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Memorandum No. 1629

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June, 2002

ISSN 0169-2690

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SPANNING 2-CONNECTED SUBGRAPHS OF ALPHABET GRAPHS, SPECIAL CLASSES OF GRID GRAPHS

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Abstract

A grid graph G is a finite induced subgraph of the infinite 2-dimensional grid defined by $Z \times Z$ and all edges between pairs of vertices from $Z \times Z$ at Euclidean distance precisely 1. A natural drawing of G is obtained by drawing its vertices in $\Re^2$ according to their coordinates. Apart from the outer face, all (inner) faces with area exceeding one (not bounded by a 4-cycle) in a natural drawing of G are called the holes of G. We define 26 classes of grid graphs called alphabet graphs, with no or a few holes. We determine which of the alphabet graphs contain a Hamilton cycle, i.e. a cycle containing all vertices, and solve the problem of determining a spanning 2-connected subgraph with as few edges as possible for all alphabet graphs.

Keywords: grid graph, alphabet graphs, Hamilton cycle, spanning 2-connected subgraph

AMS Subject Classifications: 05C40, 05C85

1 Introduction

The *infinite grid graph* G^∞ is defined by the set of vertices $V = \{(x,y) \mid x \in Z, y \in Z\}$ and the set of edges $E \subseteq V \times V$ between all pairs of vertices from V at Euclidean distance precisely 1. For any integers $s \geq 1$ and $t \geq 1$, the *rectangular grid graph* $R(s,t)$ is the (finite) subgraph of G^∞ induced by $V(s,t) = \{(x,y) \mid 1 \leq x \leq s, 1 \leq y \leq t, x \in Z, y \in Z\}$ (and just containing all edges from G^∞ between pairs of vertices from $V(s,t)$). This graph $R(s,t)$ is also known as the *product graph* $P_s \times P_t$ of two disjoint paths P_s and P_t. A *grid graph* is a graph that is isomorphic to a subgraph of $R(s,t)$ induced by a subset of $V(s,t)$ for some integers $s \geq 1$ and $t \geq 1$. It is clear that a grid graph $G = (V,E)$ is *a planar graph*, i.e. it can be drawn in the plane $\Re^2$ in such a way that the edges only intersect at the vertices of graph. In such a drawing, the regions of $\Re^2 \backslash (V \cup E)$ are called the *faces* of G. Exactly one of the faces is unbounded; this is called the *outer face*; the others are its *inner faces*. The *natural drawing* of a grid graph is just described by drawing its vertices in $\Re^2$ according to their coordinates. A *solid grid graph* is a grid graph all of whose inner faces have area one (are bounded by a cycle on four vertices) in a natural drawing. A grid graph that is not solid contains inner faces (in a natural drawing) that have area larger than one; these faces are called *holes*. A subgraph H of a graph $G = (V,E)$ is called a *spanning subgraph* if $V(H) = V$. A connected graph is called 2-*connected* if it remains connected if at most one vertex is removed. A *Hamilton cycle* in a graph $G = (V,E)$ is a cycle containing every vertex of V, i.e. a spanning 2-connected subgraph in which every vertex has degree 2 (the number of edges is $|V|$).

Itai, Papadimitriou and Szwarcfiter [2] proved that deciding whether a given grid graph has a Hamilton cycle is an NP-complete problem. This implies that the problem of finding a spanning 2-connected subgraph with as few edges as possible is also NP-hard for grid graphs. It has been conjectured that the first problem remains NP-complete when it is restricted to solid grid graphs. However, Umans and Lenhart [3] recently proved that this problem is polynomially solvable. For the second problem the complexity is not known when it is restricted to solid grid graphs. It remains an open problem –what the complexity of both problems is –when we restrict ourselves to grid graphs with a fixed number of holes.

Motivated by the above problems, we study the problem of the existence of a Hamilton cycle and the problem of determining a spanning 2-connected subgraph with as few edges as possible for a large number of classes of finite

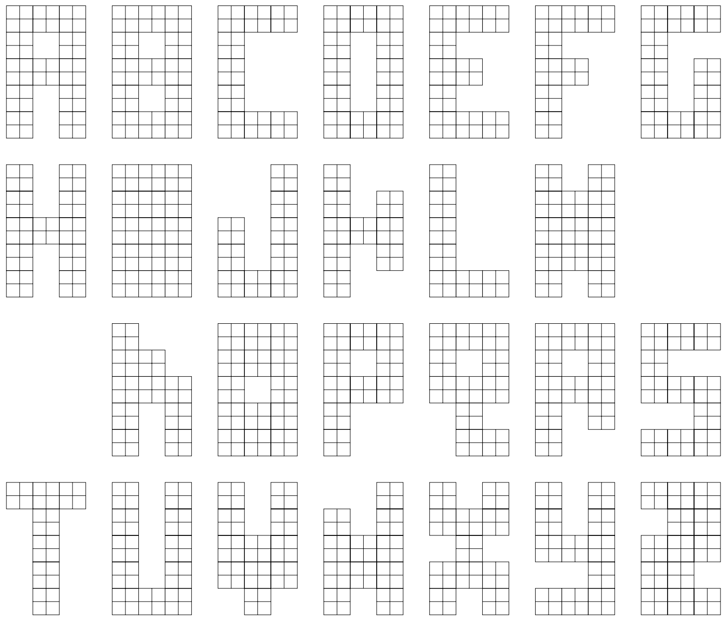

Figure 1: Alphabet graphs in order from A to Z for $n = 3$

grid graphs with no or a few holes called *alphabet graphs*. For all graphs of the defined classes we are able to solve the second problem. All solutions are of the same type : first, we use the well-known *Grinberg-condition* to derive a lower bound for the number of edges in a spanning 2-connected subgraph. Secondly, we show by construction that this lower bound is in fact the optimum value.

2 Alphabet graphs

We now introduce the 26 classes of grid graphs which we call *alphabet graphs*.

For every letter λ of the alphabet $\{a, b, \cdots, z\}$ we define a corresponding subgraph $\Lambda_{n,n}$ of $R(3n-2, 5n-4)$ for all $n \geq 3$. These alphabet graphs $\{A_{n,n}, B_{n,n}, \cdots, Z_{n,n}\}$ are shown in Figure 1 for $n = 3$. It is clear from these figures how these graphs should be extended for larger values of n. We will avoid the tedious details of defining all these 26 graph classes formally.

Notice that from these 26 classes, there is one class of alphabet graphs with two holes, namely the graph $B_{n,n}$; six classes with one hole, namely the graphs $A_{n,n}, D_{n,n}, O_{n,n}$, $P_{n,n}, Q_{n,n}$ and $R_{n,n}$; the remaining 19 classes contain no holes, i.e. are solid grid graphs.

We refer to these classes in the next result just by the capital letters, omitting the indices. Our main result characterizes which of the alphabet graphs are hamiltonian and shows that all of them contain a spanning 2-connected subgraph with at most one edge more than their number of vertices. In fact, in the proofs and constructions, we determine a spanning 2-connected subgraph with the smallest number of edges for all of the alphabet graphs (for $n = 4$ and $n = 5$; for $n = 3$ or for larger values of n it is again clear from the figures that follow how to extend these solutions). We postpone the proofs and constructions (figures) until the next section.

Theorem 1 *Let $A, B, \cdots, Z$ denote the alphabet graphs $A_{n,n}, B_{n,n}, \cdots, Z_{n,n}$ as defined above. Then:*
(i) A, D, O, P, Q and R are hamiltonian for all $n \geq 3$;
(ii) N contains a spanning 2-connected subgraph with $|V| + 1$ edges, but no Hamilton cycle for all $n \geq 3$;
(iii) The remaining alphabet graphs contain a spanning 2-connected subgraph with (at most) $|V| + 1$ edges and are hamiltonian if and only if n is an even number.

3 Proofs and constructions

A useful necessary condition for hamiltonicity is the following result due to Grinberg [1].

Lemma 2 *Suppose a planar graph G has a Hamilton cycle H. Let G be drawn in the plane, and let r_i denote the number of faces inside H bounded by i edges in this planar embedding. Let $r_i^{'}$ be the number of faces outside H bounded by i edges. Then the numbers r_i and $r_i^{'}$ satisfy the following equation.*

$$\sum_i (i-2)(r_i - r_i^{'}) = 0.$$

We use this lemma to show that N contains no Hamilton cycle, and that the alphabet graphs in (iii) have no Hamilton cycle if n is an odd number.

Corollary 3 *N contains no Hamilton cycle for all $n \geq 3$.*

Proof. The corollary is proved by contradiction. We know that there is exactly one face with $20(n-1)$ edges and there are exactly $10(n-1)^2$ faces with four edges in the planar (natural) drawing of the graph N. Now, suppose that N is hamiltonian. Then by Lemma 2 we have

$$(20(n-1)-2)(-1)+(4-2)(r_4-r_4^{'})=0.$$

Hence

$$r_4-r_4^{'}=10n-11. \tag{1}$$

It is known that the number of faces with four edges is

$$r_4+r_4^{'}=10(n-1)^2. \tag{2}$$

From equation (1) and (2) we obtain

$$2r_4=10n(n-1)-1. \tag{3}$$

This can not hold, since the right hand side of the equation (3) is odd. Hence, we conclude that N is not hamiltonian.

□

Corollary 4 *The alphabet graphs in (iii) contain no Hamilton cycle if n is an odd number.*

Proof. We will divide the proof into nine cases.

Case 1 We consider the alphabet graph B. There is exactly one face with $16(n-1)$ edges, there are two faces with $4(n-1)$ edges and there are exactly $13(n-1)^2$ faces with four edges in the planar (natural) drawing of this graph. Let B be hamiltonian. Then by Lemma 2 we have

$$(16(n-1)-2)(-1)+(4-2)(r_4-r_4^{'})+(4(n-1)-2)(r_{4(n-1)}-r_{4(n-1)}^{'})=0.$$

Hence

$$r_4-r_4^{'}=8n-9-(2n-3)(r_{4(n-1)}-r_{4(n-1)}^{'}). \tag{4}$$

It is known that the number of faces with four edges is

$$r_4+r_4^{'}=13(n-1)^2. \tag{5}$$

From equation (4) and (5) we obtain

$$2r_4=13n^2-18n+4-(2n-3)(r_{4(n-1)}-r_{4(n-1)}^{'}). \tag{6}$$

So, n is an even number since $(r_{4(n-1)}-r_{4(n-1)}^{'})$ is $-2, 0$ or 2.

Case 2 We consider the alphabet graphs C, J and K. There is exactly one face with $20(n-1)$ edges and there are exactly $9(n-1)^2$ faces with four edges in the planar (natural) drawing of every one of these graphs. Let these graphs be hamiltonian. Then by Lemma 2 and using a similar method as in the proof of Case 1, we obtain

$$2r_4 = 9n^2 - 8n - 2. \tag{7}$$

So, n is an even number.

Case 3 We consider the alphabet graph E. There is exactly one face with $22(n-1)$ edges and there are exactly $10(n-1)^2$ faces with four edges in the planar (natural) drawing of this graph. Let E be hamiltonian. Then by Lemma 2 and using a similar method as in the proof of Case 1, we obtain

$$2r_4 = 10n^2 - 9n - 2. \tag{8}$$

So, n is an even number.

Case 4 We consider the alphabet graph F. There is exactly one face with $18(n-1)$ edges and there are exactly $8(n-1)^2$ faces with four edges in the planar (natural) drawing of this graph. Let F be hamiltonian. Then by Lemma 2 and using a similar method as in the proof of Case 1, we obtain

$$2r_4 = 8n^2 - 7n - 2. \tag{9}$$

So, n is an even number.

Case 5 We consider the alphabet graphs G, H, S, U, X and Y. There is exactly one face with $24(n-1)$ edges and there are exactly $11(n-1)^2$ faces with four edges in the planar (natural) drawing of every one of these graphs. Let these graphs be hamiltonian. Then by Lemma 2 and using a similar method as in the proof of Case 1, we obtain

$$2r_4 = 11n^2 - 10n - 2. \tag{10}$$

So, n is an even number.

Case 6 We consider the alphabet graph I. There is exactly one face with $16(n-1)$ edges and there are exactly $15(n-1)^2$ faces with four edges in the planar (natural) drawing of this graph. Let I be hamiltonian. Then by Lemma 2 and using a similar method as in the proof of Case 1, we obtain

$$2r_4 = 15n^2 - 22n + 6. \tag{11}$$

So, n is an even number.

Case 7 We consider the alphabet graphs L and T. There is exactly one face with $16(n-1)$ edges and there are exactly $7(n-1)^2$ faces with four edges in the planar (natural) drawing of every one of these graphs. Let these graphs be hamiltonian. Then by Lemma 2 and using a similar method as in the proof of Case 1, we obtain

$$2r_4 = 7n^2 - 6n - 2. \tag{12}$$

So, n is an even number.

Case 8 We consider the alphabet graphs M and Z. There is exactly one face with $20(n-1)$ edges and there are exactly $13(n-1)^2$ faces with four edges in the planar (natural) drawing of every one of these graphs. Let these graphs be hamiltonian. Then by Lemma 2 and using a similar method as in the proof of Case 1, we obtain

$$2r_4 = 13n^2 - 16n + 2. \tag{13}$$

So, n is an even number.

Case 9 We consider the alphabet graphs V and W. There is exactly one face with $20(n-1)$ edges and there are exactly $11(n-1)^2$ faces with four edges in the planar (natural) drawing of every one of these graphs. Let these graphs be hamiltonian. Then by Lemma 2 and using a similar method as in the proof of Case 1, we obtain

$$2r_4 = 11n^2 - 12n. \tag{14}$$

So, n is an even number. □

We complete the proof of Theorem 1 by showing, through construction, the existence of a Hamilton cycle or a spanning 2-connected subgraph with $|V| + 1$ edges, in all cases where $n = 4$ or $n = 5$. As we mentioned before, for $n = 3$ or for larger values of n, it is not difficult to see from the patterns in the figures that follow how to extend the solutions.

Hamilton cycles for the alphabet graphs A, D, O, P, Q and R are shown in Figure 2 for $n = 4$ and in Figure 3 for $n = 5$. The patterns in Figure 2 can be used for finding Hamilton cycles for these graphs for a larger even value of n; the patterns in Figure 3 can be used for determining Hamilton cycles for these graphs for $n = 3$ or for a larger odd value of n.

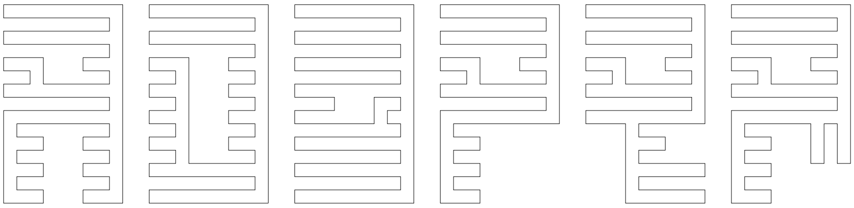

Figure 2: Hamilton cycles for the alphabet graphs A, D, O, P, Q and R for $n = 4$

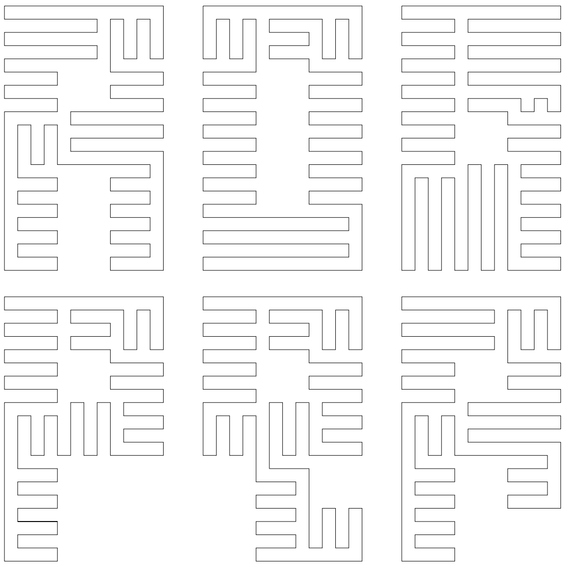

Figure 3: Hamilton cycles for the alphabet graphs A, D, O, P, Q and R for $n = 5$

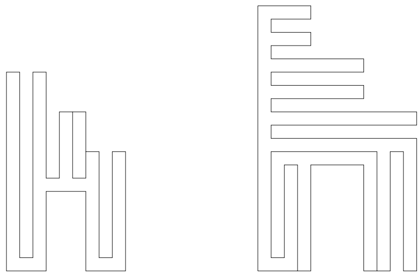

Figure 4: A spanning 2-connected subgraph with $|V| + 1$ edges for the alphabet graph N for $n = 4$ (on the left) and $n = 5$ (on the right)

In Figure 4, we show a spanning 2-connected subgraph with $|V|+1$ edges for the alphabet graph N for $n = 4$ and $n = 5$. The pattern in the left part of the figure can be used for determining a spanning 2-connected subgraph with $|V| + 1$ edges for this graph for a larger even value of n. The pattern in the right part of the figure can be used for constructing such a spanning subgraph for this graph for $n = 3$ or for a larger odd value of n.

Hamilton cycles for the remaining alphabet graphs for $n = 4$ are shown in Figure 5. The patterns in this figure can be used for finding Hamilton cycles for these graphs for a larger even value of n. Finally, in Figure 6 we show spanning 2-connected subgraphs with $|V| + 1$ edges for these graphs for $n = 5$. The patterns in this last figure can be used for constructing such spanning subgraphs for these graphs for $n = 3$ or for a larger odd value of n.

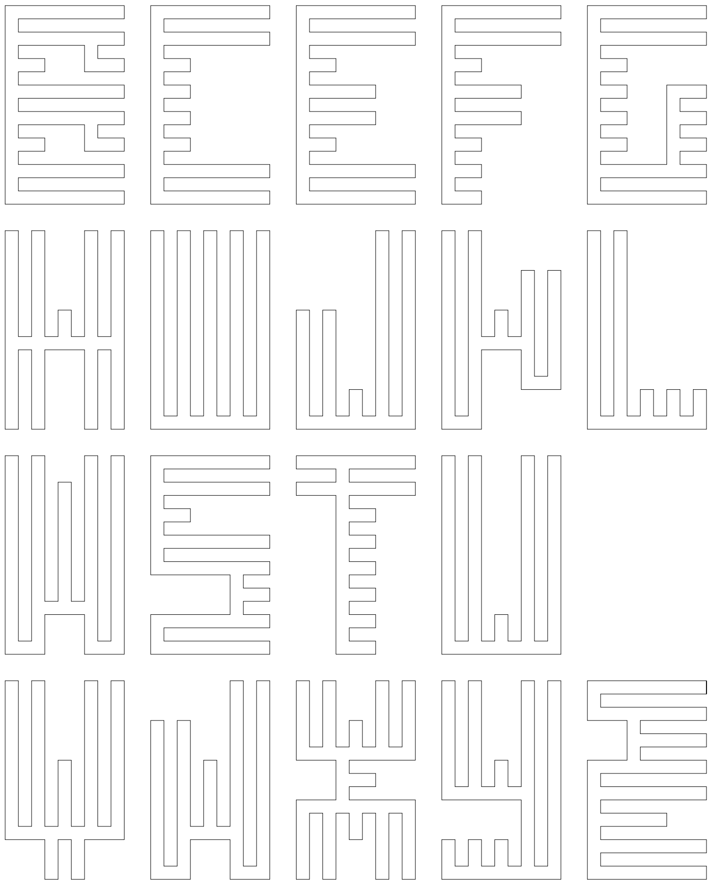

Figure 5: Hamilton cycles for the alphabet graphs in (iii) for $n = 4$

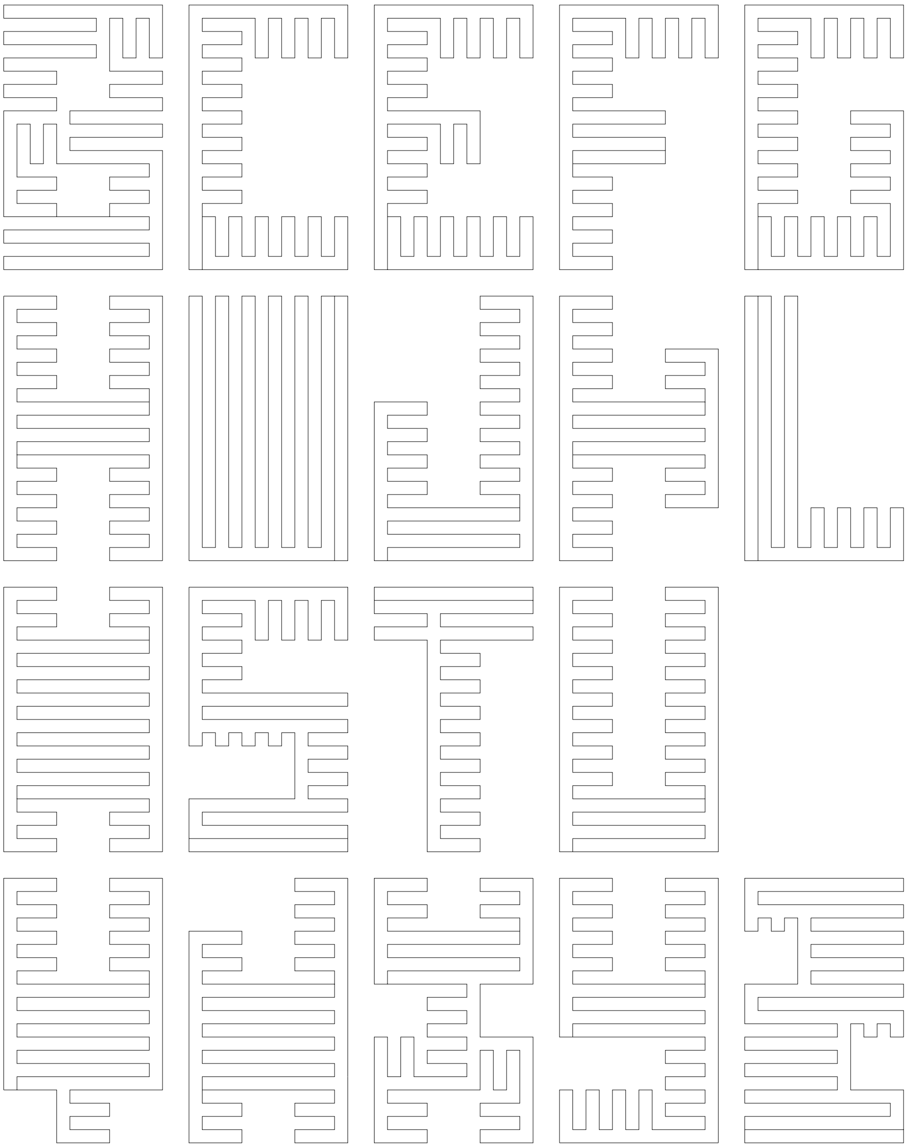

Figure 6: Spanning 2-connected subgraphs with $|V|+1$ edges for the alphabet graphs in *(iii)* for $n = 5$

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