A Closer Look at Jamnitzer's Polyhedra

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Abstract

The Renaissance artist Wentzel Jamnitzer designed series of intriguing polyhedra in perspective in his book "Perspectiva Corporum Regularium". In this paper we investigate the possible principles of the construction of the polyhedra and create 3D computer models of them. Comparing those to the originals, we get an idea of how successful he was in drawing the complex structures by imagination. Furthermore, we analyse Jamnitzer's use of linear perspective, an important key in creating such drawings.

1 Introduction

Wentzel Jamnitzer (1508-1585) was born in Vienna. Later he moved to Nuremberg where he became the one of the most famous goldsmiths of his time. His refined and richly decorated, almost baroque, masterpieces in museums are of international fame [1]. He, just as many of the Renaissance artists, had an interest in polyhedra - geometrical 3D objects build up from planar faces, each of them being a polygon. The type and variety of polygons used as faces determines families of polyhedra, with different symmetry characteristics. He must have been intrigued by the variety of them.

Jamnitzer, with the help of Jost Amman, published a book called "Perspectiva Corporum Regularium" in 1568. This work contains many geometrically interesting drawings. He even came up with a form representing a new symmetry group, the chiral icosahedral symmetry, which is quite a great accomplishment, see [2]. Among other things in his book are multiple series of polyhedra. Jamnitzer took the five convex regular polyhedra, or Platonic solids, and had drawn variations on these. Jamnitzer had drawn four series of variations on each Platonic solid.

Drawing with the use of linear perspective was mostly developed in the Renaissance. In those days drawing had to be done without the aid of modern tools of course. Jamnitzer's had to visualize the polyhedra in his mind and draw them in perspective, which is no small achievement if done right.

The goal of this paper is to research if he made some (systematical) mistakes in his drawings, if at all. Interesting is, did he imagine the complex polyhedra correctly and how was his use of perspective? For this purpose we have chosen the polyhedra series of figure 1 to analyze. The polyhedron in the upper-left corner is called *truncated octahedron* which is related to the octahedron by 'cutting off' the corners. The other five polyhedra are variations on this first one. For more information, Peter Cromwell's book *Polyhedra* gives a good discussion on Jamnitzer's work and polyhedra in general [3].

In the next section we take a look at each polyhedron, examine what they are and how they were derived. To be able to analyse Jamnitzer's drawing we have reconstructed the polyhedra virtually. In section 3 the reconstruction process is explained, what tools and methods we used. section 4 contains the detailed analysis of each polyhedron. This is done by comparing the 3D virtual model to the one drawn by Jamnitzer and by discussing how perspective was used by him. Finally we sum up our findings and outline some further work.

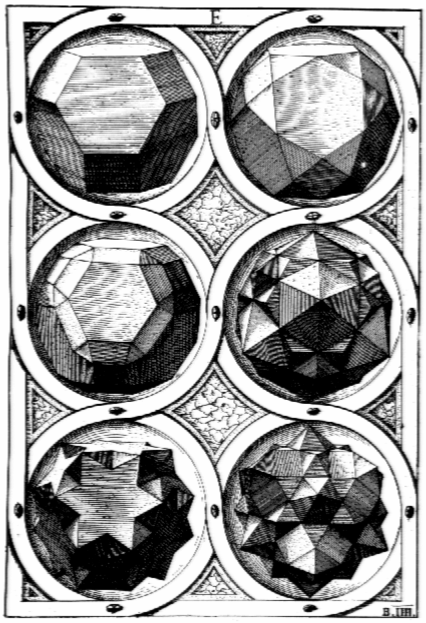

Figure 1: *One of Jamnitzer's octahedron series.*

Computer reconstructions of Jamnitzer's sculptures have previously been made by Peter Cromwell using POV-Ray [4] and by Rinus Roelofs using Rhinoceros [5]. These, however, do not make comparisons with the original.

2 Global Analysis of the Designs

In this section we will give an analysis of the types of polyhedra Jamnitzer drew in the chosen series. With the help of this information we were able to recreate the polyhedra in the 3D modelling environment. Furthermore, since Jamnitzer created variations on regular polyhedra, which are highly symmetrical, we assumed he wanted these variations to be symmetrical to a certain level. When, otherwise, there was an indication this is not the case, we explored this occurrence.

The first one is the truncated octahedron. Figure 2 shows the process of truncating the octahedron with the truncated octahedron as result. The truncation must be done in such a manner that all the edges are of equal length. This is accomplished by truncating the octahedron to one third the edge length. This polyhedron has octahedral symmetry.

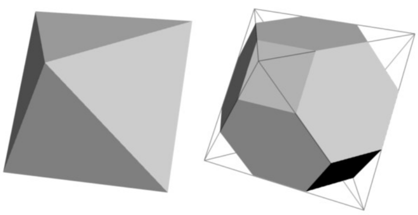

Figure 2: *Constructing the truncated octahedron.*

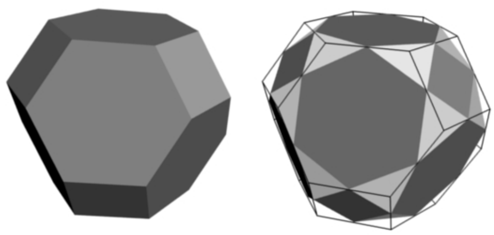

Figure 3: *Rectifying the truncated octahedron.*

The second polyhedron is gained by truncating the truncated octahedron. Looking at figure 2 we can see that the corners of the newly constructed polyhedron lie halfway on the edges of the truncated octahedron. This is illustrated in figure 3**.** This special type of truncation is also known as rectification. The resulting polyhedron can therefore be called a *rectified truncated octahedron*.

The third one is constructed similar: by rectifying the previous polyhedron. This gives a double rectified truncated octahedron or *beveled truncated octahedron*, see figure 4.

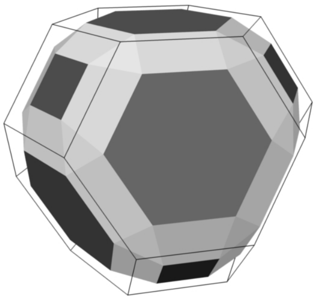

Figure 4: *Beveled truncated octahedron.*

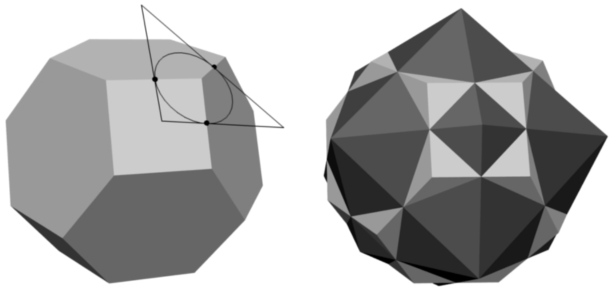

Figure 5: *Constructing the compound of the truncated octahedron (light grey) and its dual (dark grey).*

Now for the fourth polyhedron. This one appears to be the compound of the truncated octahedron and its dual the *tetrakis hexahedron*. Figure 5 shows this compound and also illustrates the creation of the first face for the dual making use of the *Dorman-Luke construction*, see [6]. An other possibility is that the tetragonal and hexagonal pyramids are independent of each other and thus creating two degrees of freedom, the height of the tetragonal pyramids and of the hexagonal ones.

Next is the fifth polyhedron. This one seems to be related to the truncated octahedron as follows. Transform every hexagonal, or six-sided, face into a hexagram shaped one. Also transform every square face into a star shaped one. And finally place faces to fill the gaps as illustrated in figure 6. The transformation to a hexagram shaped face is well defined, but the transformation of the square face into a star shaped one is not. There were several ways how we could define the length of those edges. One of the possibilities was that the edges were just as long as the ones of the hexagram shaped faces. Another was that the edges joining two light grey faces, see figure 6, were all of the same length. To be able to investigate variants, we took the length of those edges as a free variable and determined it visually. This method thereafter resulted in rejecting the two proposed methods because the length found differed about six percent from the expected values of both methods.

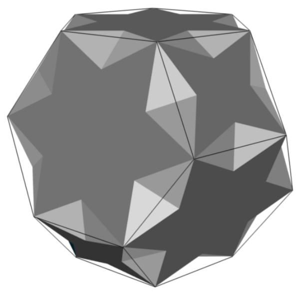

Figure 6: *The light grey faces are new, the dark grey ones are reshaped old faces.*

The last polyhedron in this series is the most difficult one. Figure 7 gives a step by step approach to see how it's constructed. First take the midpoints of the edges of a hexagonal face and construct a hexagram from them. Do this for each hexagon. Also do this for the square faces except for creating rotated squares instead of hexagrams. Now construct a new hexagon in each hexagram, using the inner six vertices, and fill the areas of the hexagrams not overlapped by these new hexagons with faces. See the second polyhedron in figure 7. Next, create right pyramids on the inner hexagons and inner squares. The two different heights of those pyramids were to be determined visually. The last step is to fill in the remaining gaps with faces as illustrated in figure 7's fourth polyhedron.

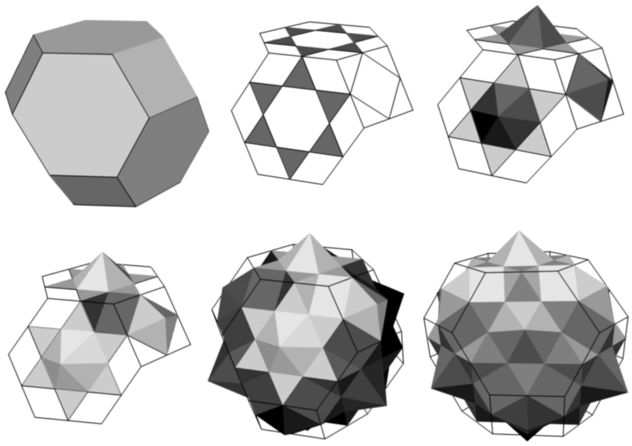

Figure 7: *Step by step reconstruction of the last polyhedron in this (truncated) octahedron series.*

3 3D Reconstruction Process

Recreating the polyhedra accurately is important for the comparison to come. This section is therefore dedicated to the methods we used to recreate and render them.

3.1 Modelling Environment and Precision

For the virtual recreation of the polyhedra we used Blender [7]. There are two main advantages, for us, for using Blender: it is free and we have some experience in using Blender. Compared to other software packages Blender is a good all around application.

The construction began with a preset model, the hexagon. After some calculations new vertices, edges and faces have been placed at their correct locations by translating, scaling and rotating them accordingly about a manually placeable pivot until the whole polyhedron was finished. The presence of symmetry made this easier since whole faces could be duplicated and located at an other point in space abiding to the symmetry.

The precision of Blender is important. If Blender is inaccurate, this will have a negative impact on the reliability of the results. Fortunately Blender is quite precise. When translating or scaling objects or vertices this may be done numerically to four decimal places precise. When the translation is done manually, it can be done with even higher precision. The same holds for rotation, except when rotating numerical, the precision is reduced to two decimal places. There are better results when using the Python programming language within Blender. This will give a precision of six decimal places for rotation. Scaling and translation will have a precision of up to fifteen decimal places, given the fact we used a length of one for the edges of the truncated octahedron.

Because rotating gives the most significant rounding errors, except when dealing with angles of up to two decimal places, we tried to minimize the number of rotations. Only in two occasions of the first model we did make use of a rounded rotation but when looking at the final location of the vertices they were up to 3 decimal places precise where they should be.

The resolution of the image used to compare our models to Jamnitzer's is also quite important. The one we used has a resolution of 430 by 624 pixels, the largest we could find of this series. Compared to this low resolution of the image the rounding errors of Blender will be insignificant. But importantly, the resolution seemed to be sufficient. A higher resolution however would have been welcome since, because of this, there might have been some things we did not notice while analyzing the results which could have been interesting. Furthermore we assume that the image hasn't been deformed by the digitalization process.

3.2 Reconstruction Methods and Viewing Parameters

The virtual scene had to be set up right. This paragraph discusses the important details involving this setup and some of the methods used.

Using the analysis of section 2, we recreated the polyhedra. However, some of the polyhedra did seem to have some degree of freedom, for example the height of the pyramids on the sixth polyhedron. This freedom was used to manually scale or translate the vertices in question to fit Jamnitzer's drawing. But, in such a way that symmetry is preserved.

Now for the positioning of the polyhedra. The polyhedron models had to be rotated, translated and scaled accordingly to Jamnitzer's drawings. To make this process easier we put the drawing of the particular polyhedron on the background of the scene. This way we didn't have to make a great number of test renders and thus be able to position the model relatively fast.

Another important issue, while we were setting up the scene, involved the angle of the camera lens. When this angle is decreased, and the object is resized accordingly, the object appears with less depth. For a good result we want this angle to match the one Jamnitzer used in his drawings as close as possible. Unfortunately we didn't find an other way to accomplish this except through trial-and-error.

Finally, it is preferable that it is easy to see the possible differences between our recreated polyhedra and Jamnitzer's original ones. This we achieved by using a partial wireframe model, using only the visible parts of the solid model for the wireframe one. Further, Jamnitzer's drawing had to be put on the rendering background, so both his drawing and our virtual model were rendered together in one picture and the differences were easy to spot.

4 Research Results

In this section the analysis has been set out regarding each polyhedron in the chosen octahedron series. First we look at the mismatches between Jamnitzer's drawings and the recreated polyhedra. After that, the perspective in Jamnitzer's drawings is analysed. The observations are used to discuss the results in section 5. We assume that Jost Amman made no mistakes in doing the engravings. Nevertheless, it might be a possible source of some mismatches.

4.1 Comparison and Analysis

The first three polyhedra matched quite good as can be seen in figures 8 through 10. The last two however do show some small mismatches. The biggest of those are circled in figures 11 and 12.

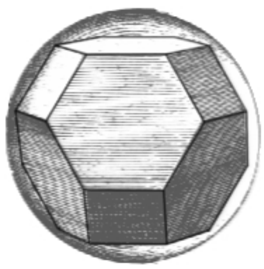

Figure 8: *Truncated octahedron.*

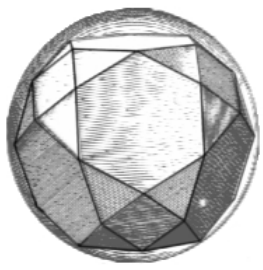

Figure 9: *Rectified truncated octahedron.*

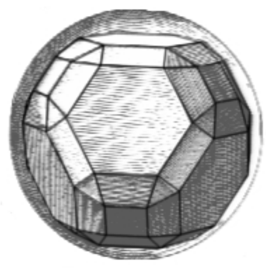

Figure 10: *Beveled truncated octahedron.*

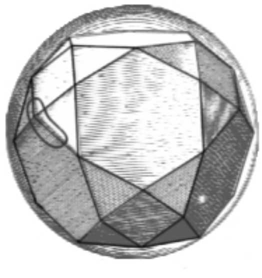

Figure 11: *A small mismatch.*

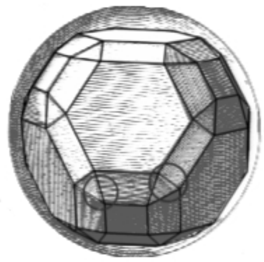

Figure 12: *More small mismatches.*

Getting the point of view right for the fourth polyhedron was hard. It didn't work. There were always multiple mismatches. See figure 13. One of those was that the hexagonal pyramids seemed to be smaller than those of the recreated compound polyhedron. Let's see where this goes wrong.

Edges are expected to be straight, so the edges of the dual are expected to be straight too. However, see figure 14, the edges corresponding to OB and OA are not straight. This meant that either the polyhedron isn't a compound of the truncated octahedron and its dual or that Jamnitzer made a mistake.

Let's explore the other possibility mentioned in section 2, that the tetragonal and hexagonal pyramids are independent of each other and thus not constructed as part of the dual. This approach resulted in figure 15. This one matches the drawing much better then the compound polyhedron, But still, some mismatches are found. Beginning in figure 16 we see that edges of the hexagonal pyramid, which are circled, don't intersect with the edge joining the hexagon with the square. This is a mistake of Jamnitzer which resulted in a mismatch with the reconstructed polyhedron in figure 18 around the same spot. Furthermore, in figure 17 a top of a hexagonal pyramid from the back being visible is circled. Jamnitzer's drawing doesn't show this however.

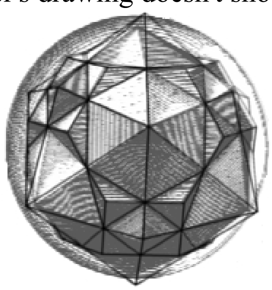

Figure 13: *Truncated octahedron - tetrakis hexahedron compound.*

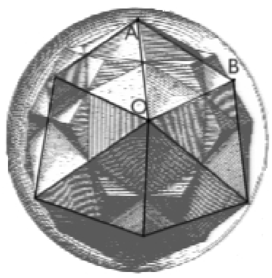

Figure 14: *The dual should have straight edges.*

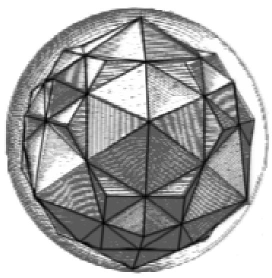

Figure 15: *Second version of the polyhedron four.*

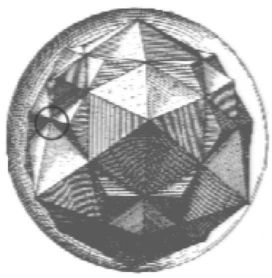

Figure 16: *Not all edges are intersecting.*

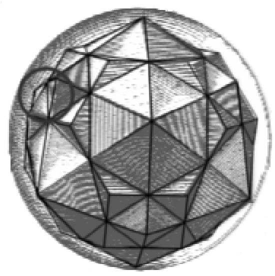

Figure 17: *The top of a hexagonal pyramid just visible.*

Looking at the comparison with the fifth polyhedron, figure 18, we noticed two more significant anomalies, see figure 19. The right one involves an edge being only partly visible, just as in figure 17. Jamnitzer seems to have had more trouble imagining those right.

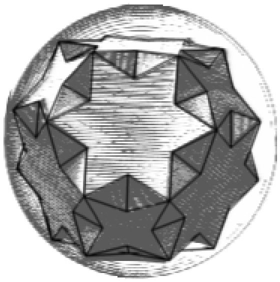

Figure 18: *Comparing the fifth polyhedron in the series.*

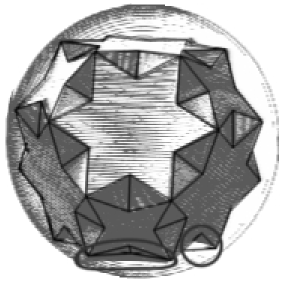

Figure 19: *Some mismatches.*

And as last, the sixth polyhedron, see figure 20. The mismatches we found are circled again, now in figure 21. All three errors are related to edges being only partly visible or expected to be visible. This supports the previous suggestion about Jamnitzer seeming to have had trouble with correctly imagining edges of polyhedra when they are just partly visible.

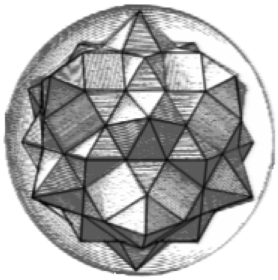

Figure 20: *Comparing the final polyhedron in the series.*

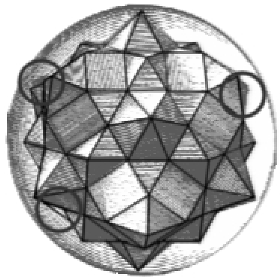

Figure 21: *Circled are some mismatches again.*

4.2 Perspective Analysed

For this analysis we draw multiple lines on Jamnitzer's drawings to see whether parallel edges converge in the distance or not. This results in figures 22 through 24. As shown it seems that most of the lines are converging very slow or are nearly parallel. This means Jamnitzer imagined the polyhedra with a very small 'camera lens' angle, even nearly orthographic. This is reflected in the camera lens angle used while reconstructing, it was always smaller than ten degrees.

There are some exceptions on the lines converging. In figure 23 we see that the solid lines show some improper linear perspective as do the solid lines in figure 24. These errors in using linear perspective in general may explain some of the small mismatches found earlier in the previous paragraph.

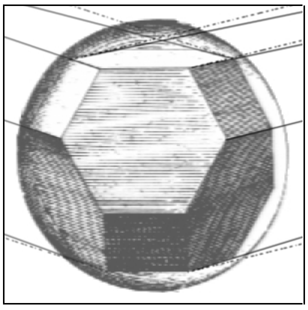

Figure 22: *The dashed lines are parallel to the ones in the middle.*

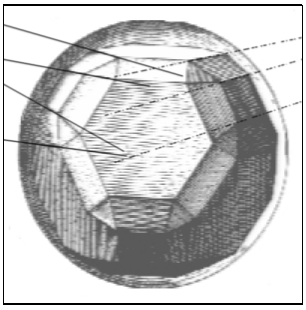

Figure 23: *Perspective on polyhedron three.*

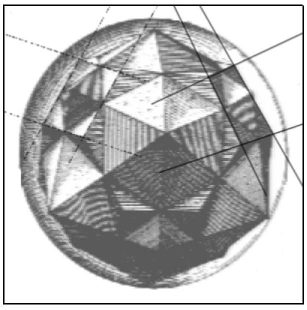

Figure 24: *Perspective on polyhedron four.*

5 Discussion

In the previous section we have shown what could have been the principle behind each design and how correctly these mental images were displayed by using linear perspective. He seemed to have had some trouble with visualizing the edges correctly when they were just partly visible. Also, not every edge is positioned quite as it should be when linear perspective is assumed. Some other small mismatches were also found, most were the results of either an error of Jamnitzer, see figure 16, or a mistake in the use of perspective.

Concluding, under the assumption that Jost Amman engraved everything perfectly, Jamnitzer made some mistakes in imagining and drawing the polyhedra, mainly the harder parts. Also his use of perspective seemed a bit flawed here and there. But nevertheless, he did a great job. Further work could be done analysing other series of Jamnitzer's polyhedra using higher resolution pictures of Jamnitzer's work.

Acknowledgements

This work was carried out as a project within the *Art and Media Technology* course. We are thankful for Rinus Roelofs, who drew our attention to the polyhedra by Jamnitzer.

References

Used image for the octahedron series, http://www.mathe.tu-freiberg.de/~hebisch/cafe/jamnitzer/perspectiva11.html

[1] Short history of Jamnitzer's life and an introduction to his book *Perspectiva Corporum Regularium*, http://www.mathe.tu-freiberg.de/~hebisch/cafe/jamnitzer/jamnitzer.html

[2] Jamnitzer's work placed in the view of the history of geometry, http://www.georgehart.com/virtual-polyhedra/jamnitzer.html

[3] P. R. Cromwell, *Polyhedra*, 1997.

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[6] M. Wenninger, *Dual Models*, pp.30. 1983.

[7] The Blender website, http://www.blender.org