# Local Approximation Schemes for Ad Hoc and Sensor Networks*

Fabian Kuhn
Computer Engineering and Networks Laboratory
ETH Zurich, Switzerland
kuhn@tik.ee.ethz.ch

Thomas Moscibroda
Computer Engineering and Networks Laboratory
ETH Zurich, Switzerland
moscitho@tik.ee.ethz.ch

Tim Nieberg
Department of Applied Mathematics
Universiteit Twente, The Netherlands
t.nieberg@utwente.nl

Roger Wattenhofer
Computer Engineering and Networks Laboratory
ETH Zurich, Switzerland
wattenhofer@tik.ee.ethz.ch

## ABSTRACT

We present two local approaches that yield polynomial-time approximation schemes (PTAS) for the Maximum Independent Set and Minimum Dominating Set problem in unit disk graphs. The algorithms run locally in each node and compute a $(1+\varepsilon)$-approximation to the problems at hand for any given $\varepsilon > 0$. The time complexity of both algorithms is $O(T_{\mathrm{MIS}} + \log^* n/\varepsilon^{O(1)})$, where $T_{\mathrm{MIS}}$ is the time required to compute a maximal independent set in the graph, and $n$ denotes the number of nodes. We then extend these results to a more general class of graphs in which the maximum number of pair-wise independent nodes in every $r$-neighborhood is at most polynomial in $r$. Such *graphs of polynomially bounded growth* are introduced as a more realistic model for wireless networks and they generalize existing models, such as unit disk graphs or coverage area graphs.

## Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: Nonnumerical Algorithms and Problems—*computations on discrete structures*;
G.2.2 [**Discrete Mathematics**]: Graph Theory—*network problems*

## General Terms

Algorithms, Theory

## Keywords

wireless ad hoc networks, maximum independent set, minimum dominating set, approximation, distributed algorithm

*This work is partially supported by the European research project Embedded WiSeNts (FP6-004400) and by the Dutch project Smart Surroundings (BSIK-03060).


*DIALM-POMC'05*, September 2, 2005, Cologne, Germany.


## 1. INTRODUCTION

Wireless ad hoc and sensor networks consist of autonomous devices communicating via radio. There is no central server or common infrastructure which could be used for the organization of the network. The coordination necessary for efficient communication has to be carried out by the nodes themselves. That is, the nodes have to build up a communication service by applying a distributed algorithm.

In the literature, numerous proposals to structure wireless networks have been presented. The main purposes of such structures are to provide the communication between adjacent nodes (MAC layer protocols, [32, 31, 14]) and between distant nodes (routing protocols, [34, 2, 36, 33, 35]), to improve the energy efficiency of the network [10], or to enable a proper initialization of the network [18, 27]. As diverse as the different structures may be, there are a few key primitives on which most of them are built. The most important such primitives are classical graph-theoretic objects such as dominating sets, independent sets, or vertex colorings. It is therefore evident that analyzing the distributed complexities of the corresponding graph-theoretic problems is an essential prerequisite for a profound understanding of the principles of ad hoc and sensor networks. In particular, we are interested in the complexities of the described problems on the kind of network topologies which can typically arise in the context of wireless networks.

Depending on the particular purpose for which the network structure is to be employed, the underlying communication model needs to be adjusted, such that the specific conditions are appropriately modeled. In contrast to papers on the initialization phase [18], we concentrate in this paper on networks that are already established and in which the nodes have already set up some basic infrastructure. Particularly, we assume that there is a MAC layer which provides reliable point-to-point connections between neighboring nodes.

Very often, wireless ad hoc and sensor networks are modeled as *unit disk graphs* (UDG). In a UDG, it is assumed that all network nodes are located on a two-dimensional plane. All wireless nodes have the same transmission range, two nodes can communicate directly with each other, whenever they are within each others transmission range. Hence, a

graph $G$ is a unit disk graph exactly if there is an assignment of coordinates in $\mathbb{R}^2$ such that there is an edge between two nodes if and only if their distance is at most 1. Many classical graph theory problems are easy to approximate on UDGs. For instance, a maximal independent set (MIS) is a constant approximation for minimum dominating set and for maximum independent set and a $(\Delta+1)$-coloring is a constant approximation for minimum vertex coloring. These simple observations suffice to obtain efficient sequential and distributed constant approximation algorithms for those three problems [26]. In the sequential scenario, however, even better solutions are known. Particularly, *polynomial approximation schemes (PTAS)* are known for various combinatorial optimization problems in unit disk graphs. That is, given any $\varepsilon > 0$, the algorithm computes in polynomial time a solution with an approximation ratio of $1+\varepsilon$. If a representation (coordinates in $\mathbb{R}^2$) of the UDG is known, polynomial-time approximation schemes for minimum dominating set and maximum independent set exist [13]. That is, for an arbitrary constant $\varepsilon > 0$, a $(1+\varepsilon)$-approximation can be computed in polynomial time. In [30, 29], a PTAS which does not rely on coordinate information has been presented.

Unfortunately, similar results are not known in distributed settings, where each node has to base its decision on information gathered in its local neighborhood via communication. In this paper, we address this by extending the work of [30, 29] to obtain the first *distributed approximation schemes* for the minimum dominating set and the maximum independent set problems in unit disk graphs. Specifically, we present an algorithm for the two problems that compute an $(1+\varepsilon)$-approximate solution in time $O(T_{\text{MIS}} + \log^* n/\varepsilon^{O(1)})$, where $\varepsilon > 0$ can be chosen arbitrarily, $T_{\text{MIS}}$ denotes the time to compute a maximal independent set (MIS) in the network graph, and $n$ is the number of nodes.

Our algorithms are not restricted to unit disk graphs. Instead, they are formulated to work on a more general family of graphs which we call *polynomially growth-bounded graphs*. A graph is said to be polynomially growth-bounded if the maximum number of pair-wise independent nodes in every $r$-neighborhood is at most polynomial in $r$. Note that the growth of a UDG, is at most quadratic in $r$ and hence, every UDG is growth-bounded. Our results carry over to general growth-bounded graphs. As shown in [21], a MIS can be computed deterministically in time $T_{\text{MIS}} \in O(\log\Delta \log^* n)$ in growth-bounded graphs, hence yielding an $O(\log\Delta \log^* n + \log^* n/\varepsilon^{O(1)})$ time algorithm in combination with our algorithm.

In [20], the UDG definition has been extended to arbitrary metric spaces, resulting in the notion of *unit ball graphs* (UBG). The nodes of a unit ball graph are assumed to be points of a metric space and two nodes are adjacent if and only if their distance in the given metric space is at most 1. If the underlying metric is doubling, that is, if it has constant doubling dimension[1], it is not hard to see that UBGs are polynomially growth-bounded (see Section 3). In [20], it has been showed that if nodes can measure the inter-node distances to their neighbors, a MIS can be computed in time $O(\log^* n)$ on a UBG if the underlying metric is doubling. Thus, for UBGs, our algorithm yields an $(1+\varepsilon)$-approximate solutions for minimum dominating set and maximum independent set in time $O(\log^* n/\varepsilon^{O(1)})$.

To the best of our knowledge, the algorithms of this paper are the first distributed approximation schemes for combinatorial optimization problems. The only previous distributed approximation schemes were given for covering and packing linear programs where solutions may be fractional and hence, the problem becomes easier because no symmetries have to be broken, e.g. [5, 23].

The remainder of this paper is organized as follows. An overview of relevant previous work in given in Section 2. We formally introduce the model of computation in Section 3. Section 4 introduces the concept of locally optimal subsets which will be the basis of the algorithms presented and analyzed in Section 5. Finally, Section 6 concludes the paper.

## 2. RELATED WORK

As described in the introduction, the distributed time-complexity of computing important network coordination structures such as dominating sets or independent sets has been studied intensively over the last years.

As for the minimum dominating set problem in general graphs, algorithms with provably efficient time-complexity and approximation guarantees have been given in [16, 24], whereas in [19], distributed time-lower bounds have been proven. For the unit disk graph, a multiplicity of algorithms have been proposed in the wireless networking community. Unfortunately, most of them have the property that in the worst case, they have either linear running time [34, 2, 35] or no approximation guarantees [36]. In [12], an algorithm was recently proposed that computes a constant approximation to the connected dominating set problem in polylogarithmic time in a model that accounts for possible collisions of messages. In an even harsher model of computation, [27] presents a polylogarithmic time algorithm for computing a maximal independent set, which is a constant approximation to the dominating set problem in unit disk graphs. Finally, the fastest currently known algorithm for the minimum dominating set problem in unit disk graphs has been given in [20], which achieves a constant approximation in time $O(\log^* n)$, albeit in a message passing model without collisions and only if nodes can measure distances to their neighbors.

While many of the above works are capable of producing solutions that approximate the optimum up to a specific constant, none of them sheds light into the time required in order to obtain an arbitrarily good approximation, i.e., an $(1+\varepsilon)$-approximation for any $\varepsilon > 0$. In other words, there is no prior work on a *distributed PTAS* for the problems of interest in this paper.

This absence is in contrast to the centralized case, in which polynomial time approximation schemes for independent and dominating set problems in geometric intersection graphs, especially in disk graphs, have been studied in great detail. However, most approaches that yield a PTAS exploit the representation, and then apply a shifting strategy introduced in [4, 15]. In [11], a PTAS for the Maximum Independent Set problem on general disk graphs is described, and in [13], the Minimum Dominating Set problem is considered for unit disk graphs together with other problems for which the shifting strategy is viable. In [7], a PTAS for the Minimum Connected Dominating Set problem is given.

[1]A metric's doubling dimension is the smallest $\rho$ such that every ball can be covered by at most $2^\rho$ balls of half the radius. For instance, the doubling dimension of the two-dimensional Euclidean plane is a small constant.

Approaches based on the shifting strategy are inherently central and cannot efficiently be adapted to work in a distributed context. The graph is partitioned into boxes or stripes, e.g. for independent set construction by removing all vertices alongside of designated boundaries. Then, a candidate solution is created by solving each component separately and combining the partial subsets. This is done for several disjunctive and exhaustive boundaries, and the best overall candidate solution is returned as desired solution. Clearly, such an approach requires some sort of centralized control for gathering the partial solutions and deciding on the best solution among these.

Hence, even when considering centralized approaches, the case when there is no geometric representation of a (unit) disk graph given is significantly different: Computing a corresponding representation is an NP-hard problem [6], in fact, it does not even admit a PTAS [22]. For this case, there also exist approximation schemes that do not rely on coordinates of the vertices given in [30] and [29].

# 3. MODEL AND DEFINITIONS

A wireless ad-hoc network is modeled by a graph $G = (V, E)$, where the vertices $V$ represent the nodes and two vertices are adjacent if they can communicate with one another over the wireless medium.

For the sake of simplicity, we present our local algorithms in a synchronous message passing model in which time is divided into rounds, and in each round, a vertex is able to send a message to all of its neighbors. However, note that at the cost of higher message complexity, our algorithms can also be employed in asynchronous settings using the notion of synchronizers [3]. In each communication round, a node can send a message to each of its neighbors in the graph. In other words, we study the classic *message-passing model* which abstracts away collisions.

In a graph $G = (V, E)$, a *dominating set* $S \subseteq V$ is a subset of the nodes such that every node is either in $S$ or has at least one neighbor in $S$. The *Minimum Dominating Set Problem* is the problem of finding a dominating set of minimal cardinality. While two nodes in a dominating set may be neighbors in $G$, an *independent set* of $G$ is a subset of $R \subseteq V$ such that no two nodes $u, v \in S$ are neighbors. In this paper, two specific independent sets will play a vital part. First, a *maximal independent set* (MIS) is an independent set $R \subseteq V$ in which every node $v \notin R$ has at least one neighbor in $R$. In other words, a MIS is an independent dominating set. Finally, a *maximum independent set* is the independent set with maximum cardinality. Note that while a *MIS* can easily be computed using a greedy approach, finding a *maximum independent set* is a strongly NP-hard problem, even when restricting the input to UDGs [8].

For any two vertices $u, v \in V$, we denote by $d(u, v)$ the shortest hop-distance between $u$ and $v$. Using this, we call $\Gamma_r(v)$ the (closed) $r$-neighborhood of $v$, that is, for any $r \geq 0$, and $v \in V$, we define $\Gamma_r(v) := \{u \in V \mid d(u, v) \leq r\}$.

The communication graphs formed by wireless networks typically have the structure that not too many nodes that are in physical proximity can be independent of one another. While the frequently studied unit disk graph captures this limit in the number of independent nodes in a neighborhood, it may be a too optimistic and rigid model. In this paper, we therefore study a more general family of graphs to model wireless networks. Specifically, we are interested in graphs that have the property of polynomially bounded growth, which we define as follows.

DEFINITION 3.1. ***(Growth-Bounded Graph)*** *An undirected graph $G = (V, E)$ is called a* growth-bounded *if there exists a function $f(r)$ such that for every $v \in V$ and $r \geq 0$, the size of the largest independent set in the $r$-neighborhood $\Gamma_r(v)$ is at most $f(r)$. Further, we say that $G$ has* polynomially *bounded growth if $f(r)$ is a polynomial $p(r)$.*

The term *polynomial growth* has been used in the past with a different connotation and we should therefore clarify the relationship between these definitions. Especially in the context of P2P networks, polynomial growth means that the number of nodes at distance $r$ from any given node depends polynomially on $r$ (e.g. [1, 17]). In our definition, however, we use the term for the purpose of modeling the wireless nature of communication in radio networks. Particularly, our definition does not restrict the *number of nodes* located in any given area, but it merely restricts the *number of independent nodes*, i.e., nodes that do not have a direct communication link between them.

Note that $f(r)$ does not depend on the number of vertices in the graph, but on the radius of the neighborhoods only. Most intersection graphs used to model wireless ad-hoc networks, including (Quasi) Unit Disk and Coverage Area Graphs satisfy the polynomially bounded growth condition [25, 28].

Consider a subset $W \subseteq V$ of vertices called clusterheads. A *clustergraph* $\overline{G}$ of radius $c$ is then given by introducing edges in the $c$-hop neighborhoods around $w \in W$. In this context, the vertices in $W$ are used to identify each cluster. In other words, a clustergraph $\overline{G} = (\overline{V}, \overline{E})$ can be described by the set $\overline{V} := W$ of vertices, and the edges are characterized by

$$(u, v) \in \overline{E} \iff d_G(u, v) \leq c$$

for any two $u, v \in W$.

The following lemma for a clustergraph that uses independent vertices is a direct consequence from the definition of a growth-bounded graph.

LEMMA 3.2. *Let $G = (V, E)$ be a graph of (polynomially) bounded growth, $I \subset V$ be an independent set on $G$, and $c$ be a constant. Then, the maximum degree of the clustergraph given by the clusters $\Gamma_c(v), v \in I$, is bounded by $O(f(2c))$.*

# 4. LOCALLY OPTIMAL SUBSETS

In this section, we show how in a sequential algorithm, an independent set or dominating set in the bounded neighborhood of a node can be computed, such that the chosen subset meets the desired approximation ratio of $1+\varepsilon, \varepsilon > 0$.

Suppose that $v_0 \in V$ has been chosen as central vertex. In the following, we may omit explicitly stating $v_0$ for local subsets etc. whenever no ambiguity occurs.

For the Maximum Independent Set problem, $v_0$ then considers its $r$-neighborhood for increasing $r = 0, 1, 2, \ldots$, and computes a maximum independent set $I_r \subset \Gamma_r(v_0)$ as long as

$$|I_{r+1}| > (1 + \varepsilon)|I_r|$$

holds. Let $\bar{r}$ denote the smallest $r$ for which the above criterion is violated. For $\bar{r}$, we have the following claim.

LEMMA 4.1. ***(Independent Set)*** *Let $G = (V, E)$ be a graph of polynomially bounded growth. There exists a constant $c = c(\varepsilon)$ such that $\bar{r} \leq c$.*

PROOF. [30]. Due to the structure of the graph $G$, we have $|I_r| \leq p(r)$. From the definition of $\bar{r}$, we have for $r < \bar{r}$ the following inequalities

$$\begin{aligned} p(r) \geq |I_r| &> (1+\varepsilon)|I_{r-1}| \\ &> \ldots > (1+\varepsilon)^r |I_0| = (1+\varepsilon)^r, \end{aligned}$$

and the claim follows. □

The above lemma also implies that the radius of the largest neighborhood from which information is needed by $v_0$ is also bounded by a constant that only depends on the desired approximation ratio. Furthermore, the computations to be performed by $v_0$ are bounded by $n^{O(p(c)^2)}$.

For the Minimum Dominating Set problem, due to the fact that any maximum independent set is also maximal, and thus dominating, we have that for a subset $D_r$ which dominates $\Gamma_r(v_0)$, the inequality $|D_r| \leq |I_r|$ holds, and that $v_0$ may therefore also compute an optimal dominating set for $\Gamma_r(v_0)$ in time $n^{O(p(r)^2)}$. Now, the vertex $v_0$ computes for $r = 0, 1, 2, \ldots$, a minimum cardinality set $D_r$ that dominates $\Gamma_r(v_0)$ as long as

$$|D_{r+2}| > (1+\varepsilon)|D_r|$$

holds. Again, denote by $\bar{r}$ the smallest $r$ which violates the above.

LEMMA 4.2. ***(Dominating Set)*** *Let $G = (V, E)$ be a graph of polynomially bounded growth. There exists a constant $c = c(\varepsilon)$ such that $\bar{r} \leq c$.*

PROOF. From a similar argumentation as in Lemma 4.1, we see that for even $r < \bar{r}$, it is

$$\begin{aligned} p(r) \geq |I_r| &\geq |D_r| > (1+\varepsilon)|D_{r-2}| \\ &> \ldots > (1+\varepsilon)^{\frac{r}{2}} |D_0| = (\sqrt{1+\varepsilon})^r. \end{aligned}$$

For uneven $r$, the same chain of inequalities holds, and the claim follows. □

A (central) PTAS for both problems is then obtained by iteration of the above approach. A central vertex is chosen, which computes a subset in its bounded neighborhood that meets the desired criterion. Then, for the independent set case, $\Gamma_{\bar{r}+1}(v_0)$ is removed from $G$, while keeping $I_{\bar{r}}$ as partial solution, and we go on by induction for the remaining vertices of $G$. For the dominating set case, $\Gamma_{\bar{r}+2}(v_0)$ is removed from $G$, and $D_{\bar{r}+2}$ is added to the partial solution, before again going on with the remaining graph. Considering the $(\bar{r}+2)$-neighborhoods is due to the fact that $D_r \subset \Gamma_r$ needs not hold, but $D_r \subset \Gamma_{r+1}$ is clearly satisfied.

In [30] and [29], it is shown that this strategy yields a PTAS for the Maximum Independent Set and Minimum Dominating Set problems respectively for graphs of polynomially bounded growth.

Of course, this approach already exhibits several local properties and can be extended towards an in-network approach in a straightforward way. This extension is based on a greedy strategy to pick a new central vertex based on, e.g., lowest ID. However, at any point in time, there may only be a single point of activity in the graph, which yields a linear number of rounds at least. In the next part, we show how to improve this towards $O(\log \Delta \cdot \log^* n)$-complexity.

**Algorithm 1** Local approximation scheme (Maximum Independent Set)

**Input:** Graph $G = (V, E)$ of poly. bounded growth, $\varepsilon > 0$, $\tilde{c} := c(\varepsilon) + 2$ (Lemma 4.1)
**Output:** $(1+\varepsilon)$-approx. Maximum Indep. Set $I$

1: Compute maximal independent set $\mathcal{I}$ on $G$;
2: Use $\mathcal{I}$ to construct cluster graph $\overline{G}$ consisting of clusters $\Gamma_{2\tilde{c}}(v), v \in \mathcal{I}$;
3: Color $\overline{G}$ using $\Delta_{\overline{G}} + 1$ colors;
4: $I := \emptyset$;
5: **for** $k = 1$ **to** $\Delta_{\overline{G}} + 1$ **do**
6:  **for every** $v \in \mathcal{I}$ with color $k$ **do**
7:  **while** $\Gamma(v) \cap V \neq \emptyset$ **do**
8:  For some $u \in \Gamma(v) \cap V$, compute maximum independent set $I_{\bar{r}} \subset \Gamma_{\bar{r}}(u) \cap V$ such that $|I_{\bar{r}+1}| \leq (1+\varepsilon)|I_{\bar{r}}|$;
9:  Inform vertices in $\Gamma_{\tilde{c}}(v)$ about $\bar{r}$ and $I_{\bar{r}}$;
10:  $I := I \cup I_{\bar{r}}(u)$;
11:  $V := V \setminus \Gamma_{\bar{r}+1}(u)$;
12:  **od**;
13:  **od**;
14: **od**;

# 5. LOCAL APPROXIMATION SCHEMES

In this section, we present the algorithms that yield a local approach to constructing a $(1+\varepsilon)$-approximate independent and dominating set in a graph. For the Maximum Independent Set problem, Algorithm 1 outlines this approach, and we now show its correctness followed by a discussion on the number of rounds needed for its completion. After that, we shortly present the necessary adjustments to be made for the algorithm to approximate a dominating set. Throughout the following, we assume the graph $G$ to be of polynomially bounded growth.

## 5.1 Maximum independent set

During the pre-processing part of Algorithm 1 (1-3), the graph is clustered using balls of radius $2\tilde{c}$ around vertices that form a maximal independent set $\mathcal{I}$ in $G$. The set $\mathcal{I}$ can be created, e.g., by a local algorithm presented in [21]. The resulting cluster graph $\overline{G}$ then has bounded degree (Lemma 3.2), which allows for an efficient $(\Delta_{\overline{G}}+1)$-coloring by a local algorithm [9].

This structure is then exploited to simultaneously create local independent sets in the neighborhoods of central vertices with the same color. These local independent sets are constructed taking into account partial solutions coming from clusters with a lower color. In the algorithm, this is taken care of by updating the set of "unconsidered" vertices $V$. Clearly, each execution of the while-loop eliminates some vertices from $V$. Furthermore, all vertices and thus the entire graph are considered in the algorithm by the following lemma.

LEMMA 5.1. *Upon completion of Algorithm 1, it is $V = \emptyset$.*

PROOF. Consider the inner while-loop (step 7), and suppose that a vertex $v \in V$ has not been eliminated. Then, $v$ cannot be adjacent to a vertex in $\mathcal{I}$, a contradiction to the maximality. □

The local independent sets $I_{\bar{r}}$ are kept, and the following lemma then establishes that the overall created subset $I$ has the independence property in $G$.

LEMMA 5.2. *The set $I$ created by Algorithm 1 is independent.*

PROOF. Looking at steps 10 and 11 of the algorithm, while keeping $I_{\bar{r}}(u) \subset \Gamma_{\bar{r}}(u)$ as partial solution, we eliminate $\Gamma_{\bar{r}+1}(u)$ from $V$. Since vertices of different colors are not considered at the same time, it suffices to show that two central vertices $v, \tilde{v} \in \mathcal{I}$ with the same color create partial subsets which do not interfere with each other.

However, this is a direct consequence of the definition of $\overline{G}$, where $v$ and $\tilde{v}$ are at least distance $2\tilde{c} = 2c(\varepsilon) + 4$ apart: Any $u \in \Gamma(v)$ and $\tilde{u} \in \Gamma(\tilde{v})$ are $2c(\varepsilon) + 2$ apart, and therefore any independent sets considered by vertices in $\Gamma(v)$ and $\Gamma(\tilde{v})$ (in sequence, respectively, for $v$ and $\tilde{v}$) are separated (Lemma 4.1). □

Further, from the above proof, we see the reason for the coloring of the clustergraph $\overline{G}$. Vertices of the same color do not interfere with each other and can thus do all the computations in parallel.

Now, it remains to show that the independent set $I$ satisfies the desired approximation ratio of $1 + \varepsilon$.

LEMMA 5.3. *Let $I^*$ denote an optimal solution to the Maximum Independent Set problem on $G$. Then, the solution $I$ created by Algorithm 1 satisfies*

$$(1 + \varepsilon)|I| \geq |I^*|.$$

PROOF. As already seen in the proof of Lemma 5.2, the neighborhoods $\Gamma_{\bar{r}+1}(u)$ (step 11), $u$ according to the algorithm (step 8), either are separated due to the coloring of the clustergraph $\overline{G}$ or are constructed in sequence when considering different colors or two vertices $u, \tilde{u} \in \Gamma(v)$ for a fixed $v \in \mathcal{I}$. Also, every vertex in $G$ is considered in the creation of the neighborhoods. Let $\tilde{\Gamma}_{\bar{r}_u+1}(u)$ denote the respective neighborhoods $\Gamma_{\bar{u}+1} \cap V$, where $V$ is considered with respect to previous steps of the algorithm, and $u \in V$ are the central vertices as chosen by step 8. Clearly, these adjusted sets $\tilde{\Gamma}_{\bar{r}_u+1}(u)$ form a partition of $G$.

Furthermore, the partition together with the criterion for the creation of the neighborhoods, i.e. so that $|I_{\bar{r}+1}| \leq (1 + \varepsilon)|I_{\bar{r}}|$ is satisfied, yields

$$\begin{aligned} |I^*| &= |\bigcup\nolimits_u (I^* \cap \tilde{\Gamma}_{\bar{r}_u+1}(u))| \\ &= \sum\nolimits_u |I^* \cap \tilde{\Gamma}_{\bar{r}_u+1}(u)| \\ &\leq \sum\nolimits_u |I_{\bar{r}_u+1}(u)| \\ &\leq \sum\nolimits_u (1 + \varepsilon)|I_{\bar{r}_u}(u)| \\ &= (1 + \varepsilon)|\bigcup\nolimits_u I_{\bar{r}_u}(u)| \\ &= (1 + \varepsilon)|I|. \end{aligned}$$

Note that the summation is always over all central vertices as chosen by the algorithm. □

Next, we look at the complexity of the algorithm in terms of number of rounds needed to complete the $(1+\varepsilon)$-approximation on a graph with polynomially bounded growth.

In the pre-processing part of Algorithm 1, the maximal independent set computation can be achieved deterministically in $T_{\text{MIS}} = O(\log \Delta \log^* n)$ communication rounds [21].

**Algorithm 2** Local approximation scheme (Minimum Dominating Set)

**Input:** Graph $G = (V, E)$ of poly. bounded growth, $\varepsilon > 0$, $\tilde{c} := c(\varepsilon) + 2$ (Lemma 4.2)
**Output:** $(1 + \varepsilon)$-approx. Min. Dominating Set $D$

```
 1: Construct MIS I and colored clustergraph Ḡ;
 2: D := ∅;
 3: for k = 1 to Δ_Ḡ + 1 do
 4:   for every v ∈ I with color k do
 5:     if Γ(v) ∩ V ≠ ∅ then
 6:       For some u ∈ Γ(v) ∩ V, compute minimum dominating set D_r̄ of Γ_r̄(u) ∩ V such that |D_{r̄+2}| ≤ (1 + ε)|D_r̄|;
 7:       Inform Γ_c̃(v) about r̄ and D_{r̄+2};
 8:       D := D ∪ D_{r̄+2}(u);
 9:       V := V \ Γ_{r̄+2}(u);
10:     fi;
11:   od;
12: od;
```

The coloring of the clustergraph $\overline{G}$, due to its bounded degree (Lemma 3.2), takes $O(\Delta_{\overline{G}} \log^* n)$ rounds [9]. The maximum degree of $\overline{G}$ depends on the constant $c = c(\varepsilon)$, since $G$ has polynomially bounded growth, it is $\Delta_{\overline{G}} = O(1/\varepsilon^{O(1)})$, where the exponent of $1/\varepsilon$ depends on the polynomial bound of the graph itself.

The second part of the algorithm (step 5-14), benefits from the fact that step 6 can be done completely parallel by the respective leaders of the same colors in the MIS. Note that every action performed is limited to the $c$-neighborhood of the respective vertices.

For the local neighborhoods of the cluster leaders, we have the following lemma.

LEMMA 5.4. *Consider step 7 of Algorithm 1. The while-loop is executed at most $f(1) = O(1)$ times for any $v \in \mathcal{I}$.*

PROOF. In $\Gamma(v)$, there are at most $f(1)$ independent vertices, and in each round, at least the first order neighborhood of the vertex $u \in \Gamma(v) \cap V$ is eliminated from $V$. □

Overall, we can therefore bound the number of communication rounds for the second part of the algorithm by $O(cf(1)\Delta_{\overline{G}}) = O(1/\varepsilon^{O(1)})$ for $\varepsilon > 0$.

## 5.2 Minimum dominating set

We go on to describe the adjustments for the case of computing an approximation of a minimum dominating set (Algorithm 2). Basically, this is just an adjustment to the computation of the local, partial solutions. A locally optimal dominating set is computed by the central vertices of the MIS $\mathcal{I}$. This is done in such a way that the combination of these local subsets dominate the whole graph $G$, while also obeying the desired approximation ratio.

Considering the second part of the algorithm, similar to the independent set case, we have the following properties for the approach for the Minimum Dominating Set problem.

LEMMA 5.5. *Algorithm 2 terminates, and at the end, we have $V = \emptyset$.*

PROOF. We need to show that every vertex eventually is eliminated from $V$. Considering step 5, it is clear that if the neighborhood of a vertex $v \in \mathcal{I}$ in $V$ is not empty, at

least $\Gamma(v)$ is removed from $V$ by step 9. That is, we have $\Gamma(v) \subset \Gamma_{\bar{r}+2}(u)$ for any $r \geq 0$ and $u \in \Gamma(v)$. The claim follows from the maximality of the independent set $\mathcal{I}$ in $G$. □

Also, the created solution has the desired domination property.

LEMMA 5.6. *At the end of Algorithm 2, the solution $D$ dominates the graph $G$.*

PROOF. The local set $D_{\bar{r}+2}$ dominates $\Gamma_{\bar{r}+2} \cap V$ which is eliminated from $V$ at every step. □

As for the cardinality of $D$ created by the algorithm, the following lemma establishes its approximation ratio.

LEMMA 5.7. *Let $D^*$ denote an optimal solution to the Minimum Dominating Set problem on $G$. Then, the set $D \subset V$ computed by Algorithm 2 satisfies*

$$(1+\varepsilon)|D^*| \geq |D|.$$

PROOF. Again, let $\tilde{\Gamma}_{\bar{r}_u}(u)$ denote the neighborhoods with respect to $V$ during the execution of the algorithm.

By inductive argumentation over $u \in \Gamma(v), v \in \mathcal{I}$, it is clear to see that the sets $\Gamma(\tilde{\Gamma}_{\bar{r}_u}(u))$ are mutually disjoint $G$. Note in this context that the $\tilde{\Gamma}_{\bar{r}_u}(u)$ created in parallel (step 4) are non-overlapping and can thus w.l.o.g. be considered in arbitrary order for central vertices of the same color. Furthermore, $\Gamma(\tilde{\Gamma}_{\bar{r}_u}(u)) \cap D^*$ dominates $\tilde{\Gamma}_{\bar{r}_u}(u)$ in $G$. Then,

$$\begin{aligned}
|D^*| &\geq |\bigcup_u (D^* \cap \Gamma(\tilde{\Gamma}_{\bar{r}_u}(u)))| \\
&= \sum_u |D^* \cap \Gamma(\tilde{\Gamma}_{\bar{r}_u}(u))| \\
&\geq \sum_u |D_{\bar{r}_u}(u)| \\
&\geq \sum_u \frac{1}{1+\varepsilon} |D_{\bar{r}_u+2}(u)| \\
&\geq \frac{1}{1+\varepsilon} |\bigcup_u D_{\bar{r}_u+2}(u)| \\
&= \frac{1}{1+\varepsilon} |D|.
\end{aligned}$$

□

The number of rounds needed in the second part of Algorithm 2, by the same argumentation as in the previous case of the independent set, is $O(1)$ for fixed $\varepsilon > 0$.

Concluding this section, we summarize by the following theorem.

THEOREM 5.8. *Let $G = (V, E)$ be a polynomially growth-bounded graph. Then, there exist local, distributed $(1+\varepsilon)$-approximation algorithms, $\varepsilon > 0$, for the Maximum Independent Set and Minimum Dominating Set problems on $G$. The number of communication rounds needed for the respective construction of the subsets is $O(T_{MIS} + \log^* n/\varepsilon^{O(1)})$.*

# 6. CONCLUSION

In this paper, we have studied a completely local approach to the problems of approximating independent and dominating sets within a factor of $1+\varepsilon$ for arbitrary $\varepsilon > 0$ on graphs of polynomially bounded growth, such as unit disk graphs. The algorithms do not rely on any positional information, e.g. they work even if nodes do not have coordinate information and cannot measure distances or angles to their neighbors.

The two algorithms presented require a running-time of $O(T_{\text{MIS}} + \log^* n/\varepsilon^{O(1)})$, where $T_{\text{MIS}}$ is the time needed to construct a maximal independent set in the graph. Considering $\varepsilon > 0$ fixed, this time is dominated by the time to compute a MIS in the graph, and the change from a MIS to a $(1+\varepsilon)$-approximate independent or dominating set respectively can be achieved in constant time.

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