Department of Applied Mathematics
Faculty of EEMCS

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl
www.math.utwente.nl/publications

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A.B. Khmelnitskaya[1] and E.B. Yanovskaya[1]

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[1]SPb Institute for Economics and Mathematics Russian Academy of Sciences, 1 Tchaikovsky St., 191187 St. Petersburg, Russia

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Anna B. Khmelnitskaya, Elena B. Yanovskaya

SPb Institute for Economics and Mathematics Russian Academy of Sciences,
1 Tchaikovsky St., 191187 St.Petersburg, Russia,
e-mail: anna@AK3141.spb.edu, eyanov@iatp20.spb.org

Abstract. We show that the Owen value for TU games with coalition structure can be characterized without additivity axiom similarly as it was done by Young for the Shapley value for general TU games. Our axiomatization via four axioms of efficiency, marginality, symmetry across coalitions, and symmetry within coalitions is obtained from the original Owen's one by replacement of the additivity and null-player axioms via marginality. We show that the alike axiomatization for the generalization of the Owen value suggested by Winter for games with level structure is valid as well.

Keywords: cooperative TU game, coalitional structure, Owen value, axiomatic characterization, marginality.

Mathematics Subject Classification 2000: 91A12

1 Introduction

We consider the Owen value for TU games with coalition structure that can be regarded as an expansion of the Shapley value for the situation when a coalition structure is involved. The Owen value was introduced in [2] via a set of axioms it determining. These axioms were vastly inspired by original Shapley's axiomatization that in turn exploits the additivity axiom. However, the additivity axiom that being a very beautiful mathematical statement does not express any fairness property. Another axiomatization of the Shapley value proposed by Young [5] via marginality, efficiency, and symmetry appears to be more attractive since all the axioms present different reasonable properties of fair division. The goal of this paper is to evolve the Young's approach to the case of the Owen value for games with coalition structure. We provide a new axiomatization for the Owen value without additivity axiom that is obtained from the original Owen's one by the replacement of additivity and null-player via marginality. We show that the similar axiomatization can be also obtained for the generalization of the Owen value suggested by Winter in [4] for games with level structure.

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Sect. 2 introduces basic definitions and notation. In Sect. 3, we present an axiomatization for the Owen value for games with coalition structure and for the Winter's generalization for games with level structure on the basis of marginality axiom.

2 Definitions and notation

First recall some definitions and notation. A *cooperative game with transferable utility (TU game)* is a pair $\langle N, v\rangle$, where $N = \{1, \dots, n\}$ is a finite set of $n \geq 2$ players and $v\colon 2^N \to \mathbb{R}$ is a *characteristic function*, defined on the power set of N, satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ (or $S \in 2^N$) of s players is called a *coalition*, and the associated real number $v(S)$ presents the *worth* of the coalition S. For simplicity of notation and if no ambiguity appears, we write v instead of $\langle N, v\rangle$ when refer to a game, and also omit the braces when writing one-player coalitions such as $\{i\}$. The set of all games with a fixed player set N we denote $\mathcal{G}_N$. For any set of games $\mathcal{G} \subseteq \mathcal{G}_N$, a *value on* $\mathcal{G}$ is a mapping $\psi\colon \mathcal{G} \to \mathbb{R}^n$ that associates with each game $v \in \mathcal{G}$ a vector $\psi(v) \in \mathbb{R}^n$, where the real number $\psi_i(v)$ represents the *payoff* to the player i in the game v.

We consider games with coalition structure. A coalition structure $\mathcal{B} = \{B_1, ..., B_m\}$ on a player set N is a partition of the player set N, i.e., $B_1 \cup ... \cup B_m = N$ and $B_i \cap B_j = \emptyset$ for $i \neq j$. Denote by $\mathfrak{B}_N$ a set of all coalition structures on N. In this context a value is an operator that assigns a vector of payoffs to any pair $(v, \mathcal{B})$ of a game and a coalitional structure on N. More precisely, for any set of games $\mathcal{G} \subseteq \mathcal{G}_N$ and any set of coalition structures $\mathfrak{B} \subseteq \mathfrak{B}_N$, a *coalitional value on* $\mathcal{G}$ *with a coalition structure from* $\mathfrak{B}$ is a mapping $\xi\colon \mathcal{G} \times \mathfrak{B} \to \mathbb{R}^n$ that associates with each pair $\langle v, \mathcal{B}\rangle$ of a game $v \in \mathcal{G}$ and a coalition structure $\mathcal{B} \in \mathfrak{B}$ a vector $\xi(v, \mathcal{B}) \in \mathbb{R}^n$, where the real number $\xi_i(v, \mathcal{B})$ represents the *payoff* to the player i in the game v with the coalition structure $\mathcal{B}$.

We say players $i, j \in N$ are *symmetric* with respect to the game $v \in \mathcal{G}$ if they make the same marginal contribution to any coalition, i.e., for any $S \subseteq N\backslash\{i, j\}$, $v(S \cup i) = v(S \cup j)$. A player i is a *null-player* in the game $v \in \mathcal{G}$ if he adds nothing to any coalition non-containing him, i.e., $v(S \cup i) = v(S)$, for every $S \subseteq N\backslash i$.

In what follows we denote the cardinality of any set A by $|A|$.

A coalitional value ξ is *efficient* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$,

$$\sum_{i \in N} \xi_i(v, \mathcal{B}) = v(N).$$

A coalitional value ξ is *marginalist* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, for every $i \in N$, $\xi_i(v, \mathcal{B})$ depends only upon the ith marginal utility vector $\{v(S \cup i) - v(S)\}_{S \subseteq N\backslash i}$, i.e.,

$$\xi_i(v, \mathcal{B}) = \phi_i(\{v(S \cup i) - v(S)\}_{S \subseteq N\backslash i}),$$

where $\phi_i \colon \mathbb{R}^{2^{n-1}} \to \mathbb{R}^1$.

A coalitional value ξ possesses the *null-player property* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, every null-player i in game v gets nothing, i.e., $\xi_i(v, \mathcal{B}) = 0$.

A coalitional value ξ is *additive* if, for any two $v, w \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$,

$$\xi_i(v + w, \mathcal{B}) = \xi_i(v, \mathcal{B}) + \xi_i(w, \mathcal{B}),$$

where $(v + w)(S) = v(S) + w(S)$, for all $S \subseteq N$.

We consider two symmetry axioms. First note that for a given game $v \in \mathcal{G}$ and coalition structure $\mathcal{B} = \{B_1, ..., B_m\} \in \mathfrak{B}$, we can define a game between coalitions or in other terms a *quotient game* $\langle M, v^{\mathcal{B}} \rangle$ with $M = \{1, \ldots, m\}$ in which each coalition B_i acts as a player. We define the quotient game $v^{\mathcal{B}}$ as:

$$v^{\mathcal{B}}(Q) = v(\bigcup_{i \in Q} B_i), \qquad \text{for all } Q \subseteq M.$$

A coalitional value ξ is *symmetric across coalitions* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, for any two symmetric in $v^{\mathcal{B}}$ players $i, j \in M$, the total payoffs for coalitions B_i, B_j are equal, i.e.,

$$\sum_{k \in B_i} \xi_k(v, \mathcal{B}) = \sum_{k \in B_j} \xi_k(v, \mathcal{B}).$$

A coalitional value ξ is *symmetric within coalitions* if, for all $v \in \mathcal{G}$ and all $\mathcal{B} \in \mathfrak{B}$, any two players who are symmetric in v and belong to the same coalition in B get the same payoffs, i.e., for any $i, j \in B_k \in \mathcal{B}$ that are symmetric in v,

$$\xi_i(v, \mathcal{B}) = \xi_j(v, \mathcal{B}).$$

The Owen value was introduced in Owen [2] as the unique efficient, additive, symmetric across coalitions, and symmetric within coalitions coalitional value that possesses the null-player property.[1] In the sequel the Owen value in a game v with a coalition structure $\mathcal{B}$ we denote $Ow(v, \mathcal{B})$. For any $v \in \mathcal{G}_N$ and any $\mathcal{B} \in \mathfrak{B}_N$, for all $i \in N$, $Ow_i(v, \mathcal{B})$ can be given by the following formula

$$Ow_i(v, \mathcal{B}) = \sum_{\substack{Q \subseteq M \\ Q \not\ni k}} \sum_{\substack{S \subseteq B_k \\ S \not\ni i}} \frac{q!\,(m - q - 1)!\,s!\,(b_k - s - 1)!}{m!\,b_k!} \cdot (v(\bigcup_{j \in Q} B_j \cup S \cup i) - v(\bigcup_{j \in Q} B_j \cup S)), \tag{1}$$

where k is such that $i \in B_k \in \mathcal{B}$.

[1] We present above the original Owen's axioms in the formulation of Winter [4].

3 Axiomatization of the Owen value via marginality

We prove below that the Owen value defined on entire set of games $\mathcal{G}_N$ with any possible coalition structure from $\mathfrak{B}_N$ can be characterized by four axioms of efficiency, marginality, symmetry across coalitions, and symmetry within coalitions. Our proof strategy by induction is similar to that in Young [5].

Theorem 3.1. *The only efficient, marginalist, symmetric across coalitions, and symmetric within coalitions coalitional value defined on $\mathcal{G}_N \times \mathfrak{B}_N$ is the Owen value.*

Proof. One can easily check that the Owen value possesses these four properties. We prove below the converse. Let ξ be an efficient, marginalist, symmetric across coalitions, and symmetric within coalitions coalitional value defined on $\mathcal{G}_N \times \mathfrak{B}_N$, and let $\mathcal{B} \in \mathfrak{B}_N$. Any game $v \in \mathcal{G}_N$ can be presented via unanimity basis $\{u_T\}_{\substack{T\subseteq N\\ T\neq\emptyset}}$ [3],

$$v = \sum_{\substack{T\subseteq N\\ T\neq\emptyset}} \lambda_T\, u_T, \tag{2}$$

where

$$u_T(S) = \begin{cases} 1, & T \subseteq S, \\ 0, & T \not\subseteq S, \end{cases} \qquad \text{for all } S \subseteq N.$$

The Owen value in the unanimity game u_T with a coalition structure $\mathcal{B}$ for any $i \in N$ is equal

$$Ow_i(u_T, \mathcal{B}) = \begin{cases} \dfrac{1}{|B(i) \cap T|\, m_T}, & i \in T, \\ 0, & i \notin T, \end{cases}$$

where $B(i)$ is such element of the coalition structure $\mathcal{B}$ that contains player i and m_T is equal to the number of coalitions in $\mathcal{B}$ that have a nonempty intersection with T, i.e., $B(i) = B_k \in \mathcal{B}\colon B_k \ni i$, and $m_T = |\{k \in M\colon B_k \cap T \neq \emptyset\}|$. Because of its additivity property the Owen value in any game v with a coalition structure $\mathcal{B}$ can be equivalently expressed as

$$Ow_i(v, \mathcal{B}) = \sum_{\emptyset\neq T\subseteq N\colon\, T\ni i} \frac{\lambda_T}{|B(i) \cap T|\, m_T}.$$

Let now the index I of a game $v \in \mathcal{G}_N$ be the minimum number of terms under summation in (2), i.e.,

$$v = \sum_{k=1}^{I} \lambda_{T_k}\, u_{T_k},$$

where all $\lambda_{T_k} \neq 0$. We proceed the remaining part of the proof by induction on this index I.

If $I = 0$, then v is identically zero on all coalitions. All players in both games v and $v^{\mathcal{B}}$ are symmetric. Therefore, by symmetry across coalitions for all $j, k \in M$,

$$\sum_{i \in B_j} \xi_i(v, \mathcal{B}) = \sum_{i \in B_k} \xi_i(v, \mathcal{B}).$$

But

$$\sum_{j \in M} \sum_{i \in B_j} \xi_i(v, \mathcal{B}) = \sum_{i \in N} \xi_i(v, \mathcal{B}),$$

and by efficiency

$$\sum_{i \in N} \xi_i(v, \mathcal{B}) = 0.$$

Thus, for all $j, k \in M$,

$$\sum_{i \in B_j} \xi_i(v, \mathcal{B}) = \sum_{i \in B_k} \xi_i(v, \mathcal{B}) = 0.$$

Whence by symmetry within coalitions it follows that for all $i \in N$,

$$\xi_i(v, \mathcal{B}) = 0,$$

i.e., $\xi_i(v, \mathcal{B})$ coincides with the Owen value if index I is equal to 0.

Assume now that $\xi(v, \mathcal{B})$ is the Owen value whenever the index of $v \in \mathcal{G}_N$ is at most I, and consider some $v \in \mathcal{G}_N$ with the index equal to $I + 1$. Let $T = \cap_{k=1}^{I+1} T_k$ and $i \notin T$. Consider the game

$$v^{(i)} = \sum_{k:\ T_k \ni i} \lambda_{T_k} u_{T_k}$$

Obviously, the index of $v^{(i)}$ is at most I and, therefore, by induction hypothesis, $\xi(v^{(i)}, \mathcal{B}) = Ow(v^{(i)}, \mathcal{B})$. For both coalitional values ξ and the Owen value, ith marginal utility vectors relevant to the games v and $v^{(i)}$ coincide and, so, by marginalism of both values, $\xi_i(v, \mathcal{B}) = \xi_i(v^{(i)}, \mathcal{B})$ and $Ow_i(v, \mathcal{B}) = Ow_i(v^{(i)}, \mathcal{B})$. Thus,

$$\xi_i(v, \mathcal{B}) = Ow_i(v, \mathcal{B}), \qquad \text{for all } \ i \notin T. \tag{3}$$

If $T \neq \emptyset$ then to complete the proof it is enough to show that the last equality is true for all $i \in T$ as well. Consider T with relevance to a coalition structure $\mathcal{B}$ and denote

$$M_T = \{j \in M \mid B_j \cap T \neq \emptyset,\ B_j \in \mathcal{B}\}.$$

Notice that if $T \neq \emptyset$ then $M_T \neq \emptyset$ and all players $j, k \in M_T$ are symmetric in the game $v^{\mathcal{B}}$. By symmetry among coalitions for both values ξ and the Owen value, for all $j, k \in M_T$,

$$\sum_{i \in B_j} \xi_i(v, \mathcal{B}) = \sum_{i \in B_k} \xi_i(v, \mathcal{B}),$$

and

$$\sum_{i\in B_j} Ow_i(v,\mathcal{B}) = \sum_{i\in B_k} Ow_i(v,\mathcal{B}).$$

Therefore, because of efficiency of both values and equality (3) it follows that for all $j \in M_T$,

$$\sum_{i\in B_j\cap T} \xi_i(v,\mathcal{B}) = \sum_{i\in B_j\cap T} Ow_i(v,\mathcal{B}).$$

All players $i \in T$ are symmetric in the game v. Hence, by symmetry within coalitions, for all $j \in M_T$ and for all $i, k \in B_j \cap T$,

$$\xi_i(v,\mathcal{B}) = \xi_k(v,\mathcal{B}),$$

and

$$Ow_i(v,\mathcal{B}) = Ow_k(v,\mathcal{B}).$$

Whence it follows that for all $i \in T$,

$$\xi_i(v,\mathcal{B}) = Ow_i(v,\mathcal{B}). \qquad \square$$

Remark 3.2. Notice that similar as in Young for the Shapley value every efficient, marginalist, symmetric across coalitions, and symmetric within coalitions coalitional value defined on $\mathcal{G}_N \times \mathfrak{B}_N$ possesses the null-player property. Indeed, from the proof of the case $I = 0$ it follows that in the null-game which is identically zero on all coalitions every efficient, symmetric across coalitions, and symmetric within coalitions coalitional value ξ gives all players nothing. But in the null-game all marginal utility vectors are null-vectors. Therefore by marginality, for any game $v \in \mathcal{G}_N$ and for every null-player $i \in N$ in v, that is the same as for all $S \subseteq N\backslash i$, $v(S \cup i) - v(S) = 0$, follows that $\xi_i(v,\mathcal{B}) = 0$, i.e., the value ξ possesses the null-player property.

Remark 3.3. It is reasonable to note that for some subclasses of games $\mathcal{G} \subset \mathcal{G}_N$, for example for the subclass $\mathcal{G}_N^{sa}$ of superadditive games or for the subclass $\mathcal{G}_N^{c}$ of constant-sum games, if it is desired to stay entirely within one of these subclasses and not in the entire set of games $\mathcal{G}_N$, the same axiomatization for the Owen value via efficiency, marginality, symmetry across coalitions, and symmetry within coalitions is still valid. It can be proved similarly to the case of the Shapley value (see [5], [1]) adapting the ideas applied in the proof of Theorem 3.1.

Winter [4] introduced a generalization of the Owen value for games with level structure. A *level structure* is a finite sequence of partitions $\mathcal{L} = (\mathcal{B}_1, ..., \mathcal{B}_p)$ such that every $\mathcal{B}_i$ is a refinement of $\mathcal{B}_{i+1}$. Denote by $\mathfrak{L}_N$ the set of all level structures on N. In this context, for any set of games $\mathcal{G} \subseteq \mathcal{G}_N$ and any set of level structures $\mathfrak{L} \subseteq \mathfrak{L}_N$, a *level structure value on $\mathcal{G}$ with a level structure from $\mathfrak{L}$* is an operator defined on $\mathcal{G} \times \mathfrak{L}$ that assigns a vector of payoffs to any pair $(v, \mathcal{L})$ of a game $v \in \mathcal{G}$ and a level structure $\mathcal{L} \in \mathfrak{L}$. It is not difficult to see that the Winter's extension of the Owen value for games with level structure admits the similar axiomatization with the replacement of two above mentioned symmetry axioms by the following two captured from [4].

A level structure value ξ is *coalitionally symmetric* if, for all $v \in \mathcal{G}$ and any level structure $\mathcal{L} = (\mathcal{B}_1, ..., \mathcal{B}_p)$, for each level $1 \leq k \leq p$ for any two symmetric in $v^{\mathcal{B}_k}$ players $i, j \in M_k$ such that $B_i, B_j \in \mathcal{B}_k$ are subsets of the same component in $\mathcal{B}_t$ for all $t > k$, the total payoffs for coalitions B_i, B_j are equal, i.e.,

$$\sum_{r \in B_i} \xi_r(v, \mathcal{L}) = \sum_{r \in B_j} \xi_r(v, \mathcal{L}).$$

A level structure value ξ is *symmetric within coalitions* if, for all $v \in \mathcal{G}$ and any level structure $\mathcal{L} = (\mathcal{B}_1, ..., \mathcal{B}_p)$, any two players i, j who are symmetric in v and for every level $1 \leq k \leq p$ simultaneously belong or not to the same non-singleton coalition in $\mathcal{B}_k$, get the same payoffs, i.e., $\xi_i(v, \mathcal{L}) = \xi_j(v, \mathcal{L})$.

Theorem 3.4. *The only efficient, marginalist, coalitionally symmetric, and symmetric within coalitions level structure value defined on $\mathcal{G}_N \times \mathfrak{L}_N$ is the Winter value for games with level structure.*

The proof of Theorem 3.4 is a straightforward generalization for the proof of Theorem 3.1 and we leave it to the reader.

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