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Characterization of well-posedness of piecewise linear systems†

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ABSTRACT: One of the basic issues in the study of hybrid systems is the well-posedness (existence and uniqueness of solutions) problem of discontinuous dynamical systems. This paper addresses this problem for a class of piecewise linear discontinuous systems under the definition of solutions of Carathéodory. The concepts of jump solutions or a sliding mode are not considered here. In this sense, the problem to be discussed is one of the most basic problems in the study of well-posedness for discontinuous dynamical systems. First, we derive necessary and sufficient conditions for bimodal systems to be well-posed, in terms of an analysis based on lexicographic inequalities and the smooth continuation property of solutions. Next, its extensions to the multi-modal case are discussed. As an application to switching control, in the case that two state feedback gains are switched according to a criterion depending on the state, we give a characterization of all admissible state feedback gains for which the closed loop system remains well-posed.

Keywords: piecewise linear systems, hybrid systems, discontinuous systems, well-posedness, lexicographic inequalities.

1 Introduction

Various approaches to modeling, analysis, and control synthesis of hybrid systems have been developed within the computer science community and the systems and control community, from different points of view (see, e.g., [1] -[6]). In the computer science community, as an extension of finite automata, several models of hybrid systems such as timed automata [7] and hybrid automata [8] have been proposed and some results on verification of their models have been obtained. In the control community, from the dynamical systems and control point of view, models of hybrid systems have been proposed (see e.g., [9], [10]), and several properties such as stability and controllability have been discussed; see [11] and [12] for controllability of switched systems and integrator hybrid systems, respectively, [13] and [14] for stability

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of general hybrid systems, and [15]-[17] for stability of piecewise linear systems. One of main concerns in these researches is how we define and analyze various kinds of properties of hybrid systems with discontinuous changes of vector fields and jumps of solutions (i.e., autonomous switchings and autonomous jumps in the terminology of [10]). However, there are still few results on the basic problem of uniqueness of solutions of piecewise linear discontinuous systems, while the existing standard theory of discontinuous dynamical systems is not quite satisfactory in spite of the fact that it is crucial for various developments of hybrid systems.

On the other hand, as an approach to modeling of hybrid systems, there is a new attempt in [18] and [19] to generalize in a natural manner dynamical properties of physical systems with jump phenomena which occur between unconstrained motion and constrained motion, such as the collision of a mass to a hard wall, so as to develop a framework modeling a class of hybrid systems. This framework is called the complementarity modeling (the corresponding system is called the complementarity system), which can describe several kinds of hybrid systems including electrical network with diodes and relay type systems as well as mechanical systems with unilateral constraints. Such an approach provides a natural and intuitive interpretation of jump phenomena in hybrid systems and make the analysis relatively easier. In fact, as the first result of the analysis in this line, several algebraic and checkable conditions for well-posedness (existence and uniqueness of solutions) of such systems have been derived in [18] -[21].

When hybrid (discontinuous) systems are considered from the above physical viewpoint, there also exist physical phenomena such as the collision to an elastic wall, whose system has a discontinuous vector field and does not exhibit jumps. Does there exist a common algebraic structure in the discontinuous vector field of such systems? Can we extend this to a general framework from the mathematical point of view? As far as we know, however, such questions have not been addressed, although an abstract condition can be found in the well-known book by Filippov [22]. When solutions without jumps are considered, there are, roughly speaking, two kinds of definitions of solutions, that is, Carathéodory's definition and Filippov's definition. The latter yields the concept of a sliding mode. In the case of physical systems such as the collision to an elastic wall, on the other hand, the solution belongs to the former, although we need to extend Carathéodory's definition, in a straightforward manner, to the case of discontinuous vector fields.

Besides from the viewpoint of a generalization of such physical systems, there are in addition the following three points we like to stress as a motivation to address the well-posedness problem in the sense of Carathéodory for discontinuous dynamical systems. First, this problem is a most fundamental one in the study of well-posedness for discontinuous dynamical systems. In other words, compared with the well-posedness problem including the concept of jump phenomena or a sliding mode, it is closest to the well-posedness problem in continuous dynamical systems. Therefore, as a first step to establish a theory of well-posedness of general hybrid systems, it will be very meaningful to clarify to what extent this basic problem can be analyzed. The second point is that it may be easier to analyze a system without jumps than with jumps. By representing a system with jumps as a limit of a system without jumps, we may obtain more results on the property of hybrid systems with jumps. A similar approach can be found in [23] -[26]. Third, in many examples of hybrid systems of practical interest, the solutions do not necessarily have jumps in the transition from one mode to the other mode, and also it may be desirable that no sliding mode exists in closed loop control systems.

In this paper, we address the well-posedness problem in the sense of Carathéodory for the class of piecewise linear discontinuous systems. We mainly concentrate on bimodal systems, and give several necessary and sufficient conditions for those systems to be well-posed, in terms of the analysis based on lexicographic inequalities and the smooth continuation property. Furthermore, some of results obtained in the bimodal case will be extended to the case of two kinds of multi-modal systems. Finally, as an application of our result, we discuss the well-posedness problem of feedback control systems with two state feedback gains switched according to a criterion depending on the state. Recently, switching control schemes have attracted considerable attention in the control community (see, e.g., [27], [28], and [29]). As one of its basic results, we give a characterization of all admissible state feedback gains provides that the corresponding closed loop system is well-posed.

The organization of this paper is as follows: In section 2, piecewise linear discontinuous systems in the bimodal case are described, together with the definition of solutions of Carathéodory. Section 3 is devoted to some mathematical preliminaries on lexicographic inequalities and smooth continuation. We give out main results on the well-posedness of bimodal systems in sections 4 and 5, and some extensions in section 6. In section 7, our results are applied to the well-posedness problem in switching control systems. Section 8 presents a brief summary and some topics for future research.

In the sequel, we will use the following notation for lexicographic inequalities: for $x \in \mathcal{R}^n$, if for some i, $x_j = 0$ $(j = 1, 2, \cdots, i-1)$, while $x_i > (<) 0$, we denote it by $x \succ (\prec) 0$. In addition, if $x = 0$ or $x \succ (\prec) 0$, we denote it $x \succeq (\preceq) 0$. We use the notation $*$ representing any fixed but unspecified number or matrix. Finally, I_n, $O_{m,n}$ and O_n denote the $n \times n$ identity matrix, the $m \times n$ zero matrix, and the $n \times n$ zero matrix, respectively.

2 Piecewise linear discontinuous systems

In this section, we describe the basic form of bimodal systems to be studied here, and give a definition of well-posedness for these bimodal systems. Next, we give an equivalent representation of bimodal systems, which will be important for further developments.

2.1 Description of bimodal system and definition of its solution

Consider the system given by

$$\Sigma_O \left\{ \begin{array}{lcll} \text{mode } 1 : \dot{x} & = & Ax, & \text{if } y = Cx \geq 0 \\ \text{mode } 2 : \dot{x} & = & Bx, & \text{if } y = Cx \leq 0 \end{array} \right. \tag{1}$$

where $x \in \mathcal{R}^n$, $y \in \mathcal{R}$, and A and B are $n \times n$ matrices (in general different). Since the two linear differential equations $\dot{x} = Ax$ and $\dot{x} = Bx$ are coupled by separating the region of $\mathcal{R}^n$ into two subspaces, i.e., $y \geq 0$ and $y \leq 0$, the system Σ_O belongs to the class of piecewise linear systems. Even when we consider the system Σ_O on any neighborhood of the origin, the argument below holds with some modification. However, for brevity, we consider the system to be defined on the whole R^n.

Furthermore, for simplicity of notation, we use $\dot{x}(t)$ in (1), although there may be a set (of measure 0) of points of time where the solution $x(t)$ is not differentiable. Formally, the system Σ_O is given by its integral form (which is called the Carathéodory equation):

$$x(t) = x(t_0) + \int_{t_0}^{t} f(x(\tau))d\tau \tag{2}$$

where $f(x)$ is the discontinuous vector field given by the right hand side of (1). We call the $x(t)$ given by (2) the solution in the sense of Carathéodory.

Then the well-posedness for the system Σ_O is defined as follows.

Definition 2.1 *The system Σ_O is said to be well-posed at x_0 if there exists a unique solution of (1) on $[0, \infty)$ in the sense of Carathéodory for the initial state x_0 in $\mathcal{R}^n$. In addition, the system Σ_O is said to be well-posed if it is well-posed at every initial state $x_0 \in \mathcal{R}^n$.*

The following result shows that we only have to prove local existence and uniqueness of solutions at every initial state in order to show the well-posedness of the system Σ_O.

Lemma 2.1 *If there exists an $\varepsilon > 0$ such that a unique solution $x(t)$ of Σ_O exists on $[0, \varepsilon)$ in the sense of Carathéodory from every initial state $x_0 \in \mathcal{R}^n$, then the system Σ_O is well-posed and the solution is absolutely continuous on any interval of $\mathcal{R}$.*

(Proof) Since there exists a local unique solution from every initial state, we can make a successively connected solution. Then the solution $x(t)$ in (2) is given by $x(t) = e^{S_i(t-t_i)}e^{S_{i-1}(t_i-t_{i-1})} \cdots e^{S_0 t_1}x(0)$ for all $t \in [t_i, t_i+\varepsilon)$, where $i \in \{0, 1, 2, \cdots\}$ is the switching number, t_j is a switching time ($t_0 = 0$), and $S_j = A$ or B ($j = 0, 1, 2, \cdots, i$). Since there exists a positive real number a such that $\max\{\| e^{At} \|, \| e^{Bt} \|\} \leq e^{at}$ for all $t \geq 0$, it follows that $\| x(t) \| \leq e^{at} \| x(0) \|$ for all $t \in [t_i, t_i + \varepsilon)$ and all $i \in \{0, 1, 2, \cdots\}$. Noting that there exists a unique solution for all $t \geq t_\infty$ even when $t_\infty < \infty$ (i.e., a finite accumulation point of switching times exists), we have $x \in \mathcal{L}_{\infty e}$ (extended $\mathcal{L}_\infty$ space). Thus there exists a unique solution $x(t)$ on $[0, \infty)$. In addition, since $f(x) \in \mathcal{L}_{1e}$ (with $f(x)$ defined by (2)) holds from $x \in \mathcal{L}_{\infty e}$, it follows from Lebesgue integral theory that the solution given by (2) is absolutely continuous on any interval of $\mathcal{R}$. □

Remark 2.1 *After section 5, we will consider other types of discontinuous systems such as multi-modal systems. For all these systems, Definition 2.1 can be straightforwardly extended and Lemma 2.1 also holds for these systems.*

It is well-known that a sufficient condition for a system given by a first-order differential equation to be well-posed is that it satisfies a global Lipschitz condition. When we apply this to the system Σ_O, it follows that a sufficient condition for well-posedness is that there exists a K such that $B = A + KC$. Note that in this case the vector field is necessarily continuous in the state x.

Now, how about the case of discontinuous vector fields? Let us consider the following example shown in Figure 1. The equations of motion of this system are given by

$$\left\{ \begin{array}{lcll} \text{mode } 1 : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, & \text{if } y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \\ \text{mode } 2 : \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} & = & \begin{bmatrix} 0 & 1 \\ -k & -d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, & \text{if } y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0. \end{array} \right. \tag{3}$$

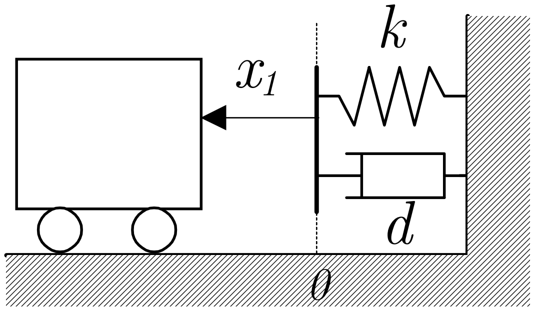

Figure 1: Collision to an elastic wall

By simple calculations, we see that this system is well-posed (without jumps and sliding modes), although the vector field is discontinuous in x when $d \neq 0$. On the other hand, we can easily find an example which is not well-posed, as shown below:

$$\begin{cases} \text{mode } 1: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, & \text{if } y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \\ \text{mode } 2: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, & \text{if } y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0. \end{cases}$$

In fact, if the initial state $x(0)$ satisfies $x_1(0) = 0$ and $x_2(0) = 1$, then the solution $x(t)$ in mode 1 belongs to the region $x_1 > 0$, and the solution $x(t)$ in mode 2 belongs to the region $x_1 < 0$. Thus there exist two solutions for this initial state.

Within the type of physical systems as given by (3), there will exist many systems with discontinuous vector fields, but which are well-posed. In the next sections, we will derive a necessary and sufficient condition for the well-posedness of the system Σ_O including such physical systems.

Remark 2.2 *Consider the system given by the equations*

$$\begin{cases} \textit{mode } 1: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, & \textit{if } y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0 \\ \textit{mode } 2: \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, & \textit{if } y = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq 0. \end{cases}$$

The system is not well-posed at $(x_1, x_2) = (0, 1)$ in the sense of Definition 2.1 because the solution $x(t)$ in mode 1 (mode 2) is included in the region in mode 2 (mode 1). However, if we use Filippov's definition, there exists a unique solution from the initial state $(x_1(0), x_2(0)) = (0, 1)$. In fact, the system Σ_O can be rewritten by $\dot{x} = \frac{1}{2}(1+u)Ax + \frac{1}{2}(1-u)Bx$, using a relay-type input of $u = sgn(y)$. Thus for $(x_1(0), x_2(0)) = (0, 1)$, there exists a unique solution given by the equivalent control input $u = 0$. Certainly, Filippov's definition is very important from a practical viewpoint as well as from a mathematical viewpoint. However, in this paper, we concentrate on the well-posedness problem in the sense of Definition 2.1.

Remark 2.3 *When we consider the case of $d \to \infty$ in the example (3), a jump in the solution will occur. Such a system can be treated within the framework of complementarity systems. Thus we conjecture that there exists some relation between complementarity systems and systems given by (1). In other words, there may be some possibility to approximate the complementarity system, i.e., the discontinuous dynamical system with jumps, by a system without jumps given by (1). Some researchers have already studied the relation between two solutions for a simple physical system as in Figure 1 (see Chapter 2 in [26]), and we plan to return to this issue in a future paper.*

2.2 Equivalent representation of the bimodal system Σ_O

For the system Σ_O, define the following row-full rank matrices:

$$T_A \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{h-1} \end{bmatrix}, \quad T_B \triangleq \begin{bmatrix} C \\ CB \\ \vdots \\ CB^{k-1} \end{bmatrix} \tag{4}$$

where h and k are the observability indexes of the pairs (C, A) and (C, B), respectively . In addition, let $\mathcal{S}_A^+$, $\mathcal{S}_A^-$, $\mathcal{S}_B^+$, and $\mathcal{S}_B^-$ be sets defined by

$$\mathcal{S}_N^+ \triangleq \{x \in \mathcal{R}^n \mid T_N x \succeq 0\}, \quad \mathcal{S}_N^- \triangleq \{x \in \mathcal{R}^n \mid T_N x \preceq 0\} \tag{5}$$

for $N = A$, B. Then noting that $T_A x = [y, \dot{y}, \cdots, y^{(h-1)}]^{\mathrm{T}}$ for the system $\dot{x} = Ax$ and $T_B x = [y, \dot{y}, \cdots, y^{(k-1)}]^{\mathrm{T}}$ for the system $\dot{x} = Bx$, we introduce the system given by

$$\Sigma_{AB} \left\{ \begin{array}{lcll} \text{mode } 1 : \dot{x} & = & Ax, & \text{if } x \in \mathcal{S}_A^+ \\ \text{mode } 2 : \dot{x} & = & Bx, & \text{if } x \in \mathcal{S}_B^- \end{array} \right. . \tag{6}$$

We call T_A and T_B the rule (or observability) matrices of the system Σ_{AB}. The well-posedness for the system Σ_{AB} is defined similar to Definition 2.1. The following result shows that the system Σ_O is well-posed if and only if the system Σ_{AB} is well-posed.

Lemma 2.2 *The system Σ_{AB} is equivalent to the original system Σ_O, i.e., both systems have the same solutions.*

(Proof) If $y(t) = Cx(t) \geq 0$ for $\dot{x} = Ax$, then $T_A x(t) \succeq 0$. Conversely, if $T_A x(t) \succ 0$ for $\dot{x} = Ax$, then $y(t) = Cx(t) \geq 0$ is obvious. When $T_A x(t) = 0$, the definition of the observability index implies that $y(t) \equiv 0$. The case of $\dot{x} = Bx$ is similar. Thus modes 1 and 2 of Σ_{AB} are equivalent to those of Σ_O, respectively, which implies that both systems have the same solutions. $\square$

Thus, we will discuss the well-posedness of the system Σ_{AB} in the next sections. Note that the claim in Lemma 2.1 is still true for the system Σ_{AB}.

3 Preliminaries on lexicographic inequalities and smooth continuation

In this section, as a preparation, we give mathematical preliminaries on lexicographic inequalities and smooth continuation for solutions of linear systems with respect to lexicographic inequalities. Most of results obtained in this section will play a central role in the study of well-posedness in the next sections.

3.1 Lemmas on lexicographic inequalities

First we give some lemmas on lexicographic inequalities. Throughout this subsection, x will be a vector in $\mathcal{R}^n$.

Lemma 3.1 *Let T be an $m \times n$ real matrix with $m \leq n$ and rank T =rank $T_1 = r$, where $T = [T_1^{\mathrm{T}}\ T_2^{\mathrm{T}}]^{\mathrm{T}}$ and $T_1 \in \mathcal{R}^{r \times n}$. Then $Tx \succeq (\preceq) 0$ if and only if $T_1 x \succeq (\preceq) 0$.*

(Proof) $Tx \succeq 0$ is equivalent to $T_1 x \succ 0$, or $T_1 x = 0$ and $T_2 x \succeq 0$. Hence, $Tx \succeq 0$ implies $T_1 x \succeq 0$. Conversely, consider $T_1 x = 0$. Then rank T =rank $T_1 = r$ yields $T_2 x = 0$. Thus $T_1 x \succeq 0$ implies $Tx \succeq 0$. The case of $Tx \preceq 0 \leftrightarrow T_1 x \preceq 0$ is similar. $\square$

This lemma shows that the row full-rank submatrix T_1 of T is enough for representing the relation of the lexicographic inequality. Thus the following result is obtained: let T be an $m \times n$ matrix and let $\bar{t}_i^{\mathrm{T}}$ be the ith row vector of T. Let also $T_i \stackrel{\triangle}{=} [\bar{t}_1\ \bar{t}_2\ \cdots \bar{t}_i]^{\mathrm{T}}$. Suppose that $\mathrm{rank}\, T_i = \mathrm{rank}\, T_{i+1} = i$. Then from Lemma 3.1, we can use, in place of T, $\tilde{T} = [\bar{t}_1\ \cdots\ \bar{t}_i\ \bar{t}_{i+2}\ \cdots\ \bar{t}_m]$ which is obtained by removing the $i+1$th column $\bar{t}_{i+1}$ from T. Hence we can assume without loss of generality that T is row-full rank, whenever we consider $Tx \succeq (\preceq) 0$.

Definition 3.1 *Let $\mathcal{L}^n$ be the set of $n \times n$ lower-triangular matrices. In addition, let $\mathcal{L}_+^n$ be the set of elements in $\mathcal{L}^n$ with all diagonal elements positive.*

The following lemma shows that the set $\mathcal{L}_+^n$ characterizes the coordinate transformations preserving the lexicographic inequality relation.

Lemma 3.2 *Let T be an $n \times n$ real matrix. Then $x \succeq (\preceq) 0 \leftrightarrow Tx \succeq (\preceq) 0$ if and only if $T \in \mathcal{L}_+^n$.*

(Proof) ($\leftarrow$) Obvious. ($\rightarrow$) First, we will prove that if $x \succeq 0 \leftrightarrow Tx \succeq 0$ holds, then T is nonsingular. So assume that T is singular and rank $T = m < n$. Then from Lemma 3.1, there exists a $T_1 \in \mathcal{R}^{m \times n}$ such that $Tx \succeq 0 \leftrightarrow T_1 x \succeq 0$. So we consider $x \succeq 0 \leftrightarrow T_1 x \succeq 0$. Let T_2 be an $(n-m) \times n$ matrix such that $\tilde{T} \stackrel{\triangle}{=} [T_1^{\mathrm{T}}\ T_2^{\mathrm{T}}]$ is nonsingular, and let $z \stackrel{\triangle}{=} [\bar{z}_1^{\mathrm{T}}\ \bar{z}_2^{\mathrm{T}}]^{\mathrm{T}}$ where $\bar{z}_i = T_i x$. Then $x = \tilde{T}^{-1} z = M_1 \bar{z}_1 + M_2 \bar{z}_2$ where $[M_1\ M_2] = \tilde{T}^{-1}$. When $\bar{z}_1 = 0$ and $\bar{z}_2$ is any vector, we obtain $x = M_2 \bar{z}_2$. In addition, since rank $M_2 = n - m$, there exists a $z_2 \in \mathcal{R}^{n-m}$ such that $x \prec 0$. This is inconsistent with the condition that $T_1 x \succeq 0 \rightarrow x \succeq 0$. Hence, T is nonsingular.

Now we define the new coordinates $z = [z_1, z_2, \cdots, z_n]^{\mathrm{T}} \stackrel{\triangle}{=} Tx$. Denote the (i,j)th element of T by t_{ij}. Suppose that, for $k \in \{1, 2, \cdots, n\}$, $x_i = 0$ $(i = 1, 2, \cdots, k-1)$, $x_k > 0$, and x_j $(j = k+1, k+2, \cdots, n)$ are arbitrary. We will prove the assertion for $\succeq$ by induction. First, let us consider $k = 1$. From

$$z_1 = t_{11} x_1 + t_{12} x_2 + \cdots + t_{1n} x_n,$$

we have $t_{1i} = 0$ $(i = 2, 3, \cdots, n)$ because $z_1 \geq 0$ and x_i $(i = 2, 3, \cdots, n)$ are arbitrary. Furthermore, if $t_{11} < 0$, then $z_1 < 0$ for $x_1 > 0$, and if $t_{11} = 0$, then T is singular. Hence we conclude $t_{11} > 0$. Next assume that, for $k = k_* \in \{1, 2, \cdots, n-1\}$, $t_{ii} > 0$ $(i = 1, 2, \cdots, k_*)$, and $t_{ij} = 0$ $(i = 1, 2, \cdots, k_*,\ j = i+1, i+2, \cdots, n)$. Under this inductive assumption, let us consider $k = k_* + 1$. From $x_1 = \cdots = x_{k_*} = 0$, it follows that

$$z_{k_*+1} = t_{k_*+1,k_*+1}x_{k_*+1} + t_{k_*+1,k_*+2}x_{k_*+2} + \cdots + t_{k_*+1,n}x_n.$$

Thus noting that $z_i = 0$ $(i = 1, 2, \cdots, k_*)$, we have $t_{k_*+1,i} = 0$ $(i = k_*+2, \cdots, n)$ since $z_{k_*+1} \geq 0$ and x_i $(i = k_*+2, \cdots, n)$ are arbitrary. In addition, similarly to the case $k = 1$, it is verified that $t_{k_*+1,k_*+1} > 0$. The proof of the assertion for $\preceq$ is similar. □

While Lemma 3.2 is concerned with the nonsingular matrices case, the following result treats the singular matrix case.

Lemma 3.3 *Let T and S be $l \times n$ and $m \times n$ real matrices with rank $T = l$, rank $S = m$, and $l \geq m$, respectively. Then the following statements are equivalent.*
(i) $Sx \succeq (\preceq) 0$ for all x satisfying $Tx \succeq (\preceq) 0$.
(ii) $S = [M \ \ 0]T$ for some $M \in \mathcal{L}_+^m$.

(Proof) (i)→(ii). Let Q be any $(n-l) \times n$ matrix such that $[T^{\mathrm{T}} \ \ Q^{\mathrm{T}}]^{\mathrm{T}} (\stackrel{\triangle}{=} \tilde{T})$ is nonsingular. We denote the new coordinates by $z \stackrel{\triangle}{=} [z_1^{\mathrm{T}} \ \ z_2^{\mathrm{T}}]^{\mathrm{T}}$, where $z_1 = Tx$ and $z_2 = Qx$. Then (i) is equivalent to that $Nz \succeq 0$ for all $z_1 \succeq 0$, where $N \stackrel{\triangle}{=} S\tilde{T}^{-1}$. Let N_1 and N_2 be $m \times l$ and $m \times (n-l)$ matrices, respectively, satisfying $N = [N_1 \ \ N_2]$. When $z_1 \succeq 0$ and z_2 is arbitrary, $N_2 = 0$ is necessary for $Nz \succeq 0$. Thus (i) is equivalent to the condition that $N_1 z_1 \succeq 0$ for all $z_1 \succeq 0$. Similarly to the proof of Lemma 3.2 and noting rank $S = m$, we can prove that $N_1 = [M \ \ 0]$ for some $M \in \mathcal{L}_+^m$. Hence it follows that $S = N\tilde{T} = N_1 T = [M \ \ 0]T$.

(ii) →(i). If $Tx \succeq 0$, then $[I_m \ \ 0]Tx \succeq 0$, which implies that $[M \ \ 0]Tx \succeq 0$ because $M \in \mathcal{L}_+^m$. Hence (ii) provides $Sx \succeq 0$. The proof of the case with $\preceq$ is similar. □

Note that $Tx \succeq (\preceq) 0$ in (i) of Lemma 3.3 can be also replaced by $Tx \succ (\prec) 0$, as can be easily seen from the proof. This fact will be used in the proof of Lemma 3.4 below. Moreover, when we describe the singular case in terms of a form corresponding to Lemma 3.2, the following corollary is obtained from Lemma 3.3.

Corollary 3.1 *Let T and S be $l \times n$ and $m \times n$ real matrices with rank $T = l$, rank $S = m$, and $l \geq m$, respectively. Then the following statements are equivalent.*
(i) $Sx \succeq (\preceq) 0 \leftrightarrow Tx \succeq (\preceq) 0$.
(ii) $l = m$ and $S = MT$ for some $M \in \mathcal{L}_+^m$.

(Proof) (i) →(ii). We can prove rank T =rank S in a similar way to the first part of the proof in Lemma 3.2. The latter part in (ii) follows from Lemma 3.3. Concerning (ii) →(i), it follows from (ii) that $Sx \succeq (\preceq) 0 \leftrightarrow MTx \succeq (\preceq) 0 \leftrightarrow Tx \succeq (\preceq) 0$, which implies (i). □

From the definition of the lexicographic inequality, it follows that for any nonsingular $n \times n$ matrix T we have the properties:

$$\{x \in \mathcal{R}^n \mid Tx \succeq 0\} \bigcup \{x \in \mathcal{R}^n \mid Tx \preceq 0\} = \mathcal{R}^n,$$

$$\{x \in \mathcal{R}^n \mid Tx \succeq 0\} \bigcap \{x \in \mathcal{R}^n \mid Tx \preceq 0\} = \{0\}.$$

The following lemma generalizes this property to the singular matrix case.

Lemma 3.4 *Let T and S be $l \times n$ and $m \times n$ real matrices with rank $T = l$, rank $S = m$, and $l \geq m$. Then the following statements are equivalent.*
(i) $\{x \in \mathcal{R}^n \mid Tx \succeq 0\} \bigcup \{x \in \mathcal{R}^n \mid Sx \preceq 0\} = \mathcal{R}^n$.
(ii) $S = [M \ \ 0]T$ for some $M \in \mathcal{L}_+^m$.

(Proof) The complement of $\{x \in \mathcal{R}^n \mid Tx \succeq 0\}$ in $\mathcal{R}^n$ is $\{x \in \mathcal{R}^n \mid Tx \prec 0\}$. Thus (i) is equivalent to (iii) $\{x \in \mathcal{R}^n \mid Sx \preceq 0\} \supseteq \{x \in \mathcal{R}^n \mid Tx \prec 0\}$. Hence we will show (ii) ↔ (iii). (iii) implies that $Sx \preceq 0$ for all x satisfying $Tx \prec 0$. From Lemma 3.3, it follows that (iii)→ (ii). The proof of (ii)→ (iii) is straightforward. □

3.2 Characterization of smooth continuation property

If all the solutions of the $n-$ dimensional linear system $\dot{x} = Ax$ locally conserve the lexicographic inequality relation, that is, for each initial state $x(0)$ satisfying $x(0) \succ (\prec) 0$, there exists an $\varepsilon > 0$ such that $x(t) \succ (\prec) 0$ for all $t \in [0, \varepsilon]$, then we say that the system has the smooth continuation property, or smooth continuation in the system is possible [18]. In this subsection, we derive a necessary and sufficient condition for this property.

Definition 3.2 *Let $\mathcal{G}_0^n$ be the set defined by*

$$\mathcal{G}_0^n \triangleq \left\{ \Gamma \in \mathcal{R}^{n\times n} \;\middle|\; \Gamma = \begin{bmatrix} * & \gamma_{12} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \gamma_{n-1,n} \\ * & \dots & \dots & \dots & * \end{bmatrix}, \gamma_{i,i+1} \geq 0, i = 1, 2, \cdots, n-1 \right\}$$

where γ_{ij} is the (i,j) element of the matrix Γ. In addition, let $\mathcal{G}_+^n$ be the set of elements in $\mathcal{G}_0^n$ with all the $(i,i+1)$ elements $\gamma_{i,i+1}$ positive.

The set $\mathcal{G}_0$ characterizes the smooth continuation property of linear systems as follows.

Lemma 3.5 *For the system $\dot{x} = Ax$, the following statements are equivalent.*
(i) The system has the smooth continuation property.
(ii) $A \in \mathcal{G}_0^n$
(iii) There exists a matrix $T \in \mathcal{L}_+^n$ such that

$$TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 & \dots & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde{A}_{p1} & \dots & \tilde{A}_{p,p-1} & \tilde{A}_{pp} \end{bmatrix} \tag{7}$$

where

$$\tilde{A}_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ * & \dots & \dots & \dots & * \end{bmatrix} \in \mathcal{R}^{n_i \times n_i}, \quad \tilde{A}_{ij} = \begin{bmatrix} * & \dots & * \\ \vdots & & \vdots \\ * & \dots & * \end{bmatrix} \in \mathcal{R}^{n_i \times n_j}, \quad \textit{for } i > j\ ,$$

and $n = n_1 + n_2 + \cdots + n_p$ ($p \in \{1, 2, \cdots, n\}$).

(Proof) (i)→ (ii). Suppose that, for $k \in \{2, \cdots, n\}$, $x_i(0) = 0$ $(i = 1, 2, \cdots, k-1)$, $x_k(0) > 0$, and x_j $(j = k+1, k+2, \cdots, n)$ take any values. We will prove the assertion by induction. First, consider $k = 2$. Let a_{ij} be the (i,j) element of A. So from

$$x_1(t) = t\{a_{12}x_2(0) + a_{13}x_3(0) + \cdots + a_{1n}x_n(0)\} + o(t^2),$$

it follows that $a_{1j} = 0$ $(j = 3, 4, \cdots, n)$. In fact, if $a_{1j} \neq 0$ for some $j \in [3, 4, \cdots, n]$, then there exists an $\varepsilon > 0$ such that $x_1(t) < 0$ for all $t \in [0, \varepsilon]$ at some $x_j(0)$, which is inconsistent with the condition (i). In addition, since $x_2(0) > 0$, no smooth continuation is possible if $a_{12} < 0$. Hence we have $a_{12} \geq 0$.

Next assume that, for $k = k_* \in \{2, 3, \cdots, n-1\}$, $a_{i,i+1} \geq 0$ and $a_{ij} = 0$ $(i = 1, 2, \cdots, k_* - 1, \quad j = i+2, i+3, \cdots, n)$. Under this assumption, let us consider $k = k_* + 1$. By inductive calculations, it is verified that

$$x_1(t) = \frac{t^{k_*}}{k_*!}\{\ \Pi_{i=1}^{k_*} a_{i,i+1}x_{k_*+1}(0) + \Pi_{i=1}^{k_*-1} a_{i,i+1}a_{k_*,k_*+2}x_{k_*+2}(0) + \cdots + \Pi_{i=1}^{k_*-1} a_{i,i+1}a_{k_*,n}x_n(0)\ \} + o(t^{k_*+1}).$$

From this, it follows that $a_{k_*,j} = 0$ $(j = k_* + 2, \cdots, n)$ and $a_{k_*,k_*+1} \geq 0$. Thus by induction, (ii) holds.

(ii)→ (iii). Suppose that, for $i = k_j$, $a_{i,i+1} = 0$ $(j = 1, 2, \cdots, s; s \leq n-1)$, and for the other i, $a_{i,i+1} > 0$. Set $k_0 = 0$ and $k_{s+1} = n$. Let us consider the coordinate transformation $z = [z_1, z_2, \cdots, z_n]^{\mathrm{T}} \triangleq Tx$ given by

$$\begin{aligned} z_{k_j+1} &\triangleq x_{k_j+1}, \\ z_{k_j+l} &\triangleq \sum_{i_{(k_j+1)}=1}^{k_j+2} \sum_{i_{(k_j+2)}=1}^{i_{(k_j+1)}+1} \cdots \sum_{i_{(k_j+l-1)}=1}^{i_{(k_j+l-2)}+1} a_{k_j+1, i_{(k_j+1)}} a_{i_{(k_j+1)}, i_{(k_j+2)}} \cdots a_{i_{(k_j+l-2)}, i_{(k_j+l-1)}} x_{i_{(k_j+l-1)}}, \\ & \qquad l = 2, \cdots, k_{j+1} - k_j, \quad j = 0, 1, \cdots, s. \end{aligned} \tag{8}$$

where $i_{k_j} = k_j + 1$ (note that $s = 0$ implies that all elements $a_{i,i+1}$ are positive). The matrix T is given by

$$T = \begin{bmatrix} T_{11} & 0 & \dots & 0 \\ T_{21} & T_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ T_{s+1,1} & \dots & T_{s+1,s} & T_{s+1,s+1} \end{bmatrix} \tag{9}$$

where

$$T_{ii} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ * & a_{k_{(i-1)}+1,k_{(i-1)}+2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & \Pi_{j=1}^{k_i-k_{i-1}-1} a_{k_{(i-1)}+j,k_{(i-1)}+j+1} \end{bmatrix} \in \mathcal{R}^{(k_i-k_{(i-1)})\times(k_i-k_{(i-1)})},$$

$$T_{ij} = \begin{bmatrix} 0 & \dots & \dots & 0 \\ * & \dots & \dots & * \\ \vdots & & & \vdots \\ * & \dots & \dots & * \end{bmatrix} \in \mathcal{R}^{(k_i-k_{(i-1)})\times(k_j-k_{(j-1)})}, \quad \text{for } i > j \ .$$

Thus from $a_{i,i+1} > 0$ for all $i \in \{1, 2, \cdots, n\}$ except for $i = k_j$, we conclude $T \in \mathcal{L}_+^n$. Furthermore, by direct computation, it is verified that TAT^{-1} satisfies (7).

(iii)→ (i). Denote the new coordinates by $z = [z_1, z_2, \cdots, z_n]^{\mathrm{T}} \triangleq Tx$. From Lemma 3.2, $T \in \mathcal{L}_+^n$ implies that $x \succ 0 \leftrightarrow z \succ 0$. Let $\bar{z}_k$ $(k = 1, 2, \cdots, p)$ be defined by

$$\bar{z}_k \triangleq \begin{bmatrix} z_{(\sum_{i=1}^{k-1} n_i)+1} \\ \vdots \\ z_{(\sum_{i=1}^{k} n_i)} \end{bmatrix}.$$

where $\bar{z}_1 = [z_1, z_2, \cdots, z_{n_1}]^{\mathrm{T}}$ for $k = 1$.

Note that $x(0) \succ 0$, namely $z(0) \succ 0$, is equivalent to $\bar{z}_i(0) = 0$ $(i = 1, 2, \cdots, k-1)$ and $\bar{z}_k(0) \succ 0$ for all $k \in \{1, 2, \cdots, p\}$. So from the structure of the A-matrix of the system, for each $k \in \{1, 2, \cdots, p\}$, there exists an $\varepsilon > 0$ such that

$$\begin{cases} \bar{z}_i(t) &= 0, \ i = 1, 2, \cdots, k-1 \\ \bar{z}_k(t) &\succ 0 \end{cases}, \quad \forall t \in [0, \varepsilon]$$

which implies that $x(t) \succ 0$ for all $t \in [0, \varepsilon]$. The case $x(0) \prec 0$ is proven in the same way. □

From Lemma 3.5, it turns out that, by the coordinate transformation given in (9), any linear system with the smooth continuation property is transformed into a system whose A-matrix is given by (7). In addition, the equivalence between (ii) and (iii) suggests that all the coordinates transformations given by elements in $\mathcal{L}_+^n$ conserve the smooth continuation property of the linear system. This is shown in the following lemma.

Lemma 3.6 *Let M be a matrix in $\mathcal{L}_+^n$ and Γ be a matrix in $\mathcal{G}_0^n(\mathcal{G}_+^n)$. Then $M\Gamma M^{-1} \in \mathcal{G}_0^n(\mathcal{G}_+^n)$.*

(Proof) Let M_k and Γ_k be $k \times k$ matrices with $M_k \in \mathcal{L}_+^k$ and $\Gamma_k \in \mathcal{G}_0^k$. When $k = 1$, we can show that $M_1\Gamma_1 M_1^{-1} \in \mathcal{G}_0^1$. Assume that $M_k\Gamma_k M_k^{-1} \in \mathcal{G}_0^k$ for some $k \in \{1, 2, \cdots, n-1\}$. Under this assumption, it is verified that $M_{k+1}\Gamma_{k+1}M_{k+1}^{-1} \in \mathcal{G}_0^{k+1}$. Thus by induction, we conclude $M\Gamma M^{-1} \in \mathcal{G}_0^n$. The proof in the case of $\mathcal{G}_+^n$ is similar. □

There is another type of the smooth continuation property, where ε in (i) of Lemma 3.5 is independent of the initial state $x(0)$. In other words, if there exists a positive constant ε such that $x(t) \succ (\prec)0$ for all $x(0)$ satisfying $x(0) \succ (\prec)0$ and all $t \in [0, \varepsilon]$, we call this the uniform smooth continuation property. The following lemma characterizes this property.

Corollary 3.2 *For the system $\dot{x} = Ax$, the following statements are equivalent.*
(i) The system has the uniform smooth continuation property.
(ii) There exists a positive constant ε such that $e^{At} \in \mathcal{L}_+^n$ for all $t \in [0, \varepsilon]$.
(iii) $x(t) \succ (\prec)0$ for all $x(0)$ satisfying $x(0) \succ (\prec)0$ and all $t \in [0, \infty)$.
(iv) $e^{At} \in \mathcal{L}_+^n$ for all $t \in [0, \infty)$.
(v) $A \in \mathcal{L}^n$.

(Proof) (i)$\leftrightarrow$ (ii), and (iii)$\leftrightarrow$(iv) are straightforward from Lemma 3.2. We will prove (iv) $\to$ (ii) $\to$ (v) $\to$ (iv). First, (iv) $\to$ (ii) is trivial. Next, (ii) $\to$ (v). Note that e^{At} is a one-parameter subgroup in $\mathcal{L}_+^n$ around $t = 0$. Thus the tangent vector at $t = 0$ is A. On the other hand, the tangent space $T_e\mathcal{L}_+^n$ at the identity matrix is $\mathcal{L}^n$. Hence $A \in \mathcal{L}^n$. Finally, (v)$\to$ (iv). If $A \in \mathcal{L}^n$, simple calculations show

$$e^{At} = \begin{bmatrix} e^{a_{11}t} & 0 & \dots & 0 \\ * & e^{a_{22}t} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \dots & * & e^{a_{nn}t} \end{bmatrix}, \qquad A = [a_{ij}]$$

which implies (iv). □

Obviously, the uniform smooth continuation property implies the smooth continuation property. However, the converse is not true. Corollary 3.2 asserts that the uniform smooth continuation property in the local sense (i.e., (i)) is equivalent to the global one (i.e., (iii)) in the case of linear systems. Moreover, (iii) shows that the sets $\{x \in \mathcal{R}^n \mid x \succ 0\}$ and $\{x \in \mathcal{R}^n \mid x \prec 0\}$ are invariant subsets of $\mathcal{R}^n$ with respect to the dynamics $\dot{x} = Ax$.

4 Characterization of well-posedness of bimodal systems

In this section, we discuss the well-posedness of Σ_O, or equivalently of Σ_{AB}. First, we give a result in the case that both pairs (C, A) and (C, B) are observable. This will clarify a fundamental issue in the algebraic structure for well-posed bimodal systems. Next, the unobservable case is treated, as a generalization of the observable case.

4.1 Observable case

In this subsection, we assume that the pairs (C, A) and (C, B) are observable, that is, T_A and T_B are nonsingular, where

$$T_A \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}, \quad T_B \triangleq \begin{bmatrix} C \\ CB \\ \vdots \\ CB^{n-1} \end{bmatrix}. \tag{10}$$

In addition, we consider the following two systems:

$$\Sigma_A \begin{cases} \text{mode 1}: & \dot{x} = Ax, \quad \text{if } x \in \mathcal{S}_A^+ \\ \text{mode 2}: & \dot{x} = Bx, \quad \text{if } x \in \mathcal{S}_A^- \end{cases}, \tag{11}$$

$$\Sigma_B \begin{cases} \text{mode 1}: & \dot{x} = Ax, \quad \text{if } x \in \mathcal{S}_B^+ \\ \text{mode 2}: & \dot{x} = Bx, \quad \text{if } x \in \mathcal{S}_B^- \end{cases} \tag{12}$$

where $\mathcal{S}_N^+$ and $\mathcal{S}_N^-$ $(N = A, B)$ are given by (5). Utilizing the fact that $\mathcal{S}_A^+ \bigcup \mathcal{S}_A^- = \mathcal{R}^n$, the system Σ_A is given by the rule matrix T_A only. The system Σ_B is defined by the rule matrix T_B in the same way. Then we come to the first main result on the well-posedness.

Theorem 4.1 *Suppose that both pairs (C, A) and (C, B) are observable. Then the following statements are equivalent.*
(i) Σ_{AB} is well-posed.
(ii) Σ_A is well-posed.
(iii) Σ_B is well-posed.
(iv) $\mathcal{S}_A^+ \bigcup \mathcal{S}_B^- = \mathcal{R}^n$ and $\mathcal{S}_A^+ \bigcap \mathcal{S}_B^- = \{0\}$.
(v) $T_B T_A^{-1} \in \mathcal{L}_+^n$.
(vi) $T_A B T_A^{-1} \in \mathcal{G}_+^n$.
(vii) $T_B A T_B^{-1} \in \mathcal{G}_+^n$.

(Proof) First, we prove (i)$\to$(v)$\to$ (iv) $\to$ (i).

(i)$\to$(v). $\mathcal{S}_A^+ \bigcup \mathcal{S}_B^- = \mathcal{R}^n$ is obviously necessary for well-posedness. From Lemma 3.4, there exists a $M \in \mathcal{L}_+^n$ such that $T_B = MT_A$. (v)$\to$ (iv) follows from Lemmas 3.2 and 3.4. (iv) $\to$ (i). Note that, since T_A and T_B are the observability matrices, $T_A A T_A^{-1} \in \mathcal{G}_+^n$ and $T_B B T_B^{-1} \in \mathcal{G}_+^n$. So from Lemma 3.5, these guarantee the smooth continuation property for each mode. Hence, (iv) implies that the system Σ_{AB} has a unique solution at every initial state.

Next, we prove (v)$\to$(ii)$\to$(vi)$\to$ (v).

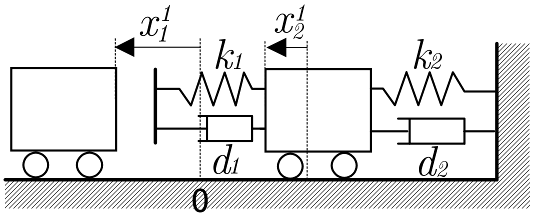

Figure 2: Elastic collision between 2 objects

(v)→(ii). Since (v) implies by Lemma 3.2 that $\mathcal{S}_A^- = \mathcal{S}_B^-$, Σ_{AB} is equivalent to Σ_A. Since Σ_{AB} is well-posed by (v), Σ_A is also well-posed. (ii)→(vi). In the new coordinates $z = [z_1\ z_2 \cdots\ z_n]^{\mathrm{T}} \triangleq T_A x$, the system Σ_A is described by

$$\tilde{\Sigma}_A \begin{cases} \text{mode } 1: & \dot{z} = T_A A T_A^{-1} z, \quad \text{if } z \succeq 0 \\ \text{mode } 2: & \dot{z} = T_A B T_A^{-1} z, \quad \text{if } z \preceq 0. \end{cases}$$

Then (ii) implies that smooth continuation is possible in each mode of Σ_A. Thus by Lemma 3.5, (ii) implies $T_A B T_A^{-1} \in \mathcal{G}_0$. Letting γ_{ij} be the (i,j) element of $\Gamma \triangleq T_A B T_A^{-1}$, and noting that $CT_A^{-1} = [1\ 0\ \cdots\ 0]$, we obtain

$$\begin{array}{ccccl} CB & = & CT_A^{-1}\Gamma T_A & = & [*\ \gamma_{12}\ \ 0\ \ \cdots\cdots\ 0]T_A, \\ CB^2 & = & CT_A^{-1}\Gamma^2 T_A & = & [*\ *\ \gamma_{12}\gamma_{23}\ 0\ \cdots\ 0]T_A, \\ \vdots & & \vdots & & \\ CB^{n-1} & = & CT_A^{-1}\Gamma^{n-1} T_A & = & [*\ \cdots\cdots\ *\ \Pi_{i=1}^{n-1}\gamma_{i,i+1}]T_A. \end{array} \tag{13}$$

From these calculations, it follows that

$$T_B = LT_A \tag{14}$$

where

$$L \triangleq \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ * & \gamma_{12} & \ddots & & \vdots \\ \vdots & \ddots & \gamma_{12}\gamma_{23} & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ * & \cdots & \cdots & * & \Pi_{i=1}^{n-1}\gamma_{i,i+1} \end{bmatrix}. \tag{15}$$

This implies that all elements $\gamma_{i,i+1}$ are positive, since T_A and T_B are nonsingular. Hence $T_A B T_A^{-1} \in \mathcal{G}_+$. (vi)→(v). In a similar way to (13), we obtain the equation (14) from (vi). Since $L \in \mathcal{L}_+^n$, (v) holds.

The proof of (v)→(iii)→(vii)→ (v) is similar. □

Remark 4.1 *From Theorem 4.1, it turns out that the well-posedness property of the bimodal system Σ_{AB} with both (C,A) and (C,B) observable is characterized by either one of the following two properties: (i) the preservation property of the lexicographic inequality relation between two rule matrices T_A and T_B, which is characterized by the set $\mathcal{L}_+^n$, and (ii) the smooth continuation property which is characterized by the set $\mathcal{G}_+^n$ (or $\mathcal{G}_0^n$). The former corresponds to the condition (iv) or (v) in Theorem 4.1, and the latter to (vi) or (vii). Note also that the well-posedness property of Σ_{AB} can be given by the equivalence between Σ_{AB}, Σ_A, and Σ_B. From (vi), it follows that a parameterization of all matrices B for which Σ_{AB} is well-posed is given by the form $B = T_A^{-1}\Gamma T_A$ for any $\Gamma \in \mathcal{G}_+^n$.*

Example 4.1 *Consider the physical system in Figure 2. The equations of motion of this system are given by*

$$\textit{mode } 1: \begin{cases} \dot{x}^1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^1 \\ \dot{x}^2 = \begin{bmatrix} 0 & 1 \\ -k_2 & -d_2 \end{bmatrix} x^2 \\ y = [1\ \ 0\ \ -1\ \ 0]x \geq 0, \end{cases} \qquad \textit{mode } 2: \begin{cases} \begin{bmatrix} \dot{x}^1 \\ \dot{x}^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1 & -d_1 & k_1 & d_1 \\ 0 & 0 & 0 & 1 \\ k_1 & d_1 & -k_1-k_2 & -d_1-d_2 \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} \\ y = [1\ \ 0\ \ -1\ \ 0]x \leq 0 \end{cases}$$

where $x = [(x^1)^{\mathrm{T}}\ \ (x^2)^{\mathrm{T}}]^{\mathrm{T}} = [x_1^1\ \ x_2^1\ \ x_1^2\ \ x_2^2]^{\mathrm{T}}$. These provide

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k_2 & -d_2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -k_1 & -d_1 & k_1 & d_1 \\ 0 & 0 & 0 & 1 \\ k_1 & d_1 & -k_1-k_2 & -d_1-d_2 \end{bmatrix},$$

$$C = [1 \quad 0 \quad -1 \quad 0].$$

Simple calculations show that the pair (C, A) is observable if and only if $k_2 \neq 0$, and also the pair (C, B) is observable if and only if $k_2 \neq 0$. Thus we here assume $k_2 \neq 0$.

From

$$T_A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & k_2 & d_2 \\ 0 & 0 & -k_2 d_2 & k_2 - d_2^2 \end{bmatrix},$$

$$T_B = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -2k_1 & -2d_1 & 2k_1 + k_2 & 2d_1 + d_2 \\ (4d_1 + d_2)k_1 & -2k_1 + (4d_1 + d_2)d_1 & -(4d_1 + d_2)k_1 - (2d_1 + d_2)k_2 & (2k_1 + k_2) - 4d_1^2 - 3d_1 d_2 - d_2^2 \end{bmatrix},$$

it follows that

$$T_B T_A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}$$

which belongs to the set $\mathcal{L}_+$. Hence the system is well-posed. We also have

$$T_A B T_A^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \\ * & * & * & * \end{bmatrix}$$

which belongs to the set $\mathcal{G}_+$.

4.2 Unobservable case

The following result is concerned with the case that both pairs are unobservable.

Theorem 4.2 *Suppose that the observability indexes of the pairs (C, A) and (C, B) are m_A and m_B, respectively, and $m_A \geq m_B$. Then the following statements are equivalent.*
(i) Σ_{AB} is well-posed.
(ii) The following conditions are satisfied.
(a) $m_A = m_B$.
(b) $T_B = MT_A$ for some $M \in \mathcal{L}_+^{m_A}$.
(c) $(A - B)x = 0$ for all $x \in$ $KerT_A$.
(iii) The following conditions are satisfied.
(a) $m_A = m_B$.
(b) $T_A B = \Gamma T_A$ for some $\Gamma \in \mathcal{G}_+^{m_A}$.
(c) $(A - B)x = 0$ for all $x \in$ $KerT_A$.

Since this theorem is a special case of Theorem 5.2 in the next section, the proof will follow from that of Theorem 5.2 (see Remark 5.3).

Remark 4.2 *If $m_A = m_B = n$, (ii) and (iii) in Theorem 4.2 generalize (v) and (vi) in Theorem 4.1, respectively. Note also that the condition $T_B = MT_A$, which is a necessary and sufficient condition for the well-posedness in the observable case, is not sufficient for the well-posedness in the unobservable case, even if $m_A = m_B$. In other words, it is required that the solutions in both modes in $KerT_A = KerT_B$ are the same. This allows us to conclude that whenever the pair (C, A) is observable and the pair (C, B) is unobservable, the system Σ_{AB} is not well-posed. However, if the number of the criterions which specify admissible regions of the state in each mode, i.e., the dimension of y in (1), is more than one, then the situation is different. The details will be given in Theorem 5.2 and Example 5.1 in the next section.*

Remark 4.3 *The conditions in Theorem 4.2 can be checked as follows. First, check the condition (iii)(a). If it is not satisfied, we conclude that the system is not well-posed. Otherwise, check (b) and (c) in (iii). So pick any matrix $\tilde{T}_A$ such that $T \triangleq [T_A^{\mathrm{T}} \ \ \tilde{T}_A^{\mathrm{T}}]^{\mathrm{T}}$ is nonsingular. Then note that (b) and (c) are equivalent to*

$$[I_{m_A} \quad 0] T B T^{-1} \begin{bmatrix} I_{m_A} \\ 0 \end{bmatrix} \in \mathcal{G}_+^{m_A} \tag{16}$$

and

$$[0 \;\; I_{n-m_A}]T(A-B)T^{-1}\begin{bmatrix} 0 \\ I_{n-m_A} \end{bmatrix} = 0, \tag{17}$$

respectively. Thus if both conditions are satisfied, we conclude that the system is well-posed. Otherwise, we conclude that the system is not well-posed. Note here that we only have to check the condition for some $\tilde{T}_A$, since the well-posedness does not depend on the choice of $\tilde{T}_A$.

Example 4.2 *Consider the system in Example 4.1 again. Assume that $k_2 = 0$ and $d_2 \neq 0$. Then since*

$$T_A = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & d_2 \end{bmatrix}, \quad T_B = \begin{bmatrix} C \\ CB \\ CB^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -2k_1 & -2d_1 & 2k_1 & 2d_1 + d_2 \end{bmatrix},$$

we have $m_A = 3$ and $m_B = 3$. Thus (iii)(a) in Theorem 4.2 is satisfied. Letting $\tilde{T}_A \triangleq [0 \;\; 0 \;\; 1 \;\; 0]$, we have

$$TAT^{-1} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -d_2 & 0 \\ \hline 0 & 0 & 1/d_2 & 0 \end{array}\right], \quad TBT^{-1} = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ -2k_1 & -2d_1 & 1 & 0 \\ k_1 d_2 & d_1 d_2 & -d_2 & 0 \\ \hline 0 & 0 & 1/d_2 & 0 \end{array}\right].$$

Using (16) and (17) in Remark 4.3, we can show that (b) and (c) in (iii) are satisfied. Therefore, the system is well-posed.

5 Well-posedness of bimodal systems with multiple criteria

In this section, we treat bimodal systems given by multiple criteria.

5.1 Description of bimodal systems with multiple criteria

Let us start with the following example:

$$\Sigma_{AB} \begin{cases} \text{mode 1}: & \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x, \quad \text{if } x \succeq 0 \\ \text{mode 2}: & \dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x, \quad \text{if } x \preceq 0. \end{cases} \tag{18}$$

Since smooth continuation in each mode is possible, that is, both A-matrices belong to $\mathcal{G}_0^2$, this system is well-posed. Then let us consider what is the original system Σ_O of this Σ_{AB}. So from mode 1, we can see that $C = [1 \;\; 0]$. However, in this case, $T_A = I_2$ and $T_B = [1 \;\; 0]$, and so (C, A) is observable but (C, B) is not observable. This implies that the system of the form (1) given by $C = [1 \;\; 0]$ is not equivalent to the system Σ_{AB}, and so is not the original system of Σ_{AB}.

How can this well-posed bimodal system be characterized by our framework? In fact, the original system for Σ_{AB} in (18) is given in terms of two criteria $Cx \geq (\leq)0$ and $\bar{C}x \geq (\leq)0$ where $C = [1 \;\; 0]$ and $\bar{C} = [0 \;\; 1]$ as follows.

$$\Sigma_O \begin{cases} \text{mode 1}: & \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} x, \quad \text{if } Cx \geq 0 \\ \text{mode 2}: & \dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x, \quad \text{if } \begin{bmatrix} C \\ \bar{C} \end{bmatrix} x \preceq 0. \end{cases} \tag{19}$$

In this section, we will generalize this example to consider the following bimodal system:

$$\Sigma_O \begin{cases} \text{mode 1}: & \dot{x} = Ax, \quad \text{if } Cx \succeq 0 \\ \text{mode 2}: & \dot{x} = Bx, \quad \text{if } Dx \preceq 0 \end{cases} \tag{20}$$

where

$$C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_p \end{bmatrix} \in \mathcal{R}^{p \times n}, \quad D = \begin{bmatrix} D_1 \\ D_2 \\ \vdots \\ D_s \end{bmatrix} \in \mathcal{R}^{s \times n},$$

and C_i^{T} and D_j^{T} are n-dimensional vectors. In this definition, note that it is at least required for well-posedness that $\{x \in \mathcal{R}^n \mid Cx \succeq 0\} \bigcup \{x \in \mathcal{R}^n \mid Dx \preceq 0\} = \mathcal{R}^n$.

First, we give an equivalent representation to the above system, as in the section 2. So we introduce the following rule matrices:

$$T_A \triangleq \begin{bmatrix} T_{A1} \\ T_{A2} \\ \vdots \\ T_{Ap} \end{bmatrix} \in \mathcal{R}^{m_A \times n}, \quad T_B \triangleq \begin{bmatrix} T_{B1} \\ T_{B2} \\ \vdots \\ T_{Bs} \end{bmatrix} \in \mathcal{R}^{m_B \times n} \tag{21}$$

where

$$T_{Ai} \triangleq \begin{bmatrix} C_i \\ C_i A \\ \vdots \\ C_i A^{h_i - 1} \end{bmatrix} \in \mathcal{R}^{h_i \times n}, \quad i = 1, 2, \cdots, p\ ,$$

$$T_{Bi} \triangleq \begin{bmatrix} D_i \\ D_i B \\ \vdots \\ D_i B^{k_i - 1} \end{bmatrix} \in \mathcal{R}^{k_i \times n}, \quad i = 1, 2, \cdots, s\ ,$$

and each h_i $(i = 1, 2, \cdots, p)$ is the maximum value of the rank such that $[T_{A1}^{\mathrm{T}}\ T_{A2}^{\mathrm{T}}\ \cdots T_{Ai}^{\mathrm{T}}]^{\mathrm{T}}$ has a row-full rank. Similarily for k_i. Note that $\sum_{i=1}^{p} h_i = m_A$ and $\sum_{i=1}^{s} k_i = m_B$, and then rank $T_A = m_A$ and rank $T_B = m_B$.

Using these rule matrixes, we consider the system given by

$$\Sigma_{AB} \begin{cases} \text{mode 1}: & \dot{x} = Ax, \quad \text{if } x \in \mathcal{S}_A^+ \\ \text{mode 2}: & \dot{x} = Bx, \quad \text{if } x \in \mathcal{S}_B^- \end{cases} \tag{22}$$

where $\mathcal{S}_N^+$ and $\mathcal{S}_N^-$ $(N = A, B)$ is defined by (5), where T_A and T_B are given by (21). Then we can prove that the system Σ_{AB} is equivalent to the original system Σ_O in a similar way to Lemma 2.2. Theorefore, we focus on the well-posedness of Σ_{AB}.

5.2 Observable case

We assume that the pairs (C, A) and (D, B) are observable, namely, $m_A = m_B = n$. Furthermore, we define the systems Σ_A and Σ_B given by (11) and (12), respectively, where T_A and T_B are given by (21). Then the first result of the multiple criteria case is obtained as follows.

Theorem 5.1 *Suppose that the pairs (C, A) and (D, B) are observable. Then the following statements are equivalent.*
(i) Σ_{AB} is well-posed.
(ii) $\mathcal{S}_A^+ \bigcup \mathcal{S}_B^- = \mathcal{R}^n$ and $\mathcal{S}_A^+ \bigcap \mathcal{S}_B^- = \{0\}$
(iii) $T_B T_A^{-1} \in \mathcal{L}_+^n$.
(iv) The following conditions are satisfied.
(a) $T_A B T_A^{-1} \in \mathcal{G}_0^n$.
(b) $D_i = [\underbrace{* \ \cdots \ *}_{\bar{k}_i}\ a\ 0\ \cdots\ 0]T_A$ for every $i \in \{1, 2, \cdots, s\}$, where $\bar{k}_i = k_1 + k_2 + \cdots + k_{i-1}$, $k_0 = 0$, and $a > 0$.
(v) The following conditions are satisfied.
(a) $T_B A T_B^{-1} \in \mathcal{G}_0^n$.
(b) $C_i = [\underbrace{* \ \cdots \ *}_{\bar{h}_i}\ b\ 0\ \cdots\ 0]T_B$ for every $i \in \{1, 2, \cdots, p\}$, where $\bar{h}_i = h_1 + h_2 + \cdots + h_{i-1}$, $h_0 = 0$, and $b > 0$.

(Proof) Noting that $T_A A T_A^{-1} \in \mathcal{G}_0$ and $T_B B T_B^{-1} \in \mathcal{G}_0$, the proof of (i) $\leftrightarrow$ (ii) $\leftrightarrow$ (iii) is given in a similar way to Theorem 4.1. Next, (iii)$\rightarrow$ (iv). From (iii), Σ_{AB} is equivalent to Σ_A. In addition, in the new coordinates $z \triangleq T_A x$, Σ_A is transformed into

$$\tilde{\Sigma}_A \begin{cases} \text{mode 1}: & \dot{z} = T_A A T_A^{-1} z, \quad \text{if } z \succeq 0 \\ \text{mode 2}: & \dot{z} = T_A B T_A^{-1} z, \quad \text{if } z \preceq 0. \end{cases}$$

Thus from Lemma 3.5, the well-posedness of $\tilde{\Sigma}_A$ implies (iv)(a). In addition, it follows from (iii) that $T_B = M T_A$ holds for some $M \in \mathcal{L}_+^n$. So letting m_{ij} be the (i, j) element of M, the relation $T_B = M T_A$ implies that, for $i \in \{1, 2, \cdots, s\}$,

$$D_i = [\ \underbrace{* \ \cdots \ *}_{\bar{k}_i}\ m_{\bar{k}_i+1, \bar{k}_i+1}\ 0\ \cdots 0]T_A.$$

Since $m_{\bar{k}_i+1, \bar{k}_i+1} > 0$, we have (iv)(b). (iv) $\rightarrow$ (iii) can be proven similar to (13) in Theorem 4.1. (iii)$\leftrightarrow$ (v) is proven in the same way as (iii)$\leftrightarrow$ (iv). $\square$

Remark 5.1 *We can also prove that (iv)(a) is equivalent to the condition that Σ_A is well-posed. Thus Σ_A is equivalent to Σ_{AB}, provided that (iv)(b) holds. For Σ_B, a similar result holds. Note that, however, this situation is a little different from the assertion in Theorem 4.1 in the sense that the condition (iv)(b) is required. In addition, from (iv)(b) or (v)(b) in Theorem 5.1, it follows that the condition $C_1 = D_1$ is necessary for the well-posedness of Σ_{AB}.*

Remark 5.2 *It follows from the proof of Theorem 5.1 that every well-posed bimodal system given by (22) can be transformed into the following canonical form:*

$$\tilde{\Sigma}_{AB} \left\{ \begin{array}{ll} \textit{mode } 1: & \dot{z} = \begin{bmatrix} \tilde{A}_{11} & 0 & \dots & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde{A}_{p1} & \dots & \tilde{A}_{p,p-1} & \tilde{A}_{pp} \end{bmatrix} z, \quad \textit{if } z \succeq 0 \\ \textit{mode } 2: & \dot{w} = \begin{bmatrix} \tilde{B}_{11} & 0 & \dots & 0 \\ \tilde{B}_{21} & \tilde{B}_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \tilde{B}_{s1} & \dots & \tilde{B}_{s,s-1} & \tilde{B}_{ss} \end{bmatrix} w, \quad \textit{if } w \preceq 0 \end{array} \right.$$

where $w = T_B T_A^{-1} z$, $T_B T_A^{-1} \in \mathcal{L}_+^n$ and

$$\tilde{A}_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ * & \dots & \dots & \dots & * \end{bmatrix} \in \mathcal{R}^{h_i \times h_i}, \quad \tilde{A}_{ij} = \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \\ * & \dots & \dots & * \end{bmatrix} \in \mathcal{R}^{h_i \times h_j}, \quad \textit{for } i > j,$$

$$\tilde{B}_{ii} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ * & \dots & \dots & \dots & * \end{bmatrix} \in \mathcal{R}^{k_i \times k_i}, \quad \tilde{B}_{ij} = \begin{bmatrix} 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \\ * & \dots & \dots & * \end{bmatrix} \in \mathcal{R}^{k_i \times k_j}, \quad \textit{for } i > j.$$

If $p = s = 1$, then this corresponds to the case of Theorem 4.1.

5.3 Unobservable case

Finally, we discuss the case that both pairs are unobservable. Let $\mathcal{T}_A$ be the set of $(n - m_A) \times n$ matrices such that $T \triangleq [T_A^{\mathrm{T}} \;\; \tilde{T}_A^{\mathrm{T}}]^{\mathrm{T}}$ is nonsingular, that is,

$$\mathcal{T}_A \triangleq \left\{ \tilde{T}_A \in \mathcal{R}^{(n-m_A)\times n} \mid T \text{ is nonsingular} \right\}. \tag{23}$$

Let also $\mathcal{T}_B$ be defined in the same way.

Theorem 5.2 *Suppose that the rank of T_A and T_B given by (21) are m_A and m_B, respectively, and $m_A \geq m_B$. Then the following statements are equivalent.*
(i) Σ_{AB} is well-posed.
(ii) The following conditions are satisfied.
(a) rank $[T_{A1}^{\mathrm{T}} \; T_{A2}^{\mathrm{T}} \; \cdots T_{Ai}^{\mathrm{T}}]^{\mathrm{T}} = m_B$ for some $i \in \{1, 2, \cdots, p\}$.
(b) $T_B = [M \;\; 0] T_A$ for some $M \in \mathcal{L}_+^{m_B}$.
(c) $(A - B)x = 0$ for all $x \in$ $KerT_B$.
(iii) The following conditions are satisfied.
(a) rank $[T_{A1}^{\mathrm{T}} \; T_{A2}^{\mathrm{T}} \; \cdots T_{Ai}^{\mathrm{T}}]^{\mathrm{T}} = m_B$ for some $i \in \{1, 2, \cdots, p\}$.
(b) $[I_{m_B} \;\; 0] T_A B = \Gamma [I_{m_B} \;\; 0] T_A$ for some $\Gamma \in \mathcal{G}_0^{m_B}$.
(c) $D_i = [\underbrace{* \; \cdots \; *}_{\bar{k}_i} \; a \; 0 \; \cdots \; 0] T_A$ for every $i \in \{1, 2, \cdots, s\}$, where $\bar{k}_i = k_1 + k_2 + \cdots + k_{i-1}$, $k_0 = 0$, and $a > 0$.
(d) $(A - B)x = 0$ for all $x \in$ $KerT_B$.

(Proof) (i) $\to$ (ii). From (i), it follows that $\mathcal{S}_A^+ \bigcup \mathcal{S}_B^- = \mathcal{R}^n$, which implies by Lemma 3.4 that T_A and T_B satisfy $T_B = [M \;\; 0]T_A$ for some $M \in \mathcal{L}_+^{m_B}$. In addition, let two new coordinates be defined by $z = [z_1^{\mathrm{T}} \;\; z_2^{\mathrm{T}}]^{\mathrm{T}} \triangleq Tx$ and $w = [w_1^{\mathrm{T}} \;\; w_2^{\mathrm{T}}]^{\mathrm{T}} \triangleq \hat{T}x$, where $T \triangleq [T_A^{\mathrm{T}} \;\; \tilde{T}_A^{\mathrm{T}}]^{\mathrm{T}}$ and $\hat{T} \triangleq [T_B^{\mathrm{T}} \;\; \tilde{T}_B^{\mathrm{T}}]^{\mathrm{T}}$ for any $\tilde{T}_A \in \mathcal{T}_A$ and any $\tilde{T}_B \in \mathcal{T}_B$. Then Σ_{AB} is transformed into

$$\tilde{\Sigma}_{AB} \left\{ \begin{array}{lll} \text{mode } 1: & \dot{z} = TAT^{-1}z, & \text{if } z_1 \succeq 0 \\ \text{mode } 2: & \dot{w} = \hat{T}B\hat{T}^{-1}w, & \text{if } w_1 \preceq 0. \end{array} \right. \tag{24}$$

Here TAT^{-1} and $\hat{T}B\hat{T}^{-1}$ are given by

$$TAT^{-1} = \left[\begin{array}{c|c} \tilde{A}_{11} & 0_{m_A, n-m_A} \\ \hline * & * \end{array} \right], \quad \tilde{A}_{11} \in \mathcal{G}_0^{m_A}, \tag{25}$$

$$\hat{T}B\hat{T}^{-1} = \left[\begin{array}{c|c} \tilde{B}_{11} & 0_{m_B, n-m_B} \\ \hline * & * \end{array} \right], \quad \tilde{B}_{11} \in \mathcal{G}_0^{m_B}. \tag{26}$$

Let z_1 be denoted by $z_1 \triangleq [z_{11}^{\mathrm{T}} \;\; z_{12}^{\mathrm{T}}]^{\mathrm{T}}$ where $z_{11} \in \mathcal{R}^{m_B}$ and $z_{12} \in \mathcal{R}^{m_A - m_B}$. So let us consider the case of $z_{11}(0) = 0$ and $z_{12}(0) \succ 0$, which also implies $w_1(0) = 0$ because $T_B = [M \;\; 0]T_A$. From (25) and (26), smooth continuation in each mode is possible from this state, and the solution in mode 2 is in the $n - m_B$ dimensional unobservable invariant subspace with $w_1(t) \equiv 0$, namely, $\mathrm{Ker}\, T_B$. Thus due to uniqueness of the solution, the solution in mode 1 must satisfy $z_{11}(t) = 0$ as far as $z_{12} \succeq 0$ holds. Hence (a) follows from this. Furthermore, the vector fields in both modes must be the same on $\mathrm{Ker}\, T_B \cap \{z \in \mathcal{R}^n \mid z_{12} \succeq 0\}$. From the property of linear systems, this implies that $Ax = Bx$ for all $x \in \mathrm{Ker}\, T_B$.

(ii) $\to$ (iii). We only have to show (b) and (c) in (iii). It follows from (ii)(b) that

$$[I_{m_B} \;\; 0]T_A B = M^{-1}T_B B = M^{-1}\tilde{B}_{11}T_B = M^{-1}\tilde{B}_{11}M[I_{m_B} \;\; 0]T_A \tag{27}$$

where $\tilde{B}_{11}$ is the same as (26). From Lemma 3.6, this implies $\Gamma \triangleq M^{-1}\tilde{B}_{11}M \in \mathcal{G}_0^{m_B}$, namely, (iii)(b). Moreover, (ii)$\to$(iii)(c) follows from (ii)(b) in the similar way to the proof (iii)$\to$(iv)(b) in Theorem 5.1.

(iii) $\to$ (i). First, we show $T_B = [M \;\; 0]T_A$ for some $M \in \mathcal{L}_+^{m_B}$. From (b) and (c) in (iii), it follows that

$$\begin{aligned} D_1 B &= a[1\ 0\ \cdots 0][I_{m_B} \;\; 0]T_A B \\ &= a[1\ 0\ \cdots 0]\Gamma[I_{m_B} \;\; 0]T_A \\ &= a[*\ \gamma_{12}\ 0\ \cdots\ 0][I_{m_B} \;\; 0]T_A \\ &= a[*\ \gamma_{12}\ 0\ \cdots\ 0]T_A. \end{aligned}$$

Thus by calculating similarly $D_1B^2, \cdots, D_1B^{k_1-1}, D_2B, \cdots, D_2B^{k_2-1}, \cdots$, and $D_sB^{k_s-1}$, we can derive $T_B = [M \;\; 0]T_A$ for some $M \in \mathcal{L}_+^{m_B}$. In addition, since $[M \;\; 0]T_A x \preceq 0 \leftrightarrow [I_{m_B} \;\; 0]T_A x \preceq 0$, Σ_{AB} is equivalent to

$$\Sigma_A \left\{ \begin{array}{lll} \text{mode } 1: & \dot{x} = Ax, & \text{if } T_A x \succeq 0 \\ \text{mode } 2: & \dot{x} = Bx, & \text{if } [I_{m_B} \;\; 0]T_A x \preceq 0. \end{array} \right. \tag{28}$$

In the new coordinates $z \triangleq [z_1^{\mathrm{T}} \;\; z_2^{\mathrm{T}}]^{\mathrm{T}} = Tx$, where $T \triangleq [T_A^{\mathrm{T}} \;\; \tilde{T}_A^{\mathrm{T}}]^{\mathrm{T}}$ for any $\tilde{T}_A \in \mathcal{T}_A$, Σ_A is transformed into

$$\tilde{\Sigma}_A \left\{ \begin{array}{lll} \text{mode } 1: & \dot{z} = TAT^{-1}z, & \text{if } z_1 \succeq 0 \\ \text{mode } 2: & \dot{z} = TBT^{-1}z, & \text{if } [I_{m_B} \;\; 0]z_1 \preceq 0. \end{array} \right. \tag{29}$$

Note here that TAT^{-1} is given by (25). On the other hand, it follows from (b) that, in mode 2,

$$\dot{z}_{11} = [I_{m_B} \;\; 0]T_A BT^{-1}z = \Gamma[I_{m_B} \;\; 0]z_1 = \Gamma z_{11}$$

where z_{11} is the m_B-dimensional vector defined by $z_1 = [z_{11}^{\mathrm{T}} \;\; z_{12}^{\mathrm{T}}]^{\mathrm{T}}$. Thus, smooth continuation in each mode is possible. Furthermore, when $z_{11}(0) = 0$ and $z_{12}(0) \succ 0$, the solutions in both modes are the same, since from (c) the vector fields in both modes are the same on $\mathrm{Ker}\, T_B$, i.e., the subspace given by $z_{11}(0) = 0$. Therefore, Σ_{AB} is well-posed. □

Remark 5.3 *When $p = 1$ and $s = 1$, Theorem 5.2 is reduced to Theorem 4.2, although $\mathcal{G}_0^{m_B}$ is replaced by $\mathcal{G}_+^{m_B}$ in (iii)(b). In the proof of Theorem 5.2, the condition (iii)(b) in Theorem 4.2 comes from the fact that $\tilde{B}_{11}$ in (27) is given by*

$$\tilde{B}_{11} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ * & \dots & \dots & \dots & * \end{bmatrix} \in \mathcal{G}_+^{m_B}.$$

Remark 5.4 *The conditions in Theorem 5.2 can be checked as described in Remark 4.3. Namely, the conditions (iii)(b) and (d) are replaced by*

$$[I_{m_B}\ \ 0]TBT^{-1}\begin{bmatrix} I_{m_B} \\ 0 \end{bmatrix} \in \mathcal{G}_0^{m_B} \tag{30}$$

and

$$[0\ \ I_{n-m_B}]T(A-B)T^{-1}\begin{bmatrix} 0 \\ I_{n-m_B} \end{bmatrix} = 0, \tag{31}$$

respectively, where $T = [T_A^{\mathrm{T}}\ \tilde{T}_A^{\mathrm{T}}]^{\mathrm{T}}$ *for some* $\tilde{T}_A \in \mathcal{T}_A$.

Example 5.1 *Let us check the well-posedness of the following simple example:*

$$\Sigma_{AB}\begin{cases} \textit{mode } 1: \quad \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x, \quad \textit{if } Cx \succeq 0 \\ \textit{mode } 2: \quad \dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} x, \quad \textit{if } Dx \leq 0 \end{cases}$$

where

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = D_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Then we obtain $m_A = 3$ *and* $m_B = 2$ *from*

$$T_A = \begin{bmatrix} C_1 \\ C_1A \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad T_B = \begin{bmatrix} D_1 \\ D_1B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Thus (ii)(a) is satisfied. From $T_B = [I_2\ \ 0]T_A$*, we obtain (ii)(b). In addition, noting* $T_AAT_A^{-1} = A$ *and* $T_ABT_A^{-1} = B$*, (c) is satisfied. Therefore, this system is well-posed, although* (C, A) *is observable and* (D, B) *is not observable.*

6 Extensions

In this section, we extend several results for the case of bimodal systems given by (1) to the case of multi-modal systems with multiple criteria and multi-modal systems based on affine-type inequalities. We will only discuss the observable case, as a first step to investigate to what extent our framework can be generalized, although the unobservable case may be extended in a similar way.

6.1 Multi-modal systems with multiple criteria

We here consider multi-modal systems with multiple criteria. For any matrix $C = [C_1^{\mathrm{T}}\ \ C_2^{\mathrm{T}}\ \cdots\ C_r^{\mathrm{T}}]^{\mathrm{T}} \in \mathcal{R}^{r\times n}$ where $r \leq n$, let the criterion vector be $y = [y_1\ y_2\ \cdots\ y_r]^{\mathrm{T}} = Cx$. We assume throughout that there exists no constant k such that $C_i = kC_j$ for each $i, j \in \{1, 2, \cdots, r\}$. Let $\mathcal{I} \subset \{1, 2, \cdots, r\}$ be the index set satisfying $y_i \geq 0$ for $i \in \mathcal{I}$ and $y_i \leq 0$ for $i \notin \mathcal{I}$. The index set $\mathcal{I}$ represents the mode (location) of the system. Note that there are 2^r possible choices for the index set $\mathcal{I}$, and so there exist 2^r modes. Moreover, let $\mathcal{C}_I$ be a subset of R^n defined by

$$\mathcal{C}_I \triangleq \{\ x \in \mathcal{R}^n \mid y_i \geq 0 \text{ for } i \in \mathcal{I},\quad y_i \leq 0 \text{ for } i \notin \mathcal{I}\ \}.$$

By numbering the index sets $\mathcal{I}$ from 1 to 2^r, we use the number $i \in \{1, 2, \cdots, 2^r\}$ in place of $\mathcal{I}$ to express the mode.

Then we consider the original 2^r-modal system Σ_O given by

$$\Sigma_O \begin{cases} \text{mode } 1: & \dot{x} = A_1x, & \text{if } x \in \mathcal{C}_1 \\ \text{mode } 2: & \dot{x} = A_2x, & \text{if } x \in \mathcal{C}_2 \\ \quad\vdots & \quad\vdots & \quad\vdots \\ \text{mode } 2^r: & \dot{x} = A_{2^r}x, & \text{if } x \in \mathcal{C}_{2^r} \end{cases} \tag{32}$$

where $x \in \mathcal{R}^n$. For example, for $r = 2$, we have the 4 modal system given by

$$\begin{cases} \text{mode } 1: & \dot{x} = A_1x, & x \in \mathcal{C}_1 = \{x \in R^n \mid y_1 \geq 0,\ y_2 \geq 0\} \\ \text{mode } 2: & \dot{x} = A_2x, & x \in \mathcal{C}_2 = \{x \in R^n \mid y_1 \geq 0,\ y_2 \leq 0\} \\ \text{mode } 3: & \dot{x} = A_3x, & x \in \mathcal{C}_3 = \{x \in R^n \mid y_1 \leq 0,\ y_2 \geq 0\} \\ \text{mode } 4: & \dot{x} = A_4x, & x \in \mathcal{C}_4 = \{x \in R^n \mid y_1 \leq 0,\ y_2 \leq 0\}. \end{cases}$$

In addition, we assume that every pair (C_i, A_k) $(i = 1, 2, \cdots, r;\ k = 1, 2, \cdots, 2^r)$ is observable. So the rule matrices

$$T^i_{A_k} \triangleq \begin{bmatrix} C_i \\ C_i A_k \\ \vdots \\ C_i A_k^{n-1} \end{bmatrix} \in \mathcal{R}^{n\times n}$$

are all nonsingular. So let $\mathcal{S}_I$ be a subset of $\mathcal{R}^n$ defined by

$$\mathcal{S}_I \triangleq \{\ x \in \mathcal{R}^n \mid T^i_{A_I} x \succeq 0 \text{ for } i \in \mathcal{I},\ \ T^i_{A_I} x \preceq 0 \text{ for } i \notin \mathcal{I}\ \}$$

Using the sets $\mathcal{S}_I$, we also define the 2^r-modal system Σ_{A_0} as follows:

$$\Sigma_{A_0} \begin{cases} \text{mode } 1: & \dot{x} = A_1 x, & \text{if } x \in \mathcal{S}_1 \\ \text{mode } 2: & \dot{x} = A_2 x, & \text{if } x \in \mathcal{S}_2 \\ \quad\vdots & \quad\vdots & \quad\vdots \\ \text{mode } 2^r: & \dot{x} = A_{2^r} x, & \text{if } x \in \mathcal{S}_{2^r}. \end{cases} \tag{33}$$

For a vector $x, y \in \mathcal{R}^n$, the notation $x \geq y$ expresses $x_i \geq y_i$ for all i. Similarly for the other notation $\leq$, $>$, and $<$. For a closed convex polyhedral cone $\mathcal{C} \triangleq \{\ x \in \mathcal{R}^n \mid Sx \geq 0\ \}$ where S is an $m \times n$ real matrix, let $\mathrm{int}\,\mathcal{C}$ be the interior of $\mathcal{C}$ and let $\partial\mathcal{C}$ be the boundary of $\mathcal{C}$.

Then the following result is a natural extension to that for bimodal systems.

Theorem 6.1 *Suppose that every pair (C_i, A_j) $(i = 1, 2, \cdots, r;\ j = 1, 2, \cdots, 2^r)$ is observable. Then the following statements are equivalent.*
(i) Σ_O is well-posed.
(ii) Σ_{A_0} is well-posed.
(iii) $\bigcup_{j=1}^{2^r} \mathcal{S}_j = \mathcal{R}^n$ and $\mathcal{S}_j \bigcap \mathcal{S}_k = \{0\}$ for all $j, k (\neq j) \in \{1, 2, \cdots, 2^r\}$.

(Proof) (i)$\rightarrow$(ii) can be proven in the same way as Lemma 2.2. (ii)$\rightarrow$(iii). It obviously follows that $\bigcup_{j=1}^{2^r} \mathcal{S}_j = \mathcal{R}^n$. In order to prove the latter part of (iii), we assume that there exists some j and $k(\neq j)$ such that $\mathcal{S}_j \bigcap \mathcal{S}_k \neq \{0\}$ and $\mathcal{S}_j \bigcap \mathcal{S}_k \neq \emptyset$. So let $x_*(\neq 0)$ be an element of $\mathcal{S}_j \bigcap \mathcal{S}_k$. Then for some $\varepsilon > 0$, the solution in mode j from the initial state x_* satisfies $x(t) \in \mathrm{int}\,\mathcal{C}_j$ for all $t \in (0, \varepsilon]$, while the solution in mode k from x_* satisfy $x(t) \in \mathrm{int}\,\mathcal{C}_k$. This implies that the solution is not unique, which is in contradiction with (ii). Hence the latter part of (iii) holds. (iii) $\rightarrow$ (ii) is obvious. $\square$

From Theorem 6.1, it turns out that the well-posedness of Σ_O is characterized by condition (iii). When is condition (iii) satisfied? We are not able to interpret condition (iii) in terms of some simple algebraic relation between the matrices $T^i_{A_j}$ as in the case of bimodal systems. However, we give below an algorithm to check condition (iii).

For brevity, we discuss the case of $r = 2$, namely, 4-modal systems. The case $r \geq 3$ can be treated in a similar way. Consider the following situation:

$$\begin{aligned} \mathcal{S}_1 &= \{\ x \in \mathcal{R}^n \mid T^1_{A_1} \succeq 0,\ \ T^2_{A_1} \succeq 0\ \}, \\ \mathcal{S}_2 &= \{\ x \in \mathcal{R}^n \mid T^1_{A_2} \succeq 0,\ \ T^2_{A_2} \preceq 0\ \}, \\ \mathcal{S}_3 &= \{\ x \in \mathcal{R}^n \mid T^1_{A_3} \preceq 0,\ \ T^2_{A_3} \succeq 0\ \}, \\ \mathcal{S}_4 &= \{\ x \in \mathcal{R}^n \mid T^1_{A_4} \preceq 0,\ \ T^2_{A_4} \preceq 0\ \}. \end{aligned}$$

In order to clarify our idea, at first, we discuss the necessity of condition (iii) in Theorem 6.1.

Since $\bigcup_{j=1} \mathcal{C}_j = \mathcal{R}^n$ and $\mathcal{C}_j \bigcap \mathcal{C}_k = \{0\}$ or $= \partial\mathcal{C}_j \bigcap \partial\mathcal{C}_k$, we do not need to check the cases $C_1 x \neq 0$ and $C_2 x \neq 0$. We only have to consider each case $C_1 x = 0$ and $C_2 x = 0$.

So suppose $C_1 x = 0$. We consider the set defined by

$$\mathcal{S}^{(1)}_j \triangleq \mathcal{S}_j \bigcap \{\ x \in \mathcal{R}^n \mid C_1 x = 0\ \}, \quad j = 1, 2, 3, 4. \tag{34}$$

Then note that the set $\mathcal{S}^{(1)}_j$ can be expressed as

$$\mathcal{S}^{(1)}_j = \{\ z \in \mathcal{R}^{n-1} \mid M^1_j z \succeq 0,\ \ M^2_j z \succeq 0\ \}, \quad j = 1, 2, 3, 4 \tag{35}$$

where M^i_j is the $(n-1) \times (n-1)$ matrix derived from $T^i_{A_j} x \succeq (\preceq) 0$ with $C_1 x = 0$. In fact, M^i_j is derived as follows. In the new coordinates $\bar{z} = [w\ \ z^{\mathrm{T}}]^{\mathrm{T}} = Tx$ where $T = [C_1^{\mathrm{T}}\ \ \tilde{T}^{\mathrm{T}}]^{\mathrm{T}}$ is nonsingular for some $\tilde{T} \in \mathcal{R}^{n-1\times n}$, and $w = C_1 x$ and $z = \tilde{T} x$, we have $T^i_{A_j} x = T^i_{A_j} T^{-1} \bar{z} \succeq (\preceq) 0$. So when $w = 0$, this yields

$$T^i_{A_j} T^{-1} \begin{bmatrix} 0_{1,n-1} \\ I_{n-1} \end{bmatrix} z \succeq (\preceq) 0. \tag{36}$$

Then by applying Lemma 3.1 to (36), we can derive a $(n-1)\times(n-1)$ matrix M_j^i in (35).

Thus it is required for condition (iii) in Theorem 6.1 to hold that

$$\bigcup_{j=1}^{4} \mathcal{S}_j^{(1)} = \mathcal{R}^{n-1} \text{ and } \mathcal{S}_j^{(1)} \bigcap \mathcal{S}_k^{(1)} = \{0\}, \quad \forall j,k \in \{1,2,3,4\}. \tag{37}$$

So letting s_j^i be the 1st row vector of M_j^i, we define

$$\mathcal{C}_j^{(1)} \triangleq \{ \ z \in \mathcal{R}^{n-1} \mid s_j^1 z \geq 0, \ s_j^2 z \geq 0 \ \}, \quad j = 1,2,3,4. \tag{38}$$

Then we can see that the following conditions are necessary for (37).

$$\bigcup_{j=1}^{4} \mathcal{C}_j^{(1)} = \mathcal{R}^{n-1}, \tag{39}$$

$$\mathcal{C}_j^{(1)} \bigcap \mathcal{C}_k^{(1)} = \{0\} \text{ or } = \partial\mathcal{C}_j^{(1)} \bigcap \partial\mathcal{C}_k^{(1)}, \quad \forall j,k \in \{1,2,3,4\}. \tag{40}$$

The condition (39) is equivalent to

$$\bigcap_{j=1}^{4} \{ \ z \in \mathcal{R}^{n-1} \mid s_j^1 z < 0, \quad s_j^2 z < 0 \ \} = \emptyset$$

and also is equivalent to

$$\bigcap_{j=1}^{4} \mathcal{Q}_j^{i_j} = \emptyset, \quad \forall i_j \in \{1,2\} \tag{41}$$

where $\mathcal{Q}_j^i \triangleq \{ \ z \in \mathcal{R}^{n-1} \mid s_j^i z < 0 \ \}$. Since this last condition can be written as $\bigcap_{j=1}^4 \mathcal{Q}_j^{i_j} = \{ \ z \in \mathcal{R}^{n-1} \mid Sz < 0 \ \} = \emptyset$ where $S = \left[\ (s_1^{i_1})^{\mathrm{T}} \quad (s_2^{i_2})^{\mathrm{T}} \quad (s_3^{i_3})^{\mathrm{T}} \quad (s_4^{i_4})^{\mathrm{T}} \ \right]^{\mathrm{T}}$, it can be checked by solving the following linear programming: min λ subject to $Sz \leq \lambda e$ or min λ subject to $Sx \leq \lambda e$ and $-e \leq x \leq e$, where e is some vector with all elements positive. Letting λ_* be an optimal solution, if $\lambda_* = 0$, then $\bigcap_{j=1}^4 \mathcal{Q}_j^{i_j} = \emptyset$, and if $\lambda_* < 0$, then $\bigcap_{j=1}^4 \mathcal{Q}_j^{i_j} \neq \emptyset$.

Concerning the condition (40), on the other hand, the following lemma is obtained.

Lemma 6.1 *Let $\mathcal{S}_i$ be a set defined by $\mathcal{S}_i \triangleq \{ \ x \in \mathcal{R}^n \mid S_i x \geq 0 \ \}$ $(i = 1,2)$ where S_i is an $m_i \times n$ real matrix. Then the following statements are equivalent.*
(i) $\mathcal{S}_1 \bigcap \mathcal{S}_2 = \{0\}$ or $= \partial\mathcal{S}_1 \bigcap \partial\mathcal{S}_2$.
(ii) $int\,\mathcal{S}_1 \bigcap int\,\mathcal{S}_2 = \emptyset$, i.e., $\{ \ x \in \mathcal{R}^n \mid S_1 x > 0 \ \} \bigcap \{ \ x \in \mathcal{R}^n \mid S_2 x > 0 \ \} = \emptyset$.

(Proof) (i)→ (ii) is trivial. (ii)→(i). We only have to show that if (ii) holds, then there also exist no elements in the intersection of the boundary of a closed convex polyhedral cone and the interior of another cone. Let s_{11} be the 1st row vector of S_1 and let $\bar{S}_1$ be the matrix such that $S_1 = [s_{11}^{\mathrm{T}} \quad \bar{S}_1^{\mathrm{T}}]^{\mathrm{T}}$. Then we will show $\mathcal{N} = \emptyset$ where

$$\mathcal{N} \triangleq \{ \ x \in \mathcal{R}^n \mid s_{11} x = 0, \ \bar{S}_1 x > 0 \ \} \bigcap \mathrm{int}\,\mathcal{S}_2.$$

Assume $\mathcal{N} \neq \emptyset$, and let x_* be an element of $\mathcal{N}$. Note that an element of $\mathcal{S}_1$ can be expressed by $x = \sum_{i=1}^{m_1} \alpha_i u_i +$ {an element of $\mathrm{Ker}\,S_1$}, where $\alpha_i \geq 0$ and $S_1 u_i = e_i$ (the ith element of e_i is 1 and the others are 0). So we denote x_* by $x_* = \sum_{i=2}^{m_1} \alpha_i u_i +$ {an element of $\mathrm{Ker}\,S_1$} where $\alpha_i > 0$. Now for $\tilde{x}_* = x_* + \varepsilon u_1$ where $\varepsilon > 0$ is sufficiently small, we have $\tilde{x}_* \in \mathrm{int}\,\mathcal{S}_1 \bigcap \mathrm{int}\,\mathcal{S}_2$, which implies that (ii) is not true. Hence, it follows that if (ii) is true, then $\mathcal{N} = \emptyset$. For any other boundary of $\mathcal{S}_i$, similar discussion holds. This completes the proof. □

Thus by Lemma 6.1, the condition (40) can be also checked using the linear programming.

Now suppose that the conditions (39) and (40) are satisfied. Then for every (j,k) satisfying $\mathcal{C}_j^{(1)} \bigcap \mathcal{C}_k^{(1)} = \partial\mathcal{C}_j^{(1)} \bigcap \partial\mathcal{C}_k^{(1)}$, we consider each case of $s_i^1 z = 0$ and $s_i^2 z = 0$ for $i = j,k$. Thus for every case, a discussion similar to the case $C_1 x = 0$ will be repeated on $\mathcal{R}^{n-2}$.

If condition (iii) is satisfied, this procedure will be repeated in every case until the corresponding set to $\mathcal{S}_j^{(1)}$ is given on $\mathcal{R}$. If not, the condition corresponding to (37) will not be satisfied at some step. The case $C_2 x = 0$ is similar.

Based on the above discussion, an algorithm for checking condition (iii) is given as follows.

Step 1: For $C_1 x = 0$, derive $\mathcal{S}_j^{(1)}$ and $\mathcal{C}_j^{(1)}$ $(j = 1,2,3,4)$.

Step 2: Check whether (39) (i.e., (41)) and (40) (i.e., the condition corresponding to (ii) in Lemma 6.1) for $\mathcal{C}_j^{(1)}$ are true or not. If both are true, then go to the next step. If not, we conclude that condition (iii) is not satisfied.
Step 3: For every (j, k) satisfying $\mathcal{C}_j^{(1)} \bigcap \mathcal{C}_k^{(1)} = \partial\mathcal{C}_j^{(1)} \bigcap \partial\mathcal{C}_k^{(1)}$, we consider $s_h^i z = 0$ for each $h = j, k$ and each $i \in \{1, 2\}$. For each case, the procedure similar to step 1 and step 2 will be repeated until either of the following two conditions is satisfied: (a) the condition corresponding to that in step 2 is not satisfied at some step, or (b) the set corresponding to $\mathcal{S}_j^{(1)}$ is given on $\mathcal{R}$ and the condition corresponding to (37) is satisfied. If (a) is true, then we conclude that condition (iii) is not satisfied. If (b) is true, go to step 4.
Step 4: For the case $C_2x = 0$, follow the same procedure. If the situation corresponding to (b) at step 3 holds, we conclude that condition (iii) is satisfied. Otherwise, it is not satisfied.

Finally, we give a simple example to capture the idea of the proposed algorithm.

Example 6.1 *Consider the 4-modal system given by*

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

and, $C_1 = [1\ 0\ 0]$ *and* $C_2 = [0\ 1\ 0]$*. Then we have*

$$\begin{aligned}
T_{A_1}^1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, & T_{A_1}^2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \\
T_{A_2}^1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, & T_{A_2}^2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}, \\
T_{A_3}^1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & -1 & -1 \end{bmatrix}, & T_{A_3}^2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \\
T_{A_4}^1 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}, & T_{A_4}^2 &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}.
\end{aligned}$$

Now we consider the case of $C_1x = 0$*. At step 1, we have*

$$M_1^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_2^1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad M_2^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$M_3^1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad M_3^2 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad M_4^1 = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad M_4^2 = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}.$$

and

$$s_1^1 = [1 \quad 0], \quad s_1^2 = [1 \quad 0], \quad s_2^1 = [-1 \quad 0], \quad s_2^2 = [1 \quad 0], \quad s_3^1 = [-1 \quad 0], \quad s_3^2 = [-1 \quad 0], \quad s_4^1 = [1 \quad 0], \quad s_4^2 = [-1 \quad 0]$$

At step 2, it can be easily verified that (39) and (40) are satisfied. At step 3, we only have to consider the case of $[1 \quad 0]z = 0$*. So from* M_j^i*, we have a new* M_j^i *as follows:*

$$M_1^1 = 1, \quad M_1^2 = 1, \quad M_2^1 = 1, \quad M_2^2 = -1, \quad M_3^1 = -1, \quad M_3^2 = -1, \quad M_4^1 = -1, \quad M_4^2 = 1.$$

Then we can see that the condition corresponding to (37) is satisfied. In the case of $C_2x = 0$*, it is also verified that every condition is satisfied. Thus we conclude that condition (iii) of Theorem 6.1 is satisfied for this system, and so the system is well-posed.*

6.2 Multi-modal systems with affine inequalities

We here start with the bimodal system given by

$$\Sigma_O(\alpha) \begin{cases} \text{mode } 1: & \dot{x} = Ax, \quad \text{if } Cx \geq \alpha \\ \text{mode } 2: & \dot{x} = Bx, \quad \text{if } Cx \leq \alpha \end{cases} \tag{42}$$

where $x \in \mathcal{R}^n$, $C \in \mathcal{R}^{1\times n}$, and $\alpha \in R$ is any given constant. Note that the inequality constraint is affine.

This system is equivalent to the following system:

$$\Sigma_{AB}(\alpha)\left\{\begin{array}{lll}\text{mode 1}: & \dot{x}=Ax, & \text{if } x\in\bar{\mathcal{S}}_A^+(\alpha)\\ \text{mode 2}: & \dot{x}=Bx, & \text{if } x\in\bar{\mathcal{S}}_B^-(\alpha)\end{array}\right. \tag{43}$$

where

$$\bar{\mathcal{S}}_A^+=\{x\in\mathcal{R}^n \mid \bar{T}_Ax\succeq\bar{\alpha}\},\quad \bar{\mathcal{S}}_B^-=\{x\in\mathcal{R}^n \mid \bar{T}_Bx\preceq\bar{\alpha}\},$$
$$\bar{T}_A=\left[\begin{array}{c}T_A\\ CA^n\end{array}\right],\quad \bar{T}_B=\left[\begin{array}{c}T_B\\ CB^n\end{array}\right]$$

and $\bar{\alpha}=[\alpha\ \ 0\ \cdots\ 0]^{\mathrm{T}}\in\mathcal{R}^{n+1}$, and T_A and T_B are defined by (10). In fact, if $Cx(t)\geq\alpha$ in $\Sigma_O(\alpha)$, then $\bar{T}_Ax(t)\succeq\bar{\alpha}$. Conversely, if $\bar{T}_Ax(t)\succ\bar{\alpha}$, then $Cx(t)\geq\alpha$, and if $\bar{T}_Ax(t)=\bar{\alpha}$, then $Cx\equiv\alpha$. The same argument holds in mode 2. Thus each mode in $\Sigma_O(\alpha)$ is identified with each mode in $\Sigma_{AB}(\alpha)$.

Denote by $(M)_{(i,j)}$ the (i,j) element of a matrix M.

Theorem 6.2 *Suppose that both pairs (C,A) and (C,B) are observable. Then for any given constant $\alpha\in\mathcal{R}$, the following statements are equivalent.*
(i) $\Sigma_O(\alpha)$ is well-posed.
(ii) $\Sigma_{AB}(\alpha)$ is well-posed.
(iii) The following conditions are satisfied.
(a) $\Sigma_{AB}(0)$ is well-posed.
(b) (b1) $(T_AAT_A^{-1})_{(n,1)}\alpha\geq 0$ and $(T_BBT_B^{-1})_{(n,1)}\alpha>0$, or (b2) $(T_AAT_A^{-1})_{(n,1)}\alpha<0$ and $(T_BBT_B^{-1})_{(n,1)}\alpha\leq 0$, or (b3) $(T_AAT_A^{-1})_{(n,1)}\alpha=0$ and $(T_BBT_B^{-1})_{(n,1)}\alpha=0$.

(Proof) (i) $\leftrightarrow$ (ii) has already been proven. (ii) $\rightarrow$ (iii). From (ii), it follows that $\bar{\mathcal{S}}_A^+ + \bar{\mathcal{S}}_B^- = \mathcal{R}^n$. In the same way as in the proof of Lemma 3.4, we get $T_B=MT_A$ for some $M\in\mathcal{L}_+^n$. Thus from Theorem 4.1, this implies (iii)(a). Then in the two new coordinates $z\stackrel{\triangle}{=}T_Ax-\tilde{\alpha}$ and $w\stackrel{\triangle}{=}T_Bx-\tilde{\alpha}$, where $\tilde{\alpha}=[\alpha\ 0\ \cdots 0]^{\mathrm{T}}\in\mathcal{R}^n$, $\Sigma_{AB}(\alpha)$ is described by

$$\tilde{\Sigma}_{AB}(\alpha)\left\{\begin{array}{lll}\text{mode 1}: & \dot{z}=T_AAT_A^{-1}z+T_AAT_A^{-1}\tilde{\alpha}, & \text{if } \left[\begin{array}{c}z\\ CA^nT_A^{-1}(z+\tilde{\alpha})\end{array}\right]\succeq 0\\ \text{mode 2}: & \dot{w}=T_BBT_B^{-1}w+T_BBT_B^{-1}\tilde{\alpha}, & \text{if } \left[\begin{array}{c}w\\ CB^nT_B^{-1}(w+\tilde{\alpha})\end{array}\right]\preceq 0.\end{array}\right. \tag{44}$$

Note here that $w=MT_Ax-\tilde{\alpha}=M(T_Ax-M^{-1}\tilde{\alpha})=Mz$ because of $M^{-1}\tilde{\alpha}=\tilde{\alpha}$.

Now let us consider $z(0)=0$, where both modes may be admissible. For mode 1, if $CA^nT_A^{-1}\tilde{\alpha}\succeq(\prec)0$, smooth continuation is (not) possible, while for mode 2, if $CB^nT_B^{-1}\tilde{\alpha}\preceq(\succ)0$, smooth continuation is (not) possible. Since the system has a unique solution, there is no situation where smooth continuation in both modes is possible at the same time, except for $z(t)=w(t)\equiv 0$. Hence (iii)(b) holds.

(iii)$\rightarrow$ (ii). (iii)(a) implies $w=Mz$ for some $M\in\mathcal{L}_+^n$, where z and w are defined above. Thus we obtain (44). In each case of $z(0)\succ 0$ and $w(0)\prec 0$, smooth continuation in only one of the two modes is possible. In addition, when $z(0)=0$, (iii)(b) guarantees smooth continuation in only one of the two modes or $z(t)=w(t)\equiv 0$. This implies (ii). □

This theorem asserts that the well-posedness of $\Sigma_O(\alpha)$ for all $\alpha\in\mathcal{R}$ is characterized by that of $\Sigma_O(0)$, provided that (iii)(b) holds. In (iii)(b), (b1) implies that, whenever $z(0)=0$, smooth continuation in mode 1 is possible, while not in mode 2. (b2) implies the converse situation of (b1). In addition, (b3) corresponds to the case that smooth continuation in both modes is possible and their solutions are the same.

Remark 6.1 *If we assume $\alpha\neq 0$ in Theorem 6.1, we can obviously remove α from the condition (iii)(b), that is, (b1) $(T_AAT_A^{-1})_{(n,1)}\geq 0$ and $(T_BBT_B^{-1})_{(n,1)}>0$, or (b2) $(T_AAT_A^{-1})_{(n,1)}<0$ and $(T_BBT_B^{-1})_{(n,1)}\leq 0$, or (b3) $(T_AAT_A^{-1})_{(n,1)}=0$ and $(T_BBT_B^{-1})_{(n,1)}=0$. Thus in the case of affine inequalities, the well-posedness of the system does not depend on α.*

Based on the above result, we consider the well-posedness of the following r-modal system:

$$\Sigma_O(\alpha_1,\alpha_2,\cdots,\alpha_{r-1})\left\{\begin{array}{lll}\text{mode 1}: & \dot{x}=A_1x, & \text{if } x\in\mathcal{C}_1\\ \text{mode 2}: & \dot{x}=A_2x, & \text{if } x\in\mathcal{C}_2\\ \vdots & \vdots & \vdots\\ \text{mode } r: & \dot{x}=A_rx, & \text{if } x\in\mathcal{C}_r\end{array}\right. \tag{45}$$

where $x\in\mathcal{R}^n$, $\alpha_1>\alpha_2>\ \cdots\ >\alpha_{r-1}$ are any real numbers, and

$$\begin{array}{rcl}\mathcal{C}_1 &=& \{x\in\mathcal{R}^n \mid Cx\geq\alpha_1\},\\ \mathcal{C}_i &=& \{x\in\mathcal{R}^n \mid \alpha_{i-1}\geq Cx\geq\alpha_i\},\quad i\in\{2,\cdots,r-1\},\\ \mathcal{C}_r &=& \{x\in\mathcal{R}^n \mid \alpha_{r-1}\geq Cx\},\end{array}$$

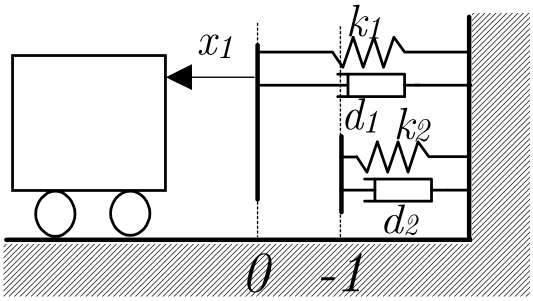

Figure 3: 3-modal system

and $C \in \mathcal{R}^{1\times n}$. Let us also introduce the bimodal system given by

$$\Sigma_O(A_i, A_{i+1}, \alpha_i) \begin{cases} \text{mode } i: & \dot{x} = A_i x, \quad \text{if } x \in \{x \in \mathcal{R}^n \mid Cx \geq \alpha_i\} \\ \text{mode } i+1: & \dot{x} = A_{i+1} x, \quad \text{if } x \in \{x \in \mathcal{R}^n \mid Cx \leq \alpha_i\} \end{cases} \tag{46}$$

for $i \in \{1, 2, \cdots, r-1\}$. Then the following fact will be useful for determining the well-posedness of $\Sigma_O(\alpha_1, \alpha_2, \cdots, \alpha_{r-1})$.

Theorem 6.3 *The multi-modal system $\Sigma_O(\alpha_1, \alpha_2, \cdots, \alpha_{r-1})$ is well-posed if and only if the bimodal system $\Sigma_O(A_i, A_{i+1}, \alpha_i)$ is well-posed for all $i \in \{1, 2, \cdots, r-1\}$.*

(Proof) $\Sigma_O(\alpha_1, \alpha_2, \cdots, \alpha_{r-1})$ is well-posed if and only if the bimodal system given by

$$\tilde{\Sigma}_O(A_i, A_{i+1}, \alpha_i) \begin{cases} \text{mode } i: & \dot{x} = A_i x, \quad \text{if } x \in \{x \in \mathcal{R}^n \mid \alpha_{i-1} > Cx \geq \alpha_i\} \\ \text{mode } i+1: & \dot{x} = A_{i+1} x, \quad \text{if } x \in \{x \in \mathcal{R}^n \mid \alpha_i \geq Cx > \alpha_{i+1}\} \end{cases} \tag{47}$$

is well-posed for all $i \in \{1, 2, \cdots, r-1\}$, where $\alpha_0 = \infty$ and $\alpha_r = -\infty$. In addition, for each $i \in \{1, 2, \cdots, r-1\}$, $\tilde{\Sigma}_O(A_i, A_{i+1}, \alpha_i)$ is well-posed if and only if $\Sigma_O(A_i, A_{i+1}, \alpha_i)$ is well-posed since we only have to consider smooth continuation in the case $Cx = \alpha_i$. □

Using Theorem 6.2, we can determine whether the multi-modal system $\Sigma_O(\alpha_1, \alpha_2, \cdots, \alpha_{r-1})$ is well-posed or not, as shown in the example below.

Example 6.2 *Consider the physical system in Figure 3. Assume that $k_1 \geq 0$ and $k_2 \geq 0$. Set $\alpha_1 = 0$ and $\alpha_2 = -1$. The dynamics of the system is given by*

$$\Sigma_O(0, -1) \begin{cases} \textit{mode } 1: & \dot{x} = A_1 x, \quad \textit{if } \; x \in \{x \in \mathcal{R}^n \mid Cx \geq 0\} \\ \textit{mode } 2: & \dot{x} = A_2 x, \quad \textit{if } \; x \in \{x \in \mathcal{R}^n \mid 0 \geq Cx \geq -1\} \\ \textit{mode } 3: & \dot{x} = A_3 x, \quad \textit{if } \; x \in \{x \in \mathcal{R}^n \mid -1 \geq Cx\} \end{cases}$$

where $x = [x_1 \;\; x_2]^{\mathrm{T}}$, $C = [1 \;\; 0]$, and

$$A_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -k_1 & -d_1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ -k_1 - k_2 & -d_1 - d_2 \end{bmatrix}.$$

Then for $\Sigma_O(A_i, A_{i+1}, \alpha_i)$ $(i = 1, 2)$ we obtain $T_{A_1} = T_{A_2} = T_{A_3} = I_2$ where T_{A_i} $(i = 1, 2, 3)$ is the rule matrix. Thus for $i = 1$, since $\alpha_1 = 0$ and $T_{A_1} T_{A_2}^{-1} \in \mathcal{L}_+^2$, $\Sigma_O(A_1, A_2, 0)$ is well-posed. For $i = 2$, on the other hand, $T_{A_2} T_{A_3}^{-1} \in \mathcal{L}_+^2$ implies (ii)(a) in Theorem 6.2. In addition, we have $(T_{A_2} A_2 T_{A_2})_{(2,1)} \alpha_2 = k_1 \geq 0$ and $(T_{A_3} A_3 T_{A_3})_{(2,1)} \alpha_2 = k_1 + k_2 \geq 0$, which implies (ii)(b) for any $k_1 \geq 0$ and $k_2 \geq 0$. Thus $\Sigma_O(A_2, A_3, -1)$ is also well-posed. Hence from Theorem 6.3, the 3-modal system $\Sigma_O(0, -1)$ is well-posed for any $k_1 \geq 0$ and $k_2 \geq 0$. From this, it turns out that the well-posedness does not depend on the value of d_1 and d_2.

Remark 6.2 *Consider the system*

$$\Sigma_O \begin{cases} \textit{mode } 1: \dot{x} &= Ax, \quad \textit{if } \; |Cx| \geq \alpha \\ \textit{mode } 2: \dot{x} &= Bx, \quad \textit{if } \; |Cx| \leq \alpha, \end{cases} \qquad \alpha > 0 \tag{48}$$

which may appear as the closed loop system resulting from the use of switching controllers. From Theorem 6.2, we can show that this system is well-posed if and only if the bimodal system

$$\Sigma_O(A, B, \alpha) \begin{cases} \textit{mode } 1: \dot{x} &= Ax, \quad \textit{if } \; Cx \geq \alpha \\ \textit{mode } 2: \dot{x} &= Bx, \quad \textit{if } \; Cx \leq \alpha \end{cases} \tag{49}$$

is well-posed. Thus the well-posedness problem for the system given by (48) is reduced to that for the system given by (49).

7 Application to well-posedness problem in control switching

The well-posedness conditions as obtained in the previous sections can be applied to several issues in hybrid systems theory. Especially, by combining a stability condition of piecewise linear systems by Johansson and Rantzer [16] with our result, we can judge stability of those systems where the existence of a unique solution without sliding modes is guaranteed.

As the other application, in this section, we discuss a well-posedness problem of switching control systems where state feedback gains are switched according to a criterion depending on the state.

Consider the following problem: let the control system be given by

$$\dot{x} = Ax + Bu \tag{50}$$

where $x \in \mathcal{R}^n$ and $u \in \mathcal{R}^m$. For this system, let us consider a state feedback controller with two modes given by

$$u = \begin{cases} K_1 x & \text{if } Cx \geq 0 \\ K_2 x & \text{if } Cx \leq 0 \end{cases} \tag{51}$$

where $C \in \mathcal{R}^{1\times n}$, and K_1 and K_2 are given according to some design specification of the closed loop system. Then the problem is to check whether this closed loop system is well-posed or not.

As a simple example, consider the system given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

and $K_1 = [k_1 \;\; k_2]$, $K_2 = [\bar{k}_1 \;\; \bar{k}_2]$, and $C = [c_1 \;\; c_2]$. Then letting T_{A+BK_1} and T_{A+BK_2} be the rule matrices (i.e., the observability matrices) for the pairs $(C, A+BK_1)$ and $(C, A+BK_2)$, and assuming that these matrices are nonsingular, we obtain

$$T_{A+BK_2} T_{A+BK_1}^{-1} = \begin{bmatrix} 1 & 0 \\ * & a \end{bmatrix}$$

where $a \triangleq \frac{c_1(c_1+c_2\bar{k}_2)-c_2^2\bar{k}_1}{c_1(c_1+c_2k_2)-c_2^2k_1}$. Thus from Theorem 4.2, we conclude that the closed loop system is well-posed if and only if $a > 0$.

This example shows the following significant fact: even if each controller stabilizes each system in the usual sense, the total system is not necessarily well-posed. For example, consider the case of $c_1 = 1$, $c_2 = 1$, $k_1 = -1$, $k_2 = -3$, $\bar{k}_1 = -1$ and $\bar{k}_2 = -1$. Then $A + BK_1$ and $A + BK_2$ are stable, but $a < 0$. Note that such a case is not rare and the stability in the usual sense for each mode does not automatically provide the well-posedness of the closed loop system.

As shown in the above example, for any given closed loop system, the well-posedness can be determined by checking the corresponding conditions derived in the previous sections. Moreover, we can give an explicit characterization of all feedback gains which guarantee the well-posedness of the closed loop systems in question.

For the closed loop system with two modes given by (50) and (51), letting $K \triangleq K_2 - K_1$ and denoting $A + BK_1$ by A again, we have

$$\Sigma_O \begin{cases} \text{mode } 1: \dot{x} &= Ax, & \text{if } y = Cx \geq 0 \\ \text{mode } 2: \dot{x} &= (A+BK)x, & \text{if } y = Cx \leq 0. \end{cases} \tag{52}$$

For single-input control systems (50), we obtain the following result.

Theorem 7.1 *Assume that the pair (C, A) is observable and the relative degree for the triple (C, A, B) is $p(\leq n)$ (i.e., $CB = CAB = \ldots = CA^{p-2}B = 0$ and $CA^{p-1}B \neq 0$). Then the following statements are equivalent.*
(i) The system Σ_O is well-posed.
(ii) $K^{\mathrm{T}} \in span\{C^{\mathrm{T}}, (CA)^{\mathrm{T}}, \cdots, (CA^{p-1})^{\mathrm{T}}\} + \{\xi \in \mathcal{R}^n \mid \xi = \gamma (CA^p)^{\mathrm{T}}, \; \gamma CA^{p-1}B > -1\}$.

(Proof) (i)→(ii). From Theorem 4.2, (i) implies that $(C, A+BK)$ is observable. Thus from Theorem 4.1, there exists an $M \in \mathcal{L}_+^n$ such that $T_{A+BK} = MT_A$, where T_{A+BK} and T_A are the observability matrices for the pairs $(C, A+BK)$ and (C, A), respectively. Noting that $C(A+BK)^i = CA^i$ $(i = 0, 1, \cdots, p-1)$ and $C(A+BK)^p = CA^p + CA^{p-1}BK$, we obtain

$$CA^p + CA^{p-1}BK = m_{p+1,1}C + m_{p+1,2}CA + \cdots + m_{p+1,p}CA^{p-1} + m_{p+1,p+1}CA^p$$

where $m_{p+1,i}$ is the $(p+1, i)$ element of M, and $m_{p+1,p+1} > 0$. This implies that

$$K = \bar{m}_{p+1,1}C + \bar{m}_{p+1,2} + \cdots + \bar{m}_{p+1,p}CA^{p-1} + \gamma CA^p$$

where $\bar{m}_{p+1,i} = m_{p+1,i}/CA^{p-1}B$ and $\gamma = (m_{p+1,p+1} - 1)/CA^{p-1}B$. From $m_{p+1,p+1} > 0$, (ii) follows.

(ii) $\rightarrow$ (i). Let $\mu \stackrel{\triangle}{=} CA^{p-1}B$ and let K be given by $K = \kappa_1 C + \kappa_2 CA + \cdots + \kappa_p CA^{p-1} + \kappa_{p+1} CA^p$ where κ_i $(i = 1, 2, \cdots, p)$ are any values and $\kappa_{p+1}\mu > -1$. Then simple calculations show that

$$T_{A+BK} = MT_A, \quad M \stackrel{\triangle}{=} \left[\begin{array}{c|cccc} I_p & & 0_{p,n-p} & & \\ \hline & 1+\kappa_{p+1}\mu & 0 & \dots & 0 \\ * & * & 1+\kappa_{p+1}\mu & \ddots & \vdots \\ & \vdots & \ddots & \ddots & 0 \\ & * & \dots & * & 1+\kappa_{p+1}\mu \end{array}\right].$$

Since $1 + \kappa_{p+1}\mu > 0$, M is nonsingular. Thus the pair $(C, A + BK)$ is observable because M and T_A are nonsingular. In addition, $M \in \mathcal{L}_+^n$. Hence by Theorem 4.1, Σ_O is well-posed. $\square$

Remark 7.1 *It follows from Theorem 7.1 that for $p = n$ the closed loop system is well-posed for any K. Note also that the case $K = \kappa_1 C$ corresponds to the vector field of the closed loop system being Lipschitz continuous.*

Remark 7.2 *Theorem 7.1 can be extended to the multi-input case. If the relative degrees for all inputs are different from each other, the extension is straightforward. On the other hand, if some relative degrees are the same, the condition for well-posedness becomes more complicated. Furthermore, Theorem 7.1 can be extend to the case of affine inequalities as given below:*

$$\Sigma_O(A, A+BK, \alpha) \left\{ \begin{array}{lcll} \textit{mode } 1: \dot{x} & = & Ax, & \textit{if } \ Cx \geq \alpha \\ \textit{mode } 2: \dot{x} & = & (A+BK)x, & \textit{if } \ Cx \leq \alpha. \end{array} \right.$$

8 Conclusion

We have discussed a well-posedness problem in the sense of Carathéodory for a class of piecewise linear discontinuous systems, and have derived necessary and sufficient conditions for those systems to be well-posed. The obtained results are based on the lexicographic inequality relation and the smooth continuation property. As an application to switching control problems, we have given a necessary and sufficient condition for two state feedback gains, which are switched according to a criterion depending on the state, to maintain the well-posedness property of the closed loop system.

There are several open problems on well-posedness of discontinuous systems to be addressed in the future. We will have to discuss well-posedness of multi-modal systems in the unobservable case as an extension of section 6. In addition, extensions to the case of nonlinear systems should be addressed. It will be also interesting to discuss some relations with well-posedness of complementarity systems as mentioned in Remark 2.3. Finally, basic results derived here such as the smooth continuation property may be useful to solve well-posedness problems arising in the framework of hybrid automata as exposed e.g., in [8].

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