Faculty of Mathematical Sciences

University of Twente

University for Technical and Social Sciences

P.O. Box 217
7500 AE Enschede
The Netherlands
Phone: +31-53-4893400
Fax: +31-53-4893114
Email: memo@math.utwente.nl

Memorandum No. 1499

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D. Bauer,[1] H.J. Broersma,
A. Morgana[2] and E. Schmeichel[3]

September 1999

ISSN 0169-2690

[1]Department of Mathematical Sciences, Stevens Institute of Technology, Hoboken, NJ 07030, U.S.A.
[2]Dipartimento di Matematica, G. Gastelnuovo, Universitá di Roma, "LA SAPIENZA", I-00185 Roma, Italy
[3]Department of Mathematics & Computer Science, San Jose State University, San Jose, CA 95192, U.S.A.

Polynomial Algorithms that Prove an NP-Hard Hypothesis Implies an NP-Hard Conclusion

D. Bauer [1] *

H. J. Broersma [2] *

A. Morgana [3]

E. Schmeichel [4]

[1] *Department of Mathematical Sciences*
Stevens Institute of Technology
Hoboken, NJ 07030, U.S.A.

[2] *Faculty of Mathematical Sciences*
University of Twente
P. O. Box 217, 7500 AE Enschede, The Netherlands

[3] *Dipartimento di Matematica, G. Castelnuovo*
Universitá di Roma, "LA SAPIENZA"
I-00185 Roma, Italia

[4] *Department of Mathematics & Computer Science*
San Jose State University
San Jose, CA 95192, U.S.A.

September 16, 1999

* Supported in part by NATO Collaborative Research Grant CRG 973085

Abstract

A number of results in hamiltonian graph theory are of the form $\mathcal{P}_1$ implies $\mathcal{P}_2$, where $\mathcal{P}_1$ is a property of graphs that is NP-hard and $\mathcal{P}_2$ is a cycle structure property of graphs that is also NP-hard. Such a theorem is the well-known Chvátal-Erdös Theorem, which states that every graph G with $\alpha \leq \kappa$ is hamiltonian. Here κ is the vertex connectivity of G and α is the cardinality of a largest set of independent vertices of G. In another paper Chvátal points out that the proof of this result is in fact a polynomial time construction that either produces a Hamilton cycle or a set of more than κ independent vertices. In this note we point out that other theorems in hamiltonian graph theory have a similar character. In particular, we present a constructive proof of the well-known theorem of Jung [12] for graphs on 16 or more vertices.

Keywords: hamiltonian graphs, toughness, complexity, NP-hardness, polynomial algorithms, constructive proofs

AMS Subject Classifications (1991): 68R10, 05C38

1 Introduction

A number of results in hamiltonian graph theory are of the form $\mathcal{P}_1$ implies $\mathcal{P}_2$, where $\mathcal{P}_1$ is a property of graphs that is NP-hard to decide and $\mathcal{P}_2$ is a cycle structure property of graphs that is also NP-hard to decide. Two such well-known theorems are the Chvátal-Erdös Theorem [7] [Theorem A below] and Jung's Theorem [12] [Theorem B below]. This raises the question of determining the practical utility of these results. However in [6], Chvátal points out that the proof of Theorem A is in fact a polynomial time construction that produces either a Hamilton cycle or a set of more than κ independent vertices. In this note we point out that other theorems in hamiltonian graph theory have a similar character. In particular, we present a constructive proof of Theorem B for graphs on at least 16 vertices that, in polynomial time, will either produce a Hamilton cycle or will produce a set of vertices whose removal indicates that G is not 1-tough. Our goal, however, is to raise the possibility that similar constructive proofs can be found for theorems in other areas of graph theory.

We begin with some useful definitions. The terminology and notation required for our proofs will be given in the next section. A good reference for any undefined terms in graph theory is [5] and in complexity theory is [8]. We consider only finite undirected graphs without loops or multiple edges. Let G be such a graph with vertex set $V(G)$ and edge set $E(G)$. Then G is *hamiltonian* if it has a *Hamilton cycle*, i.e., a cycle containing all of its vertices. We use $\kappa(G)$ for the vertex connectivity of G, $\delta(G)$ for the minimum vertex degree of G and $\alpha(G)$ to denote the cardinality of a largest set of independent vertices in G. For $1 \leq r \leq \alpha(G)$ we let

$$\sigma_r(G) = \min \left\{ \sum_{v \in S} d(v) \mid S \subseteq V(G) \text{ is an independent set with } |S| = r \right\}.$$

If $r > \alpha(G)$, we define $\sigma_r(G) = \infty$. Note that $\sigma_1(G) = \delta(G)$. Let $\omega(G)$ represent the number of components of G. We say G is 1-*tough* if $|S| \geq \omega(G - S)$ for every subset S of the vertex set $V(G)$ with $\omega(G - S) > 1$. A cycle C in G is called a *dominating cycle* if every edge of G has at least one of its endvertices on C. If no ambiguities are likely to arise, we frequently omit any explicit reference to the graph G by simply writing δ, κ, etc. We also sometimes identify a subgraph with its vertex set, e.g., use C for $V(C)$, etc.

Let

1. $\mathcal{P}_1$ be a property of graphs which is NP-hard to decide;
2. $\mathcal{P}_2$ be a cycle structure property of graphs which is NP-hard to decide; and
3. $\mathcal{C}$ be a class of graphs for which the membership decision problem is in $\mathcal{P}$.

We will consider theorems of the following type.

Theorem 1.1 *Let $G \in \mathcal{C}$. If G has property $\mathcal{P}_1$, then G has property $\mathcal{P}_2$.*

Some well-known examples of such theorems are the following.

Theorem A (Chvátal-Erdös [7])
Let G be a graph on $n \geq 3$ vertices. If $\alpha \leq \kappa$, then G is hamiltonian.

Theorem B (Jung [12])
Let G be a graph on $n \geq 11$ vertices with $\sigma_2 \geq n - 4$. If G is 1-tough, then G is hamiltonian.

Theorem C (Bauer, et al. [4])
Let G be a graph on n vertices with $\sigma_3 \geq n \geq 3$. If G is 1-tough, then G has a dominating cycle.

We wish to consider known proofs of these results from the point of view of rendering the proofs constructive in the following sense: Beginning with any cycle C in G, in polynomial time we do exactly one of the following:

1. demonstrate that G has property $\mathcal{P}_2$;
2. find a set of vertices whose existence demonstrates that property $\mathcal{P}_1$ does not hold.
3. produce a longer cycle;

In the event of (3), we begin again with the longer cycle.

An immediate consequence of the proof technique is that if G has property $\mathcal{P}_1$, then every longest cycle in G will demonstrate that G has property $\mathcal{P}_2$.

In particular, an examination of the proof of Theorem 5 in [4] indicates that such a proof exists for Theorem C. It yields the next result which also appears in [4].

Theorem D Let G be a graph on n vertices with $\sigma_3 \geq n$. If G is 1-tough, then every longest cycle in G is a dominating cycle.

The existence of a constructive proof for Theorem 1.1 is especially interesting when $\mathcal{P}_2$ implies $\mathcal{P}_1$ (e.g., as in Theorem B). In that case, both properties $\mathcal{P}_1$ and $\mathcal{P}_2$ can be recognized in polynomial time within the class of graphs $\mathcal{C}$. In particular, within the class of graphs with $\sigma_2 \geq n-4$, the properties of being 1-tough and of having a Hamilton cycle can each be recognized in polynomial time. Häggkvist [9] previously observed that for the smaller class of graphs with $\delta \geq \frac{n}{2} - 2$, the existence of a Hamilton cycle can be recognized in polynomial time. This will be discussed further in Section 4.

In Section 3 we first briefly discuss the constructive proof of Theorem A in [7], and then provide a detailed constructive proof of Theorem B (for $n \geq 16$). This later proof makes use of arguments that appear in [2] and [4].

2 Preliminary Results

Our proofs require some notation and terminology. Let C be a cycle in G. We denote by $\overrightarrow{C}$ the cycle C with a given orientation. If $u, v \in C$, then $u\overrightarrow{C}v$ denotes the consecutive vertices on C from u to v in the direction specified by $\overrightarrow{C}$. The same vertices in reverse order are given by $v\overleftarrow{C}u$. We use u^+ to denote the successor of u on $\overrightarrow{C}$ and u^- to denote its predecessor. Further define $u^{++} = (u^+)^+$ and $u^{--} = (u^-)^-$, etc. If $v \in V$, then $N(v)$ is the set of all vertices in V adjacent to v. Whenever $A \subseteq C$ we let $A^+ = \{v^+ | v \in A\}$. The sets A^- and A^{++} are defined analogously. Let $S, T \subseteq V$ and $v \in V$. Then $e(v, T)$ is the number of edges joining v to a vertex of T, and $e(S, T)$ denotes $\sum_{v \in S} e(v, T)$. We also use $d_C(v)$ to denote the number of vertices of C which are adjacent to v.

The following lemma is needed for our constructive proof of Theorem B.

Lemma 2.1 *Let C be any cycle in G, $v \in V - C$, and $A = N(v) \subseteq C$. If any of the following conditions holds, we can constructively obtain a cycle longer than C in polynomial time.*

(i) $A \cap A^+ \neq \emptyset$ *or* $A^+ \cap A^{++} \neq \emptyset$*;*

(ii) Either A^+ *or* A^- *is not independent;*

(iii) $x_1, x_2 \in A$*, and*

(a) there is a vertex $z \in x_1^+ \overrightarrow{C} x_2^+$ *such that* $x_2^+ z, x_1^+ z^+ \in E$*, or*

(b) there is a vertex $w \in x_2^+ \overrightarrow{C} x_1^+$ *such that* $x_1^+ w, x_2^+ w^+ \in E$*, or*

(c) $d_C(x_1^+) + d_C(x_2^+) > |C|$*.*

We note that (i), (ii), (iii)(a) and (iii)(b) employ standard arguments and (c) follows easily from (a) and (b). An analogous lemma holds if we replace x_1^+ by x_1^- and x_2^+ by x_2^-. This analogous lemma will also be referred to as Lemma 2.1 (iii).

(iv) $x_1, x_2 \in A$ *with* $x_2 = x_1^{+++}$*, and*

(a) there is a vertex $z \in x_2 \overrightarrow{C} x_1^-$ *such that* $x_1^{++} z, x_1^+ z^+ \in E$*, or*

(b) there is a vertex $z \in x_2 \overrightarrow{C} x_1^-$ *such that* $x_1^+ z, x_1^{++} z^+ \in E$*.*

Proof*:* If (a) is satisfied, then $x_1^+ z^+ \overrightarrow{C} x_1 v x_2 \overrightarrow{C} z x_1^{++} x_1^+$ is a cycle longer than C. If (b) is satisfied, then $x_1^+ z \overleftarrow{C} x_2 v x_1 \overleftarrow{C} z^+ x_1^{++} x_1^+$ is a cycle longer than C. □

(v) $x_1^+ \in A^+ \cap A^-, z \in N(x_1^+) \cap C$*, and*

(a) $\{z^+\} \cup A^+$ *is not an independent set of vertices, or*

(b) $\{z^-\} \cup A^-$ *is not an independent set of vertices.*

Proof*:* We prove (a); the proof of (b) uses an analogous argument. Suppose $z^+ x_j^+ \in E$, where $x_j \in A$. If $x_j^+ \in A^+ \cap x_1^{++} \overrightarrow{C} z$, then $x_j^+ z^+ \overrightarrow{C} x_1 v x_j \overleftarrow{C} x_1^+ z \overleftarrow{C} x_j^+$ is a cycle longer than C. If $x_j^+ \in A^+ \cap z^{++} \overrightarrow{C} x_1^+$, then $x_j^+ \overrightarrow{C} x_1^+ z \overleftarrow{C} x_1^{++} v x_j \overleftarrow{C} z^+ x_j^+$ is a cycle longer than C. □

3 Proofs

We begin by noting that the proof of Theorem A in [7] is constructive in the sense mentioned in the introduction. This was pointed out by Chvátal in [6]. An outline of his argument is as follows. It can be determined in polynomial time whether a graph G on $n \geq 3$ vertices has $\kappa(G) = 1$. In this case it is easy to find two independent vertices in G, thus showing that the hypothesis of Theorem A is false. Otherwise, construct a cycle C in the 2-connected graph G. If C is not a Hamilton cycle, let H

be any component of $G-C$ and $A = \bigcup_{v \in V(H)} N(v) - V(H)$. Clearly, $\kappa \leq |A| = |A^+|$. Let $v_1, v_2 \in A$. If $v_1^+ v_2^+ \in E$ or if they are joined by a path whose internal vertices lie entirely in H, then a cycle longer than C is easily constructed using standard arguments. Thus, if v_0 is any vertex in H, $A^+ \cup \{v_0\}$ is an independent set of vertices having cardinality greater than κ. Hence, in polynomial time, it is possible to either find a cycle longer than C or to find a set of more than κ independent vertices. Thus in at most n iterations we either obtain a Hamilton cycle or demonstrate that the hypothesis of Theorem A is false.

Before giving a constructive proof of Theorem B (Jung's Theorem), we need a constructive proof of the following lemma.

Lemma 3.1 *Let G be a 2-connected graph on $n \geq 16$ vertices with $\sigma_2 \geq n-4$. Then G contains a dominating cycle.*

Proof: Let C be any cycle in G, and suppose C is not a dominating cycle. Give C an orientation and let H be a nontrivial component of $G-C$. Set $A = \bigcup_{v \in V(H)} N(v) - V(H)$ and let $v_1, \ldots, v_k$ be the elements of A occurring on $\overrightarrow{C}$ in consecutive order. Since G is 2-connected, $k \geq 2$. If $v_i^+ = v_{i+1}$ for any i, $1 \leq i \leq k$ (indices modulo k), then C can easily be lengthened by at least one vertex. Let C now be the longer cycle. Furthermore, it also follows from G being 2-connected that there exist integers r and s with $1 \leq r < s \leq k$ such that v_r and v_s are connected by a path $P_{r,s}$ of length at least 3 with all internal vertices in H.

We now show that the following three conditions hold; otherwise we can constructively obtain a longer cycle in polynomial time. We then start the argument again with the new longer cycle.

(1) There exists no (v_r^+, v_s^+)-path which is internally disjoint from C; in particular, $v_r^+ v_s^+ \notin E$.

Assuming the contrary to (1), let P be a (v_r^+, v_s^+)-path, internally disjoint from C. Since $v_r^+, v_s^+ \notin A$, we have $V(P) \cap V(H) = \emptyset$. Now $v_r P_{r,s} v_s \overleftarrow{C} v_r^+ P v_s^+ \overrightarrow{C} v_r$ has length at least $|V(C)|+2$.

(2) If $v \in v_r^+ \overrightarrow{C} v_s^+$ and $v_s^+ v \in E$, then $v_r^+ v^+ \notin E$. Similarly, if $v \in v_s^+ \overrightarrow{C} v_r^+$ and $v_r^+ v \in E$, then $v_s^+ v^+ \notin E$.

To prove (2) assume, e.g. $v \in v_r^+ \overrightarrow{C} v_s^+, v_s^+ v \in E$ and $v_r^+ v^+ \in E$. By (1), $v \neq v_r^+, v_s$. If $v \in v_r^{++} \overrightarrow{C} v_s^-$, then the cycle $v_r P_{r,s} v_s \overleftarrow{C} v^+ v_r^+ \overrightarrow{C} v v_s^+ \overrightarrow{C} v_r$ has length at least $|V(C)|+2$.

(3) If $v \in v_r^+ \overrightarrow{C} v_s^+$ and $v_s^+ v \in E - E(C)$, then $v_r^+ v^{++} \notin E$. Similarly, if $v \in v_s^+ \overrightarrow{C} v_r^+$ and $v_r^+ v \in E - E(C)$, then $v_s^+ v^{++} \notin E$.

The proof of (3) is similar to the proof of (2), except now the longer cycle has length $|V(C)|+1$ instead of $|V(C)|+2$.

Using observations (1)-(3) we now obtain an upper bound for $d(u_0)+d(v_r^+)+d(v_s^+)$, where u_0 is an arbitrary vertex of H. Define

$$\begin{aligned}
R_1(v_r^+) &= \{v \in v_r^+ \overrightarrow{C} v_s \mid v_r^+ v^+ \in E\}, \\
S_1(v_s^+) &= \{v \in v_r^+ \overrightarrow{C} v_s \mid v_s^+ v \in E\}, \\
R_2(v_r^+) &= \{v \in v_s^+ \overrightarrow{C} v_r \mid v_r^+ v \in E\}, \\
S_2(v_s^+) &= \{v \in v_s^+ \overrightarrow{C} v_r \mid v_s^+ v^+ \in E\}, \\
R_3(v_r^+) &= \{v \in V - V(C) \mid v_r^+ v \in E\}, \\
S_3(v_s^+) &= \{v \in V - V(C) \mid v_s^+ v \in E\}, \\
B(v_r^+, v_s^+) &= R_1(v_r^+) \cup S_1(v_s^+) \cup R_2(v_r^+) \cup S_2(v_s^+).
\end{aligned}$$

By (2), $R_1(v_r^+) \cap S_1(v_s^+) = R_2(v_r^+) \cap S_2(v_s^+) = \emptyset$.
By (1), and the fact that $v_r^+, v_s^+ \notin A$, $R_3(v_r^+) \cap S_3(v_s^+) = V(H) \cap (R_3(v_r^+) \cup S_3(v_s^+)) = \emptyset$. Furthermore, for $i \in \{1, \dots, k\} - \{r, s\}$, either v_i^+ or v_i is not in $B(v_r^+, v_s^+)$. To see this, suppose e.g., $v_i^+ \in R_1(v_r^+) \cup S_1(v_s^+)$. Then $v_r^+ v_i^{++} \in E$, since the assumption that $v_s^+ v_i^+ \in E$ implies the existence of a cycle longer than C, containing the vertices of a (v_i, v_s)-path of length at least 2 with all internal vertices in H(cf. (1)). But then, by (3) with $v = v_i$, $v_s^+ v_i \notin E$. Since $v_r^+ v_i^+ \notin E$, it follows that $v_i \notin R_1(v_r^+) \cup S_1(v_s^+)$. Thus

$$\begin{aligned}
d(u_0) + d(v_r^+) + d(v_s^+) &= d(u_0) + |R_1(v_r^+)| + |R_2(v_r^+)| + |R_3(v_r^+)| + \\
&\quad\ |S_1(v_s^+)| + |S_2(v_s^+)| + |S_3(v_s^+)| \\
&\leq (k + |V(H)| - 1) + (|V(C)| - (k-2)) + \\
&\quad\ |R_3(v_r^+)| + |S_3(v_s^+)| \\
&\leq (k + |V(H)| - 1) + (|V(C)| - (k-2)) + \\
&\quad\ (|V| - |V(C)| - |V(H)|) \\
&= n + 1.
\end{aligned}$$

However, since $\sigma_2 \geq n-4$, we have

$$d(u_0) + d(v_r^+) + d(v_s^+) \geq \frac{3}{2}(n-4).$$

Hence $\frac{3}{2}(n-4) \leq n+1$, a contradiction since $n \geq 16$. Thus we conclude that G contains a dominating cycle. □

Constructive proof of Theorem B (Jung's Theorem)

Our constructive proof applies to all graphs on $n \geq 16$ vertices.

We begin by first constructing a dominating cycle C. This is possible by Lemma 3.1. Let v_0 be a vertex of largest degree in $V - C$ and $A = N(v_0)$. We now show that if C is not a Hamilton cycle, in polynomial time we can either

(a) Produce a longer dominating cycle; or

(b) Produce a new dominating cycle having the same length as C, but having a vertex w_0 not on the cycle such that $d(w_0) > d(v_0)$; or

(c) Produce a set of vertices whose removal shows that G is not 1-tough.

In either case (a) or (b) we let C now be the new dominating cycle and begin again. We consider several cases.

Case 1: $|C| \leq n - 3$.

Since $|A| = d(v_0) \geq \dfrac{n-4}{2}$, we immediately get a longer dominating cycle unless $C - A$ had $d(v_0)$ components consisting of all singletons and (possibly) an edge of C itself. Since $V - C$ is independent, we have

$$e(V - C - v_0, C - A) \geq |V| - |C|.$$

Otherwise

$$\begin{aligned} \omega(G - A) &\geq d(v_0) + |V - C| - e(V - C - v_0, C - A) \\ &\geq d(v_0) + |V| - |C| - (|V| - |C| - 1) \\ &= d(v_0) + 1 \\ &= |A| + 1, \end{aligned}$$

and G is not 1-tough. But then, since $e(V - C - v_0, C - A) \geq |V| - |C| > |V - C - v_0|$, there exists $w \in V - C - v_0$ with $e(w, C - A) \geq 2$. Thus w is adjacent to either two consecutive vertices of C, two vertices in A^+ or two vertices in A^-. In any case we obtain a longer dominating cycle than C.

Case 2: $|C| = n - 2$.

Let $Q = (V - C) \cup A^+$. We first assert the following:

Q is an independent set of vertices.

Let $v_1 \neq v_0 \in V - C$ and $w \in A$. Since C is a dominating cycle, it suffices by Lemma 2.1(ii) to show that $v_1 w^+ \notin E$. Suppose otherwise. We claim that v_1 is not adjacent to any vertex in $(A^+ - \{w^+\}) \cup A^{++}$. If $v_1 w^{++} \in E$ we easily obtain a longer cycle. If $v_1 s^+ \in E$, where $s^+ \in A^+ - \{w^+\}$, then $C' : v_1 s^+ \overrightarrow{C} w v_0 s \overleftarrow{C} w^+ v_1$ is a cycle longer than C and if $v_1 s^{++} \in E$, where $s^{++} \in A^{++}$, then $C'' : v_1 s^{++} \overrightarrow{C} w v_0 s \overleftarrow{C} w^+ v_1$ is a cycle longer than C. Since $(A^+ - \{w^+\}) \cap A^{++} = \emptyset$, by Lemma 2.1(i) we have $d(v_1) \leq |C| - 2d(v_0) + 1$. Since $d(v_0) + d(v_1) \geq n - 4$ and $d(v_0) \geq d(v_1)$ we have $d(v_0) \geq \frac{n-4}{2}$. Hence

$$d(v_0) + d(v_1) \leq (n-2) - \frac{n-4}{2} + 1 = \frac{n+2}{2}.$$

Since $n \geq 16$ we have $d(v_0)+d(v_1) < n-4$, a contradiction. This proves the assertion.

Since G is 1-tough, $|Q| \leq \frac{n}{2}$. Hence

$$|C| \geq \frac{n}{2} + |A^+| = \frac{n}{2} + d(v_0) \geq \frac{n}{2} + \frac{n-4}{2} = n-2,$$

and equality holds only if $d(v_0) = d(v_1) = \frac{n-4}{2}$.

We now consider two cases.

Case 2a: There exists $w \in A$ such that $w^{++}, w^{+++} \notin A$.

Let $t^+ \in A^+ \cap A^-$. By Lemma 2.1(ii), $N(t^+) \subseteq A \cup \{w^{++}\}$. But then $G - (A \cup \{w^{++}\})$ has at least $\frac{n}{2}$ components and G is not 1-tough.

Case 2b: There exist $u, w \in A$ such that $u^{++}, w^{++} \notin A$.

If $t^+ \in A^+ \cap A^-$, then by Lemma 2.1(ii), $N(t^+) \subseteq A$. Hence $G - A$ has at least $\frac{n-2}{2}$ components and again G is not 1-tough.

Case 3: $|C| = n - 1$.

First suppose $d(v_0) \neq \frac{n-3}{2}$ or $\frac{n-4}{2}$. If $d(v_0) > \frac{n-1}{2} = \frac{|C|}{2}$, we can easily construct a Hamilton cycle in G. If $d(v_0) = \frac{n-1}{2}$ or $\frac{n-2}{2}$, then $G - A$ has more than $d(v_0)$ components and G is not 1-tough. Let $x_1, x_2 \in A$. If $d(v_0) < \frac{n-7}{2}$, then $d(x_1^+), d(x_2^+) > \frac{n-1}{2}$, contradicting Lemma 2.1(iii). Hence $\frac{n-7}{2} \leq d(v_0) \leq \frac{n-5}{2}$. We now show how to construct another cycle C' of length $n-1$ with $w_0 \in V - C'$ and $d(w_0) \geq \frac{n-3}{2}$. Let $x^+ \in A^+$ and $w^{++} \in A^{++} - \{x^{++}\}$. If $x^+ w^{++} \in E$, then

C' : $x^+w^{++}\overrightarrow{C}xv_0w\overleftarrow{C}x^+$ is the required cycle and $w_0 = w^+$ is the required vertex. Thus we may assume $x^+w^{++} \notin E$ for all $w^{++} \in A^{++} - \{x^{++}\}$. Since $v_0x^+ \notin E$, it follows from Lemma 2.1(i) and (ii) that $d(x^+) \le (n-1)-2(d(v_0)-1)-1 = n-2d(v_0)$. Since $d(v_0) + d(x^+) \ge n-4$ we conclude $d(v_0) \le 4$. However $d(v_0) \ge \frac{n-7}{2}$, a contradiction for $n \ge 16$.

Case 3a: $d(v_0) = \frac{n-3}{2}$.

Case 3ai: *There exists* $z \in A$ *such that* $z^{++}, z^{+++} \notin A$.

Let $t^+ \in A^+ - \{z^+\}$. By Lemma 2.1(ii), $t^+z^+, t^+z^{+++} \notin E$. If $t^+z^{++} \in E$ then by Lemma 2.1(iii), $z^+z^{+++} \notin E$ and thus $G-(A\cup\{z^{++}\})$ has $\frac{n+1}{2}$ components, is not 1-tough. If $t^+z^{++} \notin E$ for any $t^+ \in A^+ - \{z^+\}$, then $G - A$ has $\frac{n-1}{2}$ components and again G is not 1-tough.

Case 3aii: *There exist vertices* $z, w \in A$ *such that* $z^{++}, w^{++} \notin A$.

If $z^+w^{++}, z^{++}w^+ \notin E$, then $G - A$ has $\frac{n-1}{2}$ components and G is not 1-tough. Suppose $z^+w^{++} \in E$. If $z^{+++} \ne w$ then $w^{--} \in A$. By Lemma 2.1(ii), $N(w^-) \subseteq A$ and since $v_0w^- \notin E$, $d(w^-) \ge \frac{n-5}{2}$. If $z \ne w^{+++}$, then either w^-z or $w^-w^{+++} \in E$, contradicting Lemma 2.1(iii). If $z = w^{+++}$ and all vertices in $A^+ - \{z^+, w^+\}$ are not adjacent to z, then each vertex has degree at most $\frac{n-5}{2}$. But then $d(x^+) + d(y^+) \le n-5$ for every pair of vertices $x^+, y^+ \in A^+ - \{z^+, w^+\}$, a contradiction. Hence we must have $z^{+++} = w$. By Lemma 2.1(iv), $w^+z^{++} \notin E$. Since $d(w^+) + d(z^{++}) \ge n-4$ and n is odd, either $d(w^+) \ge \frac{n-3}{2}$ or $d(z^{++}) \ge \frac{n-3}{2}$. Suppose, without loss of generality, that $d(w^+) \ge \frac{n-3}{2}$. Then $N(w^+) \subseteq A\cup\{w^{++}\}$. Hence either $w^+z \in E$ or $w^+w^{+++} \in E$. However $w^+w^{+++} \in E$ contradicts Lemma 2.1(iii) and $w^+z \in E$ contradicts Lemma 2.1(iv). Thus we conclude that $z^+w^{++} \notin E$. An analogous argument shows that $z^{++}w^+ \notin E$.

Case 3b: $d(v_0) = \frac{n-4}{2}$.

Case 3bi: *There exists* $z \in A$ *such that* $z^{++}, z^{+++}, z^{++++} \notin A$.

Let t^+ be any vertex in $A^+-\{z^+\}$. If $t^+z^{++} \in E$ then by Lemma 2.1(v), $A^+\cup\{z^{+++}\}$ is an independent set. Also $z^+z^{++++} \notin E$ by Lemma 2.1(iii). Hence $G-(A\cup\{z^{++}\})$ has $\frac{n}{2}$ components, and G is not 1-tough. Thus $t^+z^{++} \notin E$ and similarly $t^+z^{+++} \notin$

E. But this implies that $G - A$ has $\dfrac{n-2}{2}$ components, and G is not 1-tough.

Case 3bii: *There exist vertices* $z, w \in A$ *such that* $z^{++}, w^{++}, w^{+++} \notin A$.

Let t^+ be any vertex in $A^+ - \{z^+, w^+\}$. If $t^+w^{++} \in E$, then by Lemma 2.1(v), $A^+ \cup \{w^{+++}\}$ and $A^- \cup \{w^+\}$ are both independent sets of vertices. Thus $G - (A \cup \{w^{++}\})$ has $\dfrac{n}{2}$ components, a contradiction. Hence $t^+w^{++} \notin E$. Thus $N(t^+) = A$. Next we show that $w^+z^{++} \notin E$. Suppose otherwise. If $z \neq w^{++++}$ then $w^+z^{++}, z^-w \in E$ contradicts Lemma 2.1(iii). and thus $z = w^{++++}$. Since $z^+v_0 \notin E, d(z^+) \geq \dfrac{n-4}{2}$. Thus z^+ must be adjacent to either w, w^{++}, w^{+++} or z^{+++}. However if z^+z^{+++} we contradict Lemma 2.1(iii) and if either z^+w or $z^+w^{++} \in E$ we contradict Lemma 2.1(iv). If $z^+w^{+++} \in E$, then $C' : v_0zz^+w^{+++}w^{++}w^+z^{++}\overrightarrow{C}wv_0$ is a Hamilton cycle. Hence $w^+z^{++} \notin E$. Using an analogous argument we conclude $z^+w^{+++} \notin E$, and thus G is not 1-tough. For if $w^+w^{+++} \notin E$, then $G - (A \cup \{w^{++}\})$ has $\dfrac{n}{2}$ components and if $w^+w^{+++} \in E$, then by Lemma 2.1(iii), $z^+w^{++}, z^{++}w^{++} \notin E$ and $G - A$ has $\dfrac{n-2}{2}$ components.

Case 3biii: *There exist vertices* $u, z, w \in A$ *such that* $u^{++}, z^{++}, w^{++} \notin A$.

It suffices to show that $z^{++}w^+, z^+w^{++}, z^{++}u^+, z^+u^{++}, w^{++}u^+, w^+u^{++} \notin E$ since then $G - A$ has $\dfrac{n-2}{2}$ components and G is not 1-tough. We show that $z^{++}w^+$ and $z^+w^{++} \notin E$; symmetric arguments will complete the proof. We assume, without loss of generality, that $u^+ \in [w^+\overrightarrow{C}z^+]$. Suppose $z^{++}w^+ \in E$. If $w = z^{+++}$ consider any distinct pair of vertices $x^+, y^+ \in A^+ \cap A^-$. Since $N(x^+), N(y^+) \subseteq A - \{w\}$, $d(x^+) + d(y^+) < n - 4$, a contradiction. If $w \neq z^{+++}$, then since $z^+v_0 \notin E$ we have $d(z^+) \geq \dfrac{n-4}{2}$ and by Lemma 2.1(ii), z^+ must be adjacent to at least one of w, w^{++}, z^{+++} and u^{++}. However $z^+w, z^+w^{++} \notin E$ by Lemma 2.1(iv) and $z^+z^{+++} \notin E$ by Lemma 2.1(iii). If $z^+u^{++} \in E$, then $C' : z^+u^{++}\overleftarrow{C}w^+z^{++}\overrightarrow{C}wv_0u^{+++}\overrightarrow{C}z^+$ is a Hamilton cycle. Hence $z^{++}w^+ \notin E$. Now suppose $z^+w^{++} \in E$ and consider w^+. Reasoning as above, w^+ must be adjacent to at least one of z, z^{++}, w^{+++} and u^{++}. However $w^+z, w^+z^{++} \notin E$ by Lemma 2.1(iv) and $w^+w^{+++} \notin E$ by Lemma 2.1(iii). Hence $w^+u^{++} \in E$. Now by considering z^{++} we similarly conclude $z^{++}u^+ \in E$. Since $n \geq 16, A^+ - \{u^+, w^+, z^+\} \neq \emptyset$. Without loss of generality suppose $w^- \in A^+ - \{u^+, w^+, z^+\}$. Clearly $N(w^-) = A$. Thus $w^-w^{+++} \in E$. However, since $z^+w^{++} \in E$, this contradicts Lemma 2.1(iii) and completes the proof. □

4 Concluding Remarks

As mentioned earlier, our constructive proof of Theorem B shows that within the class of graphs with $\sigma_2 \geq n - 4$, the properties of being 1-tough and of having a Hamilton cycle can be recognized in polynomial time. Our proof is based on the proof of Jung's Theorem in [2] and the proof of Theorem D in [4]. At the time these results were established, the computational complexity of recognizing 1-tough graphs was not known. Consequently, a number of researchers questioned the utility of such theorems. Later it was established in [1] that recognizing t-tough graphs was indeed NP-hard, for any positive rational $t > 0$. Hence the constructive argument in this note shows that, in some sense, Theorem B and Theorem D have more than purely theoretical interest.

In fact, it can be determined in polynomial time if G is 1-tough within the larger class of graphs G on n vertices with $\sigma_2 \geq n - k$, for any fixed integer $k \geq 0$. To see this, it suffices to note that if G is not 1-tough, then G contains a set of vertices S such that $G - S$ contains at least $|S| + 1$ components. Suppose this is the case, and let T_1 and T_2 be two smallest components of $G - S$, with $t_1 = |T_1| \leq |T_2| = t_2$. Then clearly

$$n \geq t_1 + s \cdot t_2 + s$$

and by examining the degree of a vertex in T_1 and a vertex in T_2 we get

$$s + t_1 - 1 + s + t_2 - 1 \geq n - k.$$

These inequalities imply

$$k - 1 \geq (s - 1)(t_2 - 1).$$

Thus $s \geq k + 1 \Rightarrow t_2 = 1$. Hence, to determine whether G has such a set S, it suffices to first check all subsets of k or fewer vertices. If S is not one of these sets, then S must consist of a set of all vertices which are adjacent to a single vertex. All of this can be checked in time $O(n^k)$.

By contrast, in [3] we proved the results below. Let $\Omega(r)$ be the class of all graphs G on n vertices with $\delta(G) \geq rn$ and let $t \geq 1$ be any rational number.

Theorem 4.1 *Let G be a graph in $\Omega\left(\frac{t}{t+1}\right)$. Then G is t-tough.*

Theorem 4.2 *For any fixed $\epsilon > 0$ it is NP-hard to recognize t-tough graphs in $\Omega\left(\frac{t}{t+1} - \epsilon\right)$.*

A consequence of Theorem 4.2 is that, for any $\epsilon > 0$, recognizing 1-tough graphs is NP-hard within the class of graphs having $\delta \geq \frac{n}{2} - f(n)$, where $f(n) = \epsilon \cdot n$. It would be interesting to find the largest $f(n)$ for which recognizing such graphs can be done in polynomial time. All we know is that $c_1 \leq f(n) < c_2 n$ for any constants $c_1, c_2 > 0$.

We noted earlier that Häggkvist [9] has shown that within the class of graphs on n vertices with $\delta \geq \frac{n}{2} - 2$, the existence of a Hamilton cycle can be recognized in polynomial time. In fact he established the following.

Theorem 4.3 *Let $k \geq 0$ be any fixed integer. Then within the class of graphs G on n vertices with $\delta \geq \frac{n}{2} - k$, a Hamilton cycle can be recognized in time $O(n^{5k})$.*

Note that our proof of Theorem B shows that the property of having a Hamilton cycle can be recognized in polynomial time within the class of graphs G on n vertices with $\sigma_2 \geq n - 4$. We do not have an argument to extend this to the class of graphs G on n vertices with $\sigma_2 \geq n - k$ for a fixed integer $k \geq 5$.

We close by raising the possibility that constructive proofs might be found for theorems and/or conjectures in other areas of graph theory that have the form of Theorem 1.1. One such possibility is in the area of edge coloring of graphs. Consider the well-known conjecture of Goldberg [11] on the edge coloring of multigraphs. Let $\chi'(G)$ denote the edge chromatic number of a graph G.

Conjecture (Goldberg [11]) Let G be a loopless multigraph.
If $\chi'(G) > 1 + \Delta(G)$, then $\chi'(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil$.

It is known that determining $\chi'(G)$ is NP-hard [10], and it appears that determining $\max_{H \subseteq G} \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil$ is also NP-hard. Thus Goldberg's Conjecture is of the form of Theorem 1.1, and the following conjecture would yield a constructive proof of Goldberg's Conjecture.

Conjecture Let G be a loopless multigraph, and let k be any integer with $k \geq 1 + \Delta(G)$. Then we can construct in polynomial time either a k-edge-coloring of G or an induced subgraph H of G with $\left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil > k$.

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