Faculty of Mathematical Sciences

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl
www.math.utwente.nl/publications

Memorandum No. 1615

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P.H.M. Kersten and A.S. Sorin[1]

February 2002

ISSN 0169-2690

[1]Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research (JINR), 141980 Dubna, Moscow Region, Russia

January 2002
nlin.SI/0201061

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P.H.M. Kersten[(a)] and A.S. Sorin[(b)]

[(a)] *University of Twente, Faculty of Mathematical Sciences,*
P.O.Box 217,7500 AE Enschede, The Netherlands
E-Mail: kersten@math.utwente.nl

[(b)] *Bogoliubov Laboratory of Theoretical Physics,*
Joint Institute for Nuclear Research (JINR),
141980 Dubna, Moscow Region, Russia
E-Mail: sorin@thsun1.jinr.ru

Abstract

The $N = 2$ supersymmetric $\alpha = 1$ KdV hierarchy in $N = 2$ superspace is considered and its rich symmetry structure is uncovered. New nonpolynomial and nonlocal, bosonic and fermionic symmetries and Hamiltonians, bi-Hamiltonian structure as well as a recursion operator connecting all symmetries and Hamiltonian structures of the $N = 2$ $\alpha = 1$ KdV hierarchy are constructed in explicit form. It is observed that the algebra of symmetries of the $N = 2$ supersymmetric $\alpha = 1$ KdV hierarchy possesses two different subalgebras of $N = 2$ supersymmetry.

AMS Subject Classification: 35Q51; 35Q53; 35Q58
Keywords: Completely Integrable Systems;Supersymmetry; Discrete Symmetries; Recursion Operator;Bi-Hamiltonian Structure

1 Introduction

The $N = 2$ supersymmetric $\alpha = 1$ KdV equation was originally introduced in [1] as a Hamiltonian equation with the $N = 2$ superconformal algebra as a second Hamiltonian structure, and its integrability was conjectured there due to the existence of a few additional nontrivial bosonic Hamiltonians. Then its Lax–pair representation has indeed been constructed in [2], and it allowed an algoritmic reconstruction of the whole tower of highest *commutative* bosonic flows and their Hamiltonians belonging to the $N = 2$ supersymmetric $\alpha = 1$ KdV hierarchy.

Actually, besides the $N = 2$ $\alpha = 1$ KdV equation there are another two inequivalent $N = 2$ supersymmetric Hamiltonian equations with the $N = 2$ superconformal algebra as a second Hamiltonian structure (the $N = 2$ $\alpha = -2$ and $\alpha = 4$ KdV equations [3, 1]), but the $N = 2$ $\alpha = 1$ KdV equation is rather exceptional [4]. Despite knowledge of its Lax–pair description, there remains a lot of longstanding, unsolved problems which resolution would be quite important for a deeper understanding and more detailed description of the $N = 2$ $\alpha = 1$ KdV hierarchy. Thus, since the time when the $N = 2$ $\alpha = 1$ KdV equation was proposed, much efforts were made to construct a tower of its *noncommutative* bosonic and fermionic, local and nonlocal symmetries and Hamiltonians, bi-Hamiltonian structure as well as recursion operator (see, e.g. discussions in [5, 6] and references therein). Though these rather complicated problems, solved for the case of the $N = 2$ $\alpha = -2$ and $\alpha = 4$ KdV hierarchies, still wait their complete resolution for the $N = 2$ $\alpha = 1$ KdV hierarchy, a considerable progress towards their solution arose quite recently. Thus, the *puzzle* [5, 6], related to the *"nonexistence"* of higher fermionic flows of the $N = 2$ $\alpha = 1$ KdV hierarchy, was partly resolved in [7, 8] by explicit constructing a few bosonic and fermionic *nonlocal* and *nonpolynomial* flows and Hamiltonains, then their $N = 2$ superfield structure and origin were uncovered in [9]. A new property, crucial for the existence of these flows and Hamiltonians, making them distinguished compared to all flows and Hamiltonians of other supersymmetric hierarchies constructed before, is their *nonpolynomiality*. A new approach to a recursion operator treating it as a form–valued vector field which satisfies a generalized symmetry equation related to a given equation was developed in [10, 11]. Using this approach the recursion operator of the bosonic limit of the N=2 $\alpha = 1$ KdV hierarchy was derived in [12], and its structure, underlining relevance of these Hamiltonians in the bosonic limit, gives a hint towards its supersymmetric generalization.

The present Letter addresses the above–mentioned problems. We demonstrate that the existence and knowledge of the bosonic and fermionic, nonlocal and nonpolynomial Hamiltonians of the $N = 2$ $\alpha = 1$ KdV hierarchy is indeed a crucial, key point in constructing the recursion operator connecting all its symmetries and Hamiltonian structures.

The Letter is organized as follows. In Section 2 we present a short summary of the main known facts concerning the $N = 2$ $\alpha = 1$ KdV hierarchy. In Section 3 we describe a general algorithm of constructing recursion operators of integrable systems which we follow in the next sections. In Section 4 we construct a minimal, basic set of fermionic and bosonic flows as well as their Hamiltonians which is then used in Section 5 to derive the recursion operator and bi-Hamiltonian structure of the $N = 2$ $\alpha = 1$ KdV hierarchy. We also demonstrate that the algebra of symmetries of the $N = 2$ $\alpha = 1$ KdV hierarchy possesses two different subalgebras of $N = 2$ supersymmetry. In Section 6 we summarize our results.

2 The N=2 supersymmetric $\alpha = 1$ KdV hierarchy

The $N=2$ supersymmetric $\alpha=1$ KdV equation [1]

$$\frac{\partial}{\partial t_3} J = [J\,'' + 3J[D,\overline{D}\,]J + J^3]\,' \tag{1}$$

is the first nontrivial representative of the infinite tower of the commutative flows $\frac{\partial}{\partial t_{2p+1}}$

$$[\frac{\partial}{\partial t_{2m+1}}, \frac{\partial}{\partial t_{2n+1}}] = 0 \tag{2}$$

belonging to the $N=2$ supersymmetric $\alpha=1$ KdV hierarchy. The latter can be defined via the Lax–pair representation [2]

$$\frac{\partial}{\partial t_{2p+1}} L = [(L^{2p+1})_{\geq 0}, L], \quad L = \partial + [D,\overline{D}]\partial^{-1} J, \quad p \in \mathbb{N} \tag{3}$$

where the subscript ≥ 0 denotes the sum of purely differential and constant parts of the operator L^{2p+1}, $J \equiv J(Z)$ is a superfield of inverse length dimension $[J]=1$ in the $N=2$ superspace with a coordinate $Z=(z,\theta,\overline{\theta})$, $'$ denotes the derivative with respect to z and $D, \overline{D}$ are the fermionic covariant derivatives of $N=2$ supersymmetry

$$D = \frac{\partial}{\partial\theta} - \frac{1}{2}\overline{\theta}\frac{\partial}{\partial z}, \quad \overline{D} = \frac{\partial}{\partial\overline{\theta}} - \frac{1}{2}\theta\frac{\partial}{\partial z}, \quad D^2 = \overline{D}^2 = 0, \quad \left\{D,\overline{D}\right\} = -\frac{\partial}{\partial z} \equiv -\partial. \tag{4}$$

The flows (3) are Hamiltonian

$$\frac{\partial}{\partial t_{2p+1}} J = \{J, H_{2p+1}\}_2 \equiv J_2 \frac{\delta}{\delta J} H_{2p+1} \tag{5}$$

with the Hamiltonians

$$H_{2p+1} = \int dZ\ \mathcal{H}_{2p+1}, \quad \mathcal{H}_{2p+1} = res(L^{2p+1}) \tag{6}$$

and the Poisson brackets

$$\{J(Z_1), J(Z_2)\}_2 = J_2(Z_1)\delta^{N=2}(Z_1 - Z_2), \quad J_2 \equiv \frac{1}{2}[D,\overline{D}\,]\partial + \overline{D}JD + DJ\overline{D} + \partial J + J\partial \tag{7}$$

forming the $N=2$ superconformal algebra which is the second Hamiltonian structure of the $N=2$ $\alpha=1$ KdV hierarchy [1]. Here, $dZ \equiv dz d\theta d\overline{\theta}$, $\int dZ (...) \equiv \int_{-\infty}^{+\infty} dz D\overline{D}(...)\Big|_{\theta=\overline{\theta}=0}$, $res(L^p)$ is the $N=2$ supersymmetric residue, i.e. the coefficient at $[D,\overline{D}]\partial^{-1}$, and $\delta^{N=2}(Z) \equiv \theta\overline{\theta}\delta(z)$ is the delta function in $N=2$ superspace.

The flows (3) admit also the following useful representation [9]

$$\frac{\partial}{\partial t_{2p+1}} J = \mathcal{H}_{2p+1}^{'} \tag{8}$$

in terms of Hamiltonian densities $\mathcal{H}_{2p+1}$ (6) and possess the complex conjugation

$$(t_{2p+1}, z, \theta, \overline{\theta}, J)^{\star} = (-t_{2p+1}, -z, \overline{\theta}, \theta, J). \tag{9}$$

A few first representatives of the Hamiltonians H_{2p+1} (6) and flows $\frac{\partial}{\partial t_{2p+1}}$ (8) are

$$H_1 = \int dZ\ J, \quad H_3 = \int dZ\ (3J[D,\overline{D}]J + J^3) \tag{10}$$

and

$$\frac{\partial}{\partial t_1} J = J\,' \tag{11}$$

and the third flow $\frac{\partial}{\partial t_3} J$ which reproduces the $N=2$ supersymmetric $\alpha=1$ KdV equation (1).

3 General algorithm of constructing recursion operators

Let us present a short summary of the general scheme of constructing recursion operators of integrable systems, considered in [7] (for more details, see [7] and references therein), which we adapt and develop to the problems under consideration and use in next sections.

In effect, in the theory of deformations of the equation structure a classical recursion operator is a form–valued vector field which is a nontrivial infinitesimal deformation of the equation structure. For reasons of simplicity and in order to make it more accessible to the physics oriented reader, in the following the form–valuedness will be transformed to Fréchet derivatives of the associated variables and quantities.

Our starting point is the one-dimensional system of evolution scale–invariant[1] equations (with the evolution time t) for the set of superfields $u \equiv u_i(Z)$ $(i = 1, ..., n)$

$$\tfrac{\partial}{\partial t} u - \Phi = 0, \tag{12}$$

their integrals of motion

$$\mathcal{G}_\alpha = \int_{-\infty}^{+\infty} dz G_\alpha, \quad \tfrac{\partial}{\partial t} G_\alpha = F_\alpha{}', \quad \alpha = 0, 1, ... \tag{13}$$

and the corresponding symmetry equation

$$\Big(\tfrac{\partial}{\partial t} - \phi\Big) Y_\tau = 0, \quad Y_\tau := \tfrac{\partial}{\partial \tau} u \tag{14}$$

which is derived by differentiation of the system (12) over an arbitrary independent time[2] τ. Here, $\Phi \equiv \Phi(\{u\})$ is a local, while $G_i \equiv G_i(\{u\})$, $F_i \equiv F_i(\{u\})$ as well as $Y_\tau \equiv Y_\tau(\{u\})$ in general are nonlocal functionals of $\{\partial^n u,\ D\partial^n u,\ \overline{D}\partial^n u, [D, \overline{D}]\partial^n u,\ n \in \mathbb{N}\}$, and the operator ϕ is the corresponding Fréchet derivative of the functional Φ.

The symmetry equation (14) represents a complicated functional equation, and its general solution is not known. Its particular solutions are symmetries of the system (12) we started with, i.e. $[\frac{\partial}{\partial t}, \frac{\partial}{\partial \tau}] = 0$ by construction. For a more complete understanding of the original system, its hierarchy structure and solutions (tau function) it seems necessary to know as many solutions of its symmetry equation (14) as possible.

It turns out that there exist subsets of the whole set of solutions of the symmetry equation (14) for which two different representatives Y_τ and $Y_{\widetilde{\tau}}$ can consistently be related as

$$Y_{\widetilde{\tau}} := \tfrac{\partial}{\partial \widetilde{\tau}} u = P \tfrac{\partial}{\partial \tau} u + \sum_\alpha P_\alpha \tfrac{\partial}{\partial \tau} \partial^{-1} G_\alpha \equiv \Big(P + \sum_\alpha P_\alpha \partial^{-1} g_\alpha\Big) Y_\tau := R Y_\tau \tag{15}$$

where the operator g_α is the Fréchet derivative of the functional G_α, $P \equiv P(\{u, \partial^{-1}G, \partial, D, \overline{D}\})$ is a general purely differential operator over the derivatives $\{\partial,\ D,\ \overline{D}\}$ which coefficient–functions are scale–homogeneous polynomials over $\{u, \partial^{-1}G\}$ and their $\{\partial^n,\ D\partial^n,\ \ \overline{D}\partial^n,$ $[D, \overline{D}]\partial^n\}$–derivatives and $P_\alpha \equiv P_\alpha(\{u, \partial^{-1}G\})$ is a scale–homogeneous polynomial functional over its two arguments and their $\{\partial^n,\ D\partial^n,\ \ \overline{D}\partial^n,\ [D, \overline{D}]\partial^n\}$–derivatives obeying the so called deformation equation[3]

$$\Big((\tfrac{\partial}{\partial t} P) - \phi P\Big) \tfrac{\partial}{\partial \tau} u + P \tfrac{\partial}{\partial \tau} \Phi + \sum_\alpha \Big\{ \Big((p_\alpha \Phi)_0 - \phi P_\alpha\Big) \tfrac{\partial}{\partial \tau} \partial^{-1} G_\alpha + P_\alpha \tfrac{\partial}{\partial \tau} F_\alpha \Big\} = 0 \tag{16}$$

[1]This means that all parameters, involved in the equation, are dimensionless.

[2]We would like to emphasize that supersymmetric systems in general admit both bosonic or fermionic evolution times. Here, for definiteness we discuss only the former case, its generalization to the latter case is rather straightforward.

[3]Here, the brackets $(...)_0$ mean that the relevant operators act only on the superfields inside the brackets.

which is the consistency condition resulting from the requirement that both Y_τ and $Y_{\widetilde{\tau}}$ have to satisfy the symmetry equation (14). Here, the operator p_α is the Fréchet derivative of the functional P_α. By definition, the operator R, defined in eq. (15), which has a minimal inverse length dimension, is the recursion operator of the hierarchy of symmetries of the equation (12). Scale dimensions of the quantities P and P_α depend on the dimension of the recursion operator (see. eqs. (18)). Extracting the coefficients at linear–independent functionals $\frac{\partial}{\partial\tau}u$, $\frac{\partial}{\partial\tau}\partial^{-1}G_\alpha$ and their $\{\partial^n,\ D\partial^n,\ \overline{D}\partial^n, [D,\overline{D}]\partial^n u\}$–derivatives in eq. (16) and equating them to zero one can derive a complete set of self–consistent equations for coefficient–functions of the operator P and polynomials P_α which solutions specify the recursion operator R (15).

A few important remarks are in order.

First, coefficient–functions of the operator P and polynomial P_α can in general be non–polynomial functions of the dimensionless quantity $\partial^{-1}G_0$ ($[\partial^{-1}G_0]=0$) (if any).

Second, one can see from a simple dimensional consideration that the sum over α in eqs. (15–16) is usually saturated by a finite number of terms due to pure dimensional restrictions. Indeed, the inverse length dimensions of the quantities entering into eqs. (15–16) are related and bounded as $[P_\alpha]+[\mathcal{G}_\alpha]-[u]=[R]>0$, $[P_\alpha]\geq 0$, $[\mathcal{G}_\alpha]\geq 0$, so the inverse length dimension $[\mathcal{G}_\alpha]$ of the integrals $\mathcal{G}_\alpha$ (13), contributing the sum over α, is bounded both from below and above as

$$0 \leq [\mathcal{G}_\alpha] \leq [R]+[u], \tag{17}$$

but in general there exists only a finite number of integrals which inverse length dimensions belong to a finite interval $\Big[0,[R]+[u]\Big]$.

Third, a simple inspection of eq. (16) shows that the functional P_α, which is a factor of the τ–derivative of the lowest dimension superfield component of the integral[4] of highest inverse length dimension entering into eq. (15), satisfies the symmetry equation (14), so this P_α is a symmetry of eq. (12).

Summarizing the above–described algorithm, it consists of a few steps. Thus, at the first step, as an input it is necessary to define the inverse length dimension $[R]$ of the recursion operator, then to construct a complete set of integrals $\mathcal{G}_\alpha$ (13) for eq. (12) which inverse length dimensions $[\mathcal{G}_\alpha]$ satisfy the inequality (17). The next step is a rather straightforward technical derivation of general expressions for scale–homogeneous operator P and polynomials P_α according to their inverse length dimensions

$$[P]=[R], \quad [P_\alpha]=[R]-[\mathcal{G}_\alpha]+[u]. \tag{18}$$

Then substituting all the derived quantities into the deformation equation (16), extracting equations for coefficient–functions of the operator P and polynomials P_α and, at last, solving them, one can finally obtain a desirable explicit expression for the recursion operator R (15).

Because of the quite technically complicated construction of the recursion operator we described above, in what follows we actually make our choice by first imposing conditions on $\{P,\ P_\alpha\}$, which are more simple than the deformation equation (16), such that the relation (15) is satisfied only for a set of special symmetries. Then, after solving these conditions and obtaining the explicit expression for the recursion operator we prove additionally that it indeed satisfies the deformation equation (16) as well, i.e. proving that associated form–valued vector field is in fact a generalized (with respect to its form–valuedness) symmetry of equation (12). We refer the interested reader to ref. [13] for all details of the complete computations of the results of the next sections.

[4]An $N=2$ superfield integral of the form (13) has four independent superfield components in general.

4 Nonlocal Hamiltonians and flows of the $N = 2$ $\alpha = 1$ KdV hierarchy

In this section we realize the first step of the general scheme, presented in the previous section, i.e. construct Hamiltonians and symmetries of the $N = 2$ $\alpha = 1$ KdV equation (1) which are relevant in the context of the further construction of the recursion operator of the $N = 2$ $\alpha = 1$ KdV hierarchy.

4.1 Hamiltonians

To this aim let us first define a dimension of the recursion operator as well as dimensions of Hamiltonians. Remembering that the recursion operator has to connect two nearest subsequent bosonic flows $\frac{\partial}{\partial t_{2p-1}}J$ and $\frac{\partial}{\partial t_{2p+1}}J$ (8) of the $N = 2$ $\alpha = 1$ KdV hierarchy and that $[\frac{\partial}{\partial t_{2p-1}}J] = 2p$, one can easily establish its inverse length dimension $[R] = 2$. Moreover, because the flows $\frac{\partial}{\partial t_{2p+1}}J$ (8) depend on the Grassmann coordinates $\{\theta, \overline{\theta}\}$ only implicitly via its dependence on the superfield $J(Z)$ as well as the fermionic covariant derivatives $\{D, \overline{D}\}$, the recursion operator has to possess the same property as well. Then using inequality (17) and relation (18) one can evaluate dimensions of Hamiltonians

$$0, \quad \frac{1}{2}, \quad 1, \quad \frac{3}{2}, \quad 2, \quad \frac{5}{2}, \quad 3 \tag{19}$$

and the corresponding dimensions of the polynomials P_α

$$3, \quad \frac{5}{2}, \quad 2, \quad \frac{3}{2}, \quad 1, \quad \frac{1}{2}, \quad 0 \tag{20}$$

we are interested in. Keeping in mind that the polynomial P_α, which corresponds to the Hamiltonian with the highest dimension entering into eq. (15), has to satisfy the symmetry equation (14) (see the third remark at the end of the previous section), as well as the above–mentioned fact that the recursion operator does not depend explicitly on $\{\theta, \overline{\theta}\}$ and that the minimal dimension of the $\{\theta, \overline{\theta}\}$–independent flows is[5] 2 (it is the dimension of the first bosonic flow $\frac{\partial}{\partial t_1}J$ (11)), we are led to the final conclusion that for our ultimate purposes we have to know a complete set of superfield Hamiltonians with inverse length dimensions

$$0, \quad \frac{1}{2}, \quad 1 \tag{21}$$

only. We would like especially to remark that all superfield components of these Hamiltonians have in general to be included into the *Ansatz* (15) for the recursion operator.

Superfield Hamiltonians

$$I_m = \int_{-\infty}^{+\infty} dz \mathcal{I}_{m+1}, \quad m \in \frac{\mathbb{N}}{2} \tag{22}$$

[5]We have verified by explicit construction that there are no flows with dimensions $\{0, \frac{1}{2}\}$ and that existing flows with dimensions $\{1, \frac{3}{2}\}$ are $\{\theta, \overline{\theta}\}$–dependent $\Big($see the expressions for the flows $U_0 J$, $D_{\frac{1}{2}}J$ and $\overline{D}_{\frac{1}{2}}J$ ($[U_0 J] = 1$, $[D_{\frac{1}{2}}J] = [\overline{D}_{\frac{1}{2}}J] = \frac{3}{2}$) in eqs. (30–32) and discussion at the end of the next subsection$\Big)$.

with dimensions $\{0, \frac{1}{2}\}$ were constructed in [9]. Their Hamiltonian densities, which are unrestricted $N = 2$ superfields containing four independent superfield components, are

$$\mathcal{I}_1 = J, \quad \frac{\partial}{\partial t_3} J = \Big(J\,'' + 3J[D, \overline{D}\,]J + J^3 \Big)\,' \tag{23}$$

and

$$\mathcal{I}_{\frac{3}{2}} = e^{+2\partial^{-1}J} DJ, \quad \frac{\partial}{\partial t_3} \mathcal{I}_{\frac{3}{2}} = \Big(\mathcal{I}_{\frac{3}{2}}\,'' - 3J\mathcal{I}_{\frac{3}{2}}\,' + 3J^2 \mathcal{I}_{\frac{3}{2}} + 3([D, \overline{D}]J)\mathcal{I}_{\frac{3}{2}} \Big)\,' \tag{24}$$

as well as

$$\mathcal{I}^{\star}_{\frac{3}{2}} = e^{-2\partial^{-1}J} \overline{D}J, \quad \frac{\partial}{\partial t_3} \mathcal{I}^{\star}_{\frac{3}{2}} = \Big(\mathcal{I}^{\star}_{\frac{3}{2}}\,'' + 3J\mathcal{I}^{\star}_{\frac{3}{2}}\,' + 3J^2 \mathcal{I}^{\star}_{\frac{3}{2}} + 3([D, \overline{D}]J)\mathcal{I}^{\star}_{\frac{3}{2}} \Big)\,'. \tag{25}$$

What concerns to the remaining dimension 1 in (21), we have constructed the corresponding Hamiltonian I_1 (22) with the density

$$\begin{aligned}
\mathcal{I}_2 &= \mathcal{I}^{\star}_{\frac{3}{2}} \partial^{-1} \mathcal{I}_{\frac{3}{2}} + (\overline{D}J)\partial^{-1} DJ, \\
\frac{\partial}{\partial t_3} \mathcal{I}_2 &= \Big[2(\overline{D}J)DJ\,' - 2(\overline{D}J)\,' DJ + 4J(\overline{D}J)DJ \\
&- (\partial^{-1}\mathcal{I}_{\frac{3}{2}})\partial^{-1} \frac{\partial}{\partial t_3} \mathcal{I}^{\star}_{\frac{3}{2}} - (D\partial^{-1}J)\overline{D}\partial^{-1} \frac{\partial}{\partial t_3} J \Big]\,'
\end{aligned} \tag{26}$$

by "brute-force". Hereafter, the subscripts denote inverse length dimensions. The Hamiltonians $I_{\frac{1}{2}}$ and $I^{\star}_{\frac{1}{2}}$ (22) with the densities $\mathcal{I}_{\frac{3}{2}}$ (24) and $\mathcal{I}_{\frac{3}{2}}$ (25), respectively, are related by the complex conjugation (9), while the complex conjugation properties of the Hamiltonians I_0, I_1 are $I^{\star}_0 = -I_0$, $I^{\star}_1 = -I_1$. We have verified by explicit construction that I_0, $I_{\frac{3}{2}}$, $I^{\star}_{\frac{3}{2}}$ and I_1 are the only Hamiltonians with dimensions $\{0, \frac{1}{2}, 1\}$.

To close this subsection let us present Fréchet derivatives f_i $(i = \frac{1}{2}, 1)$ of the Hamiltonian densities $\mathcal{I}_{i+1}$ (24–26)

$$\begin{aligned}
& f_{\frac{1}{2}} = e^{2(\partial^{-1}J)} D + 2\mathcal{I}_{\frac{3}{2}} \partial^{-1}, \quad f^{\star}_{\frac{1}{2}} = e^{-2(\partial^{-1}J)} \overline{D} - 2\mathcal{I}^{\star}_{\frac{3}{2}} \partial^{-1}, \\
& f_1 = \mathcal{I}^{\star}_{\frac{3}{2}} \partial^{-1} f_{\frac{1}{2}} - (\partial^{-1}\mathcal{I}_{\frac{3}{2}}) f^{\star}_{\frac{1}{2}} + (\overline{D}J) D\partial^{-1} - (D\partial^{-1}J)\overline{D}
\end{aligned} \tag{27}$$

as well as their operator conjugated quantities

$$\begin{aligned}
& f^T_{\frac{1}{2}} = -De^{2(\partial^{-1}J)} - 2\partial^{-1}\mathcal{I}_{\frac{3}{2}}, \quad f^{\star T}_{\frac{1}{2}} = -\overline{D}e^{-2(\partial^{-1}J)} + 2\partial^{-1}\mathcal{I}^{\star}_{\frac{3}{2}}, \\
& f^T_1 = f^T_{\frac{1}{2}} \partial^{-1} \mathcal{I}^{\star}_{\frac{3}{2}} + f^{\star T}_{\frac{1}{2}} (\partial^{-1}\mathcal{I}_{\frac{3}{2}}) - D\partial^{-1}(\overline{D}J) - \overline{D}(D\partial^{-1}J)
\end{aligned} \tag{28}$$

which we use in what follows. Here, we use the following standard convention regarding the operator conjugation (transposition) ^{T}

$$(\partial, D, \overline{D})^T = -(\partial, D, \overline{D}), \quad (OP)^T = (-1)^{d_O d_P} P^T O^T \tag{29}$$

where O (P) is an arbitrary operator with the Grassmann parity d_O (d_P), d_O=0 $(d_O = 1)$ for bosonic (fermionic) operator O.

4.2 Flows

The flows can be derived from Hamiltonians by means of the formula (5). Let us present fermionic and bosonic flows generated by the superfield integrals with the densities $\mathcal{I}_0$, $\mathcal{I}_{\frac{1}{2}}$, $\mathcal{I}^{\star}_{\frac{1}{2}}$ and $\mathcal{I}_1$ (23–26),

$$U_0 J = (\theta\frac{\partial}{\partial\theta} - \overline{\theta}\frac{\partial}{\partial\overline{\theta}})J, \quad Q_{\frac{1}{2}} J = QJ, \quad \overline{Q}_{\frac{1}{2}} J = \overline{Q}J, \quad \tfrac{\partial}{\partial t_1} J = J\,' \tag{30}$$

and

$$D_{\frac{1}{2}} J = \mathcal{D}_{\frac{5}{2}}(\theta\overline{\theta}), \quad U_1^{(+)} J = \mathcal{D}_{\frac{5}{2}}(\theta), \quad U_1^{(-)} J = \mathcal{D}_{\frac{5}{2}}(\overline{\theta}), \quad D_{\frac{3}{2}} J = \mathcal{D}_{\frac{5}{2}}(1) \tag{31}$$

and

$$\overline{D}_{\frac{1}{2}} J = \overline{\mathcal{D}}_{\frac{5}{2}}(\theta\overline{\theta}), \quad \overline{U}_1^{(+)} J = \overline{\mathcal{D}}_{\frac{5}{2}}(\overline{\theta}), \quad \overline{U}_1^{(-)} J = \overline{\mathcal{D}}_{\frac{5}{2}}(\theta), \quad \overline{D}_{\frac{3}{2}} J = \overline{\mathcal{D}}_{\frac{5}{2}}(1) \tag{32}$$

as well as

$$U_1 J = \mathcal{D}_3(\theta\overline{\theta}), \quad Q_{\frac{3}{2}} J = \mathcal{D}_3(\overline{\theta}), \quad \overline{Q}_{\frac{3}{2}} J = \mathcal{D}_3(\theta), \quad U_2 J = \mathcal{D}_3(1), \tag{33}$$

respectively, where $D_{\frac{p}{2}}$ and $\overline{D}_{\frac{p}{2}}$, $Q_{\frac{p}{2}}$ and $\overline{Q}_{\frac{p}{2}}$ (U_p, $U_p^{(\pm)}$ and $\overline{U}_p^{(\pm)}$) ($p \in \mathbb{N}$) are new fermionic (bosonic) evolution derivatives with the following properties with respect to the complex conjugation (9):

$$D^{\star}_{\frac{p}{2}} = \overline{D}_{\frac{p}{2}}, \quad Q^{\star}_{\frac{p}{2}} = \overline{Q}_{\frac{p}{2}}, \quad U_p^{\pm\star} = \overline{U}_p^{\pm}, \quad U_p{}^{\star} = (-1)^{p+1} U_p, \quad \tfrac{\partial}{\partial t_{2p+1}}{}^{\star} = -\tfrac{\partial}{\partial t_{2p+1}}, \tag{34}$$

Q and $\overline{Q}$ are generators of the $N=2$ supersymmetry,

$$\begin{aligned} &Q = \frac{\partial}{\partial\theta} + \frac{1}{2}\overline{\theta}\frac{\partial}{\partial z}, \quad \overline{Q} = \frac{\partial}{\partial\overline{\theta}} + \frac{1}{2}\theta\frac{\partial}{\partial z}, \quad Q^2 = \overline{Q}^2 = 0, \quad \left\{Q, \overline{Q}\right\} = \partial, \\ &\{Q, D\} = \left\{Q, \overline{D}\right\} = 0, \quad \left\{\overline{Q}, D\right\} = \left\{\overline{Q}, \overline{D}\right\} = 0, \end{aligned} \tag{35}$$

and in eqs. (30–33) we have introduced the operators

$$\mathcal{D}_{\frac{5}{2}} \equiv -J_2 f^T_{\frac{1}{2}}, \quad \overline{\mathcal{D}}_{\frac{5}{2}} \equiv J_2 f^{\star T}_{\frac{1}{2}}, \quad \mathcal{D}_3 \equiv J_2 f^T_1 \tag{36}$$

where the operators J_2 and f^T_i are defined in equations (7) and (28), respectively. When deriving eqs. (30–33) we integrated by parts and made essential use of the following realization for the inverse derivative:

$$\partial_z^{-1} \equiv \frac{1}{2}\int_{-\infty}^{+\infty} dx\ \epsilon(z-x), \quad \epsilon(z-x) = -\epsilon(x-z) \equiv 1, \quad if \quad z > x. \tag{37}$$

From the derived formulae (30–33) one can easily see that only the flows, generated by the highest superfield components of the Hamiltonians (23–26), depend implicitly on the Grassmann coordinates $\{\theta, \overline{\theta}\}$, while other flows comprise the latter explicitly. The minimal dimension of the $\{\theta, \overline{\theta}\}$–independent flows is 2 and it corresponds to the first bosonic flow $\frac{\partial}{\partial t_1} J$ (30).

4.3 The algebra of flows and Hamiltonians

Using the explicit expressions (30–32) for the fermionic flows $Q_{\frac{1}{2}}$, $\overline{Q}_{\frac{1}{2}}$, $D_{\frac{1}{2}}$ and $\overline{D}_{\frac{1}{2}}$ one can calculate the folowing anticommutators

$$D_{\frac{1}{2}}^2 = \overline{D}_{\frac{1}{2}}^2 = 0, \quad \{D_{\frac{1}{2}}, \overline{D}_{\frac{1}{2}}\} = +\frac{\partial}{\partial t_1} \tag{38}$$

and

$$Q_{\frac{1}{2}}^2 = \overline{Q}_{\frac{1}{2}}^2 = 0, \quad \{Q_{\frac{1}{2}}, \overline{Q}_{\frac{1}{2}}\} = -\frac{\partial}{\partial t_1} \tag{39}$$

and observe that they form two sets of closed algebraic relations, moreover each of them displays the $N = 2$ supersymmetry. Thus we are led to the important conclusion that the $N = 2$ $\alpha = 1$ KdV hierarchy possesses actually a more rich symmetry structure, than it is indicated in its title, related to two different $N = 2$ supersymmetries with the generators $\{Q_{\frac{1}{2}},\ \overline{Q}_{\frac{1}{2}}, \frac{\partial}{\partial t_1}\}$ and $\{D_{\frac{1}{2}},\ \overline{D}_{\frac{1}{2}}, \frac{\partial}{\partial t_1}\}$.

The Poisson bracket algebra (7) can also be used to derive the following useful formula

$$\Big\{\int dZ\widetilde{\mathcal{H}}_1, \int dZ\widetilde{\mathcal{H}}_2\Big\} = \int dZ(\widetilde{f}_1 J_2 \widetilde{f}_2^T)_0 \tag{40}$$

where $\widetilde{f}_1$ and $\widetilde{f}_2^T$ are Fréchet derivatives of the Hamiltonian densities $\widetilde{\mathcal{H}}_1$ and $\widetilde{\mathcal{H}}_2$, respectively. Using this formula one can calculate the Poisson brackets between the integrals I_0, $I_{\frac{1}{2}}$, $I^\star_{\frac{1}{2}}$ and I_1 (23–26), and this algebra is isomorphic to the algebra of the corresponding flows. Repeatedly applying this procedure one can derive new nonlocal Hamiltonians. As an illustrative example we present the Hamiltonian density of the Hamiltonian I_2 $(I^\star = -I_2)$ (22)

$$\begin{aligned}\mathcal{I}_3 &= (D\mathcal{I}^\star_{\frac{3}{2}})\partial^{-1}\overline{D}\mathcal{I}_{\frac{3}{2}} + (\overline{D}\mathcal{I}^\star_{\frac{3}{2}})\partial^{-1}D\mathcal{I}_{\frac{3}{2}} + \frac{1}{2}[D,\overline{D}]J^2 - \frac{3}{2}J[D,\overline{D}]J \\ &+ J(\partial^{-1}DJ)\overline{D}J - J(DJ)\partial^{-1}\overline{D}J - ([D,\overline{D}]J)(\partial^{-1}DJ)\partial^{-1}\overline{D}J\end{aligned} \tag{41}$$

derived in this way. *It is interesting to remark that the higher superfield component of the Hamiltonian I_2 (I_0) coincides with the Hamiltonian H_3 (H_1) (10).*

A discussion of the complete, very rich superalgebraic structure of the $N = 2$ $\alpha = 1$ KdV hierarchy is out of the scope of the present Letter and will be discussed in [14].

5 Recursion operator and bi-Hamiltonian structure of the $N = 2$ $\alpha = 1$ KdV hierarchy

Now, the results of preceding sections give us all necessary inputs to construct the recursion operator of the $N = 2$ $\alpha = 1$ KdV hierarchy following the algorithm described in Section 3 and then to derive its bi-Hamiltonian structure. In this Section we present main results in a telegraphic style and refer the reader to ref. [13] for more details.

5.1 Recursion operator

We use the *Ansatz* (15) for the recursion operator R where only superfield components of the Hamiltonians I_0, $I_{\frac{1}{2}}$, $I^\star_{\frac{1}{2}}$ and I_1 (23–26) are included. Then, as it was already noted at the end of Section 3, we impose the condition that two special symmetries ($\frac{\partial}{\partial t_5}J$ and $\frac{\partial}{\partial t_7}J$ (8)) of the $N=2$ $\alpha=1$ KdV equation (1) are related by this recursion operator as

$$\frac{\partial}{\partial t_7}J \equiv res(L^7) = R\; res(L^5) \equiv R\; \frac{\partial}{\partial t_5}J \tag{42}$$

in accordance with eq. (15), and solve this condition. It is interesting to remark that this condition completely fixes all unknown coefficient–functions, involved in the *Ansatz* for the recursion operator, and allows to construct the explicit expression for the latter

$$\begin{aligned} R &= \partial^2 + 2J[D,\overline{D}\,] + \frac{3}{2}([D,\overline{D}\,]J) + J^2 \\ &+ \Big\{J\,'[D,\overline{D}\,] + \frac{1}{2}([D,\overline{D}\,]J\,') + \frac{1}{2}[D,\overline{D}\,]J\,' - D(\overline{D}J^2) - \overline{D}(DJ^2)\Big\}\partial^{-1} \\ &+ \overline{D}\Big\{\mathcal{I}^\star_{\frac{3}{2}}[(D\partial^{-1}J) - \frac{1}{2}D] - D\mathcal{I}^\star_{\frac{3}{2}}\Big\}\partial^{-1}f_{\frac{1}{2}} + D\Big\{\mathcal{I}_{\frac{3}{2}}[(\overline{D}\partial^{-1}J) + \frac{1}{2}\overline{D}] + \overline{D}\mathcal{I}_{\frac{3}{2}}\Big\}\partial^{-1}f^\star_{\frac{1}{2}} \\ &- [\overline{D}\,(DJ) + D\,(\overline{D}J)]\Big\{\partial^{-1}f_1 + (\partial^{-1}\mathcal{I}_{\frac{3}{2}})\partial^{-1}f^\star_{\frac{1}{2}} + (D\partial^{-1}J)\overline{D}\partial^{-1}\Big\} \end{aligned} \tag{43}$$

where f_i ($i=\frac{1}{2},1$) are the Fréchet derivatives defined in eqs. (27). Then we have explicitly verified that the constructed expression (43) satisfies the deformation equation (16), i.e. it indeed gives the proper recursion operator for symmetries of the $N=2$ $\alpha=1$ KdV hierarchy.

We would like to note that other conditions, similar to the condition (42), which relate flows $\frac{\partial}{\partial t_1}J$ and $\frac{\partial}{\partial t_3}J$ or/and flows $\frac{\partial}{\partial t_3}J$ and $\frac{\partial}{\partial t_5}J$ do not fix the recursion operator completely.

For completeness let us also present the recurrence relations

$$Y^a_{p+1}J \;=\; R\,Y^a_pJ \tag{44}$$

for flows Y^a_pJ of the $N=2$ supersymmetric $\alpha=1$ KdV hierarchy in the following useful form:

$$\begin{aligned} Y^a_{p+1}J &= \Big\{\partial^2 + 2J[D,\overline{D}\,] + \frac{3}{2}([D,\overline{D}\,]J) + J^2\Big\}Y^a_pJ \\ &+ \Big\{J\,'[D,\overline{D}\,] + \frac{1}{2}([D,\overline{D}\,]J\,') + \frac{1}{2}[D,\overline{D}\,]J\,' - D(\overline{D}J^2) - \overline{D}(DJ^2)\Big\}\partial^{-1}Y^a_pJ \\ &+ (-1)^{d_{Y^a}}\overline{D}\Big\{\mathcal{I}^\star_{\frac{3}{2}}[(D\partial^{-1}J) - \frac{1}{2}D] - D\mathcal{I}^\star_{\frac{3}{2}}\Big\}\partial^{-1}Y^a_p\mathcal{I}_{\frac{3}{2}} \\ &+ (-1)^{d_{Y^a}}D\Big\{\mathcal{I}_{\frac{3}{2}}[(\overline{D}\partial^{-1}J) + \frac{1}{2}\overline{D}] + \overline{D}\mathcal{I}_{\frac{3}{2}}\Big\}\partial^{-1}Y^a_p\mathcal{I}^\star_{\frac{3}{2}} \\ &- [\overline{D}(DJ) + D(\overline{D}J)]\Big\{\partial^{-1}Y^a_p\mathcal{I}_2 + (-1)^{d_{Y^a}}(\partial^{-1}\mathcal{I}_{\frac{3}{2}})\partial^{-1}Y^a_p\mathcal{I}^\star_{\frac{3}{2}} + (D\partial^{-1}J)\overline{D}\partial^{-1}Y^a_pJ\Big\} \end{aligned} \tag{45}$$

where Y^a_p is an evolution derivative from the set (34) and d_{Y^a} is its Grassmann parity.

5.2 Bi-Hamiltonian structure

We have observed that the constructed recursion operator R (43) can be represented in the factorized form

$$R = J_2J_0^{-1}, \quad J_0^{-1} = [D,\overline{D}\,]\partial^{-1} + \partial^{-1}J_2\partial^{-1} + \frac{1}{2}f^{\star T}_{\frac{1}{2}}\partial^{-1}f_{\frac{1}{2}} - \frac{1}{2}f^T_{\frac{1}{2}}\partial^{-1}f^\star_{\frac{1}{2}} \tag{46}$$

where J_2 is the second Hamiltonian structure (5), (7) and $f^T_{\frac{1}{2}}$ ($f^{\star T}_{\frac{1}{2}}$) is the operator conjugated Fréchet derivative (28). Then J_0^{-1} ($J_0J_0^{-1} = J_0^{-1}J_0 = 1$) can obviously be treated as the inverse operator of the zero Hamiltonian structure,

$$\{J(Z_1), J(Z_2)\}_0 = J_0(Z_1)\delta^{N=2}(Z_1 - Z_2), \quad J_0^{-1}\frac{\partial}{\partial t_{2p-1}}J = \frac{\delta}{\delta J}H_{2p+1}. \tag{47}$$

Therefore, we come to the conclusion that the $N=2$ supersymmetric $\alpha=1$ KdV hierarchy is a bi-Hamiltonian system. Acting k-times with the recursion operator (43) on the second Hamiltonian structure J_2 (5), (7) of the $N=2$ $\alpha=1$ KdV hierarchy, one can derive its $2(k+1)$-th Hamiltonian structure,

$$J_{2(k+1)} = R^k J_2, \quad \frac{\partial}{\partial t_{2p+1}} J = \{J, H_{2(p-k)+1}\}_{2(k+1)} \equiv J_{2(k+1)} \frac{\delta}{\delta J} H_{2(p-k)+1},$$
$$\{J(Z_1), J(Z_2)\}_{2(k+1)} = J_{2(k+1)}(Z_1)\delta^{N=2}(Z_1 - Z_2). \tag{48}$$

6 Summary

In this Letter we have adapted the general algorithm of constructing recursion operators to the case of the $N=2$ supersymmetric $\alpha=1$ KdV equation in $N=2$ superspace. Then we have constructed all basic objects which are relevant to this aim: nonpolynomial and nonlocal, bosonic and fermionic Hamiltonians (23–26) and symmetries (30–33). Furthermore we have observed that the $N=2$ $\alpha=1$ KdV hierarchy possesses a more rich symmetry structure, than it is indicated in its title, related to two subalgebras of the $N=2$ supersymmetry (38–39) of its algebra of symmetries. Finally we have constructed its recursion operator (43), recursion relations (45) as well as zero Hamiltonian structure (46–47) which were unsolved longstanding problems.

Acknowledgments. We would like to thank I.S. Krasil'shchik for useful discussions. A.S. is grateful to University of Twente for the hospitality extended to him during this research. This work was partially supported by the grants NWO NB 61-491, FOM MF 00/39, RFBR 99-02-18417, RFBR-CNRS 98-02-22034, PICS Project No. 593, Nato Grant No. PST.CLG 974874 and the Heisenberg-Landau program.

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