Department of Applied Mathematics

Faculty of EEMCS

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands

Phone: +31-53-4893400

Fax: +31-53-4893114

Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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for generalized 2D vorticity dynamics**

E. BERNSEN, O. BOKHOVE

AND J.J.W. VAN DER VEGT

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A (Dis)Continuous Finite Element Model for Generalized 2D Vorticity Dynamics

Erik Bernsen, Onno Bokhove, and Jaap J.W. van der Vegt^{*}

*Department of Applied Mathematics
University of Twente
P.O. Box 217, 7500 AE, Enschede, The Netherlands*

Abstract

A mixed continuous and discontinuous Galerkin finite element discretization is constructed for a generalized vorticity streamfunction formulation in two spatial dimensions. This formulation consists of a hyperbolic (potential) vorticity equation and a linear elliptic equation for a (transport) streamfunction. The generalized formulation includes three systems in geophysical fluid dynamics: the incompressible Euler equations, the barotropic quasi-geostrophic equations and the rigid-lid equations. Multiple connected domains are considered with impenetrable and curved boundaries such that the circulation at each connected piece of boundary must be introduced. The generalized system is shown to globally conserve energy and weighted smooth functions of the vorticity. In particular, the weighted square vorticity or enstrophy is conserved. By construction, the spatial finite-element discretization is shown to conserve energy and is L^2 -stable in the enstrophy norm. The method is verified by numerical experiments which support our error estimates. Particular attention is paid to match the continuous and discontinuous discretization. Hence, the implementation with a third-order Runge-Kutta time discretization conserves energy and is L^2 -stable in the enstrophy norm for increasing time resolution in multiple connected curved domains.

Key words: generalized vorticity streamfunction formulation, multiple connected and curved domains, discontinuous Galerkin finite elements, energy conservation and enstrophy stability, error estimates

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^{*} Corresponding author: dr. Onno Bokhove; IMPACT, Institute of Mechanics, Processes, and Control - Twente; Department of Applied Mathematics; University of Twente; P.O. Box 217; 7500 AE; Enschede; The Netherlands; o.bokhove@math.utwente.nl; Parts of this memorandum are published in [3].

1 INTRODUCTION

A mixed continuous-discontinuous Galerkin finite element model is constructed to solve several (geophysical) fluid equations in a generalized (potential) vorticity streamfunction formulation. Our algorithm is an inviscid extension of the mixed continuous-discontinuous Galerkin finite element method presented in [10] for the incompressible two-dimensional Euler equations in a vorticity streamfunction formulation. We extend this method to the generalized formulation in a multiple connected domain with curved boundaries, and implement and test the algorithm more thoroughly. In particular, the circulation boundary condition around each connected part of the boundary or each “island” requires a modification, (17), of the usual function space for the continuous test and trial functions of the streamfunction.

The generalization consists of a hyperbolic equation for the (potential) vorticity, $\xi = \xi(x, y, t)$, and a linear elliptic equation for the streamfunction, $\psi = \psi(x, y, t)$, in a bounded domain $\Omega \subset \mathbb{R}^2$ as function of the horizontal coordinates x, y and time t . It is defined as follows

$$\partial_t \xi / A + \nabla \cdot (\xi \vec{U}) = 0 \quad (1a)$$

$$\vec{U} = \nabla^\perp \psi \quad (1b)$$

$$\nabla \cdot (A \nabla \psi) - B \psi + D = \xi / A \quad (1c)$$

with $0 < A_0 < A = A(x, y) < A_1 < \infty$ and $A_{0,1}$ strictly positive finite constants, $B = B(x, y) \geq 0$ and $D = D(x, y)$. The gradient operator is given by $\nabla = [\partial_x, \partial_y]^T$ and the two-dimensional curl operator by $\nabla^\perp = [-\partial_y, \partial_x]^T$. The system (1) is completed with boundary conditions and initial conditions in Section 3.

The generalized system (1) serves as a model for several fluid flow problems by choosing A , B and D to yield the incompressible two-dimensional (2D) Euler equations [8], the quasi-geostrophic equations [14], and the rigid-lid equations [9], often used in atmosphere and ocean dynamics. In all these cases ξ represents the (potential) vorticity of the fluid, $\vec{u} = A \vec{U}$ represents the velocity and $\vec{U}$ the (mass transport) velocity of the fluid. We explicitly consider the multiple connected and curved domains required in geophysical applications. This contrasts with [10], where the domain was simply connected and all numerical tests were done in rectangular domains. In these curved domains, it is essential to use isoparametric boundary elements, otherwise higher-order accuracy is impossible.

The outline of the paper is as follows. Details on the applications are provided in Section 2. In Section 4, we show that the energy and enstrophy of the generalized system (1) are conserved quantities, as well as Casimir invariants, for

the slip flow boundary conditions introduced in Section 3. For the discretization of (1), we use the same method as in [10]. The equation for the vorticity, (1a), is discretized using a discontinuous Galerkin (DG) method, while for the elliptic equation, (1c), we use a continuous Galerkin discretization (see Section 5). In Section 6, we show that the spatial discretization of the generalized system leads to a scheme in which the energy and enstrophy are conserved and stable quantities. Moreover, discretizing time with an implicit θ - or modified midpoint-scheme, we also show energy and enstrophy to be conserved in time for $\theta = 1/2$. Numerical conservation of energy and enstrophy is essential for long-time stable integrations of geophysical systems (see, *e.g.*, [11]). Further preservation of higher-order vortical integral constraints, in the form of numerical Casimir invariants, is also desirable, but not achieved here. Error estimates of the discretization are provided in Section 7. We verify the method and its implementation in Section 8, focusing on the convergence to several exact solutions as well as the properties of energy conservation and enstrophy stability. Finally, we summarize and conclude in Section 9.

2 APPLICATIONS

The generalized vorticity streamfunction formulation given by (1) includes at least three distinct systems of interest:

- (i) The **2D Euler equations** [8] describe the flow of an incompressible fluid

$$\partial_t \xi + \nabla \cdot (\vec{u} \xi) = 0 \quad (2a)$$

$$\vec{u} = \nabla^\perp \psi \quad (2b)$$

$$\nabla^2 \psi = \xi \quad (2c)$$

with $\xi = \partial_x v - \partial_y u$ the vertical component of the vorticity of the fluid, $\vec{u} = (u, v)^T$ the velocity field and ψ the streamfunction. Defining $A = 1$, $B = D = 0$, and $\vec{U} = \vec{u}$, the Euler system (2) emerges as a special case of (1).

- (ii) The **quasi-geostrophic equations** [14] approximately describe the motion of the atmosphere or oceans at mid-latitudes

$$\partial_t \xi + \nabla \cdot (\vec{u} \xi) = 0 \quad (3a)$$

$$\vec{u} = \nabla^\perp \psi \quad (3b)$$

$$\xi = \nabla^2 \psi - \frac{f_0^2}{gD_0} \psi + \frac{f_0 h_B}{D_0} + \beta y \quad (3c)$$

with the quasi-geostrophic potential vorticity ξ , D_0 a characteristic depth of the atmosphere or ocean, g the acceleration of gravity, the bottom to-

pography at the vertical position $z = h_B = h_B(x, y)$, the Coriolis parameter $f_0 = 2\Omega_e \sin \Theta_0$ and $\beta = 2\Omega_e \cos \Theta_0 / R$, where R is the radius of the Earth, Θ_0 a characteristic value of the latitude and Ω_e the Earth's rotation speed. Defining $A = 1$, $B = f_0^2 / (g D_0)$, $D = f_0 h_B / D_0 + \beta y$, and $\vec{U} = \vec{u}$, the quasi-geostrophic system (3) follows from (1).

(iii) The **rigid-lid equations** [9] describe the vertically averaged motion of fluid between topography at $z = h_B(x, y)$ and a rigid lid at $z = h_B + H$

$$\partial_t \xi + \vec{u} \cdot \nabla \xi = 0 \quad (4a)$$

$$\vec{u} = (1/H) \nabla^\perp \psi \quad (4b)$$

$$\xi = (\nabla^\perp \cdot \vec{u} + f) / H \quad (4c)$$

with ξ the potential vorticity, $\vec{u}$ the velocity field, f the Coriolis parameter and $0 < 1/A_1 < H = H(x, y) < 1/A_0 < \infty$ the depth of the fluid. Hence, by taking $A = 1/H$, $B = 0$, $D = f = f_0 + \beta y$ and $\vec{U} = H\vec{u}$, (4) emerges from (1).

3 BOUNDARY AND INITIAL CONDITIONS

In [10] the domain was assumed to be simply connected which made it possible to consider one single Dirichlet boundary condition: $\psi|_{\partial\Omega} = 0$ with $\partial\Omega$ the boundary of Ω . In contrast, we focus on multiple connected curved domains with impenetrable walls, where, in general, we omit additional inflow and outflow boundary conditions for ease of presentation. This (part of the) boundary with slip flow boundary conditions is also denoted by $\partial\Omega_D$, on which

$$\vec{U} \cdot \hat{n} = 0 \quad (5)$$

holds with $\hat{n} = [n_x, n_y]^T$ the outward unit vector normal to the boundary. The boundary $\partial\Omega_D$ is partitioned into N separate simply connected subsets. Thus, there exist $\partial\Omega_{D_i} \subset \partial\Omega_D$ such that

$$\cup_{i=1}^N \overline{\partial\Omega_{D_i}} = \overline{\partial\Omega_D} \quad \text{and} \quad \partial\Omega_{D_i} \cap \partial\Omega_{D_j} = \emptyset \quad (6)$$

for $i, j = 1, 2, \dots, N$ and $i \neq j$, where $\partial\Omega_{D_i} \cup \partial\Omega_{D_j}$ is not a simply connected set for $i \neq j$ (see Figure 1).

On each part $\partial\Omega_{D_i}$ of the boundary, ψ is independent of x and y because $\partial\psi/\partial\hat{\tau} = \nabla\psi \cdot \hat{\tau} = -\nabla^\perp\psi \cdot \hat{n} \stackrel{(1b)}{=} -\vec{U} \cdot \hat{n} \stackrel{(5)}{=} 0$ with $\hat{\tau} = [-n_y, n_x]^T$ the unit vector tangential to $\partial\Omega$. Hence, on these boundaries

$$\psi|_{\partial\Omega_{D_i}} = f_i(t) \quad (7)$$

is a function only depending on time. Consider the circulation $\mathcal{C}_i$ around $\partial\Omega_{D_i}$, defined by

$$\mathcal{C}_i = \int_{\partial\Omega_{D_i}} \vec{u} \cdot \hat{\tau} \, d\Gamma = \int_{\partial\Omega_{D_i}} A \vec{U} \cdot \hat{\tau} \, d\Gamma \quad (8)$$

with $d\Gamma$ a line element along $\partial\Omega_D$. A relevant boundary condition at $\partial\Omega_{D_i}$, see [14], is

$$d\mathcal{C}_i/dt = 0, \quad (9)$$

whence the functions $f_i(t)$ in (7) are only implicitly defined.

Remark 1 *Defining $f_i(t)$ using (9) for each $i = 1, 2, \dots, N$ defines ψ up to a constant. Therefore, for the case $B = 0$, we should prescribe $f_1(t) = 0$ on $\partial\Omega_{D_1}$ to enforce a unique solution ψ for (1c).*

The initial conditions are specified by the vorticity field and the circulations $\mathcal{C}_i$ at $t = 0$. We calculate ψ (at $t = 0$) from (1c) using the boundary conditions. In Section 4, we show that the energy of the system (1) is a conserved quantity for the boundary conditions (5) and (9), provided that there are no parts of the boundary with inflow or outflow.

4 CONSERVATION OF ENERGY AND ENSTROPY

Energy and enstrophy are conserved quantities of the system (1) for the slip flow boundary conditions introduced in Section 3. Define the total energy $E(t)$ of the system (1) by

$$E(t) = \frac{1}{2} \|\sqrt{A} \nabla \psi\|_{\Omega}^2 + \frac{1}{2} \|\sqrt{B} \psi\|_{\Omega}^2, \quad (10)$$

where we require $A(x, y) > A_0 > 0$ and $B(x, y) \geq 0$. Here, the L^2 -norm is denoted as $\|\cdot\|^2 = (\cdot, \cdot)$, while $(\cdot, \cdot)$ denotes the usual L^2 -inner product with integration over the domain or sub domain, as indicated by a subscript.

Lemma 2 *The energy (10) of system (1) subject to slip flow boundary conditions, (5) and (9), is conserved:*

$$dE/dt = 0. \quad (11)$$

PROOF. Differentiating (1c) with respect to time and combining the result with (1a) gives $\nabla \cdot (A \nabla (\partial_t \psi)) - B \partial_t \psi = -\nabla \cdot (\vec{U} \xi)$. Multiplying this equation with ψ , integrating over the domain, and integrating by parts yields

$$-\frac{dE}{dt} = - \int_{\partial\Omega} A \psi \partial_t (\nabla \psi \cdot \hat{n}) \, d\Gamma + \int_{\Omega} \xi \vec{U} \cdot \nabla \psi \, d\Omega - \int_{\partial\Omega} \xi \psi \vec{U} \cdot \hat{n} \, d\Gamma. \quad (12)$$

The last two terms on the right-hand-side of (12) vanish by using $\vec{U} \perp \nabla \psi$ and (5). The first term on the right-hand-side of (12) cancels as well since:

$$\begin{aligned} \int_{\partial\Omega_D} A\psi \partial_t (\nabla \psi \cdot \hat{n}) \, d\Gamma &\stackrel{(6)}{=} \sum_{i=1}^N \int_{\partial\Omega_{D_i}} A\psi \partial_t (\nabla \psi \cdot \hat{n}) \, d\Gamma \\ &\stackrel{(7)}{=} \sum_{i=1}^N f_i(t) \frac{d}{dt} \int_{\partial\Omega_{D_i}} A \nabla \psi \cdot \hat{n} \, d\Gamma \stackrel{(8),(9)}{=} 0, \end{aligned}$$

where we used the relation $\nabla \psi \cdot \hat{n} = \nabla^\perp \psi \cdot \hat{\tau}$. Hence, we find (11). $\square$

The Casimirs invariants are

$$\mathcal{C}(t) = \int_{\Omega} (1/A) C_a(\xi) \, d\Omega \quad (13)$$

with an arbitrary function $C_a = C_a(\xi)$ of the generalized vorticity. For the case $C_a(\xi) = \xi^2/2$, the enstrophy $S(t)$ emerges as a particularization

$$S(t) = \frac{1}{2} \|\xi / \sqrt{A}\|_{\Omega}^2. \quad (14)$$

Lemma 3 *The Casimirs (13) of system (1), and thus the enstrophy (14), subject to the slip flow boundary condition, (5), are conserved:*

$$d\mathcal{C}/dt = 0 \quad \text{and} \quad dS/dt = 0. \quad (15)$$

PROOF. Result (15) emerges after we multiply (1a) by $C'_a(\xi) = dC_a/d\xi$ and use $\nabla \cdot (\vec{U}\xi) = \vec{U} \cdot \nabla \xi$ twice:

$$(1/A)C'_a(\xi)\partial_t \xi + C'_a(\xi)\vec{U} \cdot \nabla \xi = (1/A)\partial_t C_a(\xi) + \nabla \cdot (\vec{U}C_a(\xi)) = 0, \quad (16)$$

integrate over the domain, Ω , and use the boundary conditions. $\square$

5 FINITE ELEMENT METHOD

We use a finite element method to solve the generalized vorticity equations (1). By discretizing the streamfunction with a continuous Galerkin finite element method in Section 5.1, we enforce continuity of the normal velocity through element boundaries. This simplifies the choice of the numerical flux in the DG discretization for (1a). In Section 5.3, we discuss time discretization schemes and, in Section 6, the properties of the numerical method concerning the

conservation and stability of energy and enstrophy of the numerical solution
1

5.1 Continuous Galerkin Space Discretization

Define the space of test functions $\mathbf{W}^1(\Omega)$ as follows:

$$\mathbf{W}^1(\Omega) = \left\{ w \in H^1(\Omega) \mid \forall i \in \{1, 2, \dots, N\} \exists c_i \in \mathbb{R} : \|w - c_i\|_{\partial\Omega_{D_i}} = 0 \right\} \quad (17)$$

with the standard $H^1(\Omega)$ Sobolev space and with $\|\cdot\|_{\partial\Omega_{D_i}}$ the $L^2(\partial\Omega_{D_i})$ norm. Note that $x \rightarrow \psi(x, t) \in \mathbf{W}^1(\Omega)$ with t fixed because of the boundary condition (7).

Remark 4 *In the case $B = 0$, a Dirichlet boundary condition $\psi|_{\partial\Omega_1} = 0$ is used (see Remark 1), and instead of (17) we use*

$$\mathbf{W}^1(\Omega) = \left\{ w \in H^1(\Omega) \mid \|w\|_{\partial\Omega_{D_1}} = 0, \forall i \in \{2, \dots, N\} \exists c_i \in \mathbb{R} : \|w - c_i\|_{\partial\Omega_{D_i}} = 0 \right\},$$

which ensures the weak formulation of (1c) to have a definite solution.

We multiply (1c) by a test function $w \in \mathbf{W}^1(\Omega)$ and integrate over the domain. For the first term this gives, after integrating by parts,

$$\int_{\Omega} \nabla \cdot (A \nabla \psi) w \, d\Omega = \sum_{i=1}^N \int_{\partial\Omega_{D_i}} w A \nabla \psi \cdot \hat{n} \, d\Gamma - \int_{\Omega} A \nabla \psi \cdot \nabla w \, d\Omega.$$

Using (9) and the fact that $w \in \mathbf{W}^1(\Omega)$ is constant on each $\partial\Omega_{D_i}$, the boundary integral over $\partial\Omega_{D_i}$ ($i = 1, 2, \dots, N$) yields

$$\int_{\partial\Omega_{D_i}} w A \nabla \psi \cdot \hat{n} \, d\Gamma = \int_{\partial\Omega_{D_i}} w A \vec{U} \cdot \hat{\tau} \, d\Gamma \stackrel{(17)}{=} w|_{\partial\Omega_{D_i}} \int_{\partial\Omega_{D_i}} A \vec{U} \cdot \hat{\tau} \, d\Gamma \stackrel{(8)}{=} w|_{\partial\Omega_{D_i}} \mathcal{C}_i.$$

The weak formulation then becomes: Find $\psi \in \mathbf{W}^1(\Omega)$ such that for all $w \in \mathbf{W}^1(\Omega)$ the following holds:

$$L(\psi, w) = F_{\xi}(w) \quad (18)$$

with the operators L and F_{ξ} defined by

¹ Additional technical details are found in appendices D–G.

$$L(v, w) = \left(\sqrt{A} \nabla v, \sqrt{A} \nabla w \right)_\Omega + \left(\sqrt{B} v, \sqrt{B} w \right)_\Omega \quad (19)$$

$$F_\xi(w) = -(\xi/A, w)_\Omega + \int_\Omega Dw \, d\Omega + \sum_{i=1}^N w|_{\partial\Omega_{D_i}} \mathcal{C}_i \quad (20)$$

for $v, w \in \mathbf{W}^1(\Omega)$. Note that L is a symmetric, coercive bilinear operator. Hence, $L(w, w) \geq \alpha \|w\|_{H^1(\Omega)}^2$ with $\alpha > 0 \, \forall w \in \mathbf{W}^1(\Omega)$, where coercivity is ensured in the case $B = 0$ by Remark 4. The proof of coercivity can be obtained by a minor change of the proof in Section 5.3 in [4]. Hence, the matrix of the system of linear equations in the finite element discretization is positive definite.

To approximate ψ and w in the discretized form of (18), we define the function space

$$\mathbf{W}_h^k = \mathbf{W}^1(\Omega) \cap X_h^k, \quad (21)$$

with X_h^k a finite element space consisting of continuous functions and including at least all polynomials of degree k on each element of a triangulation $\mathcal{T}_h = \{K\}$. We replace $\psi, w \in \mathbf{W}^1(\Omega)$ in (18) with the numerical approximations $\psi_h, w_h \in \mathbf{W}_h^k$ and approximate the vorticity with $\xi_h \in \mathbf{V}_h^k$ to be defined in Section 5.2. Hence, we obtain the discretized weak formulation: Find $\psi_h \in \mathbf{W}_h^k$ such that for each $w_h \in \mathbf{W}_h^k$ the following holds:

$$L(\psi_h, w_h) = F_{\xi_h}(w_h). \quad (22)$$

5.2 Discontinuous Galerkin Space Discretization

For the DG discretization, we define the space of discontinuous test functions

$$\mathbf{V}_h^k = \left\{ v_h \mid \forall K \in \mathcal{T}_h, \exists w_h \in X_h^k : v_h|_K = w_h|_K \right\} \quad (23)$$

with $\mathcal{T}_h = \{K\}$ a triangulation of the domain and X_h^k the same continuous finite element space as in (21). We define $\mathbf{V}_h^k$ by (23), instead of

$$\tilde{\mathbf{V}}_h^k = \left\{ v_h \mid \forall K \in \mathcal{T}_h : v_h|_K \in \mathcal{P}_k(K) \right\}, \quad (24)$$

with $\mathcal{P}_k(K)$ the usual space of polynomials on K of degree equal to or less than k , because conservation of energy of the numerical solution requires $\mathbf{W}_h^k \subset \mathbf{V}_h^k$ (see Section 6). The DG discretization is obtained by multiplying (1a) with a test function $v \in \mathbf{V}_h^k$ and integrating over $K \in \mathcal{T}_h$ to obtain

$$(\partial_t \xi/A, v)_K - (\xi \vec{U}, \nabla v)_K + (\xi U_n, v)_{\partial K} = 0 \quad (25)$$

with ∂K the boundary of an element $K \in \mathcal{T}_h$, and $U_n = \vec{U} \cdot \hat{n}$ the normal component of the velocity $\vec{U}$. Subsequently, we substitute the approximations $v_h, \xi_h \in \mathbf{V}_h^k$ into (25) with the approximate velocity field $\vec{U}_h = \nabla^\perp \psi_h$.

Define the inside, $v^- = \lim_{\epsilon \uparrow 0} v(\mathbf{x} + \epsilon \hat{n})$, and outside, $v^+ = \lim_{\epsilon \downarrow 0} v(\mathbf{x} + \epsilon \hat{n})$, trace values of a function v at a boundary point $\mathbf{x} = (x, y)^T$ on ∂K . For the test function v_h on ∂K in (25), we choose the inside value v_h^- , while $\xi_h U_n$ on ∂K is replaced by a numerical flux $\hat{f}(\xi_h^+, \xi_h^-, U_n)$. Note that the normal transport velocity U_n is continuous across element boundaries because ψ_h is continuous and $U_n = \vec{U}_h \cdot \hat{n} = \nabla^\perp \psi_h \cdot \hat{n} = -\nabla \psi_h \cdot \hat{\tau} = -d\psi_h/d\hat{\tau}$. Hence, $d\psi_h/d\hat{\tau}$ and U_n are single valued on element boundaries. Only the vorticity ξ_h is multi valued.

The numerical flux satisfies the following properties:

(i) It is consistent

$$\hat{f}(\xi_h, \xi_h, U_n) = \xi_h U_n; \quad (26)$$

(ii) it is conservative:

$$\hat{f}(\xi_h^+, \xi_h^-, U_n) = -\hat{f}(\xi_h^-, \xi_h^+, -U_n), \quad (27)$$

ensuring that the flux from two neighboring elements $K_L \in \mathcal{T}_h$ to $K_R \in \mathcal{T}_h$ is opposite to the flux from K_R to K_L ; and

(iii) it is L^2 -stable in the enstrophy norm (see Section 6), that is,

$$(\xi_h^+ - \xi_h^-) U_n (\hat{\xi}_h - \bar{\xi}_h) \leq 0 \quad (28)$$

with $\bar{\xi}_h = (\xi_h^+ + \xi_h^-)/2$ and $\hat{\xi}_h = \hat{f}(\xi_h^+, \xi_h^-, U_n)/U_n$. Hence, for $U_n > 0$ the value of the numerical flux is closer to the flux, $U_n \xi^-$, evaluated on the inside of the element than outside, $U_n \xi^+$.

The following fluxes $\hat{f}$ satisfy properties (26)–(28):

$$\textbf{central} \quad \hat{f}(\xi^+, \xi^-, U_n) = \frac{\xi^+ + \xi^-}{2} U_n, \quad (29a)$$

$$\textbf{upwind} \quad \hat{f}(\xi^+, \xi^-, U_n) = U_n \begin{cases} \xi^+ & \text{if } U_n < 0 \\ \xi^- & \text{if } U_n \geq 0 \end{cases}, \quad (29b)$$

$$\textbf{Lax-Friedrichs} \quad \hat{f}(\xi^+, \xi^-, U_n) = \frac{1}{2} (U_n(\xi^+ + \xi^-) - \alpha_{LF}(\xi^+ - \xi^-)) \quad (29c)$$

with $\alpha_{LF} \geq 0$. A common choice is $\alpha_{LF} = \max |U_n|$ with a local or global maximum. For $\alpha_{LF} = 0$ and $\alpha_{LF} = |U_n|$, we obtain the central and upwind flux.

After replacing v_h with v_h^- and $\xi_h U_n$ with $\hat{f}(\xi_h^+, \xi_h^-, U_n)$ in (25), the weak formulation of (1a) is: Find $\xi_h \in \mathbf{V}_h^k$ such that for all $v_h \in \mathbf{V}_h^k$ the following equation holds:

$$p_K(\partial_t \xi_h, v_h) = R_K(\xi_h, \psi_h, v_h) \quad (30)$$

with the operators p_K and R_K defined by

$$p_K(u_h, v_h) = (u_h/A, v_h)_K \quad (31)$$

$$R_K(u_h, w_h, v_h) = \left(u_h \nabla^\perp w_h, \nabla v_h \right)_K - \int_{\partial K} v_h^- \hat{f}(u_h^+, u_h^-, \nabla^\perp w_h \cdot \hat{n}) \, d\Gamma \quad (32)$$

for $u_h, v_h \in \mathbf{V}_h^k$ and $w_h \in \mathbf{W}_h^k$. Note that we can formally write the space discretization as

$$\partial_t \xi_h = \mathcal{L}_h(\xi_h) \quad (33)$$

for an operator $\mathcal{L}_h$ following from (22) and (30).

5.3 Time Discretization

5.3.1 Explicit Scheme

For the test cases presented in Section 8, we use the explicit, third order Runge-Kutta scheme (RK3) of [16] (see also [10]). The maximum size of the time step depends on the maximum velocity and the size of the elements. We performed a linear stability analysis for a space DG finite element and RK3 time discretization of the vorticity equation with constant velocity on a regular one-dimensional mesh, see Table 1.

5.3.2 Implicit Scheme

Implicit methods usually have the advantage that the restriction on the time step is less severe, but at the expense of more computational cost per time step. We consider, inspired by the work in [19], the following θ - or modified midpoint-scheme [5] for the time discretization of (33):

$$\Delta \xi_h^n / \Delta t = \mathcal{L}_h(\xi_h^{n+\theta}) \quad (34)$$

with $\xi_h^{n+\theta} = \theta \xi_h^{n+1} + (1 - \theta) \xi_h^n$ and $\Delta \xi_h^n = \xi_h^{n+1} - \xi_h^n$, for $\theta \in [0, 1]$. Using (22) and (30), we find:

$$L(\psi_h^{n+\theta}, w_h) = F_{\xi_h^{n+\theta}}(w_h) \quad (35)$$

$$p_K(\Delta \xi_h^n, v_h) = \Delta t R_K(\xi_h^{n+\theta}, \psi_h^{n+\theta}, v_h) \quad (36)$$

for all $v_h \in \mathbf{V}_h^k$ and $w_h \in \mathbf{W}_h^k$. For each integer $n \geq 0$ and $w_h \in \mathbf{W}_h^k$, the streamfunction ψ_h^n is defined by:

$$L(\psi_h^n, w_h) = F_{\xi_h^n}(w_h). \quad (37)$$

Note that (35) then is satisfied because

$$\psi_h^{n+\theta} = \theta \psi_h^{n+1} + (1 - \theta) \psi_h^n, \quad (38)$$

and L and F are (bi-)linear operators.

6 Numerical Conservation of Energy and Enstrophy

In Section 4, we showed that energy and enstrophy are conserved quantities for the slip flow boundary conditions described in Section 3. The energy and enstrophy of the discretization of (1) have similar properties.

Define the energy, E_h , of the numerical solution of (1):

$$E_h(t) = \frac{1}{2} \|\sqrt{A} \nabla \psi_h\|_{\Omega}^2 + \frac{1}{2} \|\sqrt{B} \psi_h\|_{\Omega}^2 = \frac{1}{2} L(\psi_h, \psi_h). \quad (39)$$

Theorems 6 and 7 are the counterparts of Lemma 2 for the numerical system. We use the following lemma.

Lemma 5 *Assume that we have slip flow boundary conditions, (5), then (32) satisfies*

$$\sum_{K \in \mathcal{T}_h} R_K(v_h, w_h, w_h) = 0 \quad (40)$$

for each $v_h \in \mathbf{V}_h^k$ and $w_h \in \mathbf{W}_h^k$.

PROOF. Using definition (32) of R_K , we have

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} R_K(v_h, w_h, w_h) &= \sum_{K \in \mathcal{T}_h} \int_K v_h \nabla^\perp w_h \cdot \nabla w_h \, dK \\ &\quad - \sum_{K \in \mathcal{T}_h} \int_{\partial K} w_h^- \hat{f}(v_h^+, v_h^-, \nabla^\perp w_h \cdot \hat{n}) \, d\Gamma, \end{aligned}$$

where the interior integral vanishes because $\nabla^\perp w_h \perp \nabla w_h$. The summation over the element boundary integrals vanishes because w_h is continuous across element boundaries ($w_h^+ = w_h^-$), the conservation property of the numerical flux (27), and the slip flow boundary conditions. $\square$

Theorem 6 *Consider the solution of (22) and (30) subject to slip flow boundary conditions, (5). The energy associated with this numerical solution, as given by (39), is a conserved quantity:*

$$dE_h/dt = 0. \quad (41)$$

PROOF. We prove that (41) holds at each point in time. First choose an arbitrary, but fixed, point in time, denoted by t_0 . Differentiate (22) with respect to time and use (30) to obtain for all $w_h \in \mathbf{W}_h^k$ at $t = t_0$

$$\begin{aligned} [L(\partial_t \psi_h, w_h)]_{t=t_0} &= - \sum_{K \in \mathcal{T}_h} p_K([\partial_t \xi_h]_{t=t_0}, w_h) \\ &\stackrel{(30)}{=} - \sum_{K \in \mathcal{T}_h} R_K([\xi_h]_{t=t_0}, [\psi_h]_{t=t_0}, w_h) \end{aligned}$$

Now choose $w_h = [\psi_h]_{t=t_0}$ and apply Lemma 5 to obtain

$$\begin{aligned} \left[\frac{dE_h}{dt} \right]_{t=t_0} &= L([\partial_t \psi_h]_{t=t_0}, [\psi_h]_{t=t_0}) \\ &= - \sum_{K \in \mathcal{T}_h} R_K([\xi_h]_{t=t_0}, [\psi_h]_{t=t_0}, [\psi_h]_{t=t_0}) = 0. \end{aligned}$$

Since t_0 was chosen arbitrary, we find (41). $\square$.

Note that an essential step consists of using $w_h = \psi_h$ as a test function in the DG discretization (30). The space of test functions for the continuous Galerkin discretization must, therefore, satisfy $\mathbf{W}_h^k \subset \mathbf{V}_h^k$.

Denote the numerical energy at time level n by $E_h^n = L(\psi_h^n, \psi_h^n)/2$. The implicit time discretization scheme has the property that for certain choices of θ the numerical energy is conserved.

Theorem 7 *Consider the solution of (36) and (35) subject to slip flow boundary conditions, (5) and (9). For $\theta \in [\frac{1}{2}, 1]$, the numerical energy is stable:*

$$\Delta E_h^n \equiv E_h^{n+1} - E_h^n \leq 0. \quad (42)$$

For $\theta = \frac{1}{2}$, the numerical energy is conserved:

$$\Delta E_h^n = 0. \quad (43)$$

PROOF. We combine (37) to obtain for each $w_h \in \mathbf{W}_h^k$

$$L(\Delta \psi_h^n, w_h) = F_{\xi_h^{n+1}}(w_h) - F_{\xi_h^n}(w_h)$$

with $\Delta \psi_h^n = \psi_h^{n+1} - \psi_h^n$. Since $w_h \in \mathbf{W}_h^k \subset \mathbf{V}_h^k$, we use (20), (31), (36) and $\Delta \xi_h^n = \xi_h^{n+1} - \xi_h^n$ to write

$$F_{\xi_h^{n+1}}(w_h) - F_{\xi_h^n}(w_h) = - \sum_{K \in \mathcal{T}_h} p_K(\Delta \xi_h^n, w_h) \stackrel{(36)}{=} - \sum_{K \in \mathcal{T}_h} \Delta t R_K(\xi_h^{n+\theta}, \psi_h^{n+\theta}, w_h).$$

Combining the above two equations, choosing $w_h = \psi_h^{n+\theta}$ and applying Lemma 5 gives

$$L(\Delta\psi_h^n, \psi_h^{n+\theta}) = - \sum_{K \in \mathcal{T}_h} \Delta t R_K(\xi_h^{n+\theta}, \psi_h^{n+\theta}, \psi_h^{n+\theta}) = 0.$$

As the operator L is bilinear and symmetric, we use (38) to obtain (see [19])

$$\Delta E_h^n = L(\Delta\psi_h^n, \psi_h^{n+\theta}) - \left(\theta - \frac{1}{2}\right) L(\Delta\psi_h^n, \Delta\psi_h^n).$$

Thus, we find

$$\Delta E_h^n = - \left(\theta - \frac{1}{2}\right) L(\Delta\psi_h^n, \Delta\psi_h^n).$$

Since $L(w_h, w_h) \geq 0$ for each $w_h \in \mathbf{W}_h^k$ and $\psi_h \in \mathbf{W}_h^k$, we obtain $\Delta E_h^n \leq 0$ for $\theta \in [\frac{1}{2}, 1]$ and $\Delta E_h^n = 0$ for $\theta = \frac{1}{2}$. $\square$

Define the numerical enstrophy S_h :

$$S_h(t) = \frac{1}{2} \|\xi_h / \sqrt{A}\|^2. \quad (44)$$

Theorems 9 and 10 are the counterparts of Lemma 3 for the numerical system. We use the following lemma.

Lemma 8 *Assume that we have slip flow boundary conditions (5). For each $v_h \in \mathbf{V}_h^k$ and $w_h \in \mathbf{W}_h^k$, the operator R_K defined in (32) satisfies*

$$\sum_{K \in \mathcal{T}_h} R_K(v_h, w_h, v_h) \leq 0. \quad (45)$$

For the central flux (29a), one has

$$\sum_{K \in \mathcal{T}_h} R_K(v_h, w_h, v_h) = 0. \quad (46)$$

PROOF. Using definition (32) of R_K we have

$$\begin{aligned} & \sum_{K \in \mathcal{T}_h} R_K(v_h, w_h, v_h) \\ &= \sum_{K \in \mathcal{T}_h} \int_K v_h \nabla^\perp w_h \cdot \nabla v_h \, dK - \sum_{K \in \mathcal{T}_h} \int_{\partial K} v_h^- \hat{f}(v_h^+, v_h^-, U_n) \, d\Gamma \end{aligned} \quad (47)$$

with $U_n = \nabla^\perp w_h \cdot \hat{n}$. After integration by parts and using the boundary

conditions, (47) becomes

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} R_K(v_h, w_h, v_h) &= \sum_{K \in \mathcal{T}_h} \int_{\partial K} U_n \left(\frac{1}{2} (v_h^-)^2 - v_h^- \hat{v}_h \right) d\Gamma \\ &= \sum_{K \in \mathcal{T}_h} \int_{\partial K} \frac{1}{2} U_n [v_h] (\hat{v}_h - \bar{v}_h) d\Gamma + \sum_{K \in \mathcal{T}_h} \int_{\partial K} U_n \left(\frac{1}{2} \bar{v}_h^2 - \bar{v}_h \hat{v}_h \right) d\Gamma \end{aligned} \quad (48)$$

with $\bar{v}_h = (v_h^+ + v_h^-)/2$, $\bar{v}_h^2 = ((v_h^+)^2 + (v_h^-)^2)/2$, $[v_h] = v_h^+ - v_h^-$ and $\hat{v}_h = \hat{f}(v_h^+, v_h^-, U_n)/U_n$. The second summation in (48) vanishes because of (27) and the boundary condition $\vec{U} \cdot \hat{n} = 0$. The first term in (48) is smaller than or equal to zero because of (28). Hence, (45) emerges. For the central flux (29a) by using the equality in (28), the first term in (48) also vanishes. Hence, we obtain (46). $\square$

The following theorem shows that the enstrophy of the numerical solution of (1) is conserved or stable. Note that time is not yet discretized.

Theorem 9 *Consider the solution of (22) and (30) subject to slip flow boundary conditions (5). The enstrophy associated with this numerical solution, as given by (44), is a stable quantity:*

$$dS_h/dt \leq 0. \quad (49)$$

For the central flux (29a), (49) becomes an equality.

PROOF. We rewrite

$$dS_h/dt = \sum_{K \in \mathcal{T}_h} p_K(\partial_t \xi_h, \xi_h). \quad (50)$$

Using (30) with $v_h = \xi_h$ and Lemma 8, we obtain

$$\sum_{K \in \mathcal{T}_h} p_K(\partial_t \xi_h, \xi_h) = \sum_{K \in \mathcal{T}_h} R_K(\xi_h, \psi_h, \xi_h) \leq 0. \quad (51)$$

Combining (50) and (51) yields (49). Using the central flux and Lemma 8, the equality in (49) is obtained. $\square$

Denote the numerical enstrophy at time level n by S_h^n . Using the θ -scheme, the numerical enstrophy is stable for certain choices of θ .

Theorem 10 *Consider the solution of (35) and (36) subject to slip flow boundary conditions (5). For $\theta \in [\frac{1}{2}, 1]$, the numerical enstrophy is stable:*

$$\Delta S_h^n \equiv S_h^{n+1} - S_h^n \leq 0. \quad (52)$$

For the central flux (29a) with $\theta = \frac{1}{2}$ the enstrophy is conserved:

$$\Delta S_h^n = 0. \quad (53)$$

PROOF. Take $v_h = \xi_h^{n+\theta}$ in (36) to obtain

$$\sum_{K \in \mathcal{T}_h} p_K(\Delta \xi_h^n, \xi_h^{n+\theta}) = \sum_{K \in \mathcal{T}_h} R_K(\xi_h^{n+\theta}, \psi_h^{n+\theta}, \xi_h^{n+\theta}). \quad (54)$$

Because the operator p_K is bilinear and symmetric, we find after using (54)

$$\Delta S_h^n = \sum_{K \in \mathcal{T}_h} p_K(\Delta \xi_h^n, \xi_h^{n+\theta}) - \left(\theta - \frac{1}{2}\right) \sum_{K \in \mathcal{T}_h} p_K(\Delta \xi_h^n, \Delta \xi_h^n) \quad (55)$$

$$= \sum_{K \in \mathcal{T}_h} R_K(\xi_h^{n+\theta}, \psi_h^{n+\theta}, \xi_h^{n+\theta}) - \left(\theta - \frac{1}{2}\right) \sum_{K \in \mathcal{T}_h} p_K(\Delta \xi_h^n, \Delta \xi_h^n). \quad (56)$$

If we choose $\theta \in [\frac{1}{2}, 1]$, then we find (52) because of Lemma 8 and $p_K(v_h, v_h) \geq 0$ for each $v_h \in \mathbf{V}_h^k$. If we choose $\theta = \frac{1}{2}$, then the second term of (56) vanishes. If, additionally, the central flux (29a) is used, then the first term vanishes as well because of Lemma 8. Hence, we obtain (53). $\square$

7 ERROR ESTIMATES

In this section, we will state error estimates for the numerical discretization discussed in this paper. The error analysis for the Euler equations presented in [10] has been extended to the numerical scheme given by (22) and (30). The main differences with the analysis in [10] concern the elliptic part of the problem and the fact that we make the dependence of the error on the (generalized) vorticity explicit. In addition, we impose a slightly less restrictive condition on the vorticity field than in [10], where the condition $\xi \in H^{k+1}(\Omega)$, with $k > 1$, is required. An error estimate which imposes minimal smoothness requirements on the (generalized) vorticity field will be published elsewhere since this rather technical analysis is outside the scope of this paper ([18]).

We now state the main error estimate:

Theorem 11 *Assume that Ω is a bounded domain with Lipschitz boundary $\partial\Omega$. In addition, we assume that the coefficients in (1c) satisfy $A, B \in C^{1,1}(\overline{\Omega})$, $D \in L^\infty(\Omega)$, with*

$$A_0 = \text{ess inf}\{A(x, y) \mid (x, y) \in \Omega\} > 0, \quad B_0 = \text{ess inf}\{B(x, y) \mid (x, y) \in \Omega\} \geq 0,$$

and that the vorticity field ξ belongs to $L^2([0, T], W_\infty^1(\Omega) \cap H^{k+1}(\Omega))$, with $k \geq 1$. Then the error in the DG finite element discretization (22) and (30),

with the numerical flux given by one of (29a)–(29c), subject to the slip flow boundary condition (5), on a quasi-uniform mesh $\mathcal{T}_h$ with sufficiently small mesh size $h \leq h_0 \leq 1$, can be estimated as

$$\begin{aligned} & \|\vec{u} - \vec{u}_h\|_{L^2(\Omega)} + \|\xi - \xi_h\|_{L^2(\Omega)} \\ & \leq Ch^k \exp\left(CT \sup_{t \in [0, T]} (\|\nabla \xi(\cdot, t)\|_{L^\infty(\Omega)}, \|\xi(\cdot, t)\|_{H^{k+1}(\Omega)})\right) \\ & \quad \left(\|\xi(\cdot, 0)\|_{H^k(\Omega)}^2 + \int_0^T \|\xi(\cdot, t)\|_{H^{k+1}(\Omega)}^2 dt\right)^{\frac{1}{2}}, \end{aligned}$$

with k the order of the polynomial basis functions in the DG discretization and C a positive constant, independent of h , ξ and $\vec{u}$.

Corollary 12 *The L^∞ -norm of the error in the velocity field can be estimated as*

$$\|\vec{u} - \vec{u}_h\|_{L^\infty(\Omega)} \leq Ch^k \sup_{t \in [0, T]} \|\xi(\cdot, t)\|_{H^k(\Omega)} + Ch^{-1} \|\vec{u} - \vec{u}_h\|_{L^2(\Omega)}.$$

Remark 13 *An upper bound for the minimum mesh size h_0 is*

$$h_0 = \min\left(\frac{\sqrt{A_0}}{C_I C_R C_{AB} \sqrt{4(2b+1)}}, 1\right),$$

with $b \geq \frac{1}{2}A_0 - B_0$, C_R the regularity constant for the differential operator (1c) on the Lipschitz domain Ω , C_I the constant in the interpolation estimate for the projection from $H^2(\Omega)$ onto $\mathbf{W}_h^k$ and $C_{AB} = \max(\|A\|_{L^\infty(\Omega)}, \|B\|_{L^\infty(\Omega)})$.

Remark 14 *If $B_0 \geq \frac{1}{2}A_0 + \frac{1}{2}$ then there is no restriction on the mesh size h_0 and $h \leq 1$ is a sufficient condition for Theorem 11.*

Here $H^k(\Omega)$ denotes the Hilbert space of k -times weakly differentiable functions which are square integrable, including all derivatives up to the order k , $L^\infty(\Omega)$ is the space of essentially bounded functions, $W_\infty^1(\Omega)$ the space of essentially bounded functions with also a bounded weak derivative, and $C^{1,1}(\overline{\Omega})$ the space of Hölder continuous functions.

For a detailed proof and the definition of the function spaces and norms we refer to Appendix B.

8 VERIFICATION

In this section, we present some examples which test the numerical method. We implemented the finite element method described using the C++ programming language. Each of the applications discussed in Section 2 is tested with the

Runge-Kutta time discretization discussed in Section 5.3.1 for the central flux (29a) as well as the upwind flux (29b).

8.1 Example 1: Stuart Vortex

First, we consider the Stuart vortex, which is a stationary solution for the 2D Euler equations (see Section 2 and Appendix A.1). The coarsest mesh $\mathcal{T}_1$ used in the simulations is shown in Figure 2. The upper and lower boundary and the boundary of the island in the center of the domain coincide with streamlines of the exact solution. The left and right boundaries are periodic. On the upper and lower boundaries we prescribe a value of the streamfunction given by

$$\mathcal{C}_{low} = \mathcal{C}_{up} = \int_{\partial\Omega_{up}} A\vec{U} \cdot \hat{\tau} \, d\Gamma \approx 6.180637249. \quad (57)$$

On the boundary of the island in the center of the domain we impose the exact value of the streamfunction (see Remark 1).

We tested our implementation up to cubic basis functions, and used an unstructured quadrilateral basic grid. To obtain the orders of convergence for the L^1 -, L^2 - and L^∞ -errors, given in Tables 2 to 5, this basic grid was refined in a structured manner up to three times. For internal elements this grid refinement is straightforward, while for boundary elements the curvature of the boundary was taken into account. Note that vorticity and streamfunction converge as $O(h^{k+1})$ in the upwind case for the L^1 - and L^2 -errors, one order higher than the error estimates. Here and in the remaining tests, the absolute value of the error is larger for the central flux and the convergence is slower than for the upwind flux, but is in accordance with our error estimates. It means that the polynomial order k should be larger than one. The order of convergence for the L^∞ -error is also lower in all tests. Numerical results are shown in Figure 3 for the vorticity and streamfunction at time $t = 8$. The use of isoparametric elements at the curved boundaries is essential to reach convergence for $k \geq 2$. When the curvature at the boundary is approximated with low-order (piecewise linear) elements at the boundary, the inversion required to obtain ψ from the elliptic equation transports the local lack of accuracy instantly into the entire domain.

8.2 Example 2: A Traveling Wave Solution

Consider a traveling wave solution for the quasi-geostrophic equations on a β -plane (see Section 2 and Appendix A.2). The domain is $\Omega = [0, 2\pi]^2$ with periodic boundary conditions on the left and right boundary. On the upper

and lower boundaries we prescribe the circulation of the exact solution as $\mathcal{C}_{low} = \mathcal{C}_{up} = 0$. The solutions are seen to converge to $O(h^{k+1})$ for the vorticity and streamfunction in Tables 6 and 8 for the L^1 - and L^2 -errors using the upwind flux, and to $O(h^k)$ in Tables 7 and 9 for the central flux. For $k = 1$, we seem to have a slow convergence in L^∞ for the central flux, which may hint at an inconsistency. Numerical results are shown in Figure 4.

Theorems 6 and 9 state that the energy and enstrophy of the numerical discretization in space are conserved and stable for a spatial discretization only. To illustrate these theorems using a Runge-Kutta time discretization, it is, therefore, necessary to refine the time step. Consider the numerical solution on a 32×32 grid using various time steps for quadratic basis functions. Figures 5 and 6 show the relative change in energy and enstrophy of the numerical solution for different time steps for the upwind flux as well as the central flux. In Figure 5, the energy of the numerical solution appears to converge to a constant value for a decreasing time step. Figure 6(a) illustrates similar convergence for the enstrophy when using a central flux. However, for an upwind flux, the enstrophy is only a stable but not a conserved quantity, as follows from Figure 6(b). These results are consistent with Theorems 6 and 9. It is also verified from these results that RK3 is third order in time.

8.3 Example 3: Rigid Lid Equations

Consider a modified solution of the Stuart vortex for the rigid lid equations (see Section 2 and Appendix A.3). The left and right boundary ($x = 0$ and $x = 2\pi$) are periodic. On the upper boundary we prescribe a value for the streamfunction, while at the lower boundary we prescribe a value of the circulation given by (57). The solutions are seen to converge to $O(h^k)$ and $O(h^{k+1})$ for the vorticity and streamfunction in Tables 10 to 13. Numerical results for the streamfunction and vorticity are shown in Figure 7.

9 SUMMARY AND CONCLUSION

To summarize, the following results have been obtained:

- A generalized (potential) vorticity streamfunction formulation was defined, including not only the incompressible 2D Euler equations as in [10], but also the quasi-geostrophic and rigid lid equations. This formulation was shown to conserve energy and any weighted smooth function of the (potential) vorticity. In particular, enstrophy is conserved.

- Multiply connected and curved domains were considered, requiring the introduction of circulation around each connected piece of boundary and the use of isoparametric elements at curved boundaries for higher-order spatial discretizations. Otherwise, only a restricted set of initial conditions can be considered, and higher-order accuracy can not be obtained as any local reduction of accuracy due to improper boundary elements affects the global solution of the streamfunction.
- The generalized system was discretized using a DG finite element method for the vorticity equation, and a continuous Galerkin finite element method for the elliptic equation to determine the streamfunction. Particular attention was paid to efficiently implement the circulation boundary condition, which requires use of the modified function space (17).
- The implementation of the numerical method matched the properties of conservation of numerical energy and enstrophy. Hence, the function space used in the continuous Galerkin discretization was a subset of the discontinuous one: $W_h^k \subset V_h^k$, which was also implemented and tested numerically. It is important to stress that for geophysical applications, preservation of at least energy and enstrophy is required and generally accepted practice for long-time integrations.
- An implicit time discretization scheme was defined for which conservation of energy and stability of enstrophy for the system discretized in space and time was proven, but not tested numerically
- Three exact solutions, one for each of the above geophysical applications, were used to verify the numerical algorithm and implementation in curved and multiple connected domains. In these test cases, the explicit third order Runge-Kutta time discretization was used. Particular attention was paid to observe the tendency towards energy conservation and L^2 -stability.
- In accordance with our error estimates, the scheme was shown to converge, often an order higher than the $O(h^k)$ predicted by the analysis for k^{th} -order basis functions.

Further work will consist of extending the mixed continuous and DG finite element formulation to balanced equations in geophysical fluid dynamics (see, *e.g.*, [13]), and simulations of localized nonlinear vortical flows around separatrices in complex, curved domains. In addition, a discontinuous (space-time) Galerkin finite-element discretization of both hyperbolic and elliptic equations with energy and enstrophy conservation and stability is of interest.

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Fig. 1. An example of a domain, Ω , with slip flow boundary conditions. The boundary is partitioned into three separate parts: $\partial\Omega_{D_1}$, $\partial\Omega_{D_2}$ and $\partial\Omega_{D_3}$. Each of these three parts is a simply connected set.

Fig. 2. The coarsest mesh $\mathcal{T}_1$ with 720 elements.

Fig. 3. The (a) vorticity and (b) streamfunction field at $t = 8$ for the Stuart vortex.

Fig. 4. (a) Potential vorticity and (b) streamfunction field at $t = 12\pi$ for the traveling wave.

Fig. 5. The change in numerical energy $E = E_h$ (39), relative to the numerical energy, E_0 , at $t = 0$, on a 32×32 grid using (a) a central flux (29a) and (b) an upwind flux (29b) for quadratic basis functions and for different time steps.

Fig. 6. The change in numerical enstrophy $S = S_h$ (44), relative to the numerical enstrophy, S_0 , at $t = 0$, on a 32×32 grid using (a) a central flux and (b) an upwind flux for quadratic basis functions and for different time steps.

Fig. 7. (a) Vorticity and (b) streamfunction field at $t = 8$ for the rigid lid example.

Basis functions	CFL (Central Flux)	CFL (Upwind Flux)
constant	$\sqrt{3}$	1.25
linear	$\sqrt{3}/4$	0.409
quadratic	0.214	0.209
cubic	0.130	0.130

Table 1

CFL -condition for different orders of basis functions for the discretized, linearized one-dimensional vorticity equation with constant velocity u . The indicative time step for the RK3 time discretization in the nonlinear problem is then chosen such that $\Delta t \leq CFL d_K / |\vec{u}_h|_{max}$ with d_K the diameter of the inscribing circle of K and $||\vec{u}_h||_{max} = \max_{\vec{x} \in K} ||A\vec{U}_h||$ the maximum velocity in K .

Errors in ξ_h for the Stuart vortex example (Upwind Flux)							
k	$ \mathcal{T}_h $	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	720	1.94e-01	-	8.22e-02	-	1.22e-01	-
	2880	5.80e-02	1.74	3.21e-02	1.35	6.15e-02	0.99
	11520	1.52e-02	1.94	1.05e-02	1.61	2.60e-02	1.24
	46080	3.32e-03	2.19	2.68e-03	1.97	8.91e-03	1.55
2	720	1.47e-02	-	7.20e-03	-	1.61e-02	-
	2880	1.57e-03	3.23	9.86e-04	2.87	2.95e-03	2.44
	11520	1.51e-04	3.38	1.24e-04	2.99	5.00e-04	2.56
	46080	1.26e-05	3.58	1.33e-05	3.22	7.54e-05	2.73
3	720	1.48e-02	-	7.53e-03	-	1.69e-02	-
	2880	8.26e-04	4.17	5.60e-04	3.75	1.97e-03	3.10
	11520	4.39e-05	4.23	3.25e-05	4.11	1.82e-04	3.44
	46080	2.29e-06	4.26	1.76e-06	4.21	1.45e-05	3.65

Table 2

Errors in ξ_h for the Stuart vortex. The L^1 -, L^2 - and L^∞ -errors in ξ_h at $t = 8$ and orders of convergence for k^{th} -order basis functions with $k = 1, 2, 3$ are shown using the upwind flux.

Errors in ξ_h for the Stuart vortex example (Central Flux)							
k	$ \mathcal{T}_h $	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	720	5.83e-01	-	1.78e-01	-	4.04e-01	-
	2880	3.44e-01	0.76	1.06e-01	0.75	2.18e-01	0.89
	11520	2.18e-01	0.66	6.67e-02	0.66	2.03e-01	0.10
	46080	1.24e-01	0.81	3.82e-02	0.80	1.02e-01	0.99
2	720	1.54e-01	-	4.93e-02	-	9.45e-02	-
	2880	1.35e-02	3.51	4.75e-03	3.38	1.66e-02	2.51
	11520	1.85e-03	2.87	6.35e-04	2.90	1.86e-03	3.16
	46080	2.36e-04	2.97	7.98e-05	2.99	3.44e-04	2.43
3	720	6.34e-02	-	1.95e-02	-	4.28e-02	-
	2880	8.03e-03	2.98	2.56e-03	2.93	6.99e-03	2.61
	11520	9.21e-04	3.12	3.11e-04	3.04	1.30e-03	2.43
	46080	1.13e-04	3.02	3.89e-05	3.00	1.91e-04	2.76

Table 3

Errors in ξ_h for the Stuart vortex at $t = 8$ using the central flux.

Errors in ψ_h for the Stuart vortex example (Upwind Flux)							
k	$ \mathcal{T}_h $	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	720	8.13e-02	-	1.90e-02	-	1.26e-02	-
	2880	1.78e-02	2.19	4.20e-03	2.18	3.50e-03	1.85
	11520	4.22e-03	2.08	1.00e-03	2.06	9.58e-04	1.87
	46080	1.02e-03	2.05	2.44e-04	2.04	2.69e-04	1.83
2	720	4.99e-03	-	1.15e-03	-	8.61e-04	-
	2880	3.53e-04	3.82	8.10e-05	3.83	9.48e-05	3.18
	11520	2.42e-05	3.87	5.97e-06	3.76	1.64e-05	2.53
	46080	1.93e-06	3.65	5.51e-07	3.44	2.45e-06	2.75
3	720	4.39e-03	-	1.02e-03	-	7.41e-04	-
	2880	3.12e-04	3.81	6.96e-05	3.88	5.34e-05	3.79
	11520	1.92e-05	4.02	4.25e-06	4.03	3.86e-06	3.79
	46080	1.26e-06	3.94	2.77e-07	3.94	2.96e-07	3.71

Table 4

Errors in ψ_h for the Stuart vortex. The L^1 -, L^2 - and L^∞ -errors in ξ_h at $t = 8$ and orders of convergence for k^{th} -order basis functions are shown using the upwind flux.

Errors in ψ_h for the Stuart vortex example (Central Flux)							
k	$ \mathcal{T}_h $	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	720	9.68e-02	-	2.27e-02	-	1.26e-02	-
	2880	2.40e-02	2.01	5.70e-03	1.99	3.73e-03	1.76
	11520	5.94e-03	2.02	1.47e-03	1.96	1.18e-03	1.66
	46080	1.61e-03	1.89	3.83e-04	1.94	3.65e-04	1.69
2	720	6.04e-03	-	1.51e-03	-	1.02e-03	-
	2880	3.58e-04	4.08	8.32e-05	4.18	9.39e-05	3.44
	11520	2.40e-05	3.90	5.98e-06	3.80	1.64e-05	2.51
	46080	1.93e-06	3.63	5.52e-07	3.44	2.45e-06	2.74
3	720	4.35e-03	-	1.04e-03	-	7.90e-04	-
	2880	3.04e-04	3.84	6.84e-05	3.92	5.48e-05	3.85
	11520	1.91e-05	3.99	4.24e-06	4.01	3.85e-06	3.83
	46080	1.26e-06	3.92	2.78e-07	3.93	2.96e-07	3.70

Table 5
Errors in ψ_h for the Stuart vortex at $t = 8$ using the central flux.

Errors in ξ_h for the traveling wave example (Upwind Flux)							
k	$\mathcal{T}_h$	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	8×8	1.27e+01	-	2.50e+00	-	9.57e-01	-
	16×16	3.18e+00	1.99	6.15e-01	2.02	2.35e-01	2.03
	32×32	7.41e-01	2.10	1.44e-01	2.09	5.48e-02	2.10
	64×64	1.77e-01	2.07	3.47e-02	2.05	1.54e-02	1.83
2	8×8	8.67e-01	-	1.76e-01	-	8.04e-02	-
	16×16	7.93e-02	3.45	1.72e-02	3.35	1.25e-02	2.68
	32×32	1.28e-02	2.63	3.31e-03	2.38	2.82e-03	2.15
	64×64	1.86e-03	2.79	6.25e-04	2.41	8.69e-04	1.70
3	8×8	5.02e-02	-	1.16e-02	-	1.24e-02	-
	16×16	2.00e-03	4.65	5.28e-04	4.45	8.14e-04	3.93
	32×32	1.09e-04	4.19	2.95e-05	4.16	9.22e-05	3.14
	64×64	6.90e-06	3.98	1.81e-06	4.02	9.49e-06	3.28

Table 6

Errors in ξ_h for the traveling wave example. The L^1 -, L^2 - and L^∞ -errors in ξ_h at $t = 12\pi$ and orders of convergence for k^{th} -order basis functions are shown using the upwind flux.

Errors in ξ_h for the traveling wave example (Central Flux)							
k	$\mathcal{T}_h$	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	8×8	3.07e+01	-	5.95e+00	-	2.93e+00	-
	16×16	1.16e+01	1.41	2.39e+00	1.32	1.51e+00	0.96
	32×32	6.65e+00	0.80	1.43e+00	0.74	1.55e+00	-0.04
	64×64	9.03e-01	2.88	2.06e-01	2.79	2.70e-01	2.52
2	8×8	1.41e+00	-	2.99e-01	-	2.26e-01	-
	16×16	1.93e-01	2.87	4.06e-02	2.88	3.50e-02	2.69
	32×32	2.97e-02	2.71	6.18e-03	2.71	5.10e-03	2.78
	64×64	6.36e-03	2.22	1.34e-03	2.20	1.55e-03	1.72
3	8×8	3.35e-01	-	7.13e-02	-	6.98e-02	-
	16×16	3.09e-02	3.44	6.54e-03	3.45	7.36e-03	3.25
	32×32	4.37e-03	2.82	9.35e-04	2.81	1.15e-03	2.68
	64×64	3.20e-04	3.77	6.84e-05	3.77	9.72e-05	3.57

Table 7

Errors in ξ_h for the traveling wave example at $t = 12\pi$ using the central flux.

Errors in ψ_h for the traveling wave example (Upwind Flux)							
k	$\mathcal{T}_h$	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	8×8	4.31e+00	-	8.31e-01	-	2.67e-01	-
	16×16	1.06e+00	2.03	2.04e-01	2.03	6.47e-02	2.05
	32×32	2.45e-01	2.11	4.75e-02	2.10	1.53e-02	2.08
	64×64	5.80e-02	2.08	1.13e-02	2.07	3.66e-03	2.06
2	8×8	1.00e-01	-	1.98e-02	-	9.10e-03	-
	16×16	9.23e-03	3.44	1.77e-03	3.49	7.11e-04	3.68
	32×32	1.09e-03	3.08	1.96e-04	3.18	6.99e-05	3.35
	64×64	1.36e-04	3.00	2.43e-05	3.01	8.18e-06	3.10
3	8×8	1.04e-02	-	2.28e-03	-	1.26e-03	-
	16×16	4.53e-04	4.52	1.14e-04	4.33	7.59e-05	4.06
	32×32	2.70e-05	4.07	6.91e-06	4.04	4.72e-06	4.01
	64×64	1.67e-06	4.02	4.29e-07	4.01	2.92e-07	4.01

Table 8

Errors in ψ_h for the traveling wave. The L^1 -, L^2 - and L^∞ -errors in ξ_h at $t = 12\pi$ and orders of convergence for k^{th} -order basis functions are shown using the upwind flux.

Errors in ψ_h for the traveling wave example (Central Flux)							
k	$\mathcal{T}_h$	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	8×8	8.72e+00	-	1.63e+00	-	5.05e-01	-
	16×16	2.76e+00	1.66	5.21e-01	1.64	1.70e-01	1.57
	32×32	1.09e+00	1.34	2.05e-01	1.35	6.89e-02	1.30
	64×64	5.32e-02	4.36	1.01e-02	4.34	3.25e-03	4.41
2	8×8	1.27e-01	-	2.59e-02	-	1.16e-02	-
	16×16	8.80e-03	3.85	1.65e-03	3.97	7.05e-04	4.04
	32×32	1.10e-03	3.00	2.03e-04	3.03	8.05e-05	3.13
	64×64	1.37e-04	3.00	2.50e-05	3.02	8.22e-06	3.29
3	8×8	1.15e-02	-	2.44e-03	-	1.44e-03	-
	16×16	2.13e-03	2.43	4.19e-04	2.54	1.53e-04	3.23
	32×32	8.90e-05	4.58	1.77e-05	4.57	7.50e-06	4.35
	64×64	2.30e-06	5.27	5.03e-07	5.13	3.07e-07	4.61

Table 9

Errors in ψ_h for the traveling wave at $t = 12\pi$ using the central flux.

Errors in ξ_h for the rigid lid example (Upwind Flux)							
k	$\mathcal{T}_h$	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	8×8	3.18e-01	-	1.19e-01	-	1.97e-01	-
	16×16	8.14e-02	1.97	3.79e-02	1.65	8.88e-02	1.15
	32×32	1.92e-02	2.09	8.70e-03	2.12	2.45e-02	1.86
	64×64	4.80e-03	2.00	2.13e-03	2.03	6.23e-03	1.97
2	8×8	7.28e-02	-	3.76e-02	-	4.49e-02	-
	16×16	8.73e-03	3.06	5.21e-03	2.85	8.45e-03	2.41
	32×32	1.12e-03	2.97	6.90e-04	2.92	1.25e-03	2.75
	64×64	1.41e-04	2.98	8.78e-05	2.97	1.61e-04	2.96
3	8×8	1.36e-02	-	5.70e-03	-	1.09e-02	-
	16×16	1.08e-03	3.66	7.19e-04	2.99	1.70e-03	2.68
	32×32	5.73e-05	4.23	4.33e-05	4.05	1.89e-04	3.17
	64×64	3.46e-06	4.05	2.61e-06	4.05	1.33e-05	3.83

Table 10

Errors in ξ_h for the rigid lid example. The L^1 -, L^2 - and L^∞ -errors in ξ_h at $t = 8$ and orders of convergence for k^{th} -order basis functions using the upwind flux are shown.

Errors in ξ_h for the rigid lid example (Central Flux)							
k	$\mathcal{T}_h$	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	8×8	7.07e-01	-	1.95e-01	-	2.36e-01	-
	16×16	2.35e-01	1.59	6.93e-02	1.49	7.46e-02	1.66
	32×32	3.85e-02	2.61	1.24e-02	2.48	2.75e-02	1.44
	64×64	7.34e-03	2.39	2.57e-03	2.27	7.03e-03	1.97
2	8×8	3.05e-01	-	8.15e-02	-	7.71e-02	-
	16×16	3.40e-02	3.17	1.01e-02	3.01	1.28e-02	2.59
	32×32	4.28e-03	2.99	1.23e-03	3.04	1.55e-03	3.04
	64×64	5.41e-04	2.98	1.58e-04	2.96	2.46e-04	2.66
3	8×8	3.71e-02	-	1.04e-02	-	1.30e-02	-
	16×16	4.26e-03	3.12	1.27e-03	3.03	2.02e-03	2.68
	32×32	2.38e-04	4.16	7.05e-05	4.17	1.88e-04	3.42
	64×64	1.34e-05	4.15	4.09e-06	4.11	1.33e-05	3.83

Table 11

Errors in ξ_h for the rigid lid example at $t = 8$ using the central flux.

Errors in ψ_h for the rigid lid example (Upwind Flux)							
k	$\mathcal{T}_h$	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	8×8	4.48e-01	-	2.19e-01	-	2.55e-01	-
	16×16	1.18e-01	1.92	7.31e-02	1.58	1.30e-01	0.97
	32×32	2.98e-02	1.99	1.84e-02	1.99	4.01e-02	1.70
	64×64	7.45e-03	2.00	4.60e-03	2.00	1.06e-02	1.92
2	8×8	8.51e-02	-	5.13e-02	-	7.47e-02	-
	16×16	9.18e-03	3.21	5.68e-03	3.18	1.08e-02	2.79
	32×32	1.12e-03	3.03	6.99e-04	3.02	1.34e-03	3.02
	64×64	1.42e-04	2.99	8.80e-05	2.99	1.68e-04	3.00
3	8×8	1.36e-02	-	5.92e-03	-	1.06e-02	-
	16×16	1.07e-03	3.67	7.49e-04	2.98	1.90e-03	2.49
	32×32	5.71e-05	4.23	4.36e-05	4.10	1.98e-04	3.26
	64×64	3.46e-06	4.05	2.62e-06	4.06	1.37e-05	3.85

Table 12

Errors in ψ_h for the rigid lid example. The L^1 -, L^2 - and L^∞ -errors in ξ_h at $t = 8$ and orders of convergence for k^{th} -order basis functions are shown using the upwind flux.

Errors in ψ_h for the rigid lid example (Central Flux)							
k	$\mathcal{T}_h$	L_1 -error	order	L_2 -error	order	L_∞ -error	order
1	8×8	4.47e-01	-	2.19e-01	-	2.55e-01	-
	16×16	1.19e-01	1.92	7.36e-02	1.57	1.33e-01	0.94
	32×32	2.98e-02	1.99	1.84e-02	2.00	4.04e-02	1.72
	64×64	7.45e-03	2.00	4.60e-03	2.00	1.06e-02	1.93
2	8×8	8.64e-02	-	5.23e-02	-	7.51e-02	-
	16×16	9.22e-03	3.23	5.71e-03	3.19	1.09e-02	2.78
	32×32	1.12e-03	3.04	6.99e-04	3.03	1.34e-03	3.03
	64×64	1.42e-04	2.99	8.80e-05	2.99	1.68e-04	3.00
3	8×8	1.37e-02	-	5.94e-03	-	1.08e-02	-
	16×16	1.07e-03	3.67	7.50e-04	2.99	1.90e-03	2.51
	32×32	5.71e-05	4.23	4.36e-05	4.10	1.98e-04	3.27
	64×64	3.46e-06	4.05	2.62e-06	4.06	1.37e-05	3.85

Table 13

Errors in ψ_h for the rigid lid example at $t = 8$ using the central flux.

A EXACT SOLUTIONS

Exact solutions of (1) are given for the examples used in Section 8.

A.1 Stuart Vortex

The Stuart vortex ([17], [6]) is an exact stationary solution of the Euler equations (2) with

$$\xi(x, y, t) = 1/(a \cosh y + \sqrt{a^2 - 1} \cos x)^2 \quad \text{and} \quad (\text{A.1a})$$

$$\psi(x, y, t) = \log(a \cosh y + \sqrt{a^2 - 1} \cos x) \quad (\text{A.1b})$$

with $a \geq 1$. The domain used in Section 8.1 is

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 2\pi, K_1 \leq \psi(x, y, 0) \leq K_2\} \quad (\text{A.2})$$

with $K_i = \log(ac_i + a + \sqrt{a^2 - 1})$ for $c_1 > 0$ and $-2\sqrt{a^2 - 1}/a < c_2 < 0$. We took $c_1 = -\sqrt{5}/2$, $c_2 = 2$ and $a = 3/2$. The left and right boundaries of the domain are periodic. The remaining parts of the boundary consist of three connected parts:

$$\partial\Omega_{up} = \{(x, y) \in \Omega \mid y \geq 0, \psi(x, y, 0) = K_2\} \quad (\text{A.3a})$$

$$\partial\Omega_{down} = \{(x, y) \in \Omega \mid y \leq 0, \psi(x, y, 0) = K_2\} \quad (\text{A.3b})$$

$$\partial\Omega_{in} = \{(x, y) \in \Omega \mid \psi(x, y, 0) = K_1\}. \quad (\text{A.3c})$$

A.2 Traveling Wave Example

A traveling wave solution for the quasi-geostrophic equations (3) is

$$\xi(x, y, t) = -3 \sin(x - ct) \sin(y) - y/2 \quad (\text{A.4a})$$

$$\psi(x, y, t) = \sin(x - ct) \sin(y). \quad (\text{A.4b})$$

In Section 8.2, the wave speed $c = 1/6$, and $A = B = 1$ and $D = -y/2$.

A.3 Rigid Lid Example

An exact stationary solution of the rigid lid equations (4) has been constructed based on the Stuart vortex. In (1), we take $B = 0$, $D = 0$ and

$$A = 1/H = a \cosh y + \sqrt{a^2 - 1} \cos x. \quad (\text{A.5})$$

The stationary solution of the system is now given by

$$\xi(x, y, t) = -\psi(x, y, t) = 1 / \left(a \cosh y + \sqrt{a^2 - 1} \cos x \right) \quad (\text{A.6a})$$

with $a \geq 0$. In Section 8.3, $a = 3/2$. Note that the real velocity of the fluid (not the depth integrated velocity), given by $\vec{u} = (1/H)\nabla^\perp \psi$, is equal and opposite to the velocity of the Stuart vortex solution given in Section 8.1.

B PROOF OF THEOREM 11

In this Appendix we give a proof of the error estimate for the continuous-discontinuous Galerkin discretization for generalized vorticity dynamics, stated in Theorem 11.

We denote with Ω a bounded domain $\Omega \subset \mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$. We define the Hilbert space $H^k(\Omega)$ of k -times weakly differentiable functions which are square integrable, including all derivatives up to the order k . The norm in $H^k(\Omega)$ is defined as $\|w\|_{H^k(\Omega)} = (\sum_{|\alpha| \leq k} \|D^\alpha w\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$ and the seminorm as $|w|_{H^k(\Omega)} = (\sum_{|\alpha|=k} \|D^\alpha w\|_{L^2(\Omega)}^2)^{\frac{1}{2}}$, with $L^2(\Omega) = H^0(\Omega)$. Here $D^\alpha w$ denotes the weak derivative of order $|\alpha|$ of w with α the multi-index symbol, see [4] Section 1.2. For clarity we use in this section $\|w\|_{L^2(\Omega)}$ for the L^2 norm instead of $\|w\|_\Omega$ used elsewhere in this paper. The space of essentially bounded functions is denoted as $L^\infty(\Omega)$ and is equipped with the norm $\|w\|_{L^\infty(\Omega)} = \text{ess sup}_{(x,y) \in \Omega} |w(x, y)|$. For any nonnegative integer m , $C^m(\Omega)$ denotes the space of all functions w which, together with all their partial derivatives $D^\alpha w$ of order $|\alpha| \leq m$, are continuous in Ω . For $0 < \lambda \leq 1$, we define $C^{m,\lambda}(\overline{\Omega})$ to be the subspace of $C^m(\overline{\Omega})$, with $\overline{\Omega}$ the closure of Ω , consisting of those functions w for which for $0 \leq \alpha \leq m$, $D^\alpha w$ satisfies in Ω a Hölder condition of exponent λ , that is there exists a constant C such that

$$|D^\alpha w(x) - D^\alpha w(y)| \leq C|x - y|^\lambda, \quad \forall x, y \in \Omega.$$

Define the projection $P_K : H^{k+1}(\Omega) \mapsto \mathbf{V}_h^k$, with for each $K \in \mathcal{T}_h$

$$(P_K v, w/A)_K = (v, w/A)_K, \quad \forall w \in \mathbf{V}_h^k, \quad (\text{B.1})$$

for which the following interpolation estimates are available:

$$\|\xi - P_K \xi\|_{H^1(K)} \leq C_I h^k |\xi|_{H^{k+1}(\tilde{K})} \quad (\text{B.2})$$

$$\|\xi - P_K \xi\|_{L^2(\partial K)} \leq C h^{k+1/2} \|\xi\|_{H^{k+1}(\tilde{K})}, \quad (\text{B.3})$$

with $\tilde{K}$ the patch of elements which connect to an edge or a vertex of element K . For a proof of (B.2)-(B.3), see Theorem 4.4 and Remark 8 in [2]. We

also introduce the error in the vorticity and streamfunction, $\epsilon = \xi - \xi_h$ and $\delta = \psi - \psi_h$, respectively, and the projection of the vorticity error $\epsilon_h = P_K \epsilon = P_K \xi - \xi_h$.

The first step in the proof of Theorem 11 is to find a relation between the errors δ and ϵ . Using Theorem 5.6.8 in [4] we directly obtain a coercivity estimate for the bilinear form L defined in (19)

$$L(w, w) \geq \frac{1}{2} A_0 \|w\|_{H^1(\Omega)}^2 - b \|w\|_{L^2(\Omega)}^2, \quad \forall w \in H^1(\Omega), \quad (\text{B.4})$$

with $A_0 = \text{ess inf}\{A(x, y) \mid (x, y) \in \Omega\} > 0$ and $b \geq \frac{1}{2} A_0 - B_0$, where $B_0 = \text{ess inf}\{B(x, y) \mid (x, y) \in \Omega\}$.

If we subtract (22) from (18) then we obtain the following relation for the error in the streamfunction δ

$$L(\delta, w) = -(\epsilon/A, w)_\Omega, \quad \forall w \in \mathbf{W}_h^k. \quad (\text{B.5})$$

Taking $w = \delta$ in (B.5) and using the coercivity estimate for L , (B.4), we obtain

$$\begin{aligned} \frac{1}{2} A_0 \|\delta\|_{H^1(\Omega)}^2 &\leq L(\delta, \delta) + b \|\delta\|_{L^2(\Omega)}^2 \\ &= -(\epsilon/A, \delta)_\Omega + b \|\delta\|_{L^2(\Omega)}^2 \end{aligned} \quad (\text{B.6})$$

$$\leq \|\epsilon/A\|_{L^2(\Omega)} \|\delta\|_{L^2(\Omega)} + b \|\delta\|_{L^2(\Omega)}^2 \quad (\text{B.7})$$

$$\leq \frac{1}{2} \|\epsilon/A\|_{L^2(\Omega)}^2 + (b + \frac{1}{2}) \|\delta\|_{L^2(\Omega)}^2. \quad (\text{B.8})$$

In (B.6) we used the error equation (B.5), in (B.7) the Schwarz inequality and finally the arithmetic-geometric mean inequality in (B.8).

The next step is to provide an error bound for the $L^2(\Omega)$ norm of δ . For this we consider the adjoint equation

$$-\nabla \cdot (A \nabla w) + Bw = \phi, \quad \text{in } \Omega, \quad (\text{B.9})$$

with $\phi \in L^2(\Omega)$, and

$$\hat{n} \cdot \nabla w = 0, \quad \text{at } \partial\Omega. \quad (\text{B.10})$$

Then, for $w \in V$, with $V := H^1(\Omega)$ if $B \neq 0$ and $V := \{v \in H^1(\Omega) \mid \int_\Omega v d\Omega = 0\}$ if $B = 0$, we obtain

$$(\delta, \phi)_\Omega = -(\delta, \nabla \cdot (A \nabla w))_\Omega + (\delta, Bw)_\Omega = L(\delta, w). \quad (\text{B.11})$$

In (B.11) we use, respectively, (B.9), integrate by parts and apply the boundary condition (B.10). If we add now (B.5), where we denote $w \in \mathbf{W}_h^k$ as w_h ,

and (B.11) then we obtain

$$\begin{aligned}
(\delta, \phi)_\Omega &= L(\delta, w - w_h) + (\epsilon/A, w - w_h)_\Omega - (\epsilon/A, w)_\Omega \\
&\leq C_{AB} \|\delta\|_{H^1(\Omega)} \inf_{\chi \in \mathbf{W}_h^k} \|w - \chi\|_{H^1(\Omega)} + \|\epsilon/A\|_{L^2(\Omega)} \inf_{\chi \in \mathbf{W}_h^k} \|w - \chi\|_{L^2(\Omega)} \\
&\quad + \|\epsilon/A\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \\
&\leq C_{AB} C_I h \|\delta\|_{H^1(\Omega)} \|w\|_{H^2(\Omega)} + \|\epsilon/A\|_{L^2(\Omega)} (C_I h^2 + 1) \|w\|_{H^2(\Omega)} \quad (\text{B.12})
\end{aligned}$$

with $C_{AB} = \max(\|A\|_{L^\infty(\Omega)}, \|B\|_{L^\infty(\Omega)})$ and we used the Schwarz inequality in the second step, and finally the interpolation estimate Theorem 4.4.20, p. 109 in [4] with constant C_I . An upper bound for $\|w\|_{H^2(\Omega)}$ can be obtained using the fact that the differential operator in (B.9) is strongly elliptic (the related bilinear form L in (19) is a symmetric and coercive bilinear form on $H^1(\Omega)$, see (B.4)) and using the regularity estimate given by Theorem 4.18(ii), pp. 137-138 in [12], where we assume that $A, B \in C^{m,1}(\overline{\Omega})$ with $m \geq 0$. This, together with the boundary condition (B.10), implies that

$$\|w\|_{H^{m+2}(\Omega_1)} \leq C \|w\|_{H^1(\Omega_2)} + C \|\phi\|_{H^m(\Omega_2)}, \quad m \geq 0, \quad (\text{B.13})$$

where $\Omega_1 = G_1 \cap \Omega$, $\Omega_2 = G_2 \cap \Omega$, with G_1 and G_2 open subsets of $\mathbb{R}^2$ such that $\overline{G_1}$ is a compact subset of G_2 . Here, G_1 intersects the boundary of Ω and G_2 has a smooth boundary not necessarily completely contained in Ω . An estimate for $\|w\|_{H^1(\Omega)}$ satisfying (B.9) together with the boundary condition (B.10) can be obtained directly from Theorem 4.10(i), pp. 128-129 in [12]. Introducing this result into (B.13) we obtain

$$\|w\|_{H^{m+2}(\Omega)} \leq C_R \|\phi\|_{H^m(\Omega)}, \quad m \geq 0, \quad \forall w \in V, \quad (\text{B.14})$$

where we used that Ω can be covered by a finite number of sets Ω_1 .

If we insert (B.14), with $m = 0$, into (B.12) and set ϕ equal to δ , then we obtain the estimate

$$\|\delta\|_{L^2(\Omega)} \leq C_I C_R C_{AB} h \|\delta\|_{H^1(\Omega)} + (C_I h^2 + 1) C_R \|\epsilon/A\|_{L^2(\Omega)}. \quad (\text{B.15})$$

Next, we combine (B.8) and (B.15) and use the arithmetic-geometric mean inequality which results in

$$\begin{aligned}
\frac{1}{2} A_0 \|\delta\|_{H^1(\Omega)}^2 &\leq \left(\frac{1}{2} + (2b+1) C_R^2 (C_I h^2 + 1)^2 \right) \|\epsilon/A\|_{L^2(\Omega)}^2 \\
&\quad + (2b+1) C_I^2 C_R^2 C_{AB}^2 h^2 \|\delta\|_{H^1(\Omega)}^2. \quad (\text{B.16})
\end{aligned}$$

Assume now that the mesh size $h \leq h_0$, with $(2b+1) C_I^2 C_R^2 C_{AB}^2 h_0^2 = \frac{1}{4} A_0$, or equivalently,

$$h_0 = \frac{\sqrt{A_0}}{C_I C_R C_{AB} \sqrt{4(2b+1)}},$$

which is the condition on h_0 stated in Remark 13, then we obtain the following relation between the errors δ and ϵ

$$\|\delta\|_{H^1(\Omega)} \leq \frac{2}{\sqrt{A_0}} \left(\frac{1}{2} + (2b+1)C_R^2(C_I h_0^2 + 1)^2 \right)^{\frac{1}{2}} \|\epsilon/A\|_{L^2(\Omega)} = C_E \|\epsilon/A\|_{L^2(\Omega)}. \quad (\text{B.17})$$

Note, if $b + \frac{1}{2} \leq 0$ in (B.16), or equivalently $B_0 \geq \frac{1}{2}A_0 + \frac{1}{2}$, then $\frac{1}{2}A_0 - (2b+1)C_I^2 C_R^2 C_{AB}^2 h^2 \geq 0$ and there is no restriction on the minimum mesh size h_0 anymore, which proves Remark 14.

Consider now the error in the vorticity. Subtract (30) from (25) and use (1b), then the error equation for the vorticity is equal to

$$\begin{aligned} (\partial_t \epsilon/A, v)_K &= (\xi \nabla^\perp \psi - \xi_h \nabla^\perp \psi_h, \nabla v)_K \\ &\quad - (\xi \nabla^\perp \psi \cdot \hat{n} - \hat{\xi}_h \nabla^\perp \psi_h \cdot \hat{n}, v^-)_{\partial K}, \quad \forall K \in \mathcal{T}_h, \forall v \in \mathbf{V}_h^k, \end{aligned} \quad (\text{B.18})$$

where the superscripts $-$ and $+$ in this section refer to the trace at the element boundary ∂K taken from the inside and outside of the element, respectively. Take $v = \epsilon_h$ and use that $(\partial_t \epsilon/A, v)_K = (\partial_t \epsilon_h/A, v)_K$ for each $v \in \mathbf{V}_h^k$ and $K \in \mathcal{T}_h$ to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\epsilon_h/A, \epsilon_h)_\Omega &= \sum_{K \in \mathcal{T}_h} (\xi \nabla^\perp \psi - \xi_h \nabla^\perp \psi_h, \nabla \epsilon_h)_K \\ &\quad - \sum_{K \in \mathcal{T}_h} (\xi \nabla^\perp \psi \cdot \hat{n} - \hat{\xi}_h \nabla^\perp \psi_h \cdot \hat{n}, \epsilon_h^-)_{\partial K}. \end{aligned} \quad (\text{B.19})$$

Rewriting the first term on the right-hand-side of (B.19) results in

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} (\xi \nabla^\perp \psi - \xi_h \nabla^\perp \psi_h, \nabla \epsilon_h)_K &= \sum_{K \in \mathcal{T}_h} (\xi \nabla^\perp \delta + \epsilon \nabla^\perp \psi_h, \nabla \epsilon_h)_K \\ &= \sum_{K \in \mathcal{T}_h} \left\{ (\xi \nabla^\perp \delta, \nabla \epsilon_h)_K + (\epsilon \nabla^\perp \psi_h, \nabla \epsilon)_K - (\epsilon \nabla^\perp \psi_h, \nabla (\xi - P_K \xi))_K \right\} \\ &= \sum_{K \in \mathcal{T}_h} \left\{ -(\epsilon_h \nabla^\perp \delta, \nabla \xi)_K + (\xi \epsilon_h^-, \nabla^\perp \delta \cdot \hat{n})_{\partial K} + \frac{1}{2} ((\epsilon^-)^2, \nabla^\perp \psi_h \cdot \hat{n})_{\partial K} \right. \\ &\quad \left. - (\epsilon \nabla^\perp \psi_h, \nabla (\xi - P_K \xi))_K \right\}. \end{aligned} \quad (\text{B.20})$$

Plugging this result into (B.19) gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (\epsilon_h/A, \epsilon_h)_\Omega &= -(\epsilon_h, \nabla^\perp \delta \cdot \nabla \xi)_\Omega - \sum_{K \in \mathcal{T}_h} (\epsilon, \nabla^\perp \psi_h \cdot \nabla (\xi - P_K \xi))_K \\ &\quad + \sum_{K \in \mathcal{T}_h} \left\{ (\epsilon_h^-, \nabla^\perp \delta \cdot \hat{n})_{\partial K} + \frac{1}{2} ((\epsilon^-)^2, \nabla^\perp \psi_h \cdot \hat{n})_{\partial K} \right. \\ &\quad \left. - (\xi \nabla^\perp \psi \cdot \hat{n} - \hat{\xi}_h \nabla^\perp \psi_h \cdot \hat{n}, \epsilon_h^-)_{\partial K} \right\}. \end{aligned} \quad (\text{B.21})$$

Define $\hat{\epsilon} = \xi - \hat{\xi}_h$ and rewrite the boundary terms in (B.21) as follows

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_h} \left\{ \left(\epsilon_h^-, \nabla^\perp \delta \cdot \hat{n} \right)_{\partial K} + \frac{1}{2} \left((\epsilon^-)^2, \nabla^\perp \psi_h \cdot \hat{n} \right)_{\partial K} \right. \\
& \quad \left. - \left(\xi \nabla^\perp \psi \cdot \hat{n} - \hat{\xi}_h \nabla^\perp \psi_h \cdot \hat{n}, \epsilon_h^- \right)_{\partial K} \right\} \\
& = \sum_{K \in \mathcal{T}_h} \left(\frac{1}{2} (\epsilon^-)^2 - \epsilon^- \hat{\epsilon}, \nabla^\perp \psi_h \cdot \hat{n} \right)_{\partial K} + \sum_{K \in \mathcal{T}_h} \left(\hat{\epsilon} (\xi - (P_K \xi)^-), \nabla^\perp \psi_h \cdot \hat{n} \right)_{\partial K}
\end{aligned} \tag{B.22}$$

to obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} (\epsilon_h / A, \epsilon_h)_\Omega & = - \underbrace{\left(\epsilon_h, \nabla^\perp \delta \cdot \nabla \xi \right)_\Omega}_{(A)} - \underbrace{\sum_{K \in \mathcal{T}_h} \left(\epsilon, \nabla^\perp \psi_h \cdot \nabla (\xi - P_K \xi) \right)_K}_{(B)} \\
& + \underbrace{\sum_{K \in \mathcal{T}_h} \left(\frac{1}{2} (\epsilon^-)^2 - \epsilon^- \hat{\epsilon}, \nabla^\perp \psi_h \cdot \hat{n} \right)_{\partial K}}_{(C)} + \underbrace{\sum_{K \in \mathcal{T}_h} \left(\hat{\epsilon} (\xi - (P_K \xi)^-), \nabla^\perp \psi_h \cdot \hat{n} \right)_{\partial K}}_{(D)}.
\end{aligned} \tag{B.23}$$

Now, we evaluate the terms (A), (B), (C) and (D) separately. For (A), we apply the Schwarz inequality, the relation $\|\nabla^\perp \delta\|_{L^2(\Omega)} = \|\nabla \delta\|_{L^2(\Omega)}$ and use (B.17), with $\epsilon = \epsilon_h + \xi - P_K \xi$, followed by the arithmetic-geometric mean inequality

$$\begin{aligned}
(\epsilon_h, \nabla^\perp \delta \cdot \nabla \xi)_\Omega & \leq \|\nabla \xi\|_{L^\infty(\Omega)} \|\epsilon_h\|_{L^2(\Omega)} \|\nabla \delta\|_{L^2(\Omega)} \\
& \leq C_E \|\nabla \xi\|_{L^\infty(\Omega)} \|\epsilon_h\|_{L^2(\Omega)} \left(\|\epsilon_h / A\|_{L^2(\Omega)} + \|(\xi - P_K \xi) / A\|_{L^2(\Omega)} \right) \\
& \leq C \|\nabla \xi\|_{L^\infty(\Omega)} \left(\frac{3}{2} \|\epsilon_h\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\xi - P_K \xi\|_{L^2(\Omega)}^2 \right). \tag{B.24}
\end{aligned}$$

For (B), we use the regularity estimate Theorem 4.18(i), pp. 137-138 in [12]. Together with the boundary condition $\psi = c_D(t)$ at $\partial\Omega_D$ (7), and the condition $A, B \in C^{m,1}(\overline{\Omega})$, this implies that

$$\|\psi\|_{H^{m+2}(\Omega_1)} \leq C \|\psi\|_{H^1(\Omega_2)} + C \|\xi\|_{H^m(\Omega_2)} + c_D(t) \left(\int_{\Gamma_2} dS \right)^{\frac{1}{2}}, \quad m \geq 0, \tag{B.25}$$

with $\Gamma_2 = \partial\Omega \cap \Omega_2$. An estimate for $\|\psi\|_{H^1(\Omega)}$ satisfying (19)-(20) together with the boundary condition (7) can be obtained directly from Theorem 4.10(i), pp. 128-129 in [12]. Introducing this result into (B.25) together with the inequality showing that $c_D(t)$ is bounded in terms of the vorticity (see [18]) we obtain

$$\|\psi\|_{H^{m+2}(\Omega)} \leq C \|\xi\|_{H^m(\Omega)}, \quad m \geq 0, \quad \forall \psi \in V, \tag{B.26}$$

where we used that Ω can be covered by a finite number of sets Ω_1 . Since $H^2(\Omega)$ is embedded in $L^\infty(\Omega)$ (Theorem 4.12 [1], p. 85) and using the regularity

estimate (B.26) this also implies

$$\begin{aligned}
\|\nabla^\perp \psi_h\|_{L^\infty(\Omega)} &\leq \|\nabla^\perp \delta\|_{L^\infty(\Omega)} + \|\nabla^\perp \psi\|_{L^\infty(\Omega)} \\
&\leq \|\nabla^\perp \delta\|_{L^\infty(\Omega)} + C\|\psi\|_{H^3(\Omega)} \\
&\leq \|\nabla^\perp \delta\|_{L^\infty(\Omega)} + C\|\xi\|_{H^1(\Omega)},
\end{aligned} \tag{B.27}$$

for $A, B \in C^{1,1}(\bar{\Omega})$. We also have the relation

$$\begin{aligned}
\|\nabla^\perp \delta\|_{L^\infty(\Omega)} &= \|\nabla(\psi - \psi_h)\|_{L^\infty(\Omega)} \\
&\leq \|\nabla(\psi - P_\Omega \psi)\|_{L^\infty(\Omega)} + \|\nabla(P_\Omega \psi - \psi_h)\|_{L^\infty(\Omega)} \\
&\leq Ch\|\psi\|_{H^3(\Omega)} + Ch^{-1}\|\nabla(P_\Omega \psi - \psi_h)\|_{L^2(\Omega)} \\
&\leq Ch\|\xi\|_{H^1(\Omega)} + Ch^{-1}\left(\|\nabla \delta\|_{L^2(\Omega)} + \|\nabla(\psi - P_\Omega \psi)\|_{L^2(\Omega)}\right) \\
&\leq Ch\|\xi\|_{H^1(\Omega)} + Ch^{-1}\|\delta\|_{H^1(\Omega)}, \quad m \geq 0, h > 0
\end{aligned} \tag{B.28}$$

with P_Ω a global Lagrangian interpolant. In the second inequality we use a standard interpolation estimate for P_Ω (Theorem 4.4.20 in [4]) and an inverse inequality (Theorem 4.5.11 in [4]). Next, we use the regularity estimate (B.26) and the triangle inequality, and, finally, for $\|\nabla(\psi - P_\Omega \psi)\|_{L^2(\Omega)}$ the interpolation estimate Theorem 4.4.20 in [4] and (B.26) again. Combining (B.27)-(B.28) and using (B.17), we obtain the estimate

$$\|\nabla^\perp \psi_h\|_{L^\infty(\Omega)} \leq C\left(\|\xi\|_{H^1(\Omega)} + \frac{1}{h}\|\epsilon\|_{L^2(\Omega)}\right), \quad 0 < h \leq 1,$$

and apply the Schwarz inequality to obtain

$$\begin{aligned}
\sum_{K \in \mathcal{T}_h} \left(\epsilon, \nabla^\perp \psi_h \cdot \nabla(\xi - P_K \xi) \right)_K &\leq \sum_{K \in \mathcal{T}_h} \|\nabla^\perp \psi\|_{L^\infty(K)} \|\epsilon\|_{L^2(K)} \|\nabla(\xi - P_K \xi)\|_{L^2(K)} \\
&\leq C\|\xi\|_{H^1(\Omega)} \sum_{K \in \mathcal{T}_h} \|\nabla(\xi - P_K \xi)\|_{L^2(K)} \left(\|\epsilon_h\|_{L^2(K)} + \|\xi - P_K \xi\|_{L^2(K)} \right) + \\
&\quad C \sum_{K \in \mathcal{T}_h} \frac{1}{h} \left(\|\epsilon_h\|_{L^2(K)} + \|\xi - P_K \xi\|_{L^2(K)} \right)^2 \|\nabla(\xi - P_K \xi)\|_{L^2(K)} \\
&\leq C\|\xi\|_{H^1(\Omega)} \sum_{K \in \mathcal{T}_h} \left(\|\epsilon_h\|_{L^2(K)}^2 + \|\xi - P_K \xi\|_{H^1(K)}^2 \right) + \\
&\quad C \sum_{K \in \mathcal{T}_h} h^{k-1} \|\xi\|_{H^{k+1}(K)} \left(\|\epsilon_h\|_{L^2(K)}^2 + \|\xi - P_K \xi\|_{L^2(K)}^2 \right) \\
&\leq C\|\xi\|_{H^{k+1}(\Omega)} \sum_{K \in \mathcal{T}_h} \left(\|\epsilon_h\|_{L^2(K)}^2 + \|\xi - P_K \xi\|_{H^1(K)}^2 \right), \quad k \geq 1, 0 \leq h \leq 1,
\end{aligned} \tag{B.29}$$

where the geometric mean inequality and the interpolation estimate (B.2) are used in the third step. For (C), we use the stability condition (28) and the relation $\hat{\epsilon} = \bar{\epsilon} - (\hat{\xi}_h - \bar{\xi}_h)$ with $\hat{\xi}_h = (1/2)(\xi_h^+ + \xi_h^-)$ and $\bar{\epsilon} = (1/2)(\epsilon^+ + \epsilon^-)$ to

obtain:

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_h} \left(\frac{1}{2} (\epsilon^-)^2 - \epsilon^- \hat{\epsilon}, \nabla^\perp \psi_h \cdot \hat{n} \right)_{\partial K} \\
&= -\frac{1}{2} \underbrace{\sum_{K \in \mathcal{T}_h} (\epsilon^- \epsilon^+, \nabla^\perp \psi_h \cdot \hat{n})_{\partial K}}_{=0} + \sum_{K \in \mathcal{T}_h} (\epsilon^- (\hat{\xi}_h - \bar{\xi}_h), \nabla^\perp \psi_h \cdot \hat{n})_{\partial K} \\
&= \frac{1}{2} \sum_{K \in \mathcal{T}_h} ((\epsilon^- - \epsilon^+) (\hat{\xi}_h - \bar{\xi}_h), \nabla^\perp \psi_h \cdot \hat{n})_{\partial K} \\
&= \frac{1}{2} \sum_{K \in \mathcal{T}_h} ((\xi_h^+ - \xi_h^-) (\hat{\xi}_h - \bar{\xi}_h), \nabla^\perp \psi_h \cdot \hat{n})_{\partial K} \stackrel{(28)}{\leq} 0. \quad (\text{B.30})
\end{aligned}$$

For estimating (D), we use that for the upwind flux (29b) and central flux (29a) the following holds $\|\hat{\epsilon}\|_{L^2(\partial K)} \leq C(\|\epsilon^-\|_{L^2(\partial K)} + \|\epsilon^+\|_{L^2(\partial K)})$. Also using the trace theorem $\|\nabla^\perp \psi_h\|_{L^\infty(\partial K)} \leq C\|\nabla^\perp \psi_h\|_{L^\infty(K)}$ and the Schwarz inequality (D) then becomes

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_h} (\hat{\epsilon}(\xi - (P_K \xi)^-), \nabla^\perp \psi_h \cdot \hat{n})_{\partial K} \\
& \leq C \sum_{K \in \mathcal{T}_h} \|\nabla^\perp \psi_h\|_{L^\infty(K)} \|\xi - (P_K \xi)^-\|_{L^2(\partial K)} \|\hat{\epsilon}\|_{L^2(\partial K)} \\
& \leq C \sum_{K \in \mathcal{T}_h} \left(\|\xi\|_{H^1(K)} + \frac{1}{h} \|\epsilon\|_{L^2(K)} \right) \|\xi - (P_K \xi)^-\|_{L^2(\partial K)} \\
& \quad \left(\|\epsilon^+\|_{L^2(\partial K)} + \|\epsilon^-\|_{L^2(\partial K)} \right) \\
& \leq C \|\xi\|_{H^{k+1}(\Omega)} \sum_{K \in \mathcal{T}_h} \left(\|\xi - (P_K \xi)^-\|_{L^2(\partial K)} + h^{k-\frac{1}{2}} (\|\epsilon_h\|_{L^2(K)} + \|\xi - P_K \xi\|_{L^2(K)}) \right) \\
& \quad \left(\|\epsilon_h^+\|_{L^2(\partial K)} + \|\xi - (P_K \xi)^+\|_{L^2(\partial K)} + \|\epsilon_h^-\|_{L^2(\partial K)} + \|\xi - (P_K \xi)^-\|_{L^2(\partial K)} \right), \quad (\text{B.31})
\end{aligned}$$

with $h \leq 1$. In the last step we used the interpolation inequality (B.3). Assuming that the mesh is quasi uniform, we can combine Lemma 4.5.3 in [4] and Theorem 1.6.6 in [4] to obtain the following inverse inequality:

$$\|v^-\|_{L^2(\partial K)} \leq \frac{C}{\sqrt{h}} \|v\|_{L^2(K)}, \quad \forall v \in \mathbf{V}_h^k.$$

Applying this estimate for $v = \epsilon_h$ to (B.31) and using the arithmetic-geometric mean inequality results, under the condition $0 \leq h \leq 1$ and $k \geq 1$, in

$$\begin{aligned}
& \sum_{K \in \mathcal{T}_h} (\hat{\epsilon}(\xi - (P_K \xi)^-), \nabla^\perp \psi_h \cdot \hat{n})_{\partial K} \\
& \leq C \|\xi\|_{H^{k+1}(\Omega)} \sum_{K \in \mathcal{T}_h} \left\{ \frac{1}{h} \|\xi - (P_K \xi)^-\|_{L^2(\partial K)}^2 + \|\xi - (P_K \xi)^-\|_{L^2(K)}^2 + \|\epsilon_h\|_{L^2(K)}^2 \right\}. \quad (\text{B.32})
\end{aligned}$$

Combining the above estimates for (A), (B), (C) and (D), plugging them into (B.23), and using the interpolation estimates (B.2)–(B.3) results in:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\epsilon_h\|_{L^2(\Omega)}^2 &\leq C \sup_{t \in [0, T]} (\|\nabla \xi(\cdot, t)\|_{L^\infty(\Omega)}, \|\xi(\cdot, t)\|_{H^{k+1}(\Omega)}) \\ &\quad (\|\epsilon_h\|_{L^2(\Omega)}^2 + h^{2k} \|\xi\|_{H^{k+1}(\Omega)}^2), \end{aligned}$$

with $k \geq 1$ the polynomial order of the basis functions and $0 \leq h \leq 1$. Now we use the Gronwall inequality, which states that if $y \geq 0$ satisfies $dy(t)/dt \leq C(y(t) + h(t))$ for $0 \leq t \leq T$ with C a constant, $h(t) \geq 0$ and $h \in L^1([0, T])$ then $y(t) \leq \exp(CT) \{y(0) + \int_0^T h(s) ds\}$. Hence, we find for $t \in [0, T]$ that

$$\begin{aligned} \|\epsilon_h\|_{L^2(\Omega)}^2 &\leq \exp\left(CT \sup_{t \in [0, T]} (\|\nabla \xi(\cdot, t)\|_{L^\infty(\Omega)}, \|\xi(\cdot, t)\|_{H^{k+1}(\Omega)})\right) \\ &\quad \left(\|\epsilon_h\|_{L^2(\Omega)}^2\Big|_{t=0} + h^{2k} \int_0^T \|\xi(\cdot, t)\|_{H^{k+1}(\Omega)}^2 dt\right), \quad k \geq 1. \end{aligned}$$

If we apply for each element $K \in \mathcal{T}_h$ the projection P_K given by (B.1) to the initial condition and use the interpolation estimate (B.2) then

$$\left[\|\epsilon_h\|_{L^2(\Omega)}\right]_{t=0} \leq Ch^k \|\xi(\cdot, 0)\|_{H^k(\Omega)},$$

with $k \geq 1$, and thus for $t \in [0, T]$ we obtain

$$\begin{aligned} \|\epsilon_h\|_{L^2(\Omega)} &\leq Ch^k \exp\left(CT \sup_{t \in [0, T]} (\|\nabla \xi(\cdot, t)\|_{L^\infty(\Omega)}, \|\xi(\cdot, t)\|_{H^{k+1}(\Omega)})\right) \\ &\quad \left(\|\xi(\cdot, 0)\|_{H^k(\Omega)}^2 + \int_0^T \|\xi(\cdot, t)\|_{H^{k+1}(\Omega)}^2 dt\right)^{\frac{1}{2}}, \quad k \geq 1. \end{aligned}$$

The error in the vorticity can now be estimated using the relation

$$\|\epsilon\|_{L^2(\Omega)} \leq \|\epsilon_h\|_{L^2(\Omega)} + \|\xi - P_K \xi\|_{L^2(\Omega)} \leq \|\epsilon_h\|_{L^2(\Omega)} + h^k \sup_{t \in [0, T]} \|\xi(\cdot, t)\|_{H^{k+1}(\Omega)}, \quad (\text{B.33})$$

where in the second step we use the interpolation estimate (B.2). An estimate for the total error is obtained by combining (B.17) and (B.33) and using (1b), viz. $\vec{u} = A \nabla^\perp \psi$, $\vec{u}_h = A \nabla^\perp \psi_h$, which implies $\|\vec{u} - \vec{u}_h\|_{L^2(\Omega)} \leq C \|\psi - \psi_h\|_{H^1(\Omega)} = C \|\delta\|_{H^1(\Omega)}$, and completes the proof.

The proof of Corollary 12 follows directly from (B.28).

C REFERENCE ELEMENT, TRANSFORMATION AND BASIS FUNCTIONS

In Section 5.1, we introduced the space X_h^k , which was defined as a continuous finite element space consisting of at least k^{th} -order polynomials on each

Fig. C.1. The position and numbering of the nodes in the reference element for $k = 1, 2, 3$.

element of a triangulation $\mathcal{T}_h = \{K\}$. In this section, we define this space X_h^k more precisely. We will use serendipity elements, see [7]. In the remaining part of this section, these elements are defined as well as the basis functions on these elements.

First, introduce the rectangular reference element given by $\hat{K} = [-1, 1] \times [-1, 1]$, which is mapped to a quadrilateral shaped element $K \in \mathcal{T}_h$ using a mapping $t_K : \hat{K} \mapsto K$. To define this mapping, we introduce the shape functions and nodes on the reference element. The number of nodes, N_k , and the coordinates of the nodes for $k = 1, 2, 3$ are given in Table C.1. In Figure C.1 the nodal numbering is sketched. Note that for $k = 1, 2, 3$ all nodes are located on the element boundary. The nodes are used in the definition of the basis functions or shape functions, $\hat{\phi}_i : \hat{K} \mapsto \mathbb{R}$ ($i = 1, 2, \dots, N_k$). These functions have the property that they vanish in all but one node: $\hat{\phi}_i(\eta_j, \zeta_j) = \delta_{ij}$ with δ_{ij} the Kronecker delta function. For $k = 1$, these shape functions are defined by

$$\hat{\phi}_i(\eta, \zeta) = \frac{1}{4}(1 + \eta_i\eta)(1 + \zeta_i\zeta) \quad (i = 1, 2, 3, 4).$$

For $k = 2$, the basis functions are defined by

$$\begin{aligned} \hat{\phi}_i(\eta, \zeta) &= \frac{1}{4}(1 + \eta_i\eta)(1 + \zeta_i\zeta)(-1 + \eta_i\eta + \zeta_i\zeta) & (i = 1, 2, 3, 4) \\ \hat{\phi}_i(\eta, \zeta) &= \frac{1}{2}(1 - \eta^2)(1 + \zeta_i\zeta) & (i = 5, 7) \\ \hat{\phi}_i(\eta, \zeta) &= \frac{1}{2}(1 - \zeta^2)(1 + \eta_i\eta) & (i = 6, 8). \end{aligned}$$

	$k = 1$	$k = 2$	$k = 3$
N_k	4	8	12
$[\eta_1, \zeta_1]^T$	$[-1, -1]^T$	$[-1, -1]^T$	$[-1, -1]^T$
$[\eta_2, \zeta_2]^T$	$[1, -1]^T$	$[1, -1]^T$	$[1, -1]^T$
$[\eta_3, \zeta_3]^T$	$[1, 1]^T$	$[1, 1]^T$	$[1, 1]^T$
$[\eta_4, \zeta_4]^T$	$[-1, 1]^T$	$[-1, 1]^T$	$[-1, 1]^T$
$[\eta_5, \zeta_5]^T$	-	$[0, -1]^T$	$[-1/3, -1]^T$
$[\eta_6, \zeta_6]^T$	-	$[1, 0]^T$	$[1, -1/3]^T$
$[\eta_7, \zeta_7]^T$	-	$[0, 1]^T$	$[1/3, 1]^T$
$[\eta_8, \zeta_8]^T$	-	$[-1, 0]^T$	$[-1, 1/3]^T$
$[\eta_9, \zeta_9]^T$	-	-	$[1/3, -1]^T$
$[\eta_{10}, \zeta_{10}]^T$	-	-	$[1, 1/3]^T$
$[\eta_{11}, \zeta_{11}]^T$	-	-	$[-1/3, 1]^T$
$[\eta_{12}, \zeta_{12}]^T$	-	-	$[-1, -1/3]^T$

Table C.1

Number of nodes and their coordinates in the reference element for $k = 1, 2, 3$.

Finally, for $k = 3$, the basis functions are

$$\begin{aligned}
\hat{\phi}_i(\eta, \zeta) &= \frac{1}{32}(1 + \eta_i\eta)(1 + \zeta_i\zeta)(9(\eta^2 + \zeta^2) - 10) & (i = 1, 2, 3, 4) \\
\hat{\phi}_i(\eta, \zeta) &= \frac{9}{32}(1 + \eta_i\eta)(1 - \zeta^2)(1 + 9\zeta_i\zeta) & (i = 6, 8, 10, 12) \\
\hat{\phi}_i(\eta, \zeta) &= \frac{9}{32}(1 + \zeta_i\zeta)(1 - \eta^2)(1 + 9\eta_i\eta) & (i = 5, 7, 9, 11).
\end{aligned}$$

Now we define the mapping $t_K : \hat{K} \mapsto K$ as follows (see Figure C.2):

$$t_K = \sum_{i=1}^{N_k} \hat{\phi}_i \vec{q}_i \quad (\text{C.1})$$

with $\vec{q}_i = [x_i, y_i]^T$ the coordinates of the point in the real element corresponding to the node $\vec{\eta}_i = [\eta_i, \zeta_i]^T$ in the reference element. Note that for $k > 1$ this mapping results in element boundaries which are curved, which is necessary for a k^{th} order approximation of the domain. However we only need this k^{th} order mapping for elements on the boundary of the domain. For elements in the interior we can always use linear shape functions. The Jacobian of the

Fig. C.2. The transformation t_K is used to map the reference element $\hat{K}$ to a real element K .

transformation (C.1) is given by

$$J_K = \sum_{i=1}^{N_k} \hat{\nabla} \hat{\phi}_i \vec{q}_i^T \quad (\text{C.2})$$

with $\hat{\nabla} \hat{\phi}_i$ the gradient with respect to the coordinates in the reference element: $\hat{\nabla} = [\partial_\eta, \partial_\xi]^T$.

Using the transformation t_K we define the local basis functions on an element, $\phi_i^{(K)} : K \mapsto \mathbb{R}$ as follows

$$\phi_i^{(K)} = \hat{\phi}_i \circ t_K^{-1}. \quad (\text{C.3})$$

Define the space X_h^k as follows:

$$X_h^k = \left\{ v \in C_0(\Omega) \mid \forall K \in \mathcal{T}_h \exists \vec{c} \in \mathbb{R}^{N_k} : v|_K = \sum_{i=1}^{N_k} c_i \hat{\phi}_i \right\} \quad (\text{C.4})$$

with $C_0(\Omega)$ the set of continuous functions on Ω .

For the definition of a basis of X_h^k , we introduce a mapping $I_K : \{1, 2, \dots, N_k\} \mapsto \{1, 2, \dots, M\}$ with M the dimension of the space X_h^k . I_K maps each (local) node number of an element to a global basis function index. Using this mapping the (global) basis functions, $\varphi_i : \Omega \mapsto \mathbb{R}$ are defined by

$$\varphi_i|_K = \sum_{j=1}^{N_k} \delta_{i, I_K(j)} \phi_j^{(K)} \quad (\text{C.5})$$

with δ the Kronecker delta function. The mapping I_K should be defined in such a way that the resulting basis functions are continuous on the domain. Hence, corresponding nodes in neighboring elements map to the same global basis

function index. In other words, if for two elements, $K_A \in \mathcal{T}_h$ and $K_B \in \mathcal{T}_h$, we have $t_{K_A}(\eta_i, \zeta_i) = t_{K_B}(\eta_j, \zeta_j)$ then the mappings I_{K_A} and I_{K_B} should satisfy: $I_{K_A}(i) = I_{K_B}(j)$.

Note that we can define the space $\mathbf{W}_h^k$ with a mapping $I_K : \{1, 2, \dots, N_k\} \mapsto \mathbb{N}$ as well. In this case an extra requirement for the mapping I_K is that nodes which are located on the same no-normal-flow boundary should map to the same (global) basis function index, since a function $w \in \mathbf{W}_h^k$ should have a constant value on each no-normal-flow boundary. In other words, if for two elements, $K_A \in \mathcal{T}_h$ and $K_B \in \mathcal{T}_h$, we have that $t_{K_A}(\eta_i, \zeta_i) \in \partial\Omega_{D_k}$ and $t_{K_B}(\eta_j, \zeta_j) \in \partial\Omega_{D_k}$, for some $k \in \{1, 2, \dots, N\}$, then the mappings I_{K_A} and I_{K_B} should satisfy: $I_{K_A}(i) = I_{K_B}(j)$.

Finally, because functions in the space $\mathbf{V}_h^k$ for the DG discretization are not required to be continuous, we define a basis for $\mathbf{V}_h^k$ per element. The basis functions for such an element oriented basis are precisely given by $\phi_1^{(K)}, \phi_2^{(K)}, \dots, \phi_{N_k}^{(K)}$.

D SYSTEM OF EQUATIONS

In this Appendix, we derive the system of equations resulting from the space discretization given by (22) and (30).

In Appendix C, basis functions for the spaces $\mathbf{W}_h^k$ and $\mathbf{V}_h^k$ were defined. Using these basis functions, the approximation of the vorticity field is:

$$\xi_h|_K = \sum_{i=1}^{N_k} \xi_i^{(K)} \phi_i^{(K)} \quad (\text{D.1})$$

with time dependent expansion coefficients $\xi_i^{(K)} \in \mathbb{R}$. For the (global) streamfunction

$$\psi_h = \sum_{i=1}^M \psi_i \varphi_i \quad (\text{D.2})$$

with $\psi_i \in \mathbb{R}$ (global) expansion coefficients and M the dimension of the space $\mathbf{W}_h^k$. We define the local expansion coefficients of the streamfunction for each element $K \in \mathcal{T}_h$ and $i = 1, 2, \dots, N_k$ as follows

$$\psi_i^{(K)} = \psi_{I_K(i)}. \quad (\text{D.3})$$

Using these expansions for the streamfunction and vorticity fields, and taking $v_h = \phi_i^{(K)}$ for $i = 1, 2, \dots, N_k$ and $K \in \mathcal{T}_h$ in (30), the following (local) matrix equation of the DG discretization emerges

$$\mathbf{M}^{(K)} \frac{d}{dt} \xi^{(K)} = \mathbf{F}^{(K)} - \hat{\mathbf{F}}^{(K)} \quad (\text{D.4})$$

with $\xi^{(K)} \in \mathbb{R}^{N_k}$ the vector of local expansion coefficients of ξ_h . The matrices and vectors $\mathbf{M}^{(K)} \in \mathbb{R}^{N_k \times N_k}$, $\mathbf{F}^{(K)} \in \mathbb{R}^{N_k}$ and $\hat{\mathbf{F}}^{(K)} \in \mathbb{R}^{N_k}$ are for each $K \in \mathcal{T}_h$ given by

$$M_{ij}^{(K)} = \int_K (1/A) \phi_i^{(K)} \phi_j^{(K)} \, dK. \quad (\text{D.5a})$$

$$F_i^{(K)} = \int_K \xi_h \nabla^\perp \psi_h \cdot \nabla \phi_i^{(K)} \, dK \quad (\text{D.5b})$$

$$\hat{F}_i^{(K)} = \int_{\partial K} \left(\phi_i^{(K)} \right)^- \hat{f}(\xi_h^+, \xi_h^-, \nabla^\perp \psi_h \cdot \hat{n}) \, d\Gamma. \quad (\text{D.5c})$$

Because the matrices $\mathbf{M}^{(K)}$ are relatively small and dense matrices, we can calculate $d\xi^{(K)}/dt$ efficiently from the above matrix equation. The continuous Galerkin discretization (22) can be written as follows

$$\mathbf{L}\psi = \mathbf{S} \quad (\text{D.6})$$

with $\mathbf{L} \in \mathbb{R}^{M \times M}$ and $\mathbf{S} \in \mathbb{R}^M$ given by

$$L_{ij} = \int_\Omega A \nabla \varphi_i \cdot \nabla \varphi_j - B \varphi_i \varphi_j \, d\Omega \quad (\text{D.7a})$$

$$S_i = \int_\Omega D \varphi_i - (1/A) \xi_h \varphi_i \, d\Omega + \sum_{j=1}^N \varphi_i|_{\partial\Omega_{D_j}} \mathcal{C}_j. \quad (\text{D.7b})$$

Note that here $\mathbf{L}$ is a large sparse matrix. Because this matrix is symmetric and positive definite we can use the conjugate gradient method to solve this system of equations for a given vorticity field ξ_h .

E ELEMENTAL INTEGRALS

In the system of equations derived in Appendix D, we encountered integrals over the domain Ω and elements $K \in \mathcal{T}_h$ in definitions (D.4)–(D.7b). Using the mapping $t_K : \hat{K} \mapsto K$ these integrals can be transformed to integrals over the reference element. For every function $f : K \mapsto \mathbb{R}$, we have:

$$\int_K f \, dK = \int_{\hat{K}} (f \circ t_K) \det(J_K) \, d\hat{K}.$$

Further, we write the gradient of a function f as

$$\nabla f = J_K^{-1} \hat{\nabla} f$$

with $\hat{\nabla} = [\partial_\eta, \partial_\zeta]^T$, and then write (D.5a) and (D.5b) as follows

$$\begin{aligned} M_{ij}^{(K)} &= \int_{\hat{K}} (1/(A \circ t_K)) \hat{\phi}_i \hat{\phi}_j \det(J_K) \, d\hat{K}, \\ F_i^{(K)} &= \int_{\hat{K}} (\xi_h \circ t_K) \hat{\nabla}^\perp (\psi_h \circ t_K) \cdot \hat{\nabla} \hat{\phi}_i \, d\hat{K}. \end{aligned}$$

To calculate (D.7a) and (D.7b), we use

$$\begin{aligned} \int_K A \nabla \varphi_{I_K(i)} \cdot \nabla \varphi_{I_K(j)} + B \varphi_{I_K(i)} \varphi_{I_K(j)} \, dK &= \\ \int_{\hat{K}} \left((A \circ t_K) \hat{\nabla} \hat{\phi}_i \cdot \hat{\nabla} \hat{\phi}_j + B \hat{\phi}_i \hat{\phi}_j \right) \det(J_K) \, d\hat{K}, \\ \int_K D \varphi_{I_K(i)} - (1/A) \xi_h \varphi_{I_K(i)} \, dK &= \int_{\hat{K}} (D \circ t_K) \hat{\phi}_i - (1/(A \circ t_K)) (\xi_h \circ t_K) \hat{\phi}_i \, d\hat{K}. \end{aligned}$$

The contribution of the circulation boundary conditions in (D.7b) are calculated by finding for each such boundary the (global) basis function index i for which $\varphi_i|_{\partial\Omega_{D_j}} = 1$. This index exists and is unique. Subsequently, we add the contribution $\mathcal{C}_j$ to S_i .

In the DG discretization (D.4), we need to evaluate the integral (D.5c) of the numerical flux over the element boundary. Here we show how to calculate this integral over one face or edge of an element:

$$\int_e \phi_h^- \hat{f}(\xi_h^+, \xi_h^-, \nabla^\perp \psi_h \cdot \hat{n}) \, d\Gamma \quad (\text{E.1})$$

with $e \subset \partial K$ an edge or face of an element. In the reference element a face is defined by the coordinates of the two vertices of the face. Assume that these coordinates are given by $\vec{\eta}_p = [\eta_p, \zeta_p]^T$ and $\vec{\eta}_q = [\eta_q, \zeta_q]^T$ for some $p \in \{1, 2, 3, 4\}$ and $q = (p + 1) \bmod 4$. Now the edge, $e \subset \partial K$, can be parameterized by

$$\vec{\eta}(s) = \frac{1-s}{2} \vec{\eta}_p + \frac{1+s}{2} \vec{\eta}_q \quad (\text{E.2})$$

with $s \in [-1, 1]$. In the real element, this parameterization is given by

$$\vec{x}(s) = t_k(\vec{\eta}(s)).$$

For $d\Gamma$ in (E.1), we write $d\Gamma = \sqrt{(dx/ds)^2 + (dy/ds)^2} ds$ and for the normal component of the weighted velocity

$$U_n = \nabla^\perp \psi_h \cdot \hat{n} = \nabla \psi_h \cdot \hat{\tau} = \frac{d\psi_h(s)/ds}{\sqrt{(dx/ds)^2 + (dy/ds)^2}}.$$

Note that the following property holds for the central flux, upwind flux and Lax-Friedrich flux, given in (29a)–(29c):

$$\hat{f}(\xi_h^+, \xi_h^-, \alpha U_n) = \alpha_{LF} \hat{f}(\xi_h^+, \xi_h^-, U_n)$$

for $\alpha > 0$. Hence, (E.1) becomes

$$\int_e \phi_h^- \hat{f}(\xi_h^+, \xi_h^-, U_n) \, d\Gamma = \int_{-1}^1 \phi_h^-(s) \hat{f}(\xi_h^+(s), \xi_h^-(s), d\psi_h(s)/ds) \, ds.$$

To evaluate the basis functions at the face of an element, we use the following polynomials for $i = 0, 1, \dots, k$:

$$P_i(s) = \prod_{j=0, j \neq i}^k (s - s_j) / (s_i - s_j) \quad (\text{E.3})$$

with $s_j = -1 + 2j/k$ ($j = 0, 1, 2, \dots, k$). Evaluating a basis function at the edge e then results in

$$\phi_i^{(K)}(\vec{\eta}(s)) = \begin{cases} P_j(s) & \text{if } i = p + 4j \pmod{4} (j = 0, \dots, k+1) \\ P_k(s) & \text{if } i = q \\ 0 & \text{otherwise} \end{cases}$$

with $\vec{\eta}(s)$ defined in (E.2) and $\phi_i^{(K)}$ defined in (C.3).

F RUNGE-KUTTA TIME DISCRETIZATION

The third order Runge-Kutta scheme [16] is

$$\begin{aligned} \xi^{(0)} &= \xi_h^n \\ \xi^{(1)} &= \xi^{(0)} + \Delta t \mathcal{L}_h(\xi^{(0)}) \\ \xi^{(2)} &= \frac{3}{4}\xi^{(0)} + \frac{1}{4}\xi^{(1)} + \frac{1}{4}\Delta t \mathcal{L}_h(\xi^{(1)}) \\ \xi^{(3)} &= \frac{1}{3}\xi^{(0)} + \frac{2}{3}\xi^{(2)} + \frac{2}{3}\Delta t \mathcal{L}_h(\xi^{(2)}) \\ \xi_h^{n+1} &= \xi^{(3)} \end{aligned}$$

with Δt the size of the time step, ξ_h^n and ξ_h^{n+1} the vorticity field at two successive time steps and $\mathcal{L}_h$ defined in (33). This scheme is total variation diminishing. The total variation of a function is a measure for the oscillations in this function. According to [15] total variation diminishing schemes have the advantage of high-order accuracy in smooth regions, and few oscillations around discontinuities.

G LINEAR STABILITY ANALYSIS FOR THE 1D ADVECTION EQUATION

We performed a linear stability analysis of the third-order Runge-Kutta space DG discretization of the vorticity equation. Hence, we consider the one-dimensional advection equation on a domain $\Omega = [x_L, x_R]$

$$\partial_t \xi + u \partial_x \xi = 0$$

with the velocity $u \in \mathbb{R}$ constant and $\xi = \xi(x, t)$. We assume that Ω is subdivided in N elements $K_k = [x_{k-1}, x_k]$ ($k = 1, 2, \dots, N$) with $x_k = x_L + k\Delta x$ and $\Delta x = (x_R - x_L)/N$. We define the space of approximation functions by

$$V_p = \{v : v|_{K_k} \in \mathcal{P}_p(K_k) \text{ for each } k=1,2,\dots,N\}$$

with $\mathcal{P}_p(K_k)$ the space of polynomials on K_k of degree $\leq p$. The DG discretization is now given by: Find $\xi_h \in V_p$ such that for each $v_h \in V_p$ and each $k = 1, 2, \dots, N$ the following equation holds:

$$\int_{x_{k-1}}^{x_k} v_h \partial_t \xi_h dx - u \int_{x_{k-1}}^{x_k} \xi_h \partial_x v_h dx + u [\hat{\xi}_h]_{x=x_k} \lim_{x \uparrow x_k} v_h - u [\hat{\xi}_h]_{x=x_{k-1}} \lim_{x \downarrow x_{k-1}} v_h = 0. \quad (\text{G.1})$$

At the element boundaries, we define a numerical flux, $\hat{\xi}_h$, for which we take the central flux here: $[\hat{\xi}_h]_{x=x_k} = (1/2)(\lim_{x \uparrow x_k} \xi_h + \lim_{x \downarrow x_k} \xi_h)$.

Each element can be mapped to a reference element, $\hat{K} = [-1, 1]$, using the coordinate transformation $x = (1/2)(1 - s)x_{k-1} + (1/2)(1 + s)x_k$. Assume that on this reference element we have defined basis functions $\phi_i : \hat{K} \mapsto K_k$ ($i = 1, 2, \dots, p$). Then we can write for $x \in K_k$: $\xi_h(x) = \sum_{i=1}^p \xi_{k,i} \phi_i(s)$ with $\xi_{k,i}$ the expansion coefficients on the k -th element. Using the central flux and these basis functions, (G.1) becomes: Find coefficients $\xi_{k,i}$ such that for each $k = 1, 2, \dots, N$ and $i = 1, 2, \dots, p$ the following equations holds ($|u| > 0$):

$$\begin{aligned} \frac{\Delta x}{2u} \sum_{j=1}^p \frac{d}{dt} \xi_{k,j} \int_{-1}^1 \phi_i \phi_j ds \\ - \sum_{j=1}^p \xi_{k,j} \left(\int_{-1}^1 \phi_j \partial_s \phi_i ds - \frac{1}{2} (\phi_i(1)\phi_j(1) - \phi_i(-1)\phi_j(-1)) \right) \\ + \sum_{j=1}^p \xi_{k+1,j} \frac{1}{2} \phi_j(-1)\phi_i(1) - \sum_{j=1}^p \xi_{k-1,j} \frac{1}{2} \phi_j(1)\phi_i(-1) = 0. \quad (\text{G.2}) \end{aligned}$$

In order to analyze the stability of the numerical scheme we use the Ansatz $\xi_{k,i} = C_i \exp(\iota Pk)$ for arbitrary $C_i, P \in \mathbb{R}$ and $\iota = \sqrt{-1}$. Then we have

$$\frac{\Delta x}{u} \mathbf{M} \frac{d}{dt} \vec{\xi}_k + \mathbf{F} \vec{\xi}_k = 0 \quad (\text{G.3})$$

with $\mathbf{M} \in \mathbb{R}^{p \times p}$ and $\mathbf{F} \in \mathbb{R}^{p \times p}$ defined by

$$\begin{aligned} M_{ij} &= \frac{1}{2} \int_{-1}^1 \phi_i \phi_j ds \\ F_{ij} &= \frac{1}{2} (\phi_i \partial_s \phi_j - \phi_j \partial_s \phi_i ds) + \frac{1}{2} \exp(-\iota Pk) \phi_i(1) \phi_j(-1) \\ &\quad - \frac{1}{2} \exp(\iota Pk) \phi_i(-1) \phi_j(1). \end{aligned} \quad (\text{G.4})$$

We can write the Runge Kutta time discretization scheme given in Section F as

$$\vec{\xi}_k^n = \mathbf{L} \vec{\xi}_k^{n-1} \quad (\text{G.5})$$

with $\vec{\xi}_k^n$ the expansion coefficients at the n^{th} time level and

$$\mathbf{L} = \mathbf{I} + C\mathbf{R} + \frac{1}{2}C^2\mathbf{R}^2 + \frac{1}{6}C^3\mathbf{R}^3 \quad \text{and} \quad \mathbf{R} = \mathbf{M}^{-1}\mathbf{F}, \quad (\text{G.6})$$

where the unity matrix is denoted by $\mathbf{I}$ and $C = u\Delta t/\Delta x$. The eigenvalues of $\mathbf{L}$ are of the form $\mu = 1 + (C)\lambda + \frac{1}{2}(C)^2\lambda^2 + \frac{1}{6}C^3\lambda^3$ with λ an eigenvalue of $\mathbf{R}$. Since $\mathbf{F}$ is anti-Hermitian we have for an eigenvalue λ and eigenvector $\vec{v}$ of $\mathbf{F}$ that

$$\lambda \vec{v}^* \mathbf{M} \vec{v} = \vec{v}^* \mathbf{F} \vec{v} = -(\vec{v}^* \mathbf{F} \vec{v})^* = -\bar{\lambda} (\vec{v}^* \mathbf{M} \vec{v})^*.$$

Since $\mathbf{M}$ is positive definite we have $\vec{v}^* \mathbf{M} \vec{v} \in \mathbb{R}$ and thus the real part of the eigenvalue λ vanishes. For the absolute value of the eigenvalues of $\mathbf{L}$ we can therefore write

$$|\mu|^2 = 1 + \frac{1}{36} \left(\frac{u\Delta t}{\Delta x} \right)^4 |\lambda|^4 \left(\left(\frac{u\Delta t}{\Delta x} \right)^2 |\lambda|^2 - 3 \right).$$

To ensure stability of the numerical scheme we must have $|\mu|^2 \leq 1$ and, therefore, $C|\lambda| \leq \sqrt{3}$. Using MAPLE we calculated the eigenvalues of $\mathbf{R}$ for many values of $Pk \in [0, 2\pi]$ for different orders of approximation spaces.

A similar stability analysis can be performed if we use an upwind flux instead of a central flux. Assuming without loss of generality that $u > 0$ the upwind flux is given by: $[\hat{\xi}_h]_{x=x_k} = \lim_{x \uparrow x_k} \xi_h$. This results in a different matrix $\mathbf{F}$ in (G.3):

$$F_{ij} = \int_{-1}^1 \phi_j \partial_s \phi_i ds + \phi_i(1)\phi_j(1) - \exp(-\iota Pk) \phi_i(-1)\phi_j(1).$$

We can calculate the largest value of C for which all eigenvalues, λ_i , of $\mathbf{L}$ satisfy $|\lambda_i| \leq 1$. The results are shown in Table G.1. We note that for higher order basis functions the time step restriction for central- and upwind flux are almost the same.

H DERIVATION OF APPLICATIONS

In Section 2, we saw that the quasi-geostrophic equations, rigid lid equations and 2D incompressible Euler equations are special cases of the general system (1). Here we give more background information for these three applications. The 2D incompressible Euler equations, quasi-geostrophic equations and rigid lid equations are discussed in Appendices H.1, H.2 and H.3 respectively.

Basis functions	CFL (Central Flux)	CFL (Upwind Flux)
constant	$\sqrt{3}$	1.25
linear	$\sqrt{3}/4$	0.409
quadratic	0.214	0.209
cubic	0.130	0.130

Table G.1

The CFL-condition is given for different orders of approximation functions for the linearized one-dimensional vorticity equation with constant velocity u . Hence, the indicative time step in the nonlinear problem should be chosen such that $\Delta t \leq CFL \Delta x / |\vec{u}_h|_{max}$.

H.1 Incompressible Euler Equations

In this section we derive the incompressible 2D Euler equations in vorticity streamfunction formulation from the standard incompressible 2D Euler equations [8]. We show that these equations are a special case of (1).

Consider the incompressible 2D Euler equations on $\Omega \subset \mathbb{R}^2$ in the primitive variables $\vec{u}$ and p :

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} = -\nabla p / \rho \quad (\text{H.1})$$

$$\nabla \cdot \vec{u} = 0. \quad (\text{H.2})$$

In these equations $\vec{u} = [u, v]^T$ denotes the velocity field, p is the pressure field and ρ is the density. We assume the fluid has constant density and, hence, it is incompressible. The vertical component of the vorticity is defined as follows

$$\omega = \nabla^\perp \cdot \vec{u} = \partial_x v - \partial_y u. \quad (\text{H.3})$$

We introduce the streamfunction, ψ , defined by

$$\vec{u} = \nabla^\perp \psi. \quad (\text{H.4})$$

From (H.2), it follows that given the velocity field, $\vec{u}$, such a function ψ exists and is defined up to a constant. By combining (H.3) and (H.4) we can express the vorticity in terms of the streamfunction

$$\omega = \nabla^\perp \cdot \vec{u} = \nabla^\perp \cdot \nabla^\perp \psi = \nabla^2 \psi. \quad (\text{H.5})$$

To obtain the Euler equation in vorticity streamfunction formulation we take the 2D curl of (H.1)

$$\nabla^\perp \cdot (\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u}) = (1/\rho) \nabla^\perp \cdot \nabla p.$$

Fig. H.1. Variables used in the shallow water equations and quasi-geostrophic equations.

Using the identities $\nabla^\perp \cdot ((\vec{u} \cdot \nabla) \vec{u}) = (\nabla \cdot \vec{u} + \vec{u} \cdot \nabla) (\nabla^\perp \cdot \vec{u})$ and $\nabla^\perp \cdot \nabla p = 0$, we can simplify the above equation to obtain

$$\partial_t \omega + \nabla \cdot (\omega \vec{u}) = 0. \quad (\text{H.6})$$

The Euler equations in vorticity streamfunction formulation are now given by (H.4), (H.5) and (H.6). We see that the advantage of the vorticity streamfunction formulation is that the number of unknowns is reduced by one: instead of solving for three variables u , v and p we need to solve for only two: ω and ψ . Note that the Euler equations in vorticity streamfunction formulation are a special case of (1) if we take $A = 1$, $B = 0$ and $D = 0$ and $\vec{U} = \vec{u}$.

H.2 Barotropic Quasi-Geostrophic Equations

The barotropic quasi-geostrophic equations can be used to describe the motions in the atmosphere and oceans of the Earth in mid-latitudes. First, we consider the shallow water equations in Appendix H.2.1. In Appendix H.2.2, we deduce the quasi-geostrophic equations from the shallow water equations.

H.2.1 Shallow Water Equations

The shallow water equations describe the motion of a fluid in a thin layer of fluid on a rotating (x, y) -plane. The fluid moves between a rigid bottom at $z = h_B(x, y)$ and a free surface at $z = h(x, y, t)$ (See Figure H.1). The density, ρ , is constant and, hence, the fluid is incompressible. The (x, y) -plane is assumed to rotate around the vertical z -axis. Further, we assume that the gravity is directed along the negative z -axis. The shallow water equations are

derived from the three-dimensional (3D) Euler equations, which are given by

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} + w \partial_z \vec{u} + f [-v, u]^T = -\nabla p / \rho \quad (\text{H.7})$$

$$\partial_t w + \vec{u} \cdot \nabla w + w \partial_z w = -\partial_z p / \rho + g \quad (\text{H.8})$$

$$\nabla \cdot \vec{u} + \partial_z w = 0 \quad (\text{H.9})$$

with p the pressure, g the gravitational acceleration, $\vec{u} = [u, v]^T$ the horizontal velocity components and w the vertical velocity component. The term $f[-v, u]^T$ is due to the Coriolis force which is caused by the rotation of the plane. The parameter f is called the Coriolis parameter. If the axis of rotation is directed along the z -axis, then $f = 2\Omega$ with Ω the rate of rotation of the plane. The pressure at the free surface, $z = h(x, y, t)$, is given by the constant the atmospheric pressure, p_0 .

We assume that the horizontal velocity components are independent of z and that the momentum equation in the vertical direction reduces to the hydrostatic approximation

$$\partial_z p = \rho g. \quad (\text{H.10})$$

Both assumptions are justified by the fact that the layer of fluid is thin [14]. Hence, the vertical scales of motion are small compared to the horizontal scales of motion. Using the hydrostatic approximation (H.10) and the boundary condition, $p|_{z=h} = p_0$, for the pressure at the free surface we find

$$p = p_0 + \rho g(h - z). \quad (\text{H.11})$$

Using these assumptions the 3D Euler equations (H.7)–(H.9) can be simplified to

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} + f [-v, u]^T = -g \nabla h \quad (\text{H.12})$$

$$\partial_z p = \rho g \quad (\text{H.13})$$

$$\nabla \cdot \vec{u} + \partial_z w = 0. \quad (\text{H.14})$$

Since we assumed that $\vec{u}$ is independent of the z -coordinate it follows from (H.14) that

$$w = -z \nabla \cdot \vec{u} + \tilde{w}(x, y, t).$$

To determine $\tilde{w}(x, y, t)$ we note that the upper and lower boundaries of the fluid are material surfaces. For a surface implicitly defined by $S(x, y, z, t) = 0$ this means that $DS/Dt = 0$ for all points on the surface. The material derivative DS/Dt is defined as $DS/Dt = \partial_t S + \vec{u} \cdot \nabla S + w \partial_z S$. Applying this to the upper boundary, with $S = z - h$, and lower boundary, with $S = z - h_B$, we obtain

$$0 = \frac{D(z-h)}{Dt} \Big|_{z=h} = -\partial_t h - \vec{u} \cdot \nabla h - h \nabla \cdot \vec{u} + \tilde{w} \quad (\text{H.15})$$

$$0 = \frac{D(z-h_B)}{Dt} \Big|_{z=h_B} = -\vec{u} \cdot \nabla h_B - h_B \nabla \cdot \vec{u} + \tilde{w}, \quad (\text{H.16})$$

respectively. Subtracting (H.15) from (H.16), we find

$$0 = \partial_t H + \vec{u} \cdot \nabla H + H \nabla \cdot \vec{u} \quad (\text{H.17})$$

with $H = h - h_B$ the depth of the fluid.

The shallow water equations are now given by the momentum balance (H.12) and mass balance (H.17). This results in

$$\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} + f[-v, u]^T = -g(\nabla H + \nabla h_B) \quad (\text{H.18})$$

$$\partial_t H + \nabla \cdot (H \vec{u}) = 0. \quad (\text{H.19})$$

Now we introduce the potential vorticity, defined by

$$\xi = \frac{\nabla^\perp \cdot \vec{u} + f}{H}. \quad (\text{H.20})$$

It has the remarkable property that it is a materially conserved quantity

$$\frac{D\xi}{Dt} = 0 \quad (\text{H.21})$$

with $D/Dt = \partial_t + \vec{u} \cdot \nabla$ the material derivative. This follows because:

$$\begin{aligned} \partial_t \xi + \vec{u} \cdot \nabla \xi &= \partial_t \left(\frac{\nabla^\perp \cdot \vec{u} + f}{H} \right) + \vec{u} \cdot \nabla \left(\frac{\nabla^\perp \cdot \vec{u} + f}{H} \right) \\ &= \frac{\nabla^\perp \cdot \partial_t \vec{u} + \vec{u} \cdot \nabla (\nabla^\perp \cdot \vec{u})}{H} - \frac{\nabla^\perp \cdot \vec{u} + f}{H^2} (\partial_t H + \vec{u} \cdot \nabla H). \end{aligned}$$

Using the identity $\nabla^\perp \cdot ((\vec{u} \cdot \nabla) \vec{u}) \equiv (\nabla \cdot \vec{u} + \vec{u} \cdot \nabla) (\nabla^\perp \cdot \vec{u})$, (H.18) and (H.19), we find

$$\begin{aligned} \partial_t \xi + \vec{u} \cdot \nabla \xi &= \frac{\nabla^\perp \cdot (\partial_t \vec{u} + (\vec{u} \cdot \nabla) \vec{u} + f[-v, u]^T)}{H} \\ &\quad - \frac{\nabla^\perp \cdot \vec{u} + f}{H^2} (\partial_t H + \vec{u} \cdot \nabla H + H \nabla \cdot \vec{u}) \\ &= -\frac{g \nabla^\perp \cdot (\nabla H + \nabla h_B)}{H} - \frac{\nabla^\perp \cdot \vec{u} + f}{H^2} (\partial_t H + \nabla \cdot (\vec{u} H)) = 0. \end{aligned}$$

H.2.2 Derivation of Quasi-Geostrophic Equations

In [14], the quasi-geostrophic equations are derived from the shallow water equations using a scaling analysis, which we will summarize next. First, we define the deviation, η , of the free surface from its rest value

$$H = D_0 + \eta - h_B \quad (\text{H.22})$$

with D_0 a characteristic value for the rest depth of the fluid. Then, we introduce the dimensionless variables x' , y' , $\vec{u}'$, t' and η' :

$$[x, y] = L[x', y'], \quad t = Tt', \quad \vec{u} = U\vec{u}', \quad \eta = N_0\eta', \quad \nabla = \nabla'/L.$$

We assume that L , T , U and N_0 characterize the horizontal length scales, time scales, velocity scales and scale of deviation from the mean surface, respectively. After substituting these dimensionless variables in (H.18) and (H.19), we obtain

$$\frac{U}{T} \partial_{t'} \vec{u}' + \frac{U^2}{L} (\vec{u}' \cdot \nabla') \vec{u}' + U f[-v', u']^T + g \frac{N_0}{L} \nabla' \eta' = 0 \quad (\text{H.23})$$

$$\frac{N_0}{T} \partial_{t'} \eta' + \frac{U}{L} (D_0 + N_0 \eta' - h_B) \nabla' \cdot \vec{u}' + \frac{U}{L} \vec{u}' \cdot \nabla' (N_0 \eta' - h_B) = 0, \quad (\text{H.24})$$

respectively. Now we introduce the scaling:

$$T = \frac{L}{U}, \quad N_0 = \frac{fUL}{g}, \quad \epsilon \equiv \frac{U}{fL} \ll 1, \quad \frac{h_B}{D_0} = \epsilon \eta'_B$$

with $\eta'_B = O(1)$. After multiplying (H.23) by $1/fU$ and (H.24) by F/N_0f , with $F = f^2 L^2 / g D_0$, the rotational Froude number, we obtain

$$\epsilon (\partial_{t'} \vec{u}' + (\vec{u}' \cdot \nabla') \vec{u}') + [-v', u']^T + \nabla' \eta' = 0 \quad (\text{H.25})$$

$$\epsilon F (\partial_{t'} \eta' + \nabla' \cdot (\vec{u}' \eta')) + (1 - \epsilon \eta'_B) (\nabla' \cdot \vec{u}') - \epsilon \vec{u}' \cdot \nabla' \eta'_B = 0 \quad (\text{H.26})$$

variables. We write the unknowns $\vec{u}$ and η (omitting the primes hereafter) as

$$\vec{u} = \vec{u}_0 + \epsilon \vec{u}_1 + \epsilon^2 \vec{u}_2 + \mathcal{O}(\epsilon^3) \quad (\text{H.27})$$

$$\eta = \eta_0 + \epsilon \eta_1 + \epsilon^2 \eta_2 + \mathcal{O}(\epsilon^3). \quad (\text{H.28})$$

Substituting (H.27) and (H.28) in (H.25) and (H.26) gives:

$$\vec{0} = \vec{K}_1 + \epsilon \vec{K}_2 + O(\epsilon^2)$$

$$0 = K_3 + \epsilon K_4 + O(\epsilon^2)$$

with

$$\vec{K}_1 \equiv [-v_0, u_0]^T + \nabla \eta_0 \quad (\text{H.29})$$

$$\vec{K}_2 \equiv \partial_t \vec{u}_0 + (\vec{u}_0 \cdot \nabla) \vec{u}_0 + [-v_1, u_1]^T + \nabla \eta_1 \quad (\text{H.30})$$

$$K_3 \equiv \nabla \cdot \vec{u}_0 \quad (\text{H.31})$$

$$K_4 \equiv F(\partial_t \eta_0 + \nabla \cdot (\eta_0 \vec{u}_0)) + \nabla \cdot \vec{u}_1 - \eta_B (\nabla \cdot \vec{u}_0) - \vec{u}_0 \cdot \nabla \eta_B. \quad (\text{H.32})$$

Since (H.29) and (H.29) should hold for each ϵ we clearly must have $\vec{K}_1 = \vec{K}_2 = \vec{0}$ and $K_3 = K_4 = 0$. From $\vec{K}_1 = \vec{0}$ and (H.29) we obtain the geostrophic balance condition

$$\vec{u}_0 = \nabla^\perp \eta_0. \quad (\text{H.33})$$

Note that from this relation the condition $K_3 = 0$ follows. Using (H.33), we simplify the condition $K_4 = 0$ to obtain

$$F(\partial_t \eta_0) + \nabla \cdot \vec{u}_1 - \vec{u}_0 \cdot \nabla \eta_B = 0. \quad (\text{H.34})$$

We eliminate η_1 from $\vec{K}_2 = \vec{0}$ by taking the 2D curl on both sides of this equation and using the identity $\nabla^\perp \cdot ((\vec{u}_0 \cdot \nabla) \vec{u}_0) \equiv (\nabla \cdot \vec{u}_0 + \vec{u}_0 \cdot \nabla) (\nabla^\perp \cdot \vec{u}_0)$

$$\begin{aligned} 0 &= \nabla^\perp \cdot (\partial_t \vec{u}_0) + \nabla^\perp \cdot ((\vec{u}_0 \cdot \nabla) \vec{u}_0) + \nabla^\perp \cdot [-v_1, u_1]^T + \nabla^\perp \cdot \nabla \eta_1 \\ &= \partial_t \nabla^2 \eta_0 + (\vec{u}_0 \cdot \nabla) \nabla^2 \eta_0 + \nabla \cdot \vec{u}_1. \end{aligned} \quad (\text{H.35})$$

Next, we eliminate $\nabla \cdot \vec{u}_1$ from (H.34) and (H.35) to obtain

$$\partial_t \nabla^2 \eta_0 + (\vec{u}_0 \cdot \nabla) \nabla^2 \eta_0 = F(\partial_t \eta_0) - (\vec{u}_0 \cdot \nabla) \eta_B \quad (\text{H.36})$$

or

$$\partial_t (\nabla^2 \eta_0 - F \eta_0 + \eta_B) + \vec{u}_0 \cdot \nabla (\nabla^2 \eta_0 - F \eta_0 + \eta_B) = 0. \quad (\text{H.37})$$

After defining $\omega = \nabla^2 \eta_0 - F \eta_0 + \eta_B$, we arrive at the quasi-geostrophic equations

$$\partial_t \omega + \nabla \cdot (\omega \vec{u}) = 0 \quad (\text{H.38a})$$

$$\vec{u} = \nabla^\perp \psi \quad (\text{H.38b})$$

$$\omega = \nabla^2 \eta_0 - F \eta_0 + \eta_B. \quad (\text{H.38c})$$

Note that the quasi-geostrophic equations we derived here are different from the equation (3) presented in Section 2. In the above derivation we assumed that the Coriolis parameter, f , is constant. This can be used to describe the motion of a fluid on a rotating plane. In (3) the β -plane approximation was given in which the Coriolis parameter f is assumed to depend linearly on y .

If we take $A = 1$, $B = F$, $D = \eta_B$, $\psi = \eta_0$ and $\xi = \omega$ in (1), we see the equivalence between (1) and (H.38).

H.3 Rigid Lid Equations

The rigid lid equations can be deduced from the shallow water equations. Just like the quasi-geostrophic equations and shallow water equations they describe the motion of thin layer of fluid on a rotating plane. To deduce the rigid lid equations we assume that the surface of the fluid is at a constant height. Hence $H = h - h_B(x, y) = H(x, y)$ is fixed in time. The justification for this assumption is that variations in h are usually much smaller than variations in h_B .

The rigid lid equations are given by (4) and follow immediately from (H.21) and (H.20). Using the assumption $\partial_t H = 0$ (H.19) reduces to $\nabla \cdot (H\vec{u}) = 0$ and hence there exists a function ψ such that (4) holds.

I Gauss Integration Rules

We use Gauss integration rules to approximate the integrals over the elements and faces given in Appendix E. For the integrals over the faces, the following N -point Gauss rule is used

$$\int_{-1}^1 f(\gamma) \, d\gamma \approx \sum_{i=1}^N w_i f(\gamma_i) \quad (\text{I.1})$$

with w_i and γ_i the Gauss weights and points. These Gauss weights and points are given in Table I.1 for different values of N . The approximation given in (I.1) is exact for $(2N - 1)^{\text{th}}$ or lower order polynomials.

For the integrals over the reference element, we use the following Gauss rule

$$\int_{-1}^1 \int_{-1}^1 f(\xi, \eta) \, d\xi \, d\eta \approx \sum_{i=1}^N \sum_{j=1}^N w_i w_j f(\gamma_i, \gamma_j) \quad (\text{I.2})$$

with w_i and γ_i again given in Table I.1. This Gauss rule is exact for functions $f(\xi, \eta)$ which are $(2N - 1)^{\text{th}}$ or lower order polynomials in ξ and η .

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N	Order	i	γ_i	w_i
1	1	1	0	2
		1	0	2
2	3	1	-0.5773502691896257	1
		2	0.5773502691896257	1
3	5	1	-0.7745966692414834	0.5555555555555556
		2	0	0.8888888888888888
		3	0.7745966692414834	0.5555555555555556
4	7	1	-0.8611363115940526	0.3478548451374538
		2	-0.3399810435848563	0.6521451548625461
		3	0.3399810435848563	0.6521451548625461
		4	0.8611363115940526	0.3478548451374538
5	9	1	-0.906179845938664	0.2369268850561891
		2	-0.5384693101056831	0.4786286704993665
		3	0	0.5688888888888889
		4	0.5384693101056831	0.4786286704993665
		5	0.906179845938664	0.2369268850561891
6	11	1	-0.9324695142031521	0.1713244923791704
		2	-0.6612093864662645	0.3607615730481386
		3	-0.2386191860831969	0.467913934572691
		4	0.2386191860831969	0.467913934572691
		5	0.6612093864662645	0.3607615730481386
		6	0.9324695142031521	0.1713244923791704
7	13	1	-0.9491079123427585	0.1294849661688697
		2	-0.7415311855993945	0.2797053914892766
		3	-0.4058451513773972	0.3818300505051189
		4	0	0.4179591836734694
		5	0.4058451513773972	0.3818300505051189
		6	0.7415311855993945	0.2797053914892766
		7	0.9491079123427585	0.1294849661688697

Table I.1
Gauss points and weights.

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