollary 4.3, we show $M^{Sh}B \cdot v_\lambda^\mathcal{B} = M^{Sh}B \cdot v$. Since $v_\lambda^\mathcal{B} = M_\lambda^\mathcal{B} \cdot v$, it is sufficient to check $M^{Sh}B \cdot M_\lambda^\mathcal{B} = M^{Sh}B$. From Proposition 4.1 (1) and Lemma 3.1 respectively, we derive

$$M^{Sh}B \cdot M_\lambda^\mathcal{B} = M^{Sh}B \cdot (B^{-1}M_\lambda B) = (M^{Sh}M_\lambda)B = M^{Sh}B.$$

This completes the proof. ■

Definition 5 (cf. [2]). *A value Φ on the game space \mathcal{G} possesses the \mathcal{B} -inessential game property with reference to a given collection of constants \mathcal{B} if the value verifies $\Phi_i(N, v) = b_1^n \cdot v(\{i\})$ for all \mathcal{B} -inessential games $\langle N, v \rangle$, and for all $i \in N$. Here the game is called \mathcal{B} -inessential if its \mathcal{B} -scaled game $\langle N, \mathcal{B}v \rangle$ is inessential.*

Similar to the result in Theorem 2.7 about the convergence of the sequence of repeated associated games, the next theorem states the convergence of the sequence of repeated \mathcal{B} -associated games.

Theorem 4.5. *Let $0 < \lambda < \frac{2}{n}$. The sequence of repeated \mathcal{B} -associated games $\{\langle N, v_\lambda^{m*\mathcal{B}} \rangle\}_{m=0}^\infty$ converges to the game $\langle N, \bar{v} \rangle$, where $\bar{v} = B^{-1}PDP^{-1}B \cdot v$. Furthermore, the limit game $\langle N, \bar{v} \rangle$ is \mathcal{B} -inessential.*

Proof. By Proposition 4.1 (3), the sequence of games $\{\langle N, v_\lambda^{m*\mathcal{B}} \rangle\}_{m=0}^\infty$ converges to $\bar{v} = \lim_{m \rightarrow \infty} (M_\lambda^\mathcal{B})^m \cdot v = B^{-1}PDP^{-1}B \cdot v$. So, $\mathcal{B}\bar{v} = B \cdot \bar{v} = PDP^{-1}B \cdot v$. By Lemma 2.6 (3), the matrix PD is row-inessential, and it follows from Lemma 2.5 (2) that the matrix $PDP^{-1}B$ is row-inessential too. Hence, by Lemma 2.5 (3), the game is $\langle N, \mathcal{B}\bar{v} \rangle$ inessential, i.e. the limit game $\langle N, \bar{v} \rangle$ is \mathcal{B} -inessential. ■

Remark 3. Notice that the limit game $\langle N, \bar{v} \rangle$ of the sequence of repeated \mathcal{B} -associated games merely depends on the game $\langle N, v \rangle$ as $\bar{v} = B^{-1}PDP^{-1}B \cdot v$. And for any player $i \in N$, the worth $b_1^n \cdot \bar{v}(\{i\})$ of the \mathcal{B} -scaled version of the limit game, is just the inner product of the i -th row vector of PDP^{-1} and the column vector Bv of the \mathcal{B} -scaled version of the initial game.

So far, we have presented three properties of a value on \mathcal{G} , which are the \mathcal{B} -inessential game property, continuity, and \mathcal{B} -associated consistency. In the following we show that any efficient, symmetric, and linear value verifies these three properties.

Lemma 4.6 (cf. [2]). *For a given collection of constants \mathcal{B} and any game $\langle N, v \rangle$, the corresponding efficient, symmetric, and linear value $\psi(N, v) = Sh(N, \mathcal{B}v) = M^{Sh}B \cdot v$ verifies the \mathcal{B} -inessential game property, continuity, and \mathcal{B} -associated consistency.*

Proof by Matrix Approach. By Theorem 4.4, the value ψ satisfies \mathcal{B} -associated consistency. If the \mathcal{B} -scaled game $\langle N, \mathcal{B}v \rangle$ is inessential, then $\psi_i(N, v) = Sh_i(N, \mathcal{B}v) = (\mathcal{B}v)(\{i\}) = b_1^n \cdot v(\{i\})$ for all $i \in N$. So, ψ verifies the \mathcal{B} -inessential game property. Let us consider any convergent sequence of games $\{\langle N, v_k \rangle\}_{k=0}^\infty$, say the limit of which is the game $\langle N, \bar{v} \rangle$. Since the corresponding sequence of values $\{\psi(N, v_k)\}_{k=0}^\infty$ equals the sequence $\{M^{Sh}B \cdot v_k\}_{k=0}^\infty$, which converges to $M^{Sh}B \cdot \bar{v}$, that is the value $\psi(N, \bar{v})$ of the limit game. This proves the continuity of the value ψ . ■

Finally, we state Driessen's axiomatization of efficient, symmetric, and linear values. And we present an alternative proof based on the previous algebraic results.

Theorem 4.7 (cf. [2]). *For a given collection of constants \mathcal{B} , there exists a unique value Φ on \mathcal{G} verifying the \mathcal{B} -inessential game property, continuity, and \mathcal{B} -associated consistency ($0 < \lambda < \frac{2}{n}$), and the value Φ is the efficient, symmetric, and linear value induced by \mathcal{B} , i.e. $\Phi(N, v) = Sh(N, \mathcal{B}v)$ for all games $\langle N, v \rangle$.*

Proof by Matrix Approach. By Lemma 4.6, it is sufficient to concentrate on the unicity proof. Consider a value Φ satisfying the \mathcal{B} -inessential game property, continuity, and \mathcal{B} -associated consistency ($0 < \lambda < \frac{2}{n}$). Fix the game $\langle N, v \rangle$. We show that $\Phi(N, v) = Sh(N, \mathcal{B}v)$. By both the \mathcal{B} -associated consistency and continuity, it holds

$$\Phi(N, v) = \Phi(N, \bar{v}), \quad \text{where } \bar{v} = B^{-1}PDP^{-1}B \cdot v.$$

Since the limit game $\langle N, \bar{v} \rangle$ is shown to be \mathcal{B} -inessential in Theorem 4.5, the \mathcal{B} -inessential game property for Φ yields $\Phi_i(N, \bar{v}) = b_1^n \cdot \bar{v}(\{i\})$ for all $i \in N$. In summary, $\Phi(N, v) = b_1^n \cdot (\bar{v}(\{i\}))_{i \in N}$.

From the proof of Theorem 3.2, we have $Sh(N, v) = Sh(N, \bar{v})$, i.e. $M^{Sh} = M^{Sh}PDP^{-1}$. It follows that

$$M^{Sh}B = M^{Sh}PDP^{-1}B = M^{Sh}B \cdot B^{-1}PDP^{-1}B.$$

That is $Sh(N, \mathcal{B}v) = Sh(N, \mathcal{B}\bar{v})$. Since the game $\langle N, \mathcal{B}\bar{v} \rangle$ is inessential, we conclude that $Sh(N, \mathcal{B}v) = Sh(N, \mathcal{B}\bar{v}) = b_1^n \cdot (\bar{v}(\{i\}))_{i \in N}$. Hence $\Phi(N, v) = Sh(N, \mathcal{B}v)$. That is, $\Phi(N, v)$ agrees with the efficient, symmetric, and linear value induced by \mathcal{B} . ■

5 Conclusions about matrix analysis

The paper deals with the class of efficient, symmetric, and linear values, of which the Shapley value is the most important representative. Concerning the matrix approach for the \mathcal{B} -associated consistency of such values, especially the associated consistency of the Shapley value, the next three tables summarize the relevant matrices, games and their mutual relationships.

Matrix	Name of matrix	Value/Game	Definition
M^{Sh}	Shapley standard	$Sh(N, v) = M^{Sh}v$	Definition 2
M_λ	associated transformation	$v_\lambda^{Sh} = M_\lambda \cdot v$	Definition 3
$M_\lambda^{\mathcal{B}}$	\mathcal{B} -associated transformation	$v_\lambda^{\mathcal{B}} = M_\lambda^{\mathcal{B}} \cdot v$	Definition 4
B	\mathcal{B} -scaling diagonal	$\mathcal{B}v = B \cdot v$	Definition

Sequence	Limit	Matrix Representation	Property of Game	Statement
$\{\langle N, v_\lambda^{m*Sh} \rangle\}_{m=0}^\infty$	$\langle N, \tilde{v} \rangle$	$\tilde{v} = PDP^{-1} \cdot v$	inessential game	Theorem 2.7
$\{\langle N, v_\lambda^{m*\mathcal{B}} \rangle\}_{m=0}^\infty$	$\langle N, \bar{v} \rangle$	$\bar{v} = B^{-1}PDP^{-1}B \cdot v$	\mathcal{B} -inessential game	Theorem 4.5

Property	Hamiache	Xu-Driessen-Sun	Statement
similarity	$Sh(N, v) = Sh(N, v_\lambda^{Sh})$	$M_\lambda^{\mathcal{B}} = B^{-1}M_\lambda B$	Proposition 4.1
associated consistency		$M^{Sh} = M^{Sh}M_\lambda$	Lemma 3.1
\mathcal{B} -ass. consistency		$M^{Sh}B = M^{Sh}BM_\lambda^{\mathcal{B}}$	Theorem 4.4

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Appendix: Proof of Lemma 3.1.

Since $Sh(N, v) = M^{Sh}v$ and $Sh(N, v_\lambda^{Sh}) = M^{Sh}(M_\lambda \cdot v)$, it suffices to check $M^{Sh} = M^{Sh}M_\lambda$ for showing that the Shapley value satisfies the associated consistency, i.e. $M^{Sh}(M_\lambda - I) = \mathbf{0}$.

By the definition of M^{Sh} and M_λ , for all $i \in N$ and for all $T \subseteq N$, $T \neq \emptyset$, the entry $[M^{Sh}(M_\lambda - I)]_{i,T}$ is as follows:

$$[M^{Sh}(M_\lambda - I)]_{i,T} = \sum_{S \subseteq N, i \in S} \frac{(s-1)!(n-s)!}{n!} [M_\lambda - I]_{S,T} - \sum_{S \subseteq N, i \notin S} \frac{s!(n-s-1)!}{n!} [M_\lambda - I]_{S,T}$$

If $i \in T$ and $t \geq 2$,

$$\begin{aligned} &= \frac{(t-1)!(n-t)!}{n!} [M_\lambda - I]_{T,T} + \sum_{j \in T \setminus \{i\}} \frac{(t-2)!(n-t+1)!}{n!} [M_\lambda - I]_{T \setminus \{j\}, T} \\ &\quad - \frac{(t-1)!(n-t)!}{n!} [M_\lambda - I]_{T \setminus \{i\}, T} \\ &= \frac{(t-1)!(n-t)!}{n!} [-(n-t)\lambda] + \frac{(t-2)!(n-t+1)!}{n!} (t-1)\lambda - \frac{(t-1)!(n-t)!}{n!} \lambda \\ &= \frac{(t-1)!(n-t)!}{n!} [-(n-t)\lambda + (n-t+1)\lambda - \lambda] = 0 \end{aligned}$$

If $i \notin T$ and $t \geq 2$,

$$\begin{aligned} &= -\left\{ \frac{t!(n-t-1)!}{n!} [M_\lambda - I]_{T,T} + \sum_{j \in T} \frac{(t-1)!(n-t)!}{n!} [M_\lambda - I]_{T \setminus \{j\}, T} \right\} \\ &= -\left\{ \frac{t!(n-t-1)!}{n!} [-(n-t)\lambda] + \frac{(t-1)!(n-t)!}{n!} t\lambda \right\} \\ &= -\left\{ \frac{t!(n-t-1)!}{n!} [-(n-t)\lambda + (n-t)\lambda] \right\} = 0 \end{aligned}$$

If $T = \{i\}$,

$$\begin{aligned} &= \frac{(1-1)!(n-1)!}{n!} [M_\lambda - I]_{i,i} - \sum_{S \subseteq N, i \notin S} \frac{s!(n-s-1)!}{n!} [M_\lambda - I]_{S,i} \\ &= \frac{(1-1)!(n-1)!}{n!} [-(n-1)\lambda] - \sum_{1 \leq s \leq n-1} \frac{s!(n-s-1)!}{n!} (-\lambda) \binom{n-1}{s} \\ &= -\frac{n-1}{n} \lambda + \sum_{1 \leq s \leq n-1} \frac{1}{n} \lambda = 0 \end{aligned}$$

If $T = \{j\}$ and $j \neq i$,

$$\begin{aligned}
&= \sum_{S \subseteq N \setminus \{j\}, i \in S} \frac{(s-1)!(n-s)!}{n!} [M_\lambda - I]_{S,j} \\
&\quad - \left\{ \frac{1!(n-1-1)!}{n!} [M_\lambda - I]_{j,j} + \sum_{S \subseteq N \setminus \{i,j\}} \frac{s!(n-s-1)!}{n!} [M_\lambda - I]_{S,j} \right\} \\
&= \sum_{1 \leq s \leq n-1} \frac{(s-1)!(n-s)!}{n!} (-\lambda) \binom{n-2}{s-1} \\
&\quad - \left\{ \frac{1!(n-1-1)!}{n!} [-(n-1)\lambda] + \sum_{1 \leq s \leq n-2} \frac{s!(n-s-1)!}{n!} (-\lambda) \binom{n-2}{s} \right\} \\
&= \sum_{1 \leq s \leq n-1} \frac{(n-s)}{n(n-1)} (-\lambda) + \frac{(n-1)!}{n!} \lambda - \sum_{1 \leq s \leq n-2} \frac{n-s-1}{n(n-1)} (-\lambda) = 0
\end{aligned}$$

Four cases above imply that $M^{Sh} = M^{Sh} M_\lambda$. Thus the Shapley value satisfies the associated consistency. \blacksquare

Remark 4. Notice that a coalitional matrix M verifies the associated consistency $M^{Sh} = M^{Sh} M$ of the Shapley value, if and only if the column space of $M - I$ is a subspace of the null space of the Shapley standard matrix M^{Sh} . Without proof, we emphasize that the column space of $M_\lambda - I$ equals the null space of M^{Sh} .