Faculty of Mathematical Sciences

University of Twente
The Netherlands

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl
www.math.utwente.nl/publications

Memorandum No. 1611

The Hamiltonian index of a graph and its branch-bonds

L. Xiong[1], H.J. Broersma and X. Li[2]

December 2001

ISSN 0169-2690

[1]Department of Mathematics, Jiangxi Normal University, Nanching, 330027, P.R. of China
[2]Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an, Shaaxi 710072, P.R. China

The Hamiltonian Index of a Graph and its Branch-bonds

Liming Xiong[*]

Department of Mathematics
Jiangxi Normal University
Nanchang, 330027, P.R. of China
e-mail:lmxiong@public.nc.jx.cn

H.J. Broersma

Faculty of Mathematical Sciences
University of Twente
P.O. Box 217
7500AE Enschede, The Netherlands
e-mail: h.j.broersma@math.utwente.nl

Xueliang Li

Department of Applied Mathematics
Northwestern Polytechnical University
Xi'an, Shaaxi 710072, P.R.China

December 21, 2001

Abstract

Let G be an undirected and loopless finite graph that is not a path. The minimum m such that the iterated line graph $L^m(G)$ is hamiltonian is called the hamiltonian index of G, denoted by $h(G)$. A reduction method to determine the hamiltonian index of a graph G with $h(G) \geq 2$ is given here. With it we will establish a sharp lower bound and a sharp upper bound for $h(G)$, respectively, which improves some known results of P.A. Catlin et al. [J. Graph Theory 14 (1990)] and H.-J. Lai [Discrete Mathematics 69 (1988)]. Examples show that $h(G)$ may reach all integers between the lower bound and the upper bound.

Keywords: iterated line graph, hamiltonian index, contraction of graphs, branch-bond, reduction method.

AMS Subject Classifications (1991): 05C45, 05C35

[*]This research has been supported by the Natural Science Fund of Jiangxi Province

1 Introduction

We use [2] for terminology and notation not defined here and consider only loop less finite graphs. Let G be a graph. For each integer $i \geq 0$, define $V_i(G) = \{v \in V(G) : d_G(v) = i\}$ and $W(G) = V(G) \setminus V_2(G)$. As in [4] *a branch in G* is a nontrivial path with ends are in $W(G)$ and with internal vertices, if any, have degree 2 in G (and thus are not $W(G)$). If a branch has length 1, then it has no internal vertices. We denote by $B(G)$ the set of branches of G and by $B_1(G)$ the subset of $B(G)$ in which every branch has an end in $V_1(G)$. For any subgraph H of G, we denote by $B_H(G)$ the set of branches of G with edges are all in H. For any two subgraphs H_1 and H_2 of G, define *the distance* $d_G(H_1, H_2)$ between H_1 and H_2 to the minimal distance $d_G(v_1, v_2)$ such that $v_1 \in V(H_1)$ and $v_2 \in V(H_2)$.

The line graph of G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in$G$. The m-iterated line graph $L^m(G)$ is defined recursively by $L^0(G) = G$, $L^m(G) = L(L^{m-1}(G))$ where $L^1(G)$ denote $L(G)$. *The hamiltonian index of a graph G,* denoted by $h(G)$, is the minimum m such that $L^m(G)$ is hamiltonian. Chartrand [5] showed that if a connected graph G is not a path, then the hamiltonian index of G exists. In [6], a formula for the hamiltonian index of a tree other than a path was established. There exist many upper bounds in literature (see [4], [6], [8], [12]). The following are the simpler bounds.

Theorem 1. (Lai [8]) Let G be a connected simple graph that is not a path, and let l be the length of a longest branch of G which is not contained in a 3-cycle. Then $h(G) \leq l + 1$.

Theorem 2. (Saražin [12]) Let G be a connected simple graph on n vertices other than a path. Then $h(G) \leq n - \Delta(G)$.

Note that the graph in Theorem 2 must be simple, which is not mentioned in [11].
These known bounds are based on the following characterization of hamiltonian line graphs obtained in [7].

Theorem 3. (Harary and Nash-Williams [7]) Let G be a graph with at least three edges. Then $h(G) \leq 1$ if and only if G has a connected subgraph H in which every vertex has even degree such that $d_G(e, H) = 0$ for any edge $e \in E(G)$.

Xiong and Liu [14] characterized the graphs with n-iterated line graphs that are hamiltonian, for integer $n \geq 2$.

Theorem 4. ([14]) Let G be a connected graph that is not a 2-cycle and let n be an integer at least two. Then $h(G) \leq n$ if and only if $EU_n(G) \neq \emptyset$. Where $EU_n(G)$ denotes the set of those graphs H of G which satisfy the following conditions:

(i) any vertex of H has even degree in H;

(ii) $V_0(H) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H)$;

(iii) $d_G(H_1, H - H_1) \leq n - 1$ for any subgraph H_1 of H;

(iv) $|E(b)| \leq n + 1$ for any branch b in $B(G) \backslash B_G(H)$;

(v) $|E(b)| \leq n$ for any branch in $B_1(G)$.

Using Theorem 4, Xiong improved Theorem 2 as follows.

Theorem 5. ([13]) Let G be a connected graph other than a path. Then $h(G) \leq dia(G) - 1$.

It is important to investigate whether the line graph of a graph is hamiltonian, i.e., $h(G) \leq 1$. Since the line graph of a hamiltonian graph is again hamiltonian, the study of these graphs with $h(G) \geq 2$ is equivalent to that of the graphs with $h(G) \leq 1$.

Motivated by these observations, and in an attempt to improve existing results including Theorem 5, we will give a reduction method to determine the hamiltonian index of a graph with $h(G) \geq 2$ in Section 3. Using this method we will give a sharp lower bound and a sharp upper bound of $h(G)$ such that the distance of the two bounds is exactly 2 in Section 4. Our results generalize results known earlier in [1], [4], [8], [12]. In the next section, we will introduce a terminology called branch-bond that involves our bounds.

2 Branch-bonds

For any subset S of $B(G)$, we denote by $G - S$ the subgraph obtained from $G[E(G) \backslash E(S)]$ by deleting all internal vertices of degree 2 in any branch of S. A subset S of $B(G)$ is called a branch cut if $G - S$ has more components than G has. A minimal branch cut is called a branch-bond. If G is connected, then a branch cut S of G is a minimal subset of $B(G)$ such that $G - S$ is disconnected. It is easily shown that, for a connected graph G, a subset S of $B(G)$ is a branch-bond if and only if $G - S$ has exactly two components. We denote by $BB(G)$ the set of branch-bonds of G. A connected graph G is eulerian if every vertex of G has even degree. The following characterization of eulerian graphs is well-known [10].

Theorem 6. (McKee[10]) A connected graph is eulerian if and only if each bond contains an even number of edges.

The following characterization of eulerian graphs involving branch-bonds follows from Theorem 6.

Theorem 7. A connected graph is eulerian if and only if each branch-bond contains an even number of branches.

3 A reduction method for determining the hamiltonian index of a graph G with $h(G) \geq 2$

Before presenting our main results, we first introduce some additional notation. Catlin [3] developed a reduction method for determining whether a graph G has a spanning closed trail. This method needs a tool, the so-called graph contractions. Let G be a graph and let H be a subgraph of G. For this we give a refinement of Catlin's reduction method. The contraction of H in G, denoted by G/H, is the graph obtained from G by contracting all edges of H, i.e., replacing H by a new vertex v_H, which is called contracted vertex in G/H, such that the number of edges in G/H joining any $v \in V(G)\backslash V(H)$ to v_H in G/H equals the number of edges joining v to H in G. Note that contractions may also result in loops and multiple adjacent , but that loops can be avoided if the subgraph H is induced by a vertex subset. The following lemma follows from Theorem 7 and is needed for our proof of main results.

Lemma 8. If G is a eulerian graph and H is a subgraph of G, then G/H is also a eulerian graph.

Catlin's reduction method and Theorem 3 are useful in the study of the hamiltonian index as seen in [4], [8], [11] and [12]. However, we must consider the lower iterated line graph when we want to do that.

In the construction, the contraction graph G/H may have lower hamiltonian index than G. For example the graph G obtained from a $K_{2,3}$ by replacing each vertex of degree 2 in the $K_{2,3}$ by a triangle K_3; and let H be one of these newly added K_3's. Then G/H has hamiltonian index 1 but G has hamiltonian index 2. In order to use Theorem 4 we must guarantee that if a graph has hamiltonian index at least two then the contraction must have that also. We do this by attaching two new edge-disjoint branches of length two at the contracted vertex v_H. The attachment-contraction $G//H$ is the graph obtained from G/H by attaching two new edge-disjoint branches b'_H, b''_H of length two at v_H, where $v_{H'}$ is the contracted vertex in G/H so that the vertices of these branches are not in G or $G//H$(except v_H, in the case of $G//H$), i.e. identifying v_H to exactly one end of b'_H, b''_H such that $b'_H, b''_H \in B_1(G//H)$. If G has k vertex disjoint nontrivial subgraphs $H_1, H_2, \cdots, H_k$, then the $\{H_1, H_2, \cdots, H_k\}$-contraction of G, denoted by $G/\{H_1, H_2, \cdots, H_k\}$, is the graph obtained from G by contracting $H_1, H_2, \cdots, H_k$, respectively, and the $\{H_1, H_2, \cdots H_k\}$-attachment-contraction of G, denoted by $G//\{H_1, H_2, \cdots H_k\}$, is the graph obtained from G by attachment-contracting $H_1, H_2, \cdots, H_k$ respectively. Let $\{H'_1, H'_2, \cdots, H'_s\} \subseteq \{H_1, H_2, \cdots, H_k\}$ be a subset of subgraphs. Let $v_{H'_1}, v_{H'_2}, \cdots, v_{H'_s}$ denote the vertices in $G//\{H_1, H_2, \cdots, H_k\}$ onto which the subgraphs $H'_1, H'_2, \cdots, H'_s$ are attachment-contracted respectively. Note that a contracted vertex is viewed as a vertex in $G/\{H_1, H_2, \cdots, H_k\}$ as well as a subgraph in G. For two branches $b_1 \in B(G)$ and $b_2 \in B(G//\{H_1, H_2, \cdots, H_k\})$, we say $b_1 = b_2$ if the internal vertices of b_1 contain the internal vertices of b_2, and if the ends of b_1 belong to the contracted vertices of the ends of b_2.

Now we can state the main result of this section.

Theorem 9. Let G be a connected graph other than a path, and let $G_1, G_2, \cdots, G_k$ be all nontrivial components of $G[\{v : d_G(v) \geq 3\}] - \{e : e$ is a nontrivial cut edge of $G\}$. If $h(G) \geq 2$, then

$$h(G) = h(G//\{G_1, G_2, \cdots, G_k\}).$$

Proof.
Theorem 4 will be used. Note that $EU_k(G)$ is the same as in Theorem 4. In order to prove that a subgraph H belongs to $EU_k(G)$, we only need to check that H satisfies all conditions in the definition of $EU_k(G)$. That is, these conditions hold for the graph G and the integer k.

Let $G' = G//\{G_1, G_2, \cdots, G_k\}$. The following claim is straightforward.

Claim 1. G and G' have the same branch set of length at least 2 and the same nontrivial cut edges set, but $\{b'_{G_1}, b''_{G_1}, b'_{G_2}, b''_{G_2}, \cdots, b'_{G_k}, b''_{G_k}\} \subseteq B_1(G') \backslash B(G)$.

First, we will prove that $h(G) \leq h(G')$.

Take $H \in EU_{h(G)}(G)$. By (ii), H contains all vertices of $\bigcup_{i=1}^{k} V(G_i)$. We set $H_i = H[V(G_i)]$ for $i \in \{1, 2, \cdots, k\}$ and let $H' = H/\{H_1, H_2, \cdots, H_k\}$. Obviously H is a subgraph of G' and H contains all vertices of $\{v_{G_1}, v_{G_2}, \cdots, v_{G_k}\}$. We will prove that $H' \in EU_{h(G)}(G)$, which implies that $h(G') \leq h(G)$. By H satisfying (i), H is an union of eulerian subgraphs in G. Hence it follows from Lemma 8 that H' is also union of eulerian subgraphs of G', which implies that H' satisfies (i). It follows that H' satisfies (ii) from H satisfying (ii). In order to prove that H' satisfies (iii), it suffices to consider the case that for a subgraph $K' \subseteq H'$ we have that $d_{G'}(K', H' - K') \geq 2$. Let $K = H[V'_K \cup V''_K]$ where $V'_K = V(K') \cap V(G)$ and $V''_K = V(K') \cap \{v_{G_1}, v_{G_2}, \cdots, v_{G_k}\}$ is a set of contracted vertices. One can easily see that K is a subgraph of H and any shortest path P in G from K to $H - K$ has end vertices of degree at least 3 in G. So $P' = G'[E(P) \cap E(G')]$ is a path from K' to $H' - K'$ in G'. Hence since H satisfies (iii), $d_{G'}(K', H' - K') \leq |E(P')| \leq |E(P)| = d_G(K, H - K) \leq h(G) - 1$. So H' satisfies (iii). By Claim 1, H' satisfies both (iv) and (v). Hence $H' \in EU_{h(G)}(G')$ which implies that $h(G') \leq h(G)$.

It remains to prove that $h(G) \leq h(G')$. Obviously $h(G') \geq 2$. Hence by Theorem 4, we can take $H' \in EU_{h(G')}(G')$. We will construct a subgraph in $EU_{h(G')}(G)$ from H'. Since H' satisfies (ii), and by the definition of $G_1, G_2, \cdots,$ G_k, H' contains all vertices of $\{v_{G_1}, v_{G_2}, \cdots, v_{G_k}\}$.
Set

$$V_{bi}(H') = \{x \in V(G_i) : x \textit{ is an endvertex of a branch of } B_{H'}(G)\}$$

for $i \in \{1, 2, \cdots, k\}$ and

$$V_b = \bigcup_{i=1}^{k} V_{bi}(H).$$

We denote by $R(x)$ the number of branches of $B_{H'}(G)$, one of which has x as an end vertex. Set

$$V_{bi}^j = \{x \in V_{bi}(H') : R(x) \equiv j (mod\ 2)\}\ and\ V_b^j = \bigcup_{i=1}^{k} V_{bi}^j for\ i \in \{1, 2\}.$$

Since H' satisfies (i),

$$\sum_{x \in V_{bi}^1} R(x) + \sum_{x \in V_{bi}^2} R(x) = \sum_{x \in V_{bi}} R(x) = d_{H'}(v_{G_i})$$

is even. Since $\sum_{x \in V_{bi}^2} R(x)$ is even, it follows that $\sum_{x \in V_{bi}^1} R(x)$ is also even. Thus $|V_{bi}^1|$ is even.

Without loss of generality, assume

$$V_{bi}^1 = \{u_1^i, v_1^i, u_2^i, v_2^i, \cdots, u_{s_i}^i, v_{s_i}^i\}.$$

Since G_i is connected, we can select a shortest path, denoted by $p(u_j^i, v_j^i)$, between u_j^i and v_j^i in G_i for $i \in \{1, 2, \cdots, s_i\}$. Set

$$P(V_b^1) = \bigcup_{i=1}^{k} \bigcup_{j=1}^{S_i} \{p(u_j^i, v_j^i)\}.$$

Let H be the subgraph of G with the following vertex set

$$V(H) = (\bigcup_{i=1}^{k} V(G_i)) \bigcup (V(H') \backslash \{v_{G_1}, v_{G_2}, \cdots, v_{G_k}\})$$

and edge set

$$E(H) = E(H') \bigcup \{e \in \bigcup_{i=1}^{k} E(G_i) : |\{p \in P(V_b^1) : e \in E(p)\}| \equiv 1 (mod\ 2)\}.$$

We will prove that $H \in EU_{h(G')}(G)$, i.e., by verifying that H satisfies the conditions in the definition of $EU_{h(G')}(G)$ for a graph G and integer $h(G')$. First we prove that H satisfies (i).

Defining $E_x(G) = \{e \in E(G) :\ e$ is an edge that is incident with $x\}$, we have

$$d_{(\bigcup_{i=1}^{k} G_i) \bigcap H}(x) = \begin{cases} 2|\{P \in P(V_b^1) : x \in V(P)\}| \\ -1 - \sum_{e \in E_x(G)} 2[\frac{1}{2}|\{P \in P(V_b^1) : e \in E(P)\}|], \\ \hfill \text{if } \ x \in V_b^1 \\ 2|\{P \in P(V_b^1) : x \in V(P)\}| \\ - \sum_{e \in E_x(G)} 2[\frac{1}{2}|\{P \in P(V_b^1) : e \in E(P)\}|], \\ \hfill \text{if } \ x \in V(G) \backslash V(V_b^1). \end{cases}$$

Hence, for any vertex $x \in V(H)\bigcap(\bigcup_{i=1}^{k}(G_i)$, we have that

$$d_H(x) = d_{(\bigcup_{i=1}^{k} G_i)\bigcap H}(x) + R(x)$$

is even. For any $x \in V(H)\backslash(\bigcup_{i=1}^{k} G_i)$, we have $d_H(x) = d_G(x) = 2$. So H satisfies (i). Since H' satisfies (ii), H satisfies (ii). By Claim 1, H satisfies both (iv) and (v).

In order to prove that H satisfies (iii), we only need to consider a subgraph K of H such that $d_G(K, H-K) \geq 2$, since $h(G) \geq 2$. Hence, since

$$V(G_i) \subseteq \bigcup_{i=3}^{\Delta(G)} V_i(G) \subseteq V(H) \ for\ i \in \{1, 2, \cdots, k\},$$

$V(K) \cap V(G_i)$ is either empty or $V(G_i)$ for $i \in \{1, 2, \cdots, k\}$. Let $K_1, K_2, \cdots, K_c$ be all nontrivial components of

$$K[\{v : d_k(v) \geq 3\}] - \{e :\ e\ is\ a\ cut\ edge\ of\ G\}.$$

We obtain that $K' = K/\{K_1, K_2, \cdots, K_c\}$ is a subgraph of H'. Let $P' = x'u_1u_2...u_ty'$ be a shortest path from K' to $H'-K'$ in G'. Since $\{v_{G_1}, v_{G_2}, \cdots, v_{G_k}\} \subseteq V(H')$,

$$\{u_1, u_2, \cdots, u_t\} \cap \{v_{G_1}, v_{G_2}, \cdots, v_{G_k}\} = \emptyset.$$

Hence $\{u_1, u_2, \cdots, u_t\} \subseteq V(G)$. By the selection of K' and H, there exist two vertices $x \in V(K)$ and $y \in V(H-K)$ such that $xu_1, u_ty \in E(G)$. Hence $P = xu_1u_2 \cdots u_ty$ is a path from K to $H-K$, which implies that

$$d_G(K, H-K) \leq |E(P)| = |E(P')| = d_{G'}(K', H'-K') \leq h(G') - 1.$$

Hence $H \in EU_{h(G')}(G)$ which implies that $h(G) \leq h(G')$. □

4 Sharp upper and lower bounds for $h(G)$

A branch-bond is called odd if it consists of an odd number of branches. The length of a branch-bond $S \in BB(G)$, denoted by $l(S)$, is the length of a shortest branch in it. Define $BB_1(G)$ to be the set of branch-bonds, one of which contains only one branch such that one of its ends has degree one in G, and define $BB_2(G)$ to be the set of branch-bonds, one of which contains only one branch such that its ends have degree at least three in G, and define $BB_3(G)$ to be the set of odd branch-bonds, one of which has at least three branches in G respectively. Define

$$h_i(G) = \begin{cases} max\{l(S) : S \in BB_i(G)\} for\ i \in \{1, 2, 3\}. & if\ BB_i(G)\ is\ not\ empty \\ 0, & otherwise. \end{cases}$$

If F_1 and F_2 are two subsets of $E(G)$, then $H + F_1 - F_2$ denotes the subgraph of G obtained from $G[(E(H) \bigcup F_1)\backslash F_2]$ by adding to its vertex set any vertices of $\bigcup_{i=3}^{\Delta(G)} V_i(G)$ which were not already in its vertex set, and so that any vertices added are isolated vertices in $H + F_1 - F_2$.

The following lower bound for $h(G)$ involving odd branch-bonds can now be given.

Theorem 10. Let G be a connected graph with $h(G) \geq 1$. Then

$$h(G) \geq max\{h_1(G), h_2(G) + 1, h_3(G) - 1\} \tag{4.1}$$

Proof. If $h(G) = 1$ then, by Theorem 3, $h_1(G) \leq 1, h_2(G) \leq 0$ and $h_3(G) \leq 2$, i.e., (4.1) is true. So we can assume that G is a connected graph with $h(G) \geq 2$. We can take any $S_i \in BB_i(G)$ such that $h_i(G) = l(S_i)$ for $i \in \{1, 2, 3\}$. For any subgraph $H \in EU_{h(G)}(G), E(b) \cap E(H) = \emptyset$ for any $b \in S_1 \cup S_2$ and there exists at least a branch $b \in S_3$ such that $E(b) \cap E(H) = \emptyset$. Hence by Theorem 4, we obtain $h(G) \geq h_1(G)$ by (v), $h(G) \geq h_2(G) + 1$ by (iii) and $h(G) \geq h_3(G) - 1$ by (iv). So $h(G) \geq max\{h_1(G), h_2(G) + 1, h_3(G) - 1\}$, i.e., (4.1) holds. $\square$

We can construct an extremal graph for the equality (4.1). For an integer $t \geq 1$, let $P_1, P_2, \cdots, P_{2t+3}$ be $2t + 3$ vertex disjoint paths and let K_a, K_b be two vertex disjoint complete graphs of order at least 3. Taking two vertices $u \in V(K_a)$ and $v \in V(K_b)$, we construct a graph G_0 by identifying exactly one end vertex of $P_1, P_2, \cdots, P_{2t+1}$ respectively, identifying u and another end vertex of $P_1, P_2, \cdots, P_{2t+1}$, exactly one end vertex of P_{2t+2}, respectively, identifying v and another end vertex of P_{2t+2}, one end vertex of P_{2t+3} respectively such that $P_1, P_2, \cdots, P_{2t+3}, K_a, K_b$ are edge-disjoint subgraphs of G_0. Set

$$k_1(G_0) = max\{h_1(G_0), h_2(G_0) + 1, h_3(G_0) - 1\}.$$

Obviously $h_1(G_0) = |E(P_{2t+3})|, h_2(G_0) = |E(P_{2t+2})|$ and

$$h_3(G_0) = min\{|E(P_1)|, |E(P_2)|, |E(P_{2t+1})|\}.$$

One can easily see that $K_a \cup K_b \in EU_{k_1(G_0)}(G_0)$. By Theorem 4, $h(G_0) \leq k_1(G_0)$. From the proof of (4.1), $h(G_0) \geq k_1(G_0)$. So $h(G_0) = max\{h_1(G_0), h_2(G_0)+1, h_3(G_0) - l\}$.$\square$

Now we state our upper bound for $h(G)$.

Theorem 11. Let G be a connected graph that is not a path. Then

$$h(G) \leq max\{h_1(G), h_2(G) + 1, h_3(G) + 1\} \tag{4.2}$$

Proof. Let $k(G) = max\{h_1(G), h_2(G) + 1, h_3(G) + 1\}$. Obviously $k(G) \geq 1$. If $k(G) = 1$, i.e., $h_1(G) \leq 1$ and $h_2(G) = h_3(G) = 0$, then, by Theorem 7, $G[V(G)\backslash V_1(G)]$ is eulerian. Hence, using Theorem 3, we obtain $h(G) \leq 1$, i.e.,

(2) is true.

So we assume that $h(G) \leq 2$ and $k(G) \leq 2$. By Theorem 9, it suffices to consider the graph G such that $G[\{v : d_G(v) \geq 3\}] - \{e : e\ is\ a\ nontrivial\ cut\ edge\ of\ G\}$ has no nontrivial component. Let H be a subgraph in $EU_{h(G)}(G)$ such that H contains as many branches as possible, one of which has a number of edges greater than $k(G) - 1$. Then we can prove the following.

Claim 1. If S is a branch-bond in $BB(G)$ such that it contain at least three branches, then there exists no branch $b \in S \backslash B_H(G)$ such that $|E(b)| \geq k(G)$.

Proof of Claim 1. Otherwise there exists a branch $b_0 \geq B(G) \backslash B_H(G)$ and a branch bond $S \in BB_3(G)$ such that $|E(b_0)| \geq k(G)$ and $b_0 \in S \backslash B_H(G)$. Obviously b_0 has two ends u and v (say). Now we can select a branch-bond, denoted by $S(u, b_0)$, such that it contains b_0 and any branch of $S(u, b_0)$ has the end u.
In order to obtain a contradiction, we will take a cycle of G that contains b_0 by the following algorithm.

Algorithm b_0.

1. If $|S(u, b_0)| \equiv 0(\text{mod}2)$, then (by Theorem 7, we can) select a branch $b_1 \in S(u, b_0) \backslash (B_H(G) \cup \{b_0\})$. Otherwise (we can) select a branch $b_1 (\neq b_0) \in S(u, b_0)$ with $|E(b_1)| = l(S(u, b_0)) \leq h_3(G)$ and let $u_1 (\neq u)$ be another end of b_1. If $u_1 = v$, then $t := 1$ and stop. Otherwise $i := 1$.

2. Select a branch-bond $S(u, u_i, b_0)$ in G which contains b_0 but $b_1, b_2, \cdots, b_i$ such that either u or u_i is an end of a branch in $S(u, u_i, b_0)$. If $|S(u, u_i, b_0)| \equiv 0(mod2)$, then (by Theorem 7, we can) select a branch
$$b_{i+1} \in S(u, u_i, b_0) \backslash (B_H(G) \cup \{b_0\}).$$
Otherwise (we can) select a branch $b_{i+1} \in S(u, u_i, b_0)$ such that $b_{i+1} \neq b_0$ and $|E(b_{i+1})| = l(S(u, u_i, b_0)) \geq h_3(G)$ and let u_{i+1} be the end of b_{i+1} that is neither u nor u_i.

3. If $u_{i+1} = v$, then $t := i + 1$ and stop. Otherwise replace i by $i + 1$ and return to step 2.

Since $|B(G)|$ is finite, Algorithm b_0 will stop after a finite number of steps. Since $u_t = v$, $G[\bigcup_{i=0}^{u} E(b_i)]$ is connected. Hence we obtain the following.

Claim 2. $G[\bigcup_{i=0}^{u} E(b_i)]$ has a cycle of G, denoted by C_0, which contains b_0.

Now we construct a subgraph $H' \subseteq G$ as follows:

$$H' = H + E(C_0) \backslash E(H) - (E(H) \cap E(C_0)).$$

By the selection of $\{b_1, b_2, \cdots, b_t\}$,

$$|E(b)| \leq h_3(G) \leq k(G) - 1 \; for \; b \in B_H(G) \cap \{b_1, b_2, \cdots, b_t\}.$$

Hence, by Claim 2, H' satisfies (iii)-(iv). Obviously H' satisfies both (i) and (ii), and this implies $H' \leq EU_{h(G)}(G)$. But H' contains more branches than H does, one of which has edges number greater than $k(G)-1$, and this contradicts the selection of H, which completes the proof of Claim 1.□
For any branch b of G, if $G[E(b)]$ is not a cycle of G, then there exists a branch bond $S \in BB(G)$ with $b \in S$. Hence, by Claim 1 and the selection of $k(G), H \in EU_{k(G)}(G)$ which implies that $h(G) \leq k(G)$.□

We can construct a family of extremal graphs for Theorem 11. From the extremal graph of Theorem 10, we only need to construct an extremal graph G_0 with $h(G_0) = h_3(G_0)+1$. In fact, in the following construction we can construct a family of graphs G_0 such that $h(G_0)$ can take all integers between $h_3(G_0) - 1$ and $h_3(G_0) + 1$. Let $k \geq 1$ be an integer and let $H = K_{2,2k+1}$ be a complete bipartite graph with $V^1(H) = \{x, y\}$ and $V^2(H) = \{u_1, u_2, \cdots, u_{2k+1}\}$. Let $1 \leq l_1 \ll l_2 \leq l_3 \ll l_4$ be four integers. Obtain G_0 by subdividing $xu_1, xu_2, \cdots, xu_{2k}$ into $2k$ branches of length $l_4, yu_1, yu_2, \cdots, yu_{2k}$ into $2k$ branches of length l_1, xu_{2k+1} into a branch b of length l_2, yu_{2k+1} into a branch b' of length l_3 respectively and by replacing each vertex of $V^2(H)$ by a K_4. One can easily see that $h_3(G_0) = l_3$ and that $G_0[E(G_0)\backslash(E(b) \cup E(b'))]$ has a subgraph in $EU_{max\{l_3-1,l_2+1\}}(G_0)$. By Theorem 4, $h(G_0) \leq max\{l_3 - 1, l_2 + 1\}$. By an argument similar to the one in the proof of (4.1), $h(G_0) \geq max\{l_3 - 1, l_2 + 1\}$. Hence $h(G_0) = max\{l_3 - 1, l_2 + 1\}$.
Clearly

$$h(G_0) = max\{l_3 - 1, l_2 + 1\} = \begin{cases} l_3 - 1, & if \;\; l_2 \leq l_3 - 2, \\ l_3, & if \;\; l_2 = l_3 - 1, \\ l_3 + 1, & if \;\; l_2 = l_3. \end{cases}$$

Hence $h(G_0)$ may have all integers from $h_3(G_0) - 1$ to $h_3(G_0) + 1$ according to different integers l_2 and l_3.

5 Analysis of known results

Theorems 5 and 11 show two upper bounds for the hamiltonian index of a graph. Clearly $h_i(G) \leq dia(G)$ for $i \in \{1, 2, 3\}$ and there exists a graph with large diameter and small $h_3(G)$. For example, the graph obtained by replacing each edge of a path by an odd branch-bond, which contains at least three branches. Hence the upper bound in Theorem 5 is not better than the one in Theorem 11. It seems that the upper bound in Theorem 11 is better than the one in Theorem 5. However this is not true. We investigate the graph F_t obtained from $K_{2,2t+1}$ by replacing each edge of $K_{2,2t+1}$ by a path of length s. Clearly $h_3(F_t) = s = dia(F_t)$ but $h(F_t) = s - 1 = h_3(G) - 1 = dia(F_t) - 1$.

The following relation between the two bounds in Theorems 5 and 11 is obtained.

Theorem 12. Let G be a connected graph other that a path with $h(G) \geq 1$. If $h_3(G) = dia(G)$, then
$$h(G) = dia(G) - 1.$$

Proof. Follows from Theorems 5 and 10. $\square$

Obviously Theorem 1 is a consequence of Theorem 11. Although Theorem 2 is not a consequence of Theorem 11, one easily checks that $h_i(G) \leq n - \Delta(G)$ for any $i \in \{1, 2, 3\}$. Hence, from Theorem 11, we have that $h(G) \leq n - \Delta(G) + 1$. Moreover if $h_3(G) \leq n - \Delta(G)$, then Theorem 11 is better than Theorem 2. The following consequences of Theorem 11 are easily obtained.

Corollary 13. (Catlin et al. [4]) Let G be a connected graph that is neither a path nor a 2-cycle. Then

$$h(G) \leq \max_{\{v_1,v_2\} \subseteq W(G)} \min_{P} X(P) + 1$$

where $X(P)$ denotes the length $|E(b)|$ of the longest branch b in $B_P(G)$ and P is a subgraph induced by all branches in G whose end-vertices are u and v.

Proof. Let S be a branch bond in $BB(G)$ with $l(S) = max\{h_2(G)+1, h_3(G)+1\}$. Then any path of G between two vertices u and v in two components of $G - S$ respectively must have a branch in S. Hence

$$max\{h_1(G), h_2(G) + 1, h_3(G) + 1\} \leq \max_{\{v_1,v_2\} \subseteq W(G)} \min_{P} X(P) + 1.$$

This relation and Theorem 11 give Corollary 13.$\square$

Corollary 14. (Chartrand and Wall [6]). If T is a tree which is not a path, then
$$h(T) = max\{h_1(T), h_2(T) + 1\}.$$

Proof. If T is a tree, then $h_3(T) = 0$. Hence by Theorems 10 and 11, we obtain Corollary 14 is true.$\square$

Corollary 15. (Balakrishnan and Paulraja [1]) Let G be a connected graph with at least four edges. If the only 2-degree cut sets of G are the singleton subsets which are neighbors of end vertices of G, then $h(G) \leq 2$.

Proof. One can easily check that $h_1(G) \leq 2, h_2(G) \leq 1$ and $h_3(G) \leq 1$. Hence this corollary follows from Theorem 11.

Corollary 16. (Lesniak-Foster and Williamson [9]) Let G be a connected graph with at least four edges. If every vertex of degree two is adjacent to an end vertex, then $h(G) \leq 2$.

Proof. From the condition of this corollary, we know $h_1(G) \leq 2, h_2(G) = 0$ and $h_3(G) = 0$. Hence this corollary follows from Theorem 11.$\square$

Corollary 17. (Chartrand and Wall [6]) Let G be a connected graph other than a path. If $\delta(G) \geq 3$, then $h(G) \leq 2$.

Proof. Obvious.$\square$

References

[1] R. Balakrishnan and P. Paulraja, *Line graphs subdivision graphs*, Journal of Combinatorics, Information and System Sciences, Vol.10, Nos.1-2 (1985) 33-35.

[2] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, American Elsevier, New York, (1996).

[3] P.A. Catlin, *A reduction method to find spanning eulerian subgraphs*, J. Graph Theory, 12 (1988) 29-45.

[4] P.A. Catlin, Iqbalunnisa, T.N. Janakiraman and N. Srinivasan, *Hamilton cycles and closed trails in iterated line graphs*, J. Graph Theory, 14 (1990) 347-364.

[5] G. Chartrand, *On hamiltonian line graphs*, Trans. Amer. Math. Soc., 134 (1968) 559-566.

[6] G. Chartrand and C.E. Wall, *On the hamiltonian index of a graph*, Studia Sci. Math. Hung., 8 (1973) 43-48.

[7] F. Harary and C.St. J.A. Nash-Williams, *On eulerian and hamiltonian graphs and line graphs*, Canad. Math. Bull., 8 (1965) 701-709.

[8] H.J. Lai, *On the hamiltonian index*, Discrete Math., 69(1988) 43-53.

[9] L. Lesniak-Foster and J.E. Williamson, *On spanning and dominating circuits in graphs*. Can. Math. Bull., 20 (1997) 215-220.

[10] T.A. Mckee, *Recharacterizing eulerian: intimations of new duality*, Discrete Math., 51 (1984) 237-242.

[11] M.L. Saražin, *On the hamiltonian index of a graph*, Discrete Math., 122 (1993) 373-376.

[12] L. Saražin, *A simple upper bound for the hamiltonian index of a graph*, Discrete Math., 134 (1994) 85-91.

[13] L. Xiong, *The hamiltonian index of a graph,* Graphs and Combinatorics, To appear.

[14] L. Xiong and Z. Liu, *Hamiltonian iterated line graphs*, preprint.