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Memorandum No. 1481

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February 1999

ISSN 0169-2690

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Forbidden Subgraphs that Imply Hamiltonian-Connectedness

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January 27, 1999

The first four authors would like to dedicate this paper to Henk Jan Veldman, a valued colleague and beloved friend who died October 12, 1998.

Abstract

It is proven that if G is a 3-connected claw-free graph which is also Z_3-free (where Z_3 is a triangle with a path of length 3 attached), P_6-free (where P_6 is a path with 6 vertices) or H_1-free (where H_1 consists of two disjoint triangles connected by an edge), then G is hamiltonian-connected. Also, examples will be described that determine a finite family of graphs $\mathcal{L}$ such that if a 3-connected graph being claw-free and L-free implies G is hamiltonian-connected, then $L \in \mathcal{L}$.

Keywords: hamiltonian-connected, forbidden subgraph, claw-free graph.

AMS Subject Classifications (1991): 05C45, 05C38, 05C35

*Research is partially supported by ONR Grant No. N00014-94-J-1085

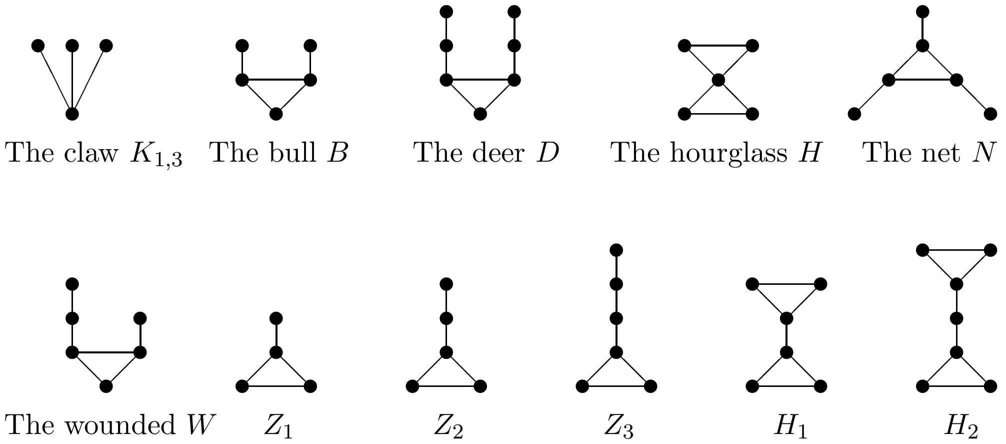

Figure 1: Frequently used forbidden subgraphs.

1 Introduction

We use Bondy & Murty [2] for terminology and notation not defined here and consider finite simple graphs only. A graph G with $n \geq 3$ vertices is *hamiltonian* if G contains a cycle of length n, and it is *hamiltonian-connected* if between each pair of vertices of G there is a Hamilton path, i.e. a path on n vertices. If H is a given graph, then a graph G is called *H-free* if G contains no induced subgraph isomorphic to H. The graph H is said to be a *forbidden* subgraph.

We first describe some graphs that will be frequently used as forbidden subgraphs. Specifically, we denote by P_k and C_k the path and the cycle on k vertices, by C the claw $K_{1,3}$, by B the bull, by D the deer, by H the hourglass, by N the net, by W the wounded, by Z_k the graph obtained by identifying a vertex of K_3 with an endvertex of P_{k+1}, and by H_k the graph obtained by joining two vertex disjoint triangles by a path of length k (see Figure 1).

The next result was obtained in Shepherd [8], and the following one in Faudree & Gould [6]. Note that in both cases 3-connectedness is assumed. This is natural since the forbidden subgraph conditons, being local conditions, do not imply 3-connectedness, and any hamiltonian-connected graph (except K_1, K_2, K_3) must be 3-connected.

Theorem 1 (Shepherd [8])
If a 3-connected graph G is claw-free and N-free, then G is hamiltonian-connected.

Theorem 2 (Faudree & Gould [6])
If a 3-connected graph G is claw-free and Z_2-free, then G is hamiltonian-connected.

We will extend this collection of pairs of forbidden graphs ensuring hamiltonian-connectedness of 3-connected graphs by proving the following result, which gives three new independent forbidden pairs. The proof of the result is postponed to Section 2.

Theorem 3
If G is a 3-connected claw-free graph, then G is hamiltonian-connected if any of the following holds.

(a) G is Z_3-free,

(b) G is P_6-free,

(c) G is H_1-free.

Chen & Gould [4] recently announced they proved that every 3-connected claw-free graph is hamiltonian-connected provided it is Z_3-free, P_6-free or W-free. In Bedrossian [1] all forbidden pairs of connected graphs ensuring that a graph is hamiltonian are characterized, and the same was done for pancyclicity. The same type of characterization was done for other hamiltonian properties in Faudree & Gould [6]. A survey of results of this kind can be found in Faudree [5]. Also, in [6] the following theorem was proved. It gives some context to the previous results on pairs of forbidden graphs ensuring hamiltonian-connectedness of 3-connected graphs.

Theorem 4 (Faudree & Gould [6])
Let X and Y be connected graphs with $X, Y \neq P_3$, and let G be a 3-connected graph. If G being X-free and Y-free implies G is hamiltonian-connected, then, up to symmetry, $X = K_{1,3}$, and Y satisfies each of the following conditions.

(a) $\Delta(Y) \leq 3$.

(b) A longest induced path in Y has at most 12 vertices.

(c) Y contains no cycles of length at least 4.

(d) All triangles in Y are vertex disjoint.

(e) Y is claw-free.

One implication of Theorem 4 is that there are only a finite number of forbidden pairs of graphs implying hamiltonian-connected of 3-connected graphs. However, the gap between Theorem 4 and the positive results in Theorems 1, 2, and 3 is still substantial. The following result will reduce, but not eliminate, that gap somewhat. The proof is postponed to Section 3.

Theorem 5
Let X and Y be connected graphs with $X, Y \neq P_3$, and let G be a 3-connected graph. If G being X-free and Y-free implies G is hamiltonian-connected, then $X = K_{1,3}$, and Y satisfies each of the following conditions.

(a) $\Delta(Y) \leq 3$.

(b) The longest induced path in Y *has at most 9 vertices.*

(c) Y *contains no cycles of length at least 4.*

(d) The distance between two distinct triangles in Y *is either* 1 *or at least* 3.

(e) There are at most two triangles in Y.

(f) Y *is claw-free.*

2 Forbidden pairs that imply hamiltonian-connectedness

Since the proofs of the results in this section have many common features and have the same basic structure, we will describe that structure in general, introduce some special notation, and make some general observations that will be used throughout all of the proofs. This will eliminate the need to do this in each individual situation.

In what follows, an (x, y)-path P is said to be *maximal* if there is no (x, y)-path Q such that $V(P) \subsetneq V(Q)$.

The set up of most of the proofs in this section will be to consider a maximal (x, y)-path P that is not a Hamilton path, between some pair of vertices x and y, and then show that P can be extended, contradicting the maximality of P. The following lemma will be useful in selecting such maximal paths.

Lemma 6
For any pair of vertices x *and* y *in a* 3*-connected claw-free graph* G*, there is a maximal* (x, y)*-path* P *such that* $N(x) \subseteq V(P)$.

Proof Let $P = x_1x_2 \ldots x_m$ with $x = x_1$ and $y = x_m$ be a maximal (x, y)-path with the property that it contains a maximum number of vertices of $N(x)$. If $N(x) \subseteq V(P)$, then we are done. Hence, we may assume there is a vertex $z \in N(x) \setminus V(P)$. We will exhibit an (x, y)-path Q that contains $(N(x) \cap V(P)) \cup \{z\}$. This will give a contradiction, since any maximal path (x, y)-path Q' that contains the vertices of Q would have more vertices in $N(x)$ than P.

Since G is 3-connected, there exist three vertex disjoint (z, P)-paths, which will be denoted by Q_1, Q_2 and Q_3. We may assume that Q_1 has endvertex x_1. Let x_r and x_s (with $1 < r < s$) be the endvertices of Q_2 and Q_3, respectively. If z has more than three adjacencies on P, then select x_r and x_s to be the last two adjacencies of z on P. Let S be the set of vertices in $N(x) \cap V(P)$ that are not adjacent to z. Note that to avoid an induced claw centered at x, the vertices in S form a complete graph. Also note that $N(x) \cap N(z) \cap V(P) \subseteq x_1\overrightarrow{P}x_r \cup \{x_s\}$.

If $S \cap x_{r+1}\overrightarrow{P}x_{s-1} = \emptyset$, then $Q = x_1\overrightarrow{P}x_r\overleftarrow{Q_2}z\overrightarrow{Q_3}x_s\overrightarrow{P}x_m$ is the required path, since this path contains z as well as $N(x) \cap V(P)$.

If $S \cap x_{r+1}\overrightarrow{P}x_{s-1} \neq \emptyset$, then select i and j such that x_i is the smallest indexed vertex in $S \cap x_{r+1}\overrightarrow{P}x_{s-1}$ and x_j is the largest. It is possible that $i = j$. By the maximality of P and since G is claw-free, $x_2x_i \in E(G)$. Then $Q = x_1x_j\overleftarrow{P}x_ix_2\overrightarrow{P}x_r\overleftarrow{Q_2}z\overrightarrow{Q_3}x_s\overrightarrow{P}x_m$ is the required path. ∎

In the next proofs we start with a graph G that is 3-connected and claw-free, and for which there is no Hamilton path between some pair of vertices x and y of G. By Lemma 6 we can select a maximal (x, y)-path $P = x_1x_2 \ldots x_m$ with $x = x_1$ and $y = x_m$ such that $N(x) \subseteq V(P)$. Since P is not a Hamilton path, there is a vertex z not on P. Since G is 3-connected, there exist three vertex disjoint (z, P)-paths, and at least two of these paths will terminate in interior vertices of P. Let x_i, x_j and x_k (with $1 < i < j < k \leq m$) be the endvertices on P of these paths and denote the paths by Q_i, Q_j and Q_k respectively. We can choose z and the paths Q_i, Q_j, Q_k in such a way that

(i) $|E(Q_i)| = 1$,

(ii) $|E(Q_j)|$ is minimum subject to (i),

(iii) $|E(Q_k)|$ is minimum subject to (i) and (ii).

For $\ell = i, j, k$, the path Q_ℓ will be denoted by $zv_\ell \cdots u_\ell x_\ell$ realizing of course that the path might be just an edge. For shortness we will use Q to denote the path $x_i\overleftarrow{Q_i}z\overrightarrow{Q_j}x_j$. By the way the paths are chosen, we conclude that Q is an induced path except possibly for the edge x_ix_j.

The maximality of P and G being claw-free implies that $x_{i-1}x_{i+1} \in E(G)$, for otherwise there would be an induced claw centered at x_i. Likewise, $x_{j-1}x_{j+1} \in E(G)$. Note that $j - i \geq 4$, for otherwise the path P could be extended; for example if $j - i = 3$, then $x_1\overrightarrow{P}x_{i-1}x_{i+1}x_i\overrightarrow{Q}x_jx_{j-1}x_{j+1}\overrightarrow{P}x_m$ is such a path. Also, observe that $x_ix_{j-2} \notin E(G)$, for otherwise the path P can be extended to the path $x_1\overrightarrow{P}x_{i-1}x_{i+1}\overrightarrow{P}x_{j-2}x_i\overrightarrow{Q}x_jx_{j-1}x_{j+1}\overrightarrow{P}x_m$.

Select the smallest r_1 with $i < r_1 < j$ such that $x_ix_{r_1} \in E(G)$, but $x_ix_{r_1+1} \notin E(G)$. By the previous remarks, such an r_1 exists. Likewise, select the smallest s_1 with $j < s_1 < k$ such that $x_jx_{s_1} \in E(G)$, but $x_jx_{s_1+1} \notin E(G)$. There are no edges between $x_i\overrightarrow{P}x_{r_1+1}$ and $x_j\overrightarrow{P}x_{s_1+1}$, except possibly for x_ix_j: the existence of any of the edges gives an extension of P; for example, if $x_{r_1+1}x_{s_1+1} \in E(G)$, then P can be extended to the path $x_1\overrightarrow{P}x_{i-1}x_{i+1}\overrightarrow{P}x_{r_1}x_i\overrightarrow{Q}x_jx_{s_1}\overleftarrow{P}x_{j+1}x_{j-1}$ $\overleftarrow{P}x_{r_1+1}x_{s_1+1}\overrightarrow{P}x_m$. In the same way select a largest r_2 with $i < r_2 < j$ such that $x_jx_{r_2} \in E(G)$, but $x_jx_{r_2-1} \notin E(G)$. By symmetry and the previous remarks, such an r_2 exists. Also, if $x_k \neq x_m$, in the same way an s_2 associated with the vertex x_k can be defined. Also, by a symmetry argument we know that there are no edges between $x_{r_2-1}\overrightarrow{P}x_j$ and $x_{s_2-1}\overrightarrow{P}x_k$ except possibly for x_jx_k.

The proof of the next theorem is just an adaptation of the corresponding result for hamiltonicity which appeared in BROERSMA & VELDMAN [3]. Lemma 6 made this adaptation much easier, since it assured the existence of a maximal path P, a vertex z not on P, and two vertex disjoint paths from z to the interior vertices of P.

Theorem 7
If a 3-connected graph G is claw-free and P_6-free, then G is hamiltonian-connected.

Proof Assume that G is a 3-connected, claw-free graph, and there is no Hamilton path between some pair of vertices x and y of G. We will show that G must contain an induced copy of P_6. We choose a maximal (x, y)-path $P = x_1x_2 \dots x_m$ with $x = x_1$ and $y = x_m$ subject to the condition that $N(x) \subseteq V(P)$. We choose a vertex $z \in V(G) \setminus V(P)$ and three vertex disjoint (z, P)-paths as in the general discussion. All of the notation and observations of the general discussion are assumed.

If $x_ix_j \in E(G)$, then from the general observations we get that $G[\{x_{r_1+1}, x_{r_1}, x_i, x_j, x_{s_1}, x_{s_1+1}\}] \cong P_6$. Otherwise, the path $x_{r_1+1}x_{r_1}x_i\overrightarrow{Q}x_jx_{s_1}$ is an induced path with at least six vertices. Hence in both cases G contains an induced P_6. ■

The proof of the next theorem is also an adaptation of the corresponding result in FAUDREE ET AL. [7] for hamiltonian graphs. However, in this case no restriction needs to be placed on the order of the graph.

Theorem 8
If a 3-connected graph G is claw-free and Z_3-free, then G is hamiltonian-connected.

Proof Assume that G is a 3-connected, claw-free graph, and there is no Hamilton path between some pair of vertices x and y of G. We will show that G must contain an induced copy of Z_3. We choose a maximal (x, y)-path $P = x_1x_2 \dots x_m$ with $x = x_1$ and $y = x_m$ subject to the condition that $N(x) \subseteq V(P)$. We choose a vertex $z \in V(G) \setminus V(P)$ and three vertex disjoint (z, P)-paths as in the general discussion. All of the notation and observations of the general discussion are assumed.

We first show that $|E(Q_j)| = 1$. If $|E(Q_j)| \geq 2$, then $x_ix_j \notin E(G)$, since otherwise $G[\{x_i; x_{i-1}, z, x_j\}] \cong K_{1,3}$. But then $G[\{x_{i-1}, x_{i+1}, x_i\} \cup V(Q_j)]$ contains an induced Z_3. Hence we may assume $E|(Q_j)| = 1$, i.e. $zx_j \in E(G)$.

Now assume $x_ix_{i+2} \in E(G)$. To avoid $G[\{x_{i+1}, x_{i+2}, x_i; z, x_j, x_{j+1}\}] \cong Z_3$, we have $x_ix_j \in E(G)$. But then $G[\{x_{i+1}, x_{i+2}, x_i; x_j, x_{s_1}, x_{s_1+1}\}] \cong Z_3$. Hence $x_ix_{i+2} \notin E(G)$.

Next assume $x_jx_{j-2} \in E(G)$. Then we may assume $j - i \geq 5$; otherwise obviously there exists a (x, y)-path contradicting the choice of P. To avoid $G[\{x_{j-2}, x_{j-1}, x_j; z, x_i, x_{i-1}\}] \cong Z_3$, we have $x_ix_j \in E(G)$. But then $G[\{x_{j-1}, x_{j+1}, x_j; x_i, x_{i+1}, x_{i+2}\}] \cong Z_3$, since $x_{i+1}x_{j-1} \notin$

$E(G)$ (otherwise $x_1\overrightarrow{P}x_izx_j, x_{j-2}\overleftarrow{P}x_{i+1}x_{j-1}x_{j+1}\overrightarrow{P}x_m$ contradicts the choice of P) and similarly $x_{i+2}x_{j-1} \notin E(G)$. Hence $x_jx_{j-2} \notin E(G)$.

Case 1 $j - i \geq 5$.
First we assume $x_ix_j \notin E(G)$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, x_j, x_{j-1}\}] \cong Z_3$, we have $x_{i+1}x_{j-1} \in E(G)$. To avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{j-1}\}] \cong K_{1,3}$, we have $x_{i+2}x_{j-1} \in E(G)$ and by symmetry $x_{i+1}x_{j-2} \in E(G)$. To avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{j-2}\}] \cong K_{1,3}$, we have $x_{i+2}x_{j-2} \in E(G)$. But then $G[\{x_{j-2}, x_{i+2}, x_{i+1}; x_i, z, x_j\}] \cong Z_3$. Hence we may assume $x_ix_j \in E(G)$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; x_j, x_{j-1}, x_{j-2}\}] \cong Z_3$, we have $x_{i+1}x_{j-1} \in E(G)$ or $x_{i+1}x_{j-2} \in E(G)$. If $x_{i+1}x_{j-1} \in E(G)$, then, to avoid $G[\{x_{j-1}; x_{i+1}, x_{j-2}, x_j\}] \cong K_{1,3}$, we also have $x_{i+1}x_{j-2} \in E(G)$. By symmetry we have $x_{i+2}x_{j-1} \in E(G)$. To avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{j-2}\}] \cong K_{1,3}$, we have $x_{i+2}x_{j-2} \in E(G)$ and to avoid $G[\{x_{j-2}, x_{i+2}, x_{i+1}; x_i, x_j, x_{j+1}\}] \cong Z_3$, we have $x_{j-2}x_{j+1} \in E(G)$. But then $x_1\overrightarrow{P}x_{i-1}x_{i+1}x_izx_jx_{j-1}x_{i+2}\overrightarrow{P}x_{j-2}x_{j+1}\overrightarrow{P}x_m$ contradicts the choice of P.

Case 2 $j - i = 4$.

Case 2.1 $x_ix_j \notin E(G)$.
To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, x_j, x_{j-1}\}] \cong Z_3$, we have $x_{i+1}x_{j-1} \in E(G)$, and similarly $x_{i-1}x_{j+1} \in E(G)$. To avoid $G[\{x_{j+1}; x_j, x_{j+2}, x_{i-1}\}] \cong K_{1,3}$, we have $x_{i-1}x_{j+2} \in E(G)$, since $x_jx_{j+2} \notin E(G)$ (otherwise $x_1\overrightarrow{P}x_{i-1}x_{j+1}x_{j-1}\overleftarrow{P}x_izx_jx_{j+2}\overrightarrow{P}x_m$ contradicts the choice of P).

If $|E(Q_k)| \geq 2$, then to avoid $G[\{x_{j-1}, x_{i+2}, x_{i+1}; x_i, z, v_k\}] \cong Z_3$, we have $x_iv_k \in E(G)$, and similarly $x_jv_k \in E(G)$. But then v_k contradicts the choice of z. Hence $|E(Q_k)| = 1$, i.e. $zx_k \in E(G)$.

To avoid $G[\{x_{j-1}, x_{i+2}, x_{i+1}; x_i, z, x_k\}] \cong Z_3$, we have $x_ix_k \in E(G)$ or $x_{i+1}x_k \in E(G)$, since $x_{i+2}x_k \notin E(G)$ (otherwise to avoid $G[\{x_k; x_{k-1}, z, x_{i+2}\}] \cong K_{1,3}$ also $x_{i+2}x_{k-1} \notin E(G)$, yielding a path which contradicts the choice of P) and similarly $x_{j-1}x_k \notin E(G)$. If $x_{i+1}x_k \in E(G)$, then also $x_{i+1}x_{k-1} \in E(G)$ and to avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{k-1}\}] \cong K_{1,3}$, we have $x_{i+2}x_{k-1} \in E(G)$, yielding a path which contradicts the choice of P. Hence $x_{i+1}x_k \notin E(G)$ and thus $x_ix_k \in E(G)$. But then $G[\{x_k, z, x_i; x_{i+1}, x_{j-1}, x_{j+1}\}] \cong Z_3$, since $x_{j+1}x_k \notin E(G)$ (otherwise also $x_{j+1}x_{k-1} \in E(G)$ and hence $x_1\overrightarrow{P}x_{i-1}x_{j+2}\overrightarrow{P}x_{k-1}x_{j+1}\overleftarrow{P}x_izx_k\overrightarrow{P}x_m$ contradicts the choice of P).

Case 2.2 $x_ix_j \in E(G)$.
First assume $|E(Q_k)| \geq 2$. If $x_iv \in E(G)$ for some vertex $v \in V(Q_k) \setminus \{z, x_k\}$, then, to avoid $G[\{x_i; x_{i-1}, v, x_j\}] \cong K_{1,3}$, also $x_jv \in E(G)$, which would contradict the choice of z. Hence $x_iv \notin E(G)$ for every $v \in V(Q_k) \setminus \{z, x_k\}$ and similarly $x_jv \notin E(G)$ for every $v \in V(Q_k) \setminus \{z, x_k\}$. Hence $G[\{x_{i-1}, x_{i+1}, x_i; z, v_k, v_k^+\}] \cong Z_3$, unless $|E(Q_k)| = 2$ and $x_{i-1}x_k \in E(G)$, $x_ix_k \in E(G)$ or $x_{i+1}x_k \in E(G)$. However, if $x_{i-1}x_k \in E(G)$, then to avoid $G[\{x_k; x_{i-1}, x_{k-1}, v_k\}] \cong K_{1,3}$, we have $x_{i-1}x_{k-1} \in E(G)$, yielding a path which contradicts the choice of P. Hence $x_{i-1}x_k \notin E(G)$ and similarly $x_ix_k \notin E(G)$. Finally, if $x_{i+1}x_k \in E(G)$,

then also $x_{i+1}x_{k-1} \in E(G)$ and $G[\{x_{i+1}, x_{k-1}, x_k; v_k, z, x_j\}] \cong Z_3$. Hence $x_{i+1}x_k \notin E(G)$ and $G[\{x_{i-1}, x_{i+1}, x_i; z, v_k, v_k^+\}] \cong Z_3$.

Now we may assume $|E(Q_k)| = 1$, i.e. $zx_k \in E(G)$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, x_k, x_{k-1}\}] \cong Z_3$, we have $x_ix_k \in E(G)$, $x_{i+1}x_k \in E(G)$ or $x_{i+1}x_{k-1} \in E(G)$. If $x_{i+1}x_k \in E(G)$, then also $x_{i+1}x_{k-1} \in E(G)$ and to avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{k-1}\}] \cong K_{1,3}$ we have $x_{i+2}x_{k-1} \in E(G)$, yielding a path which contradicts the choice of P. Hence $x_{i+1}x_k, x_{i+1}x_{k-1} \notin E(G)$. If $x_ix_k \in E(G)$, then to avoid $G[\{x_i; x_{i-1}, x_j, x_k\}] \cong K_{1,3}$, we have $x_jx_k \in E(G)$. However, then $G[\{z, x_k, x_j; x_{j-1}, x_{i+2}, x_{i+1}\}] \cong Z_3$ if $x_{i+1}x_{j-1} \notin E(G)$ and $G[\{z, x_k, x_j; x_{j-1}, x_{i+1}, x_{i-1}\}] \cong Z_3$ if $x_{i+1}x_{j-1} \in E(G)$. ■

The following result gives a pair of forbidden graphs that implies a graph is hamiltonian-connected in the presence of 3-connectedness but does not imply a graph is hamiltonian in the presence of 2-connectedness.

Theorem 9
If a 3-connected graph G is claw-free and H_1-free, then G is hamiltonian-connected.

Proof Assume that G is a 3-connected, claw-free graph, and there is no Hamilton path between some pair of vertices x and y of G. We will show that G must contain an induced copy of H_1. We choose a maximal (x, y)-path $P = x_1x_2 \ldots x_m$ with $x = x_1$ and $y = x_m$ subject to the condition that $N(x) \subseteq V(P)$. We choose a vertex $z \in V(G) \setminus V(P)$ and three vertex disjoint (z, P)-paths as in the general discussion. All of the notation and observations of the general discussion are assumed.

We claim that we can choose z in such a way that $|E(Q_j)| = 1$, and that $|E(Q_k)| = 1$ if $x_k \neq x_m$. Suppose $|E(Q_j)| \geq 2$ and consider z and the successor v_j of z on Q_j. By the choice of z, $x_iv_j \notin E(G)$. Since G is 3-connected, claw-free and $zv_j^+ \notin E(G)$, there exists a triangle T containing z and v_j or there exists a triangle T containing v_j and v_j^+. We distinguish a number of cases.

Case a.1 z, v_j and a vertex of Q_k are in a common triangle.
Let $t \in V(Q_k) \setminus \{z\}$ be the third vertex of T. By the choice of Q_k, we have $t = v_k$. If $v_k \neq x_k$, then $G[\{x_{i-1}, x_{i+1}, x_i; z, v_j, v_k\}] \cong H_1$, since $x_iv_j \notin E(G)$ (otherwise v_j contradicts the choice of z) and $x_it \notin E(G)$ (otherwise t contradicts the choice of z). Hence $v_k = x_k$.

To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, v_j, x_k\}] \cong H_1$, we must have at least one of x_kx_{i-1}, x_kx_i and $x_{i+1}x_k$ in $E(G)$. Then, since $x_{i-1}x_k \notin E(G)$ (otherwise to avoid $G[\{x_k; x_{i-1}, z, x_{k-1}\}] \cong K_{1,3}$, we have $x_{i-1}x_{k-1} \in E(G)$ yielding a path $x_1\overrightarrow{P}x_{i-1}x_{k-1}\overleftarrow{P}x_izx_k\overrightarrow{P}x_m$ which contradicts the choice of P) and $x_ix_k \notin E(G)$ (otherwise to avoid $G[\{x_k; x_i, v_j, x_{k-1}\}] \cong K_{1,3}$, we have $x_ix_{k-1} \in E(G)$, also yielding a path which contradicts the choice of P), we get $x_{i+1}x_k \in E(G)$, implying also $x_{i+1}x_{k-1} \in E(G)$.

If $v_jx_j \in E(G)$ (i.e. $|E(Q_j)| = 2$), then to avoid $G[\{x_{j-1}, x_{j+1}, x_j; v_j, z, x_k\}] \cong H_1$, we similarly have that $x_{j+1}x_k \in E(G)$, and get a contradiction since $G[\{x_k; x_{i+1}, x_{j+1}, z\}] \cong K_{1,3}$.

Hence we may assume $v_jx_j \notin E(G)$ and thus $v_j^+ \notin V(P)$ (where v_j^+ is the successor of v_j on Q_j). Since $v_jv_j^{++} \notin E(G)$, there exists a triangle T' containing v_j and v_j^+ or there exists a triangle T' containing v_j^+ and v_j^{++}. Note that $v_j^+x_k \notin E(G)$ (otherwise $G[\{x_k; z, v_j^+, x_{k-1}\}] \cong K_{1,3}$).

(i) Suppose v_j and v_j^+ are in a common triangle T' with some vertex t'. Then $t' \notin \{x_i, x_j, x_k, z\}$, while also $t' \notin V(P) \setminus \{x_i, x_j, x_m\}$; otherwise if $t' \in x_1\overrightarrow{P}x_{i-1}$, then v_j contradicts the choice of z, if $t' \in x_{i+1}\overrightarrow{P}x_{j-1}$, then the path zv_jt' contradicts the choice of Q_j, and if $t' \in x_{k+1}\overrightarrow{P}x_m$, then the paths zx_k and zv_jt' contradict the choice of Q_j and Q_k. Hence $t' \notin V(P) \cup \{z\}$. To avoid $G[\{x_{i+1}, x_{k-1}, x_k; v_j, v_j^+, t'\}] \cong H_1$, we have $x_kt' \in E(G)$, and to avoid $G[\{x_k; x_{k-1}, z, t'\}] \cong K_{1,3}$, we have $zt' \in E(G)$. But then $G[\{x_{i-1}, x_{i+1}, x_i; z, t', v_j\}] \cong H_1$, since $x_it' \notin E(G)$;otherwise t' contradicts the choice of z.

(ii) If v_j^+ is not in a common triangle with v_j, then there exists a triangle T' containing v_j^+ and v_j^{++}. Again let t' be the third vertex of T'. If $t' = x_k$, then $G[\{x_k; z, v_j^+, x_{k-1}\}] \cong K_{1,3}$. Hence $t' \neq x_m$ and also $t' \notin \{x_i, z\}$. If $t' \in x_1\overrightarrow{P}x_{i-1}$ or $t' \in x_{k+1}\overrightarrow{P}x_m$ we easily get contradictions with the chosen path system. If $t' \in x_{i+1}\overrightarrow{P}x_{j-1}$, then also $v_j^{++} = x_j$, giving a contradiction since v_j^+ contradicts the choice of z. Hence $t' \notin V(P) \cup \{z\}$. Now $G[\{t', v_j^{++}, v_j^+; v_j, z, x_k\}] \cong H_1$ unless $v_j^{++}x_k \in E(G)$ and $v_j^{++} = x_j$. But then $G[\{x_k; x_{i+1}, x_j, v_j\}] \cong K_{1,3}$.

Case a.2 z, v_j are in a common triangle T with some vertex t, and Case a.1 does not apply. Then, by the choice of z, $V(T) \cap V(P) = \emptyset$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, v_j, t\}] \cong H_1$, we have $x_it \in E(G)$. To avoid $G[\{z; x_i, v_j, v_k\}] \cong K_{1,3}$ (with possibly $v_k = x_k$), we have $x_iv_k \in E(G)$, since $v_jv_k \notin E(G)$; otherwise we would be in Case a.1. To avoid $G[\{x_i; x_{i-1}, t, v_k\}] \cong K_{1,3}$, we have $tv_k \in E(G)$. If $v_jx_j \in E(G)$, then $G[\{x_{j-1}, x_{j+1}, x_j; v_j, z, t\}] \cong H_1$. Hence $v_j^+ \neq x_j$. We use that v_j^+ is in a triangle with v_j or with v_j^{++}.

(i) Suppose v_j^+ and v_j are in a common triangle T' with some vertex t'.

Clearly, $t' \neq z, x_i$. We easily see that $t' \notin x_1\overrightarrow{P}x_{k-1}$. Now suppose $t' = x_k$. Then $G[\{x_k; x_{k-1}, v_j^+, u_k\}] \cong K_{1,3}$, unless $v_j^+u_k \in E(G)$ and $u_k \neq z, v_k$. To avoid $G[\{x_k; x_{k-1}, v_j, u_k\}] \cong K_{1,3}$, we have $v_ju_k \in E(G)$. Then $G[\{x_i, v_k, t; v_j, u_k, x_k\}] \cong H_1$, unless $v_ku_k \in E(G)$. But then $G[\{z, t, v_k; u_k, v_j^+, x_k\}] \cong H_1$. Hence $t' \neq x_k$. If $t' \in x_{k+1}\overrightarrow{P}x_m$, then to avoid $G[\{x_i, v_k, t; v_j, v_j^+, t'\}] \cong H_1$, we have $v_kt' \in E(G)$. But then $v_k = x_k$ or $v_kx_k \in E(G)$. In both cases we easily obtain path systems contradicting the chosen path system. Hence $t' \notin V(P)$.

If $t' \notin V(P)$, then consider $G[\{v_j^+, t', v_j; t, x_i, v_k\}]$ (with possibly $v_k = x_k$). If $t' \notin V(Q_k)$, then to avoid an induced H_1, we have $tt' \in E(G)$. But then $G[\{x_{i-1}, x_{i+1}, x_i; t, v_j, t'\}] \cong H_1$. Hence $t' \in V(Q_k) \setminus \{z, v_k\}$. Then to avoid an H_1, we have $t' = v_k^+$. Then $v_k^+ \neq x_k$;

otherwise $G[\{x_k; x_{k-1}, v_k^+, u_j^+\}] \cong K_{1,3}$. Considering $G[\{v_k^+; v_k, v_k^{++}, v_j\}]$, we get that $v_j v_k^{++} \in E(G)$. To avoid $G[\{v_k^+; v_k, v_k^{++}, v_j^+\}] \cong K_{1,3}$, we have $v_j^+ v_k^{++} \in E(G)$. But then $G[\{x_i, v_k, t; v_j, v_j^+, v_k^{++}\}] \cong H_1$.

(ii) If v_j^+ is not in a common triangle with v_j, then considering a triangle T with $V(T) = \{v_j^+, v_j^{++}, t'\}$, we easily obtain that $G[\{z, t, v_j; v_j^+, v_j^{++}, t'\}] \cong H_1$.

Case b z and v_j are not in a common triangle.
Hence v_j and v_j^+ are in a triangle T with some vertex t. Note that to avoid $G[\{z; x_i, v_j, v_k\}] \cong K_{1,3}$, we have $x_i v_k \in E(G)$ with possibly $v_k = x_k$.

(i) First suppose $t \notin V(P)$. Using that no induced claw is centered at x_i and that $zv_j^+ \notin E(G)$, we obtain $G[\{x_i, v_k, z; v_j, v_j^+, t\}] \cong H_1$ unless $t = v_k^+$. If $t = v_k^+$, then $v_k^+ \neq x_k$; otherwise $G[\{x_k; x_{k-1}, v_j, v_k\}] \cong K_{1,3}$ (using $v_j v_k \notin E(G)$). Considering $G[\{v_k^+; v_k, v_k^{++}, v_j^+\}]$, with possibly $x_k = v_k^{++}$, we get $v_j^+ v_k^{++} \in E(G)$. Now $G[\{x_i, z, v_k; v_k^+, v_j^+, v_k^{++}\}] \cong H_1$, unless $v_j^+ = x_j$ and $x_i x_j \in E(G)$. But then $G[\{x_i; x_{i+1}, z, x_j\}] \cong K_{1,3}$.

(ii) Now suppose $t \in V(P)$. If $t = x_k$, then $v_k \neq x_k$ (since z and v_j are not in a common triangle). No induced claw centered at x_k gives that $G[\{x_i, v_k, z; v_j, v_j^+, x_k\}] \cong H_1$, unless $v_j^+ = x_j$ and $x_i x_j \in E(G)$; in the latter case $G[\{z, v_k, x_i; x_j, x_{j-1}, x_{j+1}\}] \cong H_1$. Hence $t \neq x_k$. If $t \in x_1 \overrightarrow{P} x_{k-1}$, then v_j contradicts the choice of z. If $t \in x_{k+1} \overrightarrow{P} x_m$ (assuming $x_k \neq x_m$), and $v_j^{++} \neq x_j$, then to avoid $G[\{x_i, v_k, z; v_j, v_j^+, t\}] \cong H_1$, we have $v_k t \in E(G)$. But then $G[\{t; t^-, v_k, v_j\}] \cong K_{1,3}$. If $t \in x_{k+1} \overrightarrow{P} x_m$ (assuming $x_k \neq x_m$), and $v_j^{++} = x_j$, then to avoid $G[\{x_i, v_k, z; v_j, x_j, t\}] \cong H_1$ we have $x_i x_j \in E(G)$ or $x_i t \in E(G)$, both giving an induced claw as contradiction, or $v_k t \in E(G)$. In the latter case $G[\{t; t^-, v_k, v_j\}] \cong K_{1,3}$.

We now show that $|E(Q_k)| = 1$ if $x_k \neq x_m$. This is not difficult if $x_i x_j \notin E(G)$: consider any neighbor z' of z in $V(G) \setminus V(P)$. Then, considering $G[\{z; z', x_i, x_j\}]$, to avoid an induced claw, we get that one of $z' x_i$ and $z' x_j$ is an edge. But then considering $G[\{x_{j-1}, x_{j+1}, x_j; z, z', x_i\}]$ or $G[\{x_{i-1}, x_{i+1}, x_i; z, z', x_j\}]$ we obtain both edges. This implies all vertices in the component of $G - V(P)$ containing z have x_i and x_j as neighbors. Hence we can choose a vertex z with three neighbors on P.

Now assume $x_i x_j \in E(G)$, and assume $x_k \neq x_m$ and $|E(Q_k)| \geq 2$. Then z has no third neighbor on P. Let p denote the successor of z on Q_k. Since $\delta \geq 3$, p is in a triangle by claw-freeness. If px_i or px_j is an edge, then both edges are in; otherwise we obtain a claw induced by $\{x_i; p, x_{i+1}, x_j\}$ or $\{x_j; p, x_{j+1}, x_i\}$. But then we contradict the choice of z. Hence $px_i, px_j \notin E(G)$. We distinguish four subcases.

(i) p and z are in a common triangle with a vertex $t \notin V(P)$. Clearly, by the choice of Q_k, $t \notin V(Q_k)$. To avoid $G[\{p, t, z; x_i, x_{i+1}, x_{i-1}\} \cong H_1]$, we have $tx_i \in E(G)$, and similarly

$tx_j \in E(G)$. Suppose first that $x_k = p^+$. To avoid $G[\{z, t, p; x_k, x_{k-1}, x_{k+1}\}] \cong H_1$, we have $tx_k \in E(G)$ (Note that $zx_k \notin E(G)$ by yhe choice of z). But then t contradicts the choice of z (since $zx_i, zx_j, zx_k \in E(G)$). Hence we may assume $p^+ \neq x_k$. We use that p^+ is in a common triangle with p or p^{++}.

(a) p and p^+ are in a common triangle with some vertex t'. Similar arguments as for p show $p^+x_i, p^+x_j \notin E(G)$. If $t' \notin V(P)$, then the choice of z implies $t'x_i, t'x_j \notin E(G)$ and $t'z \notin E(G)$; if $t' \in V(P)$, then also $t'z \notin E(G)$. Now to avoid $G[\{t', p^+, p; z, x_i, x_j\}] \cong H_1$, we conclude that $t' \in V(P)$ and that t' is adjacent to x_i or x_j. Both cases yield a claw induced by $\{x_i; z, x_k, x_{i+1}\}$ or $\{x_j; z, x_k, x_{j+1}\}$, a contradiction.

(b) p and p^+ are not in a common triangle. Hence p^+ and p^{++} are in a common triangle with some vertex t'. Using the choice of z and Q_k, to avoid $G[\{z, t, p; p^+, p^{++}, t'\}] \cong H_1$, we have $t't \in E(G)$, hence $t' \notin V(P)$. To avoid $G[\{t; t', p, x_i\}] \cong K_{1,3}$, we conclude that $x_it' \in E(G)$, and similarly $x_jt' \in E(G)$, contradicting the choice of z.

(ii) p and z are in a common triangle with a vertex $t \in V(P)$. Together with $px_i, px_j \notin E(G)$ we contradict the assumption that z has no third neighbor on P.

(iii) p and z are not in a common triangle, but p and p^+ are in a common triangle with a vertex $t \notin V(P)$. Clearly, the assumption implies $tz \notin E(G)$, and by the choice of Q_k, $zp^+ \notin E(G)$. Hence also $tx_i, tx_j \notin E(G)$. As before $px_i, px_j \notin E(G)$ and similarly $p^+x_i, p^+x_j \notin E(G)$ unless $p^+ = x_k$. To avoid $G[\{t, p^+, p; z, x_i, x_j\}] \cong H_1$, we conclude $p^+ = x_k$ and x_kx_i or x_kx_j is an edge. This yields a claw induced by $\{x_i; x_{i+1}, x_k, z\}$ or $\{x_j; x_{j+1}, x_k, z\}$.

(iv) p and z are not in a triangle, and p and p^+ are not in a triangle with some vertex of $V(G)\backslash V(P)$. Hence p and p^+ are in a common triangle with some vertex $t \in V(P)$. Since $px_i, px_j \notin E(G)$, the choice of Q_k implies $p^+ \in V(P)$. Consider $G[\{x_i, x_j, z; p, x_k, t\}]$. If $x_ix_k \in E(G)$, then $G[\{x_k; p, x_j, x_{j-1}\}] \cong K_{1,3}$. By similar arguments, to avoid an H_1, we conclude $t = x_m$ and tx_i or tx_j is an edge. If $tx_i \in E(G)$, we obtain $G[\{x_{i-1}, x_{i+1}, x_i; t, p, x_k\}] \cong H_1$; the case $tx_j \in E(G)$ is similar.

Case 1 $x_ix_j \notin E(G)$.
Since $zx_i, zx_j, zx_k \in E(G)$ and $x_ix_j \notin E(G)$, claw-freeness implies $x_ix_k \in E(G)$ or $x_jx_k \in E(G)$.

First assume $x_ix_k \in E(G)$. If also $x_jx_k \in E(G)$, then to avoid $G[\{x_k; x_i, x_j, x_{k-1}\}] \cong K_{1,3}$, we have $x_ix_{k-1} \in E(G)$ or $x_jx_{k-1} \in E(G)$, both contradicting the choice of P. So $x_jx_k \notin E(G)$. If $x_kx_{j-1} \in E(G)$, then also $x_{k-1}x_{j-1} \in E(G)$, contradicting the choice of P. Hence $x_{k-1}x_{j-1}, x_kx_{j-1} \notin E(G)$. To avoid $G[\{x_i, x_k, z; x_j, x_{j-1}, x_{j+1}\}] \cong H_1$, we have $x_kx_{j+1} \in E(G)$, and hence also $x_{k-1}x_{j+1} \in E(G)$. Since $x_ix_{k-1} \notin E(G)$, we have $x_{i-1}x_k \notin E(G)$.

Since $x_{i-1}x_k \notin E(G)$, we have $x_{i-1}x_{j+1} \notin E(G)$ (otherwise $G[\{x_{j+1}, x_{i-1}, x_j, x_k\}] \cong K_{1,3}$). If $x_{i+1}x_{k-1} \in E(G)$, then $x_1\overrightarrow{P}x_izx_j\overleftarrow{P}x_{i+1}x_{k-1}\overleftarrow{P}x_{j+1}x_k\overrightarrow{P}x_m$ contradicts the choice of P. Hence $x_{i+1}x_{k-1} \notin E(G)$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; x_k, x_{k-1}, xj+1\}] \cong H_1$, we have $x_{i+1}x_k \in E(G)$. But then $G[\{x_k, x_{i+1}, z, x_{k-1}\}] \cong K_{1,3}$, a contradiction. We conclude that $x_ix_k \notin E(G)$ and $x_jx_k \in E(G)$.

To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, x_j, x_k\}] \cong H_1$, we have $x_{i+1}x_k \in E(G)$ and hence also $x_{i+1}x_{k-1} \in E(G)$. This also implies $x_k = x_m$. By the choice of P, we have $x_ix_{i+2} \notin E(G)$. To avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_k\}] \cong K_{1,3}$, we have $x_{i+2}x_k \in E(G)$ and to avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{k-1}\}] \cong K_{1,3}$, we have $x_{i+2}x_{k-1} \in E(G)$. If $x_kx_{j+1} \in E(G)$, then $G[\{x_k; x_{i+1}, x_{j+1}, z\}] \cong K_{1,3}$. If $x_{i+1}x_{j-1} \in E(G)$, then $x_1\overrightarrow{P}x_{i+1}x_{j-1}\overleftarrow{P}x_{i+2}x_{k-1}\overleftarrow{P}x_jzx_k$ contradicts the choice of P. To avoid $G[\{x_{i+1}, x_{i+2}, x_k; x_j, x_{j-1}, x_{j+1}\}] \cong H_1$, we have $x_{i+2}x_{j-1} \in E(G) \setminus E(P)$ (i.e. $x_{i+3} \neq x_{j-1}$). If $x_{i+1}x_{i+3} \in E(G)$, then $x_1\overrightarrow{P}x_izx_j\overrightarrow{P}x_{k-1}x_{i+2}x_{j-1}\overleftarrow{P}x_{i+3}x_{i+1}x_k$ contradicts the choice of P. Hence $x_{i+1}x_{i+3} \notin E(G)$, implying $x_{i+3}x_{j-1} \in E(G)$ (otherwise $G[\{x_{i+2}; x_{i+1}, x_{i+3}, x_{j-1}\}] \cong K_{1,3}$). If $x_ix_{i+3} \in E(G)$, then $x_1\overrightarrow{P}x_{i-1}x_{i+1}x_ix_{i+3}\overrightarrow{P}x_{j-1}x_{i+2}x_{k-1}\overleftarrow{P}x_jzx_k$ contradicts the choice of P, and if $x_{i-1}x_{i+3} \in E(G)$ so does $x_1\overrightarrow{P}x_{i-1}x_{i+3}\overrightarrow{P}x_{k-1}x_{i+2}x_{i+1}x_izx_k$. If $x_{i-1}x_{i+2} \in E(G)$, then, to avoid $G[\{x_{i+2}; x_{i-1}, x_{i+3}, x_{k-1}\}] \cong K_{1,3}$, we have $x_{i+3}x_{k-1} \in E(G)$ and $x_1\overrightarrow{P}x_{i+2}x_{j-1}\overleftarrow{P}x_{i+3}x_{k-1}\overleftarrow{P}x_jzx_k$ contradicts the choice of P. Hence $G[\{x_{i-1}, x_{i+1}, x_i; x_{i+2}, x_{i+3}, x_{j-1}\}] \cong K_{1,3}$.

Case 2 $x_ix_j \in E(G)$.
To avoid $G[\{x_{i-1}, x_{i+1}, x_i; x_j, x_{j-1}, x_{j+1}\}] \cong H_1$, we have either $x_{i-1}x_{j+1} \in E(G)$ or $x_{i+1}x_{j-1} \in E(G)$, since the other edges are not present by standard arguments.

Case 2.1 $x_{i-1}x_{j+1} \in E(G)$.
To avoid $G[\{x_{j+1}; x_j, x_{j+2}, x_{i-1}\}] \cong K_{1,3}$, we have $x_{i-1}x_{j+2} \in E(G)$, since $x_{i-1}x_j \notin E(G)$ (standard) and $x_jx_{j+2} \notin E(G)$ (otherwise $x\overrightarrow{P}x_{i-1}x_{j+1}x_{j-1}\overleftarrow{P}x_izx_jx_{j+2}\overrightarrow{P}y$ contradicts the choice of P).

We first show $zx_k \in E(G)$. Assuming the contrary we have $v_k \neq x_k$. Since $\delta \geq 3$ and G is claw-free, v_k belongs to a triangle.

Case a There exists a triangle T containing v_k and z.
Let q be the third vertex of T.

Case a.1 $q \notin V(P)$.
If $x_iv_k \in E(G)$, then, to avoid $G[\{x_i; x_{i+1}, x_j, v_k\}] \cong K_{1,3}$, also $x_jv_k \in E(G)$, which contradicts the choice of z (v_k would have been a better choice). Hence, to avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, v_k, q\}] \cong H_1$, we have $x_iq \in E(G)$. But then $G[\{x_{j+1}, x_{j+2}, x_{i-1}; x_i, z, q\}] \cong H_1$.

Case a.2 $q \in V(P)$.
By the way x_k was chosen, we have $q = x_i$ or $q = x_j$. If $q = x_i$, then $G[\{x_{j+1}, x_{j+2}, x_{i-1}; x_i, z, v_k\}] \cong H_1$. If $q = x_j$, then, to avoid $G[\{x_j; x_i, v_k, x_{j+1}\}] \cong K_{1,3}$, we have $x_iv_k \in E(G)$, giving the same H_1 as a contradiction.

Case b Every triangle T containing v_k does not contain z.
Let q_1 and q_2 be the two other vertices of T. If $q_1, q_2 \notin V(P)$, then $G[\{x_i, x_j, z; v_k, q_1, q_2\}] \cong H_1$; otherwise, if for example $q_1 z \in E(G)$, there would be a triangle T containing v_k and z, and if $q_1 x_i \in E(G)$, then $G[\{x_i; z, q_1, x_{i+1}\} \cong K_{1,3}$. Also, if $q_1 \in V(P)$ (and/or $q_2 \in V(P)$), then $G[\{x_i, x_j, z; v_k, q_1, q_2\}] \cong H_1$; otherwise, if for example $q_1 x_j \in E(G)$, then $G[\{q_1; x_j, v_k, q_1^-\}] \cong K_{1,3}$.

Case 2.1.1 $x_1 \neq x_{i-1}$.
To avoid $G[\{x_{i-1}; x_{i-2}, x_i, x_{i+1}\}] \cong K_{1,3}$, we have $x_{i-2}x_{j+1} \in E(G)$, and to avoid $G[\{x_{i-1}; x_{i-2}, x_i, x_{i+2}\}] \cong K_{1,3}$, we have $x_{i-2}x_{j+2} \in E(G)$. But then $G[\{x_i, z, x_j; x_{j+1}, x_{j+2}, x_{i-2}\}] \cong H_1$.

Case 2.1.2 $x_1 = x_{i-1}$.

Case 2.1.2.1 $x_k \neq x_m$.
To avoid $G[\{x_i, x_j, z; x_j, x_{j-1}, x_{j+1}\}] \cong H_1$, we have $x_i x_k \in E(G)$ or $x_j x_k \in E(G)$. First assume $x_j x_k \in E(G)$. To avoid $G[\{x_{j-1}, x_{j+1}, x_j; x_k, x_{k-1}, x_{k+1}\}] \cong H_1$, we have $x_{j-1}x_{k+1} \in E(G)$ or $x_{j+1}x_{k-1} \in E(G)$. However, if $x_{j-1}x_{k+1} \in E(G)$, then $x_1 x_{j+2} \overrightarrow{P} x_{k-1} x_{j+1} \overleftarrow{P} x_i z x_k x_{k+1}$ contradicts the choice of P; if $x_{j-1}x_{k+1} \in E(G)$, so does $x_1 x_{j+1} \overrightarrow{P} x_k z x_j x_i \overrightarrow{P} x_{j-1} x_{k+1}$. Hence $x_i x_k \in E(G)$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; x_k, x_{k-1}, x_{k+1}\}] \cong H_1$, we have $x_{i+1}x_{k-1} \in E(G)$ or $x_{i-1}x_{k+1} \in E(G)$. However, if $x_{i-1}x_{k-1} \in E(G)$, then $x_1 x_{j+1} \overrightarrow{P} x_{k-1} x_{i+1} \overrightarrow{P} x_j x_i z x_k \overrightarrow{P} x_m$; if $x_{i-1}x_{k+1} \in E(G)$, then $G[\{x_1; x_i, x_{j+1}, x_{k+1}\}] \cong K_{1,3}$.

Case 2.1.2.2 $x_k = x_m$.
We distinguish between the cases that $x_j x_k \in E(G)$ and $x_j x_k \notin E(G)$.

Case 2.1.2.2.a $x_j x_m \in E(G)$.
To avoid $G[\{x_1, x_{j+2}, x_{j+1}; x_j, z, x_m\}] \cong H_1$, we have $x_{j+2}x_m \in E(G)$, since $x_1 x_m \notin E(G)$ (standard) and $x_{j+1}x_m \notin E(G)$ (otherwise also $x_{j+1}x_{m-1} \in E(G)$, giving a path $x_1 x_{j+2} \overrightarrow{P} x_{m-1} x_{j+1} \overleftarrow{P} x_i z y$ which contradicts the choice of P) while the other possible edges are not present by standard arguments.

First assume $x_{j+3} \neq x_{m-1}$. To avoid $G[\{x_m; x_{m-1}, x_{j+2}, z\}] \cong K_{1,3}$, we have $x_{j+2}x_{m-1} \in E(G)$, and to avoid $G[\{x_{j+2}; x_1, x_{j+3}, x_{m-1}\}] \cong K_{1,3}$, we have $x_{j+2}x_{m-1} \in E(G)$. But then $G[\{x_{i+2}, x_i, x_1; x_{j+2}, x_{j+3}, x_{m-1}\}] \cong H_1$, since $x_1 x_{j+3} \notin E(G)$ (otherwise $x_1 x_{j+3} \overrightarrow{P} x_{m-1} x_{j+2} \overleftarrow{P} x_i z x_m$ contradicts the choice of P), $x_i x_{j+3} \notin E(G)$ (otherwise $x_1 x_{j+2} x_{m-1} \overleftarrow{P} x_{j+3} x_i \overrightarrow{P} x_{j-1} x_{j+1} x_j z x_m$ contradicts the choice of P), $x_{i+1}x_{j+3} \notin E(G)$ (otherwise $x_1 x_{j+1} x_{j+2} x_{m-1} \overleftarrow{P} x_{j+3} x_{i+1} \overrightarrow{P} x_j x_i z x_m$ contradicts the choice of P) and $x_{i+1}x_{m-1} \notin E(G)$ (otherwise $x_1 x_{j+1} \overrightarrow{P} x_{m-1} x_{i+1} \overrightarrow{P} x_j x_i z x_m$ contradicts the choice of P), while the other possible edges are not present by standard arguments.

Hence we may assume that $x_{j+3} = x_{m-1}$. Let $p \in V(G) \setminus \{x_{j+2}, x_m\}$ be a neighbor of x_{j+3}. We first show that we can choose p on P. Suppose there does not exist such a vertex p on P and let T be a triangle containing p and containing a maximum number of vertices of

P. Let q_1 and q_2 be the other vertices of T. To avoid $G[\{x_{j+3}; x_{j+2}, x_m, p\}] \cong K_{1,3}$, we have $x_{j+3}y \in E(G)$.

If $V(T) \cap V(P) = \emptyset$, then $G[\{q_1, q_2, p; x_{j+3}, x_{j+2}, x_m\} \cong H_1$.

If $|V(T) \cap V(P)| = 2$, then $q_1 \neq x_{j+3}$ (since q_2 is a neighbor of q_1 it would have been possible to choose p on P) and $q_2 \neq x_{j+3}$ (similar). But then p contradicts the choice of z.

If $|V(T) \cap V(P)| = 1$, let q_1 be the vertex not on P and let q_2 be the vertex on P. One easily shows that $q_2 \notin \{x_1, x_i, x_{i+1}, x_{j-1}, x_j, x_{j+1}, x_{j+2}, y\}$ by obtaining (x, y)-paths contradicting the choice of P. If $q_2 = x_{j+3}$, then $G[\{x_1, x_{j+1}, x_{j+2}; q_2, q_1, p\}] \cong H_1$. If $q_2 \in x_{i+2}\overrightarrow{P}x_{j-2}$, then to avoid $G[\{q_2; q_2^-, q_++, q_1\}] \cong K_{1,3}$, we have $q_2^- q_2^+ \in E(G)$. However, then $G[\{q_2, q_1, p; x_{j+3}, x_{j+2}, x_m\}] \cong H_1$, since $q_2x_{j+2} \notin E(G)$ (otherwise $x_1\overrightarrow{P}q_2^- q_2^+\overrightarrow{P}x_{j+2}q_2px_{j+3}x_m$ contradicts the choice of P), $q_2x_{j+3} \notin E(G)$ by assumption and $q_2x_m \notin E(G)$ (otherwise also $q_2x_{j+3} \notin E(G)$ by a standard observation).

Hence we may assume that we can choose p on P, and one easily shows that $p \in x_{i+2}\overrightarrow{P}x_{j-2}$. To avoid $G[\{p; p^-, p^+, x_{j+3}\}] \cong K_{1,3}$, we have $p^-p^+ \in E(G)$, since $p^-x_{j+3} \notin E(G)$ (otherwise $x_1x_{j+2}\overleftarrow{P}px_{j+3}p^-\overleftarrow{P}x_izx_m$ contradicts the choice of P) and $p^+x_{j+3} \notin E(G)$ (similar). We may assume that $px_{j+2} \notin E(G)$ (otherwise by considering the path $x_1\overrightarrow{P}p^-p^+\overrightarrow{P}x_{j+2}px_{j+3}x_m$ we are back in the case that $x_{j+3} \neq x_{m-1}$) and $px_m \notin E(G)$ (similar). Hence, to avoid $G[\{x_{j+3}; p, x_{j+2}, x_m\}] \cong K_{1,3}$, we have $x_{j+2}x_m \in E(G)$. However, then $G[\{p^-, p^+, p; x_{j+3}, x_{j+2}, x_m\}] \cong H_1$, since $p^-x_{j+2} \notin E(G)$ (otherwise $x_1x_{j+1}\overleftarrow{P}px_{j+3}x_{j+2}p^-\overleftarrow{P}x_izx_m$ contradicts the choice of P), $p^-x_m \notin E(G)$ (otherwise also $p^-x_{j+3} \in E(G)$), $p^+x_{j+2} \notin E(G)$ (otherwise $x_1x_{j+1}x_{j+2}p^+\overrightarrow{P}x_jzx_i\overrightarrow{P}px_{j+3}x_m$ contradicts the choice of P) and $p^+x_m \notin E(G)$ (otherwise also $p^+x_{j+3} \in E(G)$).

Case 2.1.2.2.b $x_jx_m \notin E(G)$.
Let $p \in V(G) \setminus \{z, x_{m-1}\}$ be a neighbor of x_m. We first show that we can choose p on P. Suppose there does not exist such a vertex p on P. To avoid $G[\{x_m; x_{m-1}, z, p\}] \cong K_{1,3}$, we have $pz \in E(G)$. If $px_i \in E(G)$, then $G[\{p, z, x_i; x_{i-1}, x_{j+1}, x_{j+2}\}] \cong H_1$. Hence we have $px_i \notin E(G)$. Since $x_{i-1}x_{k-1} \notin E(G)$, also $x_{i-1}x_k \notin E(G)$, and since $x_{i+1}x_{k-1} \notin E(G)$, also $x_{i+1}x_k \notin E(G)$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, p, x_k\}] \cong H_1$, we have $x_ix_k \in E(G)$. However, then $G[\{x_m, x_i, x_{m-1}, p\}] \cong K_{1,3}$.

Hence we may assume that we can choose p on P. If $x_ix_m \in E(G)$, then to avoid $G[\{x_i, x_{i+1}, x_j, x_m\}] \cong K_{1,3}$, we have $x_{i+1}x_m \in E(G)$, and hence also $x_{m-1}x_{i+1} \in E(G)$, yielding a path $x_1x_{j+1}\overrightarrow{P}x_{m-1}x_{i+1}\overrightarrow{P}x_jx_izx_m$, contradicting the choice of P. Hence $x_ix_m, x_{i+1}x_m \notin E(G)$. If $x_{i-1}x_m \in E(G)$, then also $x_{i-1}x_{m-1} \in E(G)$, a contradiction. Hence $x_{i-1}x_m \notin E(G)$, and similarly $x_{j-1}x_m \notin E(G)$. If $x_{j+1}x_m \in E(G)$, then also $x_{j+1}x_{m-1} \in E(G)$, yielding a contradicting path $x_1x_{j+2}\overrightarrow{P}x_{m-1}x_{j+1}\overleftarrow{P}x_izx_m$. The above observations leave two cases for the location of p.

(i) $p \in x_{i+2}\overrightarrow{P}x_{j-2}$. We choose $p \in N(x_k)$ as close to x_{j-1} as possible. To avoid $G[\{x_m; p, z, x_{m-1}\}] \cong K_{1,3}$, we have $px_{m-1} \in E(G)$. To avoid $G[\{x_i, x_j, z; x_m, x_{m-1}, p\}] \cong H_1$, we have $px_i \in E(G)$ or $px_j \in E(G)$. If $px_i \in E(G)$, then also $px_1 \in E(G)$ (otherwise $G[\{x_i; x_1, p, z\}] \cong K_{1,3}$). Since $px_{m-1} \in E(G)$, the choice of P implies $p^+x_1 \notin E(G)$. To avoid $G[\{p; x_1, p^+, x_m\}] \cong K_{1,3}$, we have $p^+x_m \in E(G)$, contradicting the choice of P. Next assume $px_j \in E(G)$. Then $p^+ \neq x_{j-1}$. To avoid $G[\{p; p^+, x_j, x_m\}] \cong K_{1,3}$ we have $p^+x_j \in E(G)$, and to avoid $G[\{x_j, p, z, x_{j+1}\}] \cong K_{1,3}$, we have $p^+x_{j+1} \in E(G)$. However, then $x_1\overrightarrow{P}px_{m-1}\overleftarrow{P}x_{j+1}p^+\overrightarrow{P}x_jzx_m$ contradicts the choice of P.

(ii) $p \in x_{j+2}\overrightarrow{P}x_{k-2}$. We choose $p \in N(x_k)$ as close to x_{j+1} as possible. We again have $px_{m-1} \in E(G)$ and $px_i \in E(G)$ or $px_j \in E(G)$. If $px_i \in E(G)$, then to avoid $G[\{p; x_i, p^-, x_m\}] \cong K_{1,3}$, we have $p^-x_i \in E(G)$ and $p \neq x_{j+2}$. To avoid $G[\{x_i; z, x_{i+1}, p^-\}] \cong K_{1,3}$, we have $x_{i+1}p^- \in E(G)$. But then $x_1x_{j+1}\overrightarrow{P}p^-x_{i+1}\overrightarrow{P}x_jzx_ip\overrightarrow{P}x_m$ contradicts the choice of P.

If $px_j \in E(G)$, then also $px_{j-1}, px_{j+1} \in E(G)$. If $p^- = x_{j+1}$, then $x_1x_{j+1}x_jzx_i\overrightarrow{P}x_{j-1}p\overrightarrow{P}x_m$ contradicts the choice of P. If $p^- \neq x_{j+1}$, then to avoid $G[\{p; x_j, p^-, x_m\}] \cong K_{1,3}$, we have $p^-x_j \in E(G)$, and to avoid $G[\{x_j; x_{j-1}, z, p^-\}] \cong K_{1,3}$, also $p^-x_{j-1} \in E(G)$. But then $x_1x_{j+1}\overrightarrow{P}p^-x_{j-1}\overleftarrow{P}x_izx_jp\overrightarrow{P}x_k$ contradicts the choice of P.

Case 2.2 $x_{i-1}x_{j+1} \notin E(G)$ (hence $x_{i+1}x_{j-1} \in E(G)$).

Case 2.2.1 $j - i \geq 5$.

To avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{j-1}\}] \cong K_{1,3}$, we have $x_{i+2}x_{j-1} \in E(G)$, since $x_ix_{i+2} \notin E(G)$ (contradicting path: $x_1\overrightarrow{P}x_{i-1}x_{i+1}x_{j-1}\overleftarrow{P}x_{i+2}x_izx_j\overrightarrow{P}x_m$). By symmetry, we also have $x_{i+1}x_{j-2} \in E(G)$. To avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{j-2}\}] \cong K_{1,3}$, we have $x_{i+2}x_{j-2} \in E(G)$. However, then $G[\{x_i, z, x_j; x_{j-1}, x_{j-2}, x_{i+2}\}] \cong H_1$.

Case 2.2.2 $j - i = 4$.

We use that x_{i+2} has a neighbor $p \notin \{x_{i-1}, x_i, x_{i+1}, x_{i+2}, x_{j-1}, x_j, x_{j+1}\}$.

We first show we can choose $p \in V(P)$. Supposing this is not the case consider a triangle T containing p. Let q_1 and q_2 be the other vertices of T. First suppose $V(T) \cap V(P) = \emptyset$. If $q_1x_{i+2} \in E(G)$, then $G[\{x_{i-1}, x_i, x_{i+1}; x_{i+2}, p, q_1\}] \cong H_1$. Hence $q_1x_{i+2}, q_2x_{i+2} \notin E(G)$. But then $G[\{q_1, q_2, p; x_{i+2}, x_{i+1}, x_{j-1}\}] \cong H_1$. Hence $|V(T) \cap V(P)| \geq 1$. Let q_1 denote a neighbor of p in $(V(P) \cap V(T)) \setminus \{x_{i+2}\}$. Then $x_{i+2}q_1 \notin E(G)$ by assumption. If $x_{j-1}q \in E(G)$, then also $x_{j-1}q_1^- \in E(G)$ (otherwise $G[\{q_1; q_1^-, x_{j-1}, p\}] \cong K_{1,3}$), and we easily find a path contradicting the choice of P. A similar observation shows $x_{i+1}q_1 \notin E(G)$. But then $G[\{x_{i+1}, x_{j-1}, x_{i+2}; p, q_1, q_2\}] \cong H_1$.

Hence we can choose $p \in V(P)$. If x_{i+2} has two successive neighbors on P, it is obvious that we can find a path contradicting the choice of P. Hence, if p^- and p^+ exist, we get that $p^-p^+ \in E(G)$. We deal with the cases that $p \in \{x_1, x_m\}$ later.

To avoid $G[\{x_{i+1}, x_{j-1}, x_{i+2}; p, p^-, p^+\}] \cong H_1$, we have $x_{i+1}p \in E(G)$ or $x_{j-1}p \in E(G)$. If $x_{i+1}p \in E(G)$ and $p \in x_{j+1}\overrightarrow{P}x_{m-1}$, then by considering the path $x_1\overrightarrow{P}x_{i+1}px_{i+2}\overrightarrow{P}p^-p^+\overrightarrow{P}x_m$, we are back in Case 2.2.1. But then $G[\{x_{i+1}, x_{j-1}, x_{i+2}; p, p^-, p^+\}] \cong H_1$.

Now suppose $p = x_m$. Then $x_m \neq x_k$, since otherwise $G[\{x_m; x_{i+2}, z, x_{m-1}] \cong K_{1,3}$. Note that $x_k \neq x_{m-1}$ (otherwise $x\overrightarrow{P}x_{i-1}x_{i+1}x_izx_k\overleftarrow{P}x_{i+2}x_m$ contradicts the choice of P). To avoid $G[\{x_i, x_j, z; x_k, x_{k-1}, x_{k+1}\}] \cong H_1$, we have $x_ix_k \in E(G)$ or $x_jx_k \in E(G)$. First assume $x_jx_k \in E(G)$. Like in the beginning of Case 2, we have $x_{j-1}x_{k+1} \in E(G)$ or $x_{j+1}x_{k-1} \in E(G)$. If $x_{j-1}x_{k+1} \in E(G)$, also $x_{j-2}x_{k+1} \in E(G)$. However, since $x_{j-2} = x_{i+2}$ this contradicts the fact that $x_k \neq x_{m-1}$. If $x_{j+1}x_{k-1} \in E(G)$, then like in the beginning of this case, we have $k - j = 4$. To avoid $G[\{x_{i+1}, x_{i+2}, x_{j-1}; x_{j+1}, x_{j+2}, x_{j+3}\}] \cong H_1$, we have $x_{i+1}x_{j+3} \in E(G)$. But then $G[\{x_{i-1}, x_i, x_{i+1}; x_{j+3}, x_{j+1}, x_{j+2}\}] \cong H_1$. Hence we may assume that $x_jx_k \notin E(G)$ and $x_ix_k \in E(G)$. But then $G[\{x_i; x_{i-1}, x_j, x_k\}] \cong K_{1,3}$.

For the final subcase suppose $\{x_1\} = N(x_{i+2}) \setminus \{x_{i+1}, x_{j-1}\}$. By the choice of P, $N(x_1) \subseteq V(P)$ and $x_2 \neq x_{i-1}$. All neighbors of x_1 except for possibly $x_{i+1}, x_{i+2}, x_{j-1}$ are also neighbors of x_2, otherwise we obtain an induced claw centered at x_1. If $x_1x_i \in E(G)$, then $x_2x_i \in E(G)$ and to avoid $G[\{x_i; x_2, z, x_{i+1}\}] \cong K_{1,3}$, we have $x_2x_{i+1} \in E(G)$, contradicting the choice of P. Hence $x_1x_i \notin E(G)$ and similarly $x_1x_j \notin E(G)$.

If $x_1x_{i+1} \in E(G)$, then $G[\{x_1, x_{i+1}, x_{i+2}; x_i, z, x_j\}] \cong H_1$; if $x_1x_{j-1} \in E(G)$, then $G[\{x_1, x_{i+2}, x_{j-1}; x_j, x_i, z\}] \cong H_1$. Now assume $x_1x_{i+1}, x_1, x_{j-1} \notin E(G)$. Hence x_1 has some neighbor $q \neq x_i, x_{i+1}, x_{i+2}, x_{j-1}, x_j$ which is also a neighbor of x_2. To avoid $G[\{q, x_2, x_1; x_{i+2}, x_{i+1}, x_{j-1}\}] \cong H_1$, we have $qx_{i+1} \in E(G)$ or $qx_{j-1} \in E(G)$.

First suppose $q \in x_3\overrightarrow{P}x_{i-1}$ and $qx_{i+1} \in E(G)$. Then to avoid $G[\{x_{i+1}; q, x_i, x_{i+2}\}] \cong K_{1,3}$, we have $qx_i \in E(G)$. To avoid $G[\{x_1, x_2, q; x_i, z, x_j\}] \cong H_1$, we have $qx_j \in E(G)$. But then $G[\{q; x_2, x_{i+1}, x_j\}] \cong K_{1,3}$. Next suppose $q \in x_3\overrightarrow{P}x_{i-1}$ and $qx_{i+1} \notin E(G)$. Then $qx_{j-1} \in E(G)$ and to avoid $G[\{x_{j-1}; q, x_{i+2}, x_j\}] \cong K_{1,3}$, we have $qx_j \in E(G)$. To avoid $G[\{x_1, x_2, q; x_j, z, x_i\}] \cong H_1$, we have $qx_i \in E(G)$. But then $G[\{q; x_2, x_i, x_{j-1}\}] \cong K_{1,3}$.

We now may assume $q \notin x_3\overrightarrow{P}x_{i-1}$, hence $q \in x_{j+1}\overrightarrow{P}x_m$. We choose q as close to x_m as possible, and deal with the subcase $qx_{j-1} \in E(G)$ first.

If $q = x_m$, then, as before, we can repeat the previous cases with x_j, x_k instead of x_i, x_j, and obtain an induced H_1, unless $x_k = x_m$; but in the latter case $G[\{x_m; x_2, u_k, x_{j-1}\}] \cong K_{1,3}$. Hence $q \neq x_m$. To avoid $G[\{x_1, x_2, q; x_{j-1}, x_j, x_{j+1}\}] \cong H_1$, we have $qx_j \in E(G)$ or $qx_{j+1} \in E(G)$, both implying $qx_{j+1} \in E(G)$. To avoid $G[\{q; x_1, x_{j+1}, q^+\}] \cong K_{1,3}$, we have $x_{j+1}q^+ \in E(G)$, yielding $x_1x_{i+2}x_{j-1}x_jzx_ix_{i+1}x_{i-1}\overleftarrow{P}x_2q\overleftarrow{P}x_{j+1}q^+\overrightarrow{P}x_m$, a contradiction. For the remaining case we assume $qx_{j-1} \notin E(G)$, hence $qx_{i+1} \in E(G)$. By similar arguments as before we may assume $q \neq x_m$. To avoid $G[\{q; q^+, x_1, x_{i+1}\}] \cong K_{1,3}$, we have $x_{i+1}q^+ \in E(G)$. If $q^+ = x_m$, then by similar arguments as before $x_m = x_k$ and $x_1x_{i+2}\overrightarrow{P}x_{k-1}x_2\overrightarrow{P}x_{i-1}x_{i+1}x_izQ_kx_k$ gives a contradiction. In the final case the path $P' = x_1x_{i+2}\overrightarrow{P}qx_2\overrightarrow{P}x_{i+1}q^+\overrightarrow{P}x_m$ has the same properties as P, also with respect to the choice of z. But z has two internal vertices $x_{i'}$ and $x_{j'}$ of P' with $j' - i' \geq 5$ as neighbors, so repeating the above arguments with respect to $P', x_{i'}, x_{j'}$ we will obtain an induced H_1. This completes the proof of Theorem 9. ■

3 Possible forbidden pairs and hamiltonian-connectedness

We start by defining eight graphs which are 3-connected but not hamiltonian-connected. Let $m \geq 4$ be an integer, M_i be a K_m in which three vertices x_i, y_i and z_i are marked and $M = \cup_{i=1}^{8} M_i$.

- $G_1 = K_{m,m}$.
- G_2 is obtained from a cycle $C = x_1x_2 \dots x_{2m}$, by adding the edges $x_i x_{m+i}$ $(i = 1, \dots, m)$.
- G_3 is an arbitrary 3-connected C_4-free bipartite graph.
- G_4 is obtained from M_1 by adding two vertices a and b and all (six) edges between a, b and x_1, y_1, z_1.
- G_5 is obtained from a cycle $C = x_1x_2 \dots x_{6m}$ by adding the edges $x_{3i-2}x_{3i}$ $(i = 1, \dots, 2m)$ and the edges $x_{3i-1}x_{3m+3i-1}$ $(i = 1, \dots, m)$.
- G_6 is obtained from a cycle $C = x_1x_2 \dots x_{4m}$ by adding the edges $x_{2i-1}x_{2i+1}$ $(i = 1, \dots, 2m-1)$, $x_{4m-1}x_1$ and $x_{2i}x_{2m+2i}$ $(i = 1, \dots, m)$.
- G_7 is obtained from G_5 by replacing every triangle $x_{3i-2}x_{3i-1}x_{3i}$ $(i = 1, \dots, 2m)$ by the graph G' of Figure 2.
- G_8 is obtained from M by indentifying each vertex x_i with y_{i+1} $(i = 1, \dots, 7)$, x_8 with y_1 and each vertex z_i with z_{i+4} $(i = 1, \dots, 4)$.

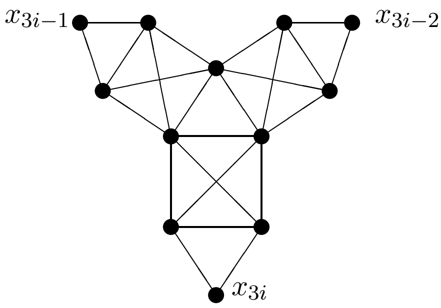

Figure 2: The graph G'.

Since the graphs $G_1, \dots G_8$ are not hamiltonian-connected, each of them must contain an induced copy of either X or Y. The graphs G_1, G_2, G_3, G_4 all contain a claw, but the last four graphs G_5, G_6, G_7, G_8 are all claw-free.

We will first show that one of the graphs X or Y must be $K_{1,3}$. Assume that this is not true. Assume, without loss of generality, that $X \subset G_1$. Then X must either contain an induced C_4 or it must be a generalized claw $K_{1,r}$ for $r \geq 4$. First consider the case when $C_4 \subset X$. Then Y must be an induced subgraph of both G_3 and G_4, since neither of these graphs contains an induced C_4. However, the only induced subgraph common to both G_3 and G_4 is the claw $K_{1,3}$. If $X = K_{1,r}$ for $r \geq 4$, then Y must be an induced subgraph of both G_2 and G_4, since neither of these graphs has an induced $K_{1,4}$. Again, the only induced subgraph common to both G_2 and G_4 is the claw $K_{1,3}$. Therefore, without loss of generality, we can assume that $X = K_{1,3}$.

Since G_5, G_6, G_7, G_8 are all claw-free, Y must be an induced subgraph of each of these graphs. Since G_5 is claw-free and $\Delta(G_5) = 3$, Y must satisfy both *(a)* and *(f)*. There is no induced P_{10} in G_8, so *(b)* is satisfied. The shortest induced cycle in G_5 besides C_3 is a C_8, the longest induced cycle in G_8 is a C_8, and G_6 contains no induced C_8. Thus *(c)* is satisfied. In G_5 the distance between distinct triangles is either one or at least three. This implies that *(d)* is satisfied. The graph G_7 does not contain an induced copy of the graph S obtained from a P_5 by placing a triangle on the first and third edge (S is an H_1 with an edge attached to a vertex of degree two). Therefore, if Y contains three triangles, then each pair of triangles would have to be at distance at least three. This would imply an induced P_{10}, which is not true. Thus *(e)* is satisfied. This completes the proof of Theorem 5. ■

4 Open question

The obvious question is the following.

Question A
What is the characterization of those pairs of connected graphs X and Y such that being X-free and Y-free implies that a 3-connected graph is hamiltonian-connected?

A simpler question, but one that is critical to answering Question A is the following.

Question B
What is the largest k such that a 3-connected claw-free and P_k-free graph is hamiltonian-connected?

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