Hybrid Systems Described by the Complementarity Formalism

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In recognition of the fact that many systems contain both continuous and discrete aspects, considerable study has been devoted recently to "hybrid systems." The formulation of equations of motion for hybrid systems in explicit form, including the condition/event rules and the description of the continuous dynamics for every possible mode, is in many cases a formidable task, and there is a clear need for devices that enable the modeler to work in what might be called a "high-level language." A formalism that can be used for this purpose is the so-called *complementarity formalism* [5, 8, 9]. The formalism is applicable to a broad class of physical hybrid systems, as well as to hybrid systems described by an underlying dynamics subject to piecewise-linear constraints. This paper surveys some of the key issues treated in [8, 9], and discusses some possible extensions.

Complementary-slackness hybrid systems

We consider systems that are described by general differential-algebraic equations (DAE's)

$$G(z(t), \dot{z}(t)) = 0, \quad z \in \mathbb{R}^N, \tag{1}$$

together with a "complementary" set of inequality constraints defined as follows. Let

$$\begin{aligned} e &= E(z(t)), \quad e \in \mathbb{R}^k \\ f &= F(z(t)), \quad f \in \mathbb{R}^k \end{aligned} \tag{2}$$

be two mappings, and consider the "complementary-slackness conditions" (the terminology stems from optimization theory)

$$e(t) \geq 0, \quad f(t) \geq 0, \quad e^T(t)f(t) = 0 \tag{3}$$

where the inequalities are understood componentwise. The conditions on $e(t)$ and $f(t)$ imply that for each index i in the index set $K := \{1, \cdots, k\}$ and each

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time instant t we must have either $e_i(t) = 0$ and $f_i(t) \geq 0$, or $f_i(t) = 0$ and $e_i(t) \geq 0$. Thus for every subset $I \subset K$ we obtain a different set of DAE's

$$\begin{aligned} G(z(t), \dot{z}(t)) &= 0 \\ E_i(z(t)) &= 0, \quad i \in I \\ F_i(z(t)) &= 0, \quad i \in K \backslash I \end{aligned} \tag{4}$$

together with feasibility conditions

$$\begin{aligned} E_i(z(t)) &\geq 0, \quad i \in K \backslash I \\ F_i(z(t)) &\geq 0, \quad i \in I \end{aligned} \tag{5}$$

The dynamics described by (4) will be called a *mode* of the system, and the mode corresponding to a subset $I \subset K$ will be simply denoted as "mode I." Thus we have obtained a multi-mode (or hybrid) system with, in principle, 2^k different modes, which each have to satisfy a set of additional feasibility conditions (5). This special class of hybrid systems has been introduced in [8] as *"complementary-slackness systems"*, and analysed in [8, 9]. Note that, loosely speaking, the mappings E and F defined in (2) can be regarded as some kind of "guards" or "system invariants", as they are often appearing in the literature on hybrid systems. However a main difference is that the mappings E and F (as well as the underlying dynamics $G(z(t), \dot{z}(t)) = 0$) are "globally" defined, that is, do not depend on the particular mode, and in fact *define* the different modes as in (4). Note that in fact for every I the functions E_i, $i \in I$, and F_i, $i \in K \backslash I$ define the mode I as in (4), with "complementary" guards E_i, $i \in K \backslash I$, and F_i, $i \in I$, given by (5).

The basic motivation for studying complementary-slackness hybrid systems in [8, 9] is two-fold. *First*, a rich class of physical hybrid systems can be directly modelled as complementary-slackness systems: e.g. electrical circuits with diodes, mechanical systems with stops, hydraulic systems with one-way valves. We refer to [11] for related developments in modeling physical systems with switches. Furthermore, with a little bit more effort also discontinuous physical phenomena as (ideal) Coulomb friction and backlash, and control elements as relays, can be modelled this way [9]. *Secondly*, the class of complementary-slackness systems has an appealing mathematical structure which suggests some natural rules for mode-switching and re-initialization, and which admits the derivation of strong theorems concerning the resulting hybrid dynamics. A major mathematical tool in this is the *Linear Complementarity Problem* (LCP): Given a vector $q \in \mathbb{R}^k$ and an $k \times k$ matrix M, find k-vectors e and f such that

$$\text{LCP:} \quad e = q + Mf, \quad e \geq 0, \quad f \geq 0, \quad e^T f = 0 \tag{6}$$

The LCP has been studied extensively, and a wealth of theoretical results and computational methods is available, see e.g. [2]. A basic result is that the LCP has a unique solution e, f if the principal minors of M are all positive. The strong relation of complementary-slackness systems with the LCP becomes more clear by considering as a special case of (1), (2) "input-output" systems

$$\begin{aligned} \dot{x}(t) &= g(x(t), f(t)) \\ e(t) &= h(x(t), f(t)) \end{aligned} \tag{7}$$

subject to the complementary-slackness conditions (3). One may regard (7), (3) as a "dynamical" (nonlinear) version of the LCP. The recognition of the close connection of complementary-slackness systems with the LCP also sheds light on the fundamental question which class of hybrid systems can be modelled as complementary-slackness systems. In fact, it has been shown in [3] that any piecewise linear n-dimensional set of equations (under a mild "nonsingularity" assumption) is equivalent to a certain LCP (whose order k is typically larger than n). This implies that generally any (linear or nonlinear) dynamics subject to piecewise linear constraints can be modelled, in principle, as a complementary-slackness system. (See also [10] for a discussion on the relation of hybrid and piecewise-linear systems.) The modelling of Coulomb friction and relay elements by complementary-slackness conditions in [9] can be understood in this way. The power of the LCP for the modeling and analysis of mechanical systems with inequality constraints and Coulomb friction has been already advocated by Lötstedt [5], and for the modeling of static electrical circuits in [6].

Mode-selection and re-initialization in the complementarity framework

Let us now discuss how the complementarity framework may be used for suggesting natural rules for mode-selection and re-initialization, and for proving strong theorems concerning the resulting hybrid dynamics.

First we discuss the *mode-selection problem* for complementary-slackness systems of the form (7), (3). To simplify discussion we assume that every mode I is *autonomous* in the sense that from each continuous state that is consistent for mode I there is a unique solution of the dynamics of the mode I (not necessarily satisfying the feasibility conditions (5)) on a time-interval of positive length. We say that *smooth continuation* from a continuous state is possible *in mode* I if also the feasibility conditions (5) are met on some time-interval $[0, \epsilon]$, $\epsilon > 0$. We consider an initial continuous state for which smooth continuation is possible in at least one of the modes; so no re-initialization is necessary. By successively differentiating the "output" equations $e = h(x, f)$ along the dynamics $\dot{x} = g(x, f)$ one obtains sets of equations (linear in the highest derivatives of e and f) which can be brought inductively into the format of an LCP (in the unknowns $e, \dot{e}, \ddot{e}, \cdots, f, \dot{f}, \ddot{f}, \cdots$.) This allows the derivation of strong theoretical results concerning *uniqueness* of smooth continuation of complementary slackness hybrid systems, such as mechanical systems subject to multiple (independent) geometric inequality constraints, and passive electrical circuits containing diodes. At the same time it gives, via the LCP, a numerical recipe for *computing* the unique mode of smooth continuation, which is clearly of much importance e.g. for simulation purposes. All this is detailed in [9]. Note that in the above analysis of unique smooth continuation the discrete part of the state (that is, the mode the system is currently in) is taken to be "sub-ordinated" to the continuous state. This has been done on the basis of physical considerations. For instance, we believe that it would not be reasonable from a physical point of view to include in the initial conditions for an electrical circuit with diodes any information as to which diodes are voltage- or current-blocking; one should be able to derive this information from the continuous state components (assuming no hysteretic effects). This is in some contrast with a more standard formulation of hybrid systems in which the system "knows" which mode it is in.

If a consistent continuous initial condition does not admit smooth continuation in any of the modes then (and only then) the system has to be re-initialized to another state (in accordance with the Principle of Constraints formulated in [4, p.79]: "Constraints shall be maintained by forces, so long as this is possible; otherwise, and only otherwise, by impulses"). This *re-initialization* can be split into two parts: (a) a *switch rule* which determines to which mode the system will be re-initialized, (b) a *jump rule* which defines the new continuous state, consistent with the mode just determined.

A *jump rule* was proposed in [8] for linear dynamics and some special nonlinear dynamics (such as mechanical systems), based on geometric considerations. Indeed, with any linear set of *autonomous* DAE's

$$E\dot{z} = Az, \quad z \in \mathbb{R}^N \tag{8}$$

one can associate two *complementary* subspaces V and T in $\mathbb{R}^N$, where V is the set of consistent points, and T is directly related to the impulsive behavior of the system. Suppose now that the switch rule has determined a mode, whose dynamics is described by (8). Then the *jump rule* is simply to project the initial continuous state z_0 *along* T to a point $z_0' \in V$. (Applied to mechanical systems this corresponds to the application of impulsive forces to the system.)

With regard to *switch rules* the following options are available. In [8] the following switch rule was proposed. Let z_0 be a consistent point for some mode I. Consider the unique solution from z_0 in mode I, and detect *all* the feasibility conditions (5) that will be immediately violated. (Since by assumption there is no smooth continuation in mode I at least *one* feasibility condition is going to be violated.) Let Γ_1 be the subset of $K \backslash I$ for which the first set of feasibility conditions in (5) are going to be violated, and let Γ_2 be the subset of I for which the second set of feasibility conditions in (5) are going to be violated. Then determine the new mode J as

$$J := (I \backslash \Gamma_2) \cup \Gamma_1 \tag{9}$$

For the bimodal case (i.e. $k = 1$) the resulting hybrid dynamics was analyzed in detail in [8]. For $k > 1$ there are some alternatives to the above switch rule, which are partly motivated by the LCP as well as by the theory of (inelastic) mechanical collisions (see e.g. [1]). Indeed, let us consider a mechanical system with n degrees of freedom $q = (q_1, \cdots, q_n)$ having kinetic energy $\frac{1}{2}\dot{q}^T M(q)\dot{q}$;

$M(q) > 0$ being the generalized mass matrix. Suppose the system is subject to k geometric inequality constraints

$$e_i = E_i(q) \geq 0, \quad i \in K = \{1, \cdots, k\} \tag{10}$$

If the i-th inequality constraint is *active*, that is $E_i(q) = 0$, then the system will experience a constraint force of the form $\frac{\partial E_i}{\partial q}(q)\lambda_i$, with $\frac{\partial E_i}{\partial q}(q)$ the column-vector of partial derivatives of E_i and λ_i a Lagrangian multiplier. Let us now consider an initial continuous state $(q, \dot{q})$ of the system, from which no smooth continuation is possible in any of the possible modes (corresponding to some of the inequality constraints being active and the resulting constraint forces). Define the set of active indices $\bar{K} = \{i \in K \mid E_i(q) = 0\}$ and the sub-vector of generalized velocities

$$y := C(q)\dot{q} \tag{11}$$

where the i-th row of $C(q)$ is the gradient vector of E_i, $i \in \bar{K}$. In order to describe the inelastic collision we consider the LCP (in the unknowns y^+, λ)

$$\begin{aligned} y^+ &= y + C(q)M^{-1}(q)C^T(q)\lambda \\ y^+ &\geq 0, \ \lambda \geq 0, \ (y^+)^T\lambda = 0 \end{aligned} \tag{12}$$

Here λ can be interpreted as the vector of *impulsive forces.* Since $C(q)M^{-1}(q)C^T$ $(q) > 0$ this LCP has a unique solution, which determines a new mode as

$$J_c := \{i \in \bar{K} \mid \lambda_i > 0\} \tag{13}$$

In general the new mode J_c will be *different* from the mode J obtained in (10). On the other hand, given the new mode J_c the *jump rule* for these mechanical systems

$$(q, \dot{q}) \mapsto (q, \dot{q}^+ := \dot{q} + M^{-1}(q)C^T(q)\lambda), \tag{14}$$

coincides with the jump rule discussed before, consisting of projection along T onto V (with now T and V corresponding to the DAE's describing mode J_c). Interestingly enough, the new velocity vector $\dot{q}^+$ may be equivalently characterized as the solution of the quadratic programming problem

$$\min_{\{\dot{q}^+ \mid C(q)\dot{q}^+ \geq 0\}} \frac{1}{2}(\dot{q}^+ - \dot{q})^T M(q)(\dot{q}^+ - \dot{q}), \tag{15}$$

which is sometimes taken as the starting point for describing multiple inelastic collisions, see [1, 7]. An appealing feature of both switch rules (9) and (13), together with the resulting jump (14), is that the energy of the mechanical system will always decrease at the switching instant. This is a promising starting point for stability analysis. The extension of the "LCP-based" switching rule (13) to general complementary-slackness systems is currently under investigation.

Conclusion

We have argued that a sizeable class of hybrid (physical) systems can be modelled as complementary-slackness hybrid systems. We have indicated that the complementarity-formalism suggests some natural rules for mode-selection and re-initialization, which provide a "high-level language" for describing the full hybrid dynamics of such systems, and in this way simplify their specification. We have argued that the LCP is a powerful tool for proving existence and uniqueness of solutions of the hybrid dynamics and for actually *computing* the mode of continuation and the re-initialization.

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