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A DETERMINISTIC APPROACH
TOWARDS SELF-ORGANIZATION
IN CONTINUOUS MEDIA

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Another extension of the principle partly resolves the apparent discrepancy between these asymptotic results and the appearance of confined vortices as observed in long time numerical simulations.

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Following a brief description of the principle as it can be found in the physical literature, a more precise dynamical version is presented in such a way that a deterministic investigation thereof becomes appealing. It is shown that recent asymptotic results rigorously prove the dynamical s.o.-principle for the 2 D Navier-Stokes equations.

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1. INTRODUCTION.

In this paper we describe how the attractive physical idea of "self-organization" can be formulated in a more mathematical way that is tracktable for deterministic investigations. For continuous systems for which it is valid, the self-organization principle describes low dimensional behaviour in a very explicit way. Roughly speaking, the principle describes the asymptotics of a dissipative system with so-called quasi-stationary states. These states are completely determined by the conservative part of the system only, in fact by an optimization principle for the constants of the motion. In this respect this attack differs essentially from the methods that use inertial-forms and -manifolds.

By introducing dynamics in the variational characterization of the quasi-stationary states, a variational principle is obtained that can describe the asymptotics of such dissipative systems. This will be called the dynamical s.o.-principle.

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We show that the recent asymptotic results for 2D N-S equations of Foias and Saut [1], rigorously prove the physical statements about self-organization in this case. These results were obtained directly from the deterministic equation, whereas physical arguments usually rely on statistical methods ([3-8], see also [2]).

A generalization of the self-organization principle indicates that confined Eulerian vortices may appear at intermediate times, thereby providing some arguments towards an explanation for their observation in numerical simulations. It is to be noted that the proposed generalization can be interpreted as a natural but essential extension to certain non-linear problems of the eigenfunction expansion in linear problems.

The organization of the paper is as follows. In section 2 we briefly describe the basic physical ideas. In section 3 dynamics is introduced in the s.o.-principle. Section 4 shows that the asymptotic results of Foias and Saut [1], can be formulated in this setting. Also, the methods that lead to these results are motivated by the principle. In section 5 we deal with the generalization of the principle which leads to the notion of extremal representation. Two specific examples are considered, viz. the spectral decomposition for linear operators and a non-linear representation with (interacting) solitary waves for completely integrable systems such as the KdV equation. In section 6, using a simple example of the generalization, Eulerian vortices are shown to be "close" to the asymptotic states of the 2D N-S equations.

2. PHYSICAL IDEAS.

For several specific dissipative systems the notion of self-organization appears in the literature, most notably for 2D Navier-Stokes equation [4-8], 2 and 3D MHD-equations [9,10], non-linear wave equations etc. [3,11-13]. Although closely related to such other notions as "spectral condensation" and "coherent states", self-organization seems to refer to a more specific property and allows for a more specific description thereof. The following general description of the idea is based on the review paper by Hasegawa [3].

Consider a dissipative system that can be thought of as a conservative system to which there is added dissipation (The dissipation needs not to be small in any way, so the following is not to be understood as a perturbation argument).

Assume that the underlying conservative system has at least two integrals, i.e. two constants of the motion, say A and B. While being constant when evaluated at different times when the state evolves according to the

conservative system, they will vary in time, in general, when evaluated at states that evolve according to the total system as a consequence of the dissipation.

Now assume that the effect of the dissipation is selective, in the sense that it causes the functional A to change (decrease, say) faster than the functional B. This "selective dissipation" introduces two time-scales in the problem, a fast time and a slow time associated with the decrease in A and that in B respectively. It is then argued that on the slow time scale the functional A must have a relatively small value. This leads to the statement of self-organization: for large values of time, the state of the system can well be approximated by a so-called quasi-stationary state, which is a state that minimizes A on level sets of B. The quasi-stationarity obviously refers to the slow time scale at which B is approximately constant.

Of course, stated in such generality this description of the self-organization phenomenon cannot be expected to be true without further requirements, but it emphasizes the different rôles that are played by the conservative part of the system - providing the integrals A and B and determining the quasi-stationary states - and by the dissipation - causing the selective dissipation of the integrals.

We will not go into the details of the physical methods to obtain the additional requirements. Suffices it to say that nonlinear mode interactions, leading to a normal or an inverse cascade for the spectral densities of the governing functionals, play a dominant rôle. Typically statistical methods, averaging over ensembles of flows, are used. We present a description of self-organization based on a deterministic approach in the next section.

3. THE DYNAMICAL SELF-ORGANIZATION PRINCIPLE.

Guided by the physical ideas we will now formulate the s.o.-principle in a more precise way.

First, assume that for all solutions $u(t)$ of the system, the function $B(u(t))$ continuously approaches a definite value, say β_0 . Since we are aiming at an asymptotic result, the interest is in a neighbourhood of β_0 , one-sided if the approach to β_0 is monotonically (as is often the case). Further, let for any relevant β the minimization problem

$$(3.1) \quad \text{Min } \{A(u) \mid B(u) = \beta\}$$

be well-posed, i.e. the minimum is finite and there exists at least one solution, $\phi(\beta)$ say. The value of this minimum defines the value function and will

be denoted by $A(\beta)$, and $\phi(\beta)$ will be called a quasi-stationary state, q.s.s. for short. Except from some technical continuity conditions for A and B on a suitably chosen function space, the well-posedness requires certain coercivity properties for A on level sets of B .

REMARK 3.1. In many physical examples, $A(\beta)$ is continuous and in fact differentiable. In that respect, note that the Lagrange-multiplier that enters in the equation for the q.s.s. $\phi(\beta)$, i.e.

$$(3.2) \quad \frac{\delta A}{\delta u}(\phi) = \lambda \frac{\delta B}{\delta u}(\phi) \quad , \quad B(\phi) = \beta$$

is tangent to the curve $\beta \rightarrow A(\beta)$ at β :

$$(3.3) \quad \lambda = \frac{d}{d\beta} A(\beta).$$

REMARK 3.2. Another way to look at the q.s.s.'s is as follows. It is a general fact that if some dynamical system has an integral, I say, then the null-set of the first variation of I , i.e. $\{\phi \mid \frac{\delta I}{\delta u}(\phi) = 0\}$ is an invariant set for the flow of this system ([14]). Therefore, since the minimizers of (3.1) satisfy $\frac{\delta}{\delta u}(A - \lambda B) = 0$ for some $\lambda \in \mathbb{R}$, it follows that if they are all isolated, in particular if there is only one minimizer, each one must be a time-independent solution of the conservative part of the system. If the set of minimizers is connected, a specific minimizer needs not to be a time-independent solution but is then a state on the trajectory of a special solution of the conservative system (see example 5.3).

We will now propose a modification of the s.o.-principle by introducing dynamics in the minimization problem (3.1) to account for the fact that the functional B is changing as a function of time.

We will say that a dynamical system evolves asymptotically according to the dynamical s.o.-principle with respect to a set U (and the integrals A and B) if for any solution $u(t)$ with initial data $u(0) \in U$ there exist a function $t \rightarrow \beta(t)$ and a family of q.s.s.'s $\phi(\beta(t))$ such that

$$(3.4) \quad u(t) = \phi(\beta(t)) + \xi(t) \quad ,$$

where ξ is a function that decreases to zero as $t \rightarrow \infty$ like

$$(3.5) \quad \frac{|\xi(t)|}{\|\phi(\beta(t))\|} \rightarrow 0 \quad ,$$

for some norm $\|\cdot\|$.

This description specifies the idea that at any (large) time t_0 the state of the system $u(t_0)$ is approximately described by a q.s.s. of the conservative part, viz. $\phi(\beta(t_0))$, but at different times by a different state, determined by the value of the constraint B at that time. Since, for each value of β , $\phi(\beta)$ is obtained as a minimizer, this description can be interpreted as a variational principle for the asymptotics of a dissipative system: a principle of dissipating constraint minimizers.

An illuminating way to interpret the s.o.-process is obtained by looking at an integral-diagram. In this diagram (see Fig. 1), the values of A and B are depicted along the vertical and horizontal axis respectively. The curve $(\beta, A(\beta))$ is the q.s.s.-curve; below this curve no physical states are possible by the definition of A .

For a given solution $u(t)$ the successive values of A and B determine the integral curve $t \rightarrow (B(u(t)), A(u(t)))$.

FIGURE 1

The integral diagram for the case $\beta_0 = 0$, $A(\beta_0) = 0$, and monotonic decrease of B . The quasi-steady-state curve consists of solutions of the minimization problem (3.1), $\beta \in \mathbb{R}_+$. The dotted line represents an integral curve: tangency to the q.s.s.-curve at the origin is a consequence of self-organization.

FIGURE 2

The integral diagram for 2D N-S. The q.s.s.-curve is the straight line $W = \mu_1 E$, supporting the Taylor vortex (4.10) with $\underline{k} = (1, 1)$. Tangency of an integral curve is equivalent to selforganization.

Given some continuity properties for A and B, the asymptotic approach (3.4) generally implies that the integral curves, for initial data in U, are tangent to the q.s.s. curve at β_0 .

On the other hand, tangency of the integral curves to the q.s.s.-curve may imply the asymptotic approach (3.4) provided several stringent coercivity conditions for A and B are satisfied.

Note that the required tangency at β_0 in fact specifies the way in which the statement "A decreases faster than B" has to be understood. To investigate this property in specific cases, the functionals

$$Q(u) := \frac{A(u)}{B(u)} \quad \text{and} \quad \Lambda(u) := \frac{\dot{A}(u)}{\dot{B}(u)}$$

turn out to be quite useful. The quantity Q determines the direction vector to a point at the integral curve, whereas Λ , the dissipation rate quotient, is the tangent to this curve at the point.

REMARK 3.3. Simple linear examples that can demonstrate the ideas above are 1D dissipative wave equations like

$$u_t = u_{xxx} - \sigma Du ,$$

for $u = u(x,t)$, periodic in $x \in (-\pi, \pi)$ with $\int u = 0$. For $\sigma = 0$, a conservative system is obtained with many quadratic integrals, such as $A = \int u_x^2$ and $B = \int u^2$. A Fourier analysis clearly shows that for $\sigma > 0$, different choices of the operator D, e.g. $D_1 = -\partial_x^2$, $D_0 = \text{identity}$, $D_{-1} = (-\partial_x^2)^{-1}$, leads to completely different space-like patterns for large time; of the three examples, only D_1 leads to s.o. as described above.

In the next section we treat the 2D Navier-Stokes equations along the lines indicated above. The same approach has been used to prove s.o. in some non-linear dissipative wave equations, like the KdV-Burgers equation; these results will be presented elsewhere. A generalization, involving more constraints, will be presented in sections 5 and 6.

4. SELF-ORGANIZATION IN 2D NAVIER-STOKES EQUATIONS.

In this section we show that the asymptotic results of Foias and Saut ([1], see also [2]) rigorously prove the dynamical s.o.-principle defined in the foregoing section. This illustrates and justifies the deterministic approach and makes the physical statements obtained by spectral considerations rigorous and precise.

For ease of presentation we consider a simple 2D geometry, a periodic square $[-\pi, \pi]^2$, and we will look for solutions with a certain symmetry around the axes. If $\underline{v}$ denotes the velocity in the x, y -plane, the vorticity is given by $\underline{\omega} = \text{curl } \underline{v} = (0, 0, \omega)$. The component ω is related to Stokes stream function ψ by $-\Delta\psi = \omega$.

The special configuration mentioned above is obtained for the boundary conditions

$$(4.1) \quad \psi = 0 \text{ and } \omega = 0 \text{ on } \partial R,$$

where R is the quarter-cell $R = [0, \pi]^2$. Odd reflections of ψ about the axes then produces a solution on the whole square.

The Navier-Stokes equations can be written like

$$(4.2) \quad \begin{aligned} \nabla \cdot \underline{v} &= 0 \\ \omega_t + \underline{v} \cdot \nabla \omega &= \nu \Delta \omega \end{aligned} \quad \text{on } R,$$

where ν is the viscosity coefficient. In the absence of dissipation, i.e. if $\nu = 0$, the Euler equations result:

$$(4.3) \quad \begin{aligned} \nabla \cdot \underline{v} &= 0 \\ \omega_t + \underline{v} \cdot \nabla \omega &= 0 \end{aligned} \quad \text{on } R.$$

As is readily verified, the Euler equations have as integral the energy

$$(4.4) \quad E = \int \frac{1}{2} \underline{v}^2 dx$$

(integration over R is understood here and in the following), as well as integrals of the form

$$(4.5) \quad \int F(\omega) dx,$$

for arbitrary functions F . Choosing $F = \frac{1}{2} \omega^2$, the integral

$$(4.6) \quad W = \int \frac{1}{2} \omega^2 dz$$

is usually called the enstrophy. In this section we will use the functionals E

and W to describe the dynamical s.o.-principle.

From a mathematical point of view, the minimization problem

$$(4.7) \quad \text{Min } \{W(\omega) \mid E(\omega) = e\}$$

is well-posed for $e \geq 0$ (for the boundary conditions (4.1) and $\omega \in L_2(R)$). The solution of (4.7) is unique up to sign and given by

$$\Omega(e) = \pm \sqrt{e} \hat{\Omega},$$

where $\hat{\Omega}$ is the positive eigenfunction, normalized such that $E(\hat{\Omega}) = 1$, corresponding to the lowest eigenvalue μ_1 of the linear eigenvalue problem

$$(4.8) \quad \begin{aligned} -\Delta \omega &= \lambda \omega & \text{in } R \\ \omega &= 0 & \text{on } \partial R. \end{aligned}$$

The value function $W(e) := W(\Omega(e))$ is then simply $\mu_1 e$, so

$$(4.9) \quad \pm \sqrt{e} \hat{\Omega} = \Omega(e) \in \text{Min } \{W(\omega) \mid E(\omega) = e\} = W(e) = \mu_1 e.$$

Other eigenfunctions $\Omega_{\underline{k}}$ and eigenvalues $\lambda_{\underline{k}}$, $\underline{k} \in \mathbb{N} \times \mathbb{N}$, are of some interest too in the following, and can be obtained explicitly. The sequence of increasing eigenvalues will be denoted by μ_j , $j \geq 1$. Specifically, $\mu_1 = \lambda_{(1,1)} = 1$, and $\hat{\Omega} = a \sin x \cdot \sin y$, where a is determined by the normalization.

These eigenfunctions are exact, time independent solutions of the Euler equations. The q.s.s.-curve in the E-W diagram is given by $(e, \mu_1 e)$, $e \in \mathbb{R}_+$ (see Fig. 2)

REMARK 4.1 Remarkable is the fact that for N-S the dissipation is such that the functions

$$(4.10) \quad \Omega_{\underline{k}} \exp(-\nu \lambda_{\underline{k}} t)$$

are exact solutions of the N-S equation, as has been observed already by Taylor (see [15]); we will call these solution (planar) Taylor vortices.

Guided by the s.o.-principle, let us now investigate the selective

dissipation of E and W . For $v \neq 0$, the change in E and W is easily found to be

$$(4.11) \quad \dot{E} = -2vW \quad \text{and} \quad \dot{W} = v \int (\nabla w)^2.$$

Hence, both E and W decrease monotonically to zero as $t \rightarrow \infty$ for any solution. The functionals Q and Λ defined by $Q = W/E$ and $\Lambda = \dot{W}/\dot{E}$ are given by

$$(4.12) \quad Q = \frac{\int w^2}{\int v^2} \quad \text{and} \quad \Lambda = \frac{\int (\nabla w)^2}{\int w^2}.$$

Note that both Q and Λ are homogeneous of degree zero and so depend only on the spatial structure $w/\|w\|$ and not on the amplitude. From the definition of μ_1 it follows that

$$(4.13) \quad Q(w) \geq \mu_1 \quad \text{for all } w.$$

A simple application of Cauchy-Schwartz inequality proves the following result:

$$\begin{aligned} \text{LEMMA 4.2} \quad \Lambda(w) - Q(w) &\geq 0 \text{ for all } w, \\ &= 0 \text{ iff } w \text{ is an eigenfunction } \underline{\omega}_k \text{ of (4.8)} \end{aligned}$$

With (4.11) the change in Q is easily found to be

$$(4.14) \quad \dot{Q} = -2vQ(\Lambda - Q).$$

The lemma therefore implies that Q is constant, $= \lambda_{\underline{k}}$, for a Taylor vortex (4.10), and monotonically decreasing for all other solutions.

Consequently, with (4.13), Q decreases to some definite value $\mu \geq \mu_1$. In fact, it can be shown that

$$(4.15) \quad \lim_{t \rightarrow \infty} Q(w(t)) = \mu_{\underline{k}} \text{ for some } k \in \mathbb{N}.$$

(See [1,2] for details; roughly: if $Q \neq \mu$, then $\dot{Q} \neq 0$ and hence $\Lambda - Q \neq 0$ which can only be true for some eigenfunction $\underline{\omega}_k$, and then $\mu = \lambda_{\underline{k}}$).

This limiting behaviour led Foias and Saut to define invariant sets like

$$(4.16) \quad I_k := \{w_0 \mid \text{the solution } w(t) \text{ with initial value } w_0 \text{ satisfies } \lim_{t \rightarrow \infty} Q(w(t)) = \mu_{\underline{k}}\}.$$

In an intuitive sense, all the sets I_k , $k \geq 2$, are "unstable" as a consequence of the monotonic decrease of Q .

The results above imply that for $\omega_0 \in I_1$ it holds that $Q \rightarrow \mu_1$ and $\Lambda \rightarrow \mu_1$ as $t \rightarrow \infty$. Hence we have obtained that these integral curves are tangent to the q.s.s.-curve in the E-W diagram. (see Fig. 2).

With some more functional analytic work it can be proved that then the solution $\omega(t)$ tends to the corresponding Taylor vortex. Summarizing the results of [1,2], the next theorem holds.

THEOREM 4.3 If $\omega(t)$ is the solution of the N-S equations with initial value $\omega_0 \in I_1$, then

$$(4.17) \quad \omega(t) = \Omega(e(t)) + O(e^{-\nu \sigma t}),$$

where

$$(4.18) \quad e(t) = c_0 \cdot \exp(-2\nu\mu_1 t) \text{ for some constant } c_0 > 0,$$

and

$$\sigma > \mu_1.$$

This result rigorously proves the dynamical s.o.-principle: at each t the solution ω is approximated by a solution of the minimization problem (4.7), i.e. by an exact solution of the Euler equations, but for different values of t by a different solution, determined by the value of $e(t)$.

Note that the function $e(t)$ satisfies the differential equation:

$$(4.19) \quad \dot{e}(t) = -2\nu\mu_1 e(t)$$

which determines the constraint dynamics in this case. This equation could have been obtained directly by adapting the equation $\dot{E} = -2\nu W$, valid for solutions of N-S., to the specific classes of functions $\Omega(e)$:

$$\dot{e} = -2\nu W(\Omega(e)) = -2\nu\mu_1 e.$$

REMARK 4.4 It is not clear at the moment how the initial data for e should be chosen such that the correct asymptotic value c_0 in (4.18) is obtained. Clearly c_0 depends on the initial condition; generally $e(0) \neq E(\omega_0)$, except

Taylor vortices.

ARK 4.5 The asymptotics described above does not seem to agree with results of "long-time" numerical simulations of 2D N-S. In fact, it has been observed ([16,17]) that the large time behaviour indicates the appearance of confined vortices, i.e. a vorticity distribution with a support that consists of a number of (interacting) disjoint bounded sets. The precise mechanism that is driving this fragmentation of vorticity is not clear yet. It seems that these confined vortices are formed already at the initial stages of the evolution, and that the qualitative numerical results are rather independent of the fineness of the grid (supporting the relevance of the calculation; [18]).

The apparent discrepancy between the predicted asymptotic behaviour and the numerical results has been an important motivation to investigate the generation described in section 6.

EXTREMAL REPRESENTATIONS.

The variational formulation for a quasi-stationary state, in the notation of section 3, $\text{Min} \{A \mid B = \beta\}$, can be interpreted in the presence of another integral C as

$$1) \quad \text{Min} \{A(u) \mid B(u) = \beta\} = \text{Min}_\gamma [\text{Min} \{A(u) \mid B(u) = \beta, C(u) = \gamma\}],$$

as if the value of C has already been optimized. Generally this cannot be expected to be true, since in a dynamical system the changes in B and C are related and will be optimized simultaneously, possibly with different speeds. Therefore it is clear that

$$2) \quad \text{Min} \{A(u) \mid B(u) = \beta(t), C(u) = \gamma(t)\}$$

describe for a suitable constraint dynamics $t \rightarrow (\beta(t), \gamma(t))$ more accurately the intermediate stages of the dynamics.

We will show in the next section how this idea can be exploited for the 2D N-S equations.

Here we will point at a more general and more fundamental way to envisage the use of such extremal problems as a nonlinear method to approximate functions by functions from the set of extremizers. Such an extremal representation is a generalization of the eigenfunction expansion for linear problems.

Stated in general, consider on a function space U a set of functionals, $\{A, C_k, 0 \leq k \leq N-1\}$, possibly infinitely many: $N = \infty$. Assuming that the

minimization problem

$$(5.3) \quad \text{Min } \{A(u) \mid C_k(u) = \gamma_k, 0 \leq k \leq N-1\}$$

is well-posed for values of the constraints $(\gamma_0, \dots, \gamma_{N-1})$ in some set $C_N \subset \mathbb{R}^N$, let $S(\gamma_0, \dots, \gamma_{N-1})$ denote the set of minimizers of (5.3) for the given values γ_k of the constraints, and let S_N be the union of all these extremizers if all values of the constraints are considered:

$$(5.4) \quad S_N = \{S(\gamma_0, \dots, \gamma_{N-1}) \mid (\gamma_0, \dots, \gamma_{N-1}) \in C_N\}.$$

PROPOSITION 5.1 The sets S_N are non-decreasing: $S_M \subset S_N$ for $M \leq N$.

Indeed, generalizing the idea of (5.1), if $\phi \in S(\gamma_0, \dots, \gamma_{M-1})$, define $\hat{\gamma}_k := C_k(\phi)$ for $M \leq k \leq N-1$. Then $\phi \in S(\gamma_0, \dots, \gamma_{M-1}; \hat{\gamma}_M, \dots, \hat{\gamma}_{N-1})$.

With the ultimate aim in mind to use (5.3) for an extension of the (dynamical) s.o.-principle, the relevance of this generalization will be clear, if not already from the simple example (5.1) above and from the application in the next section.

It would therefore be interesting to be able to investigate these sets S_N in detail. For instance, to know which functions u can be well approximated, and in what sense, by an extremizer $\phi(\gamma_0, \dots, \gamma_{N-1})$ of (5.3) and if, in case infinitely many constraints are considered, the set $\bigcup_N S_N$ is complete in some sense.

We will present two specific examples for which such questions can be answered. Although we use only facts that are well-known, the interpretation in the present context is interesting.

EXAMPLE 5.2 Eigenfunction expansion.

Let L be an unbounded self-adjoint operator on a Hilbert space H , with inner product $(\cdot, \cdot)$, which has a complete set of eigenfunctions $\phi_1, \phi_2, \dots$ and corresponding increasing eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k \rightarrow \infty$. Then any function $u \in H$ can be represented like $u = \sum_{k=1}^{\infty} \hat{u}_k \phi_k$ with $\hat{u}_k = (u, \phi_k)$. In a well-defined way, u is approximated by a finite sum $u_N = \sum_{k=1}^N \hat{u}_k \phi_k$. Define linear functionals C_k on H by $C_k(u) = (u, \phi_k)$, $k \in N$. Then, in an almost trivial way, u_N can also be interpreted as an extremal representation for u , since u_N can also be considered as the (unique) solution of the minimization problem.

$$(5.5) \quad \text{Min } \{(v, Lv) \mid C_k(v) = \gamma_k, 1 \leq k \leq N\}$$

when the constraints γ_k are chosen appropriately.

EXAMPLE 5.3 Completely integrable systems.

For continuous systems for which an infinite number of integrals are known, the extremal representation is more interesting. For definiteness we consider as a specific example the Korteweg-de Vries equation

$$(5.6) \quad u_t + u_{xxx} + 6uu_x = 0, \quad x \in \mathbb{R},$$

for which infinitely many integrals are known, $I_{-1}, I_0, I_1, \dots$. The first of these are given by

$$I_{-1}(u) = \int u dx, \quad I_0(u) = \frac{1}{2} \int u^2 dx, \quad I_1(u) = \int \left(\frac{1}{2} u_x^2 - u^3 \right) dx,$$

which can be interpreted as the total mass, horizontal momentum and energy respectively.

To understand the dynamics of smooth functions decaying sufficiently fast at infinity, the variational problems

$$(5.7) \quad \text{Max } \{I_0(u) \mid I_1(u) = \gamma_1, \dots, I_N(u) = \gamma_N\}$$

are extremely relevant. In fact, for $N = 1$, the solution set consists of all translates of the wave profile of a solitary wave, an exact travelling wave solution of (5.6); the profile, such as its amplitude, as well as the dynamic propagation speed are determined by the value of γ_1 . For $N \geq 2$, (5.7) describes the interaction of N solitary waves, i.e. the solution set $S(\gamma_1 \dots \gamma_N)$ consists of all functions $\{u(x, t) \mid t \in \mathbb{R}\}$ where $u(x, t)$ satisfies (5.6) and consists for $t \rightarrow \pm \infty$ of N widely separated solitary waves which are determined by the values of $\gamma_1, \dots, \gamma_N$.

From the inverse scattering theory for (5.6) it is known that each initial value $u_0(x)$ develops as $t \rightarrow \infty$ into a specific number of well defined solitary waves, which are determined by the discrete eigenvalues of the Schrödinger scattering problem with potential $u_0(x)$, superimposed with a perturbation that is determined by the reflection and transmission coefficients of the continuous spectrum. See e.g. [19] and the references therein for the original papers.

These results specify in this case in what sense an arbitrary function can be approximated by an element from the sets S_N .

REMARK 5.4. For the application of these ideas to the 2D Navier-Stokes equations, more integrals of the Euler-equation should be added to (4.7). For a related problem, exploiting the invariance of $\int F(\omega)$ for each function F , an interesting basis of integrals has been proposed by A. Eydeland and B. Turkington (private communication, to be published). Together with linear- and angular-momentum integrals, a rather large variety of extremal vorticity distributions may be expected. The example in the next section can be considered to be a first step in the determination of such extremal distributions.

6. CONFINED VORTICES IN 2D N-S.

For the 2D Navier-Stokes equations (4.2) we will describe a simple generalization of the s.o.-principle by adding constraints to the minimization problem (4.7). This will show that confined vortices are obtained if the added constraint does not have an optimal value.

Specifically, let C be the circulation:

$$(6.1) \quad C = \int \omega dx,$$

and consider instead of (4.7) the minimization problem

$$(6.2) \quad \text{Min } \{W(\omega) \mid E(\omega) = e; C(\omega) = \gamma; \omega \geq 0 \text{ on } R\}, \quad e \geq 0, \gamma \geq 0.$$

The value function will be denoted by $W_2(e, \gamma)$ and a solution by $\Omega(e, \gamma)$:

$$(6.3) \quad \Omega(e, \gamma) \in \text{Min } \{W \mid E = e, C = \gamma, \omega \geq 0\} = W_2(e, \gamma).$$

Note that the requirement $\omega \geq 0$ could equally well be described by an integral constraint: $\int \omega_- = 0$, where ω_- denotes the negative part of ω , and then (6.2) fits the formulation of section 5.

Since $\int F(\omega) dx$ is an integral for the Euler equation for any function F , the solution $\Omega(e, \gamma)$ is an exact, time-independent solution of the Euler equations (see Remark 3.2).

The positive q.s.s.'s $\Omega(e)$ of section 4, the minimizers of (4.7), are obtained as solutions of (6.2) by taking the optimal choice of γ in (6.2), viz.

$$(6.4) \quad \gamma^*(e) = C(\Omega(e)).$$

Hence,

$$(6.5) \quad \mu_1 e = W(e) = W_2(e, \mu^*(e)) = \min_{\gamma} W_2(e, \gamma).$$

In a three-dimensional C-E-W integral diagram, the q.s.s.'s now form a space curve, lying in the plane $W = \mu_1 E$ and with projections that are parabola in the C-E and C-W planes (see Fig. 3).

The s.o.-principle proved in section 4 implies that each integral curve (for solutions with initial data in I_1) is tangent to this space curve at the origin.

FIGURE 3

The extended integral diagram for 2D N-S. The q.s.s.-curve is now given by $E \in \mathbb{R}_+$, $W = \mu_1 E$ and $C = \gamma^*(E)$. Confined vortices, solutions of (6.2), are obtained for $0 < \gamma < \gamma^*(E)$.

Now consider non-optimal values of γ . In particular, let

$$(6.6) \quad 0 < \gamma < \gamma^*(e).$$

Then the minimization problem (6.2) has a unique solution $\Omega(e, \gamma)$ which satisfies a free boundary problem and which is a confined vortex. In fact, standard results from the calculus of variations show that $\Omega = \Omega(e, \gamma)$ satisfies the equation

$$(6.7) \quad \Omega = (\lambda\psi - \sigma)_+ \quad \text{in } R,$$

where ψ is the streamfunction, $(\)_+$ denotes the positive part and λ and σ are multipliers due to the constraints E and C respectively. Equation (6.7) implies that Ω satisfies

$$-\Delta\Omega = \lambda\Omega, \quad \text{in } \{x \in R \mid \Omega(x) > 0\}.$$

Hence γ is positive. On the other hand, $-\sigma = \frac{\partial W_2}{\partial \gamma}$ is negative for $\gamma \in (0, \gamma^*(e))$. Hence, since $\psi = 0$ on ∂R , $\lambda\psi - \sigma$ is negative on ∂R . From this it follows that the support of Ω is a set that is strictly contained in R , which is what is meant by saying that Ω is a confined vortex.

Remembering that R is a quartercell, we have thus obtained a periodic distribution of vortices with alternating signs.

The support of $\Omega(e, \gamma)$ is expanding for increasing $\gamma \rightarrow \gamma^*(e)$ at fixed e , and for decreasing $e \rightarrow \gamma^{*-1}(\gamma)$ at fixed γ . The limiting function is precisely $\hat{\Omega}(e)$, which is the non-confined q.s.s., a multiple of $\hat{\Omega}$, which has support all of R .

The results above show that even for values of γ arbitrary close to $\gamma^*(e)$ there exist exact solution of the Euler equation consisting of confined vortices.

Guided by the extension of the dynamical s.o.-principle, one may look for one parameter families of such solutions:

$$(6.8) \quad t \rightarrow \Omega(e(t), \gamma(t)).$$

It is shown in [20] that dynamical equations for the functions $e(t)$ and $\gamma(t)$, the constraint dynamics, can be formulated such that (6.8) is an approximate solution of the N-S equations with initial value $\Omega(e_0, \gamma_0)$ (for $\gamma_0 < \gamma^*(e_0)$). In particular, the space curve $t \rightarrow (\gamma(t), e(t), w(t))$, with

$w(t) = w_2(e(t), \gamma(t))$, is tangent to the q.s.s.-curve, which implies that $\Omega(e(t), \gamma(t))$ has the required asymptotic behaviour.

This result can be interpreted as an approximate description of the dissipation of confined vortices. Equally well, it resolves (for this special configuration of vortices) the apparent discrepancy between the asymptotic behaviour and observed numerical vorticity distributions, as described in Remark 4.5. More general confined vorticity distributions will be dealt with along the lines of this paper in a future publication.

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