Synchronization in Time-varying Networks of Non-Introspective Agents without Exchange of Controller States

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Abstract—We consider two synchronization problems, namely, state synchronization for networks of identical linear agents, and regulated output synchronization for networks of non-identical linear agents, under a time-varying topology. Agents are assumed to be multiple-input multiple-output (MIMO), minimum-phase, right-invertible and non-introspective (i.e., they only receive relative output information from their neighboring agents, but lack independent information about their own states or outputs). The time-varying topology is assumed to switch among a finite number of directed graphs containing a directed spanning tree with an arbitrarily a priori given non-vanishing dwell time. Moreover agents are unable to exchange of controller states with their neighboring agents. For solving each problem, we propose a distributed dynamic protocol based on a combination of low-gain and high-gain design techniques.

I. INTRODUCTION

The problem of achieving synchronization for networks, where the goal is to design a distributed protocol, such that some variables of interest, either state or output trajectory, becomes the same asymptotically, has been substantially studied in last decade, e.g., [1]–[5]. The essential difficulty is the lack of a central authority with the ability to control based on limited information about itself and its surroundings—typically in the form of measurements of its own state or output relative to that of neighboring agents.

The synchronization literature can be generally split into two categories depending whether the agent models are identical or not. The synchronization problem for networks where the agent models are identical has been considered e.g., [3], [6]–[10]. A key idea in the work of [7], [9] is the development of a distributed observer. This observer makes an additional use of the network by allowing the agents to exchange information with their neighboring agents about their internal observer estimates. On the other hand, if the agents are asymptotic null controllable with bounded control gains design techniques.

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in [19] from single-input single-output case to multi-input multi-output right invertible case. Second, regarding the communication topology, we extend the results in [19] from a class of fixed topologies to time-varying topologies. The time-varying topology switches among a finite number of directed graphs that contains a directed spanning tree with non-vanishing dwell time. We start by considering the state synchronization problem in a network of identical agents. For this case, we relax the assumption that the agent dynamics is uniform rank and invariant zeros free in [21]. This extension is a challenging problem as pointed out by [21] since the existence of the invariant zeros prevents the speed-up of the network dynamics which is needed for removing the exchange of controller states. We design a decentralized controller such that the output tracks a network of non-identical agents, where the goal is to regulate the outputs of all the agents identical. However, if one aims for solving regulated output problem, therefore it is necessary that the agent dynamics are unobservable and asymptotically stable modes play no role asymptotically, without loss of generality, we assume that the pair (S, R) is observable and all the eigenvalues of S are in the closed right-half complex plane. In order to achieve our goal, in addition to the information given by (3), it is necessary that a non-empty subset of agents must have knowledge of their output relative to the reference.

\[ \sum_{j=1}^{N} a_{ij}(t), \quad \zeta_i = \sum_{j=1}^{N} a_{ij}(t)(y_i - y_j), \quad (2) \]

Assumption 2: (i) \( G(t) \) switches from a finite set \( G = \{ \delta^* \}_{\delta \in \mathcal{P}}, \) where \( \mathcal{P} = \{1, \ldots, M\} \) is an index set, M indicates its cardinality, and \( \delta^* \), \( i \in \mathcal{P} \), is a digraph which contains a directed spanning tree, whose associated Laplacian matrix is \( L_\delta \). (ii) \( G(t) \) is right continuous, i.e., \( G(t) \) remains fixed for \( t \in [t_k, t_{k+1}) \), \( k \in \mathbb{Z} \) and switches at \( t = t_k \). (iii) the switching time sequence \( \{ t_k \} \) with \( t_0 = 0 \) has a dwell time \( \tau_d > 0 \), i.e., \( t_{k+1} - t_k \geq \tau_d \).

Since Assumption 2 is satisfied, the time-varying topology \( G(t) \) can be modeled as \( G(t) \) with \( \sigma(t) : [0, \infty) \rightarrow \mathcal{P} \) being a piecewise constant and right-continuous function of \( t \), switches at time \( t = t_k \), \( k \in \mathbb{Z} \), and has a dwell time \( \tau_d \).

**Problem 1:** Consider a multi-agent system described by (1) and (3). Assume Assumptions 1 and 2 with any a priori given dwell time \( \tau_d \) are satisfied. The state synchronization problem under a directed switching topology \( G(t) \) is to find, if possible, for any given dwell time \( \tau_d > 0 \), a distributed linear dynamical controller, such that \( \lim_{t \to \infty} (x_i(t) - x_j(t)) = 0 \) for all \( i, j \in \mathcal{V} \).

**B. Regulated Output Synchronization**

In previous section, we consider the state synchronization problem, therefore it is necessary that the agent dynamics are identical. However, if one aims for solving regulated output synchronization, to be defined shortly, it is not necessary to restrict to the case where the agent dynamics are identical. Therefore, we consider a network of \( N \) non-identical linear time-invariant agents

\[ \begin{align*}
\dot{x}_i &= A_i x_i + B_i u_i, \\
y_i &= C_i x_i, \\
\end{align*} \]

where \( x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m, \) and \( y_i \in \mathbb{R}^p \).

**Assumption 3:** For each agent \( i \in \mathcal{V} \), (i)the pair \( (A_i, B_i) \) is stabilizable, (ii) the pair \( (A_i, C_i) \) is detectable, (iii) the triple \( (A_i, B_i, C_i) \) is minimum-phase, (iv) the triple \( (A_i, B_i, C_i) \) is right invertible, and (v) the maximum order of the infinite zeros of the triple \( (A_i, B_i, C_i) \) is \( \rho_{id} \).

Our focus here will be on regulated output synchronization, where the goal is to regulate the outputs of all the agents towards a desired reference trajectory \( y_r(t) \) generated by an autonomous system

\[ \begin{align*}
\dot{\omega} &= S \omega, \\
\omega(0) &= \omega_0, \\
y_r &= R \omega, \\
\end{align*} \]

where \( \omega \in \mathbb{R}^m \) and \( y_r \in \mathbb{R}^p \).

Since unobservable and asymptotically stable modes play no role asymptotically, without loss of generality, we assume that the pair \( (S, R) \) is observable and all the eigenvalues of \( S \) are in the closed right-half complex plane.
trajectory $y_r$ generated by the exosystem system. Specifically, each agent has access to the quantity

$$\psi_i = \iota_i(y_i - y_r),$$

and $C_d \in \mathbb{R}^{p \times p}$ has the special form

$$A_d = \begin{bmatrix} 0 & I_p(p-1) \\ 0 & 0 \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C_d = \begin{bmatrix} I_p & 0 \end{bmatrix}. \quad (12)$$

where $\iota_i$ means that there is an edge from agent 0 to agent $i$, and switches at time $t = t_k$ for $k \in \mathbb{Z}$ and $t_0 = 0$ has a dwell time $\tau_d > 0$, i.e., $t_{k+1} - t_k \geq \tau_d$.

Since Assumption 5 is satisfied, the time-varying topology $\mathcal{G}(t)$ can be modeled as $\mathcal{G}_{\sigma(t)}$ with $\sigma(t) : [0, \infty) \rightarrow \mathcal{P}$ being a piecewise constant and right-continuous function of $t$, switches at time $t = t_k$, $k \in \mathbb{Z}$, and has a dwell time $\tau_d$.

For each $i \in \mathcal{P}$, both digraphs $\mathcal{G}_i$ and $\mathcal{G}_{\sigma(t)}$ are fixed, and hence $L_{\sigma(t)}$ and $L_{\sigma(i)}$ are fixed. It then follows from [15, Lemma 7] that the matrix $L(t) + \text{diag} \{t_1; \ldots; t_N\}$ in (8) for each $i \in \mathcal{P}$ has all its eigenvalues in the open right-half complex plane.

**Problem 2:** Consider a multi-agent system described by (4), (5) and (7). Assume Assumptions 3 and 5 with any a priori given dwell time $\tau_d$ are satisfied. The regulated output synchronization problem under a directed switching topology $\mathcal{G}(t)$ is to find, if possible, for any given dwell time $\tau_d > 0$, a distributed linear dynamical controller, such that $\lim_{t \rightarrow \infty} (y_i(t) - y_r(t)) = 0$ for all $i \in \mathcal{V}$.

## III. STATE SYNCHRONIZATION

The main result of this section is given in the following theorem.

**Theorem 1:** There exist a family of linear time-invariant dynamic controllers, parameterized in terms of low-and-high gain parameters $\delta, \varepsilon \in [0,1]$, of the form

$$\begin{align*}
\dot{x}_i &= A_{ic}(\delta, \varepsilon)x_i + B_{ic}(\delta, \varepsilon)\zeta_i, \quad (10a) \\
u_i &= C_{ic}(\delta, \varepsilon)x_i + D_{ic}(\delta, \varepsilon)\zeta_i, \quad i \in \mathcal{V}, \quad (10b)
\end{align*}$$

where $x_i \in \mathbb{R}^{q}$ is the controller state for agent $i$, such that Problem 1 is solvable for any given dwell time $\tau_d > 0$. That is, for any given $\tau_d$, there exists an $\delta^* \in (0,1)$ such that, for each $\delta \in (0, \delta^*)$, there exists an $\varepsilon^* \in (0,1)$ such that, for all $\varepsilon \in (0, \varepsilon^* (\delta, \tau_d)]$, $\lim_{t \rightarrow \infty} (x_i(t) - x_j(t)) = 0$ for all $i, j \in \mathcal{V}$.

**Proof:** The proof follows from Lemmas 1 and 2 developed in the next two subsections. In Lemma 1, we shall show that state synchronization is achieved for a special agent dynamics specified later. In Lemma 2, we shall show that the state synchronization problem for the general agent dynamics can be converted to this special case by first augmenting the agents with dynamic pre-compensators.

### A. Special Case

In this subsection, we consider special case where the agent dynamics (1) satisfies the following assumption.

**Assumption 6:** (i) the triple $(A, B, C)$ is minimum-phase, invertible, and uniform rank with its order of infinite zeros equal to $p$. (ii) the invariant zeros of the triple $(A, B, C)$ are output-decoupling zeros.

We will show that Problem 1 is solvable for this special case by explicitly constructing the distributed controllers. Our controller design is essentially based on the controller design in [19] for the fixed directed topology case. In what follows, we will present the design with some appropriate modification to the switching topology case.

Since Assumption 6 is satisfied, without loss of generality, we can assume that the agent dynamics (1) is given in the special coordinate basis (SCB) form [24], and thus $x_i$ can be partitioned as $x_i = [x_{id}; x_{ic}]$, where $x_{id} \in \mathbb{R}^{p - \rho}$ is the invariant zero dynamics and $x_{ic} \in \mathbb{R}^{\rho}$ is the infinite-zero dynamics. We then obtain the following dynamics

$$\begin{align*}
\dot{x}_i &= A_{ic}x_{ic} + L_{id}y_i, \quad (11a) \\
\dot{x}_{id} &= A_{id}x_{id} + B_{id}(u_i + E_{dd}x_{dd}), \quad (11b) \\
y_i &= C_{id}x_{id}. \quad (11c)
\end{align*}$$

for some appropriate matrices $L_{id}$ and $E_{dd}$. The matrices $A_{id} \in \mathbb{R}^{\rho \times \rho}$, $B_{id} \in \mathbb{R}^{\rho \times p}$, and $C_{id} \in \mathbb{R}^{p \times \rho}$ have the special form

$$A_{id} = \begin{bmatrix} 0 \\ I_p(\rho-1) \end{bmatrix}, \quad B_{id} = \begin{bmatrix} 0 \\ I_p \end{bmatrix}, \quad C_{id} = \begin{bmatrix} I_p & 0 \end{bmatrix}. \quad (12)$$
Furthermore, the eigenvalues of the matrix $A_{\delta}$ are the invariant zeros of the triple $(A,B,C)$, which are all in the open left-half complex plane due to the minimum-phase assumption.

Let $\delta$ and $\epsilon$ denote a low-gain and high-gain parameter, respectively. Note that the triple $(A_{\delta},B_{\delta},C_{\delta})$ is controllable and observable. Let therefore $K$ be chosen such that $A_{\delta} - KC_{\delta}$ is Hurwitz. Moreover, let $P_{\delta} = P_{\delta}^* > 0$ be the unique solution of the algebraic Riccati equation (ARE)

$$P_{\delta}A_{\delta} + A_{\delta}^*P_{\delta} - \tau P_{\delta}B_{\delta}^*P_{\delta} + \delta I_{pp} = 0,$$

where $\tau > 0$ is the lower bound on the real parts of the non-zero eigenvalues of all the Laplacian matrices $L_i$, $i \in \mathcal{P}$, associated with the fixed digraph $\mathcal{G}_i$.

Next, define $F_\delta = -B_{\delta}^*P_{\delta}$ and $S_{\epsilon} = \text{blkdiag}\{I_p, I_p, \ldots; I_p^{-1}; I_p\}$, $K_{\epsilon} = \epsilon^{-1}S_{\epsilon}^{-1}K$ and $F_{\delta,\epsilon} = \epsilon^{-\rho}F_{\delta}S_{\epsilon}$. We then design the following dynamic controller for each agent $i' \in \mathcal{Y}$.

$$\dot{\hat{x}}_{id} = A_{\delta}\hat{x}_{id} + B_{\delta}E_{dd}\hat{x}_{id} + K_{\epsilon}(\zeta_{id} - C_{\delta}\hat{x}_{id}),$$

$$u_i = \hat{x}_{id},$$

(14a)

(14b)

**Lemma 1:** Consider the network with identical agents dynamics (11) and controller (14). Under Assumptions 2 and 6, Problem 1 is solvable for any given dwell time $\tau_i$. That is, there exists a $\delta' \in (0,1]$ such that for all $\delta \in (0,\delta')$, there exists $\epsilon^0(\delta, \tau_i) \in (0,1]$ such that for all $\epsilon \in (0, \epsilon^0(\delta, \tau_i))$, $\lim_{t \rightarrow \infty} |\hat{x}_{id}(t) - x_{id}(t)| = 0$ for all $i, j \in \mathcal{Y}$.

**Proof:** The proof is based on the proof of [19, Theorem 1] in conjunction with the multiple Lyapunov idea in [25]. Let us first briefly recall some preliminary and necessary results from the proof of [19, Theorem 1].

Define the relative state dynamics $\hat{x}_{ia} = x_{Na} - x_{ia}$, $\bar{x}_{id} = x_{Nd} - x_{id}$, and $\hat{x}_{id} := \bar{x}_{Nd} - \bar{x}_{id}$ for $i \in \{1, \ldots, N-1\}$. We then obtain the following dynamics according to (11) and (14).

$$\dot{\hat{x}}_{ia} = A_{\delta}\hat{x}_{ia} + L_{ad}C_{ad}\hat{x}_{ia},$$

$$\dot{\hat{x}}_{id} = A_{\delta}\hat{x}_{id} + B_{\delta}E_{dd}\hat{x}_{id} + E_{dd}\bar{x}_{id},$$

(15a)

(15b)

Since $\sum_{i=1}^{N} \epsilon_{ij}(t) \equiv 0$ for all $i \in \mathcal{Y}$ and all $t \geq 0$, we have

$$\zeta_{N} - \zeta_{i} = \sum_{j=1}^{N} (\epsilon_{ij}(t) - \epsilon_{ji}(t)) (\gamma_{N} - \gamma_{i}) = \sum_{j=1}^{N} (\epsilon_{ij}(t)) \gamma_{i},$$

where $\tilde{\epsilon}_{ij}(t) = \epsilon_{ij}(t) - \epsilon_{ji}(t), i, j \in \{1, \ldots, N-1\}$. This together with (14) implies that

$$\dot{\hat{x}}_{id} = A_{\delta}\hat{x}_{id} + B_{\delta}E_{dd}\hat{x}_{id} + \sum_{j=1}^{N-1} \tilde{\epsilon}_{ij}(t) K_{\epsilon}\bar{x}_{id} - K_{\epsilon}\bar{x}_{id}. $$

(16)

We first note that if $\tilde{x}_{id} \rightarrow 0$ as $t \rightarrow \infty$ exponentially fast, it then follows from [26, Lemma B.1] and the fact that all the eigenvalues of $A_{\delta}$ are in the open left-half complex plane, that $\lim_{t \rightarrow \infty} \tilde{x}_{id}(t) = 0$.

From the proof of [19, Theorem 1], we know that for each fixed directed graph $\mathcal{G}_i$, $i \in \mathcal{P}$, there exist a non-singular state transformation $T_i$, such that

$$\eta_i = T_i[\hat{x}_{id}; \hat{x}_{id}; \ldots; \hat{x}_{i(N-1)d}; \hat{x}_{i(N-1)d}]$$

(17)

satisfies the following dynamics

$$\epsilon \eta_{id} = \tilde{A}_{\delta,\epsilon} \eta_{id} + \tilde{W}_{\delta,\epsilon} \eta_{id},$$

(18)

where

$$\tilde{A}_{\delta,\epsilon} = I_{k} - \epsilon \otimes \begin{bmatrix} A_{\delta} & 0 \\ 0 & A_{\delta} - KC_{\delta} \end{bmatrix} + J_{\epsilon} \otimes \begin{bmatrix} B_{\delta}F_{\delta} & -B_{\delta}F_{\delta} \\ B_{\delta}F_{\delta} & -B_{\delta}F_{\delta} \end{bmatrix},$$

$\tilde{W}_{\delta,\epsilon}$ is of $O(\epsilon)$, and $J_{\epsilon} \in \mathbb{R}^{(N-1) \times (N-1)}$ is the Jordan canonical form of $L_i := [\tilde{A}_{ij}^\epsilon]$, where $i, j \in \{1, \ldots, N-1\}$, for the fixed directed graph $\mathcal{G}_i$, $i \in \mathcal{P}$.

Finally, we choose $\epsilon = \min_{i \in \mathcal{P}} \frac{\lambda_{\max}(\tilde{P}(\delta, \nu)) \lambda_{\max}(\tilde{P}(\delta, \nu))}{\lambda_{\min}(\tilde{P}(\delta, \nu))} \epsilon^{0(\delta, \tau_i)}$, such that for each fixed $\epsilon \in (0, \epsilon^0)$, $\lim_{t \rightarrow \infty} \|\eta_i(t)\| = 0$. Thus $\lim_{t \rightarrow \infty} \tilde{x}_{id}(t) = 0$ since the state transformation $T_i$ in (17) is non-singular for all $t \geq t_0$. Hence, state synchronization is achieved.

**B. General Case**

In this section, we will show that how to recover the general case to the special case by designing the pre-compensators so that the augmented system satisfies the Assumption 6 as shown in the following lemma whose proof can be found in [27].

**Lemma 2:** Consider the agent dynamics described by (1). Assume that Assumption 1 is satisfied. Then for any given integer $p \geq P_d + \ell^*$, where $P_d$ is the maximum order of the infinite zeros given in Assumption 1, and $\ell^*$ is the minimum
polynomial degree 2 of the system matrix of the invariant zero dynamics, there exists a dynamic pre-compensator for each agent \( i \in Y' \) of the form
\[
\dot{z}_i = A_ip_2 \zeta_i + B_ip_2 v_i, \quad (19a)
\]
\[
u_i = C_ip_2 \zeta_i + D_ip_2 v_i, \quad (19b)
\]
where \( v_i \in \mathbb{R}^p \) is a new input, such that the augmented system of (1) and (19) satisfies Assumption 6.

IV. REGULATED OUTPUT SYNCHRONIZATION

The main result of this section is given in the following theorem.

**Theorem 2**: There exist a family of linear time-invariant dynamic controllers, parameterized in terms of low-and-high gain parameters \( \delta, \epsilon \in (0,1) \), of the form
\[
\dot{\chi}_i = A_i(\delta, \epsilon) \chi_i + B_i(\delta, \epsilon) \tilde{p}, \quad (20a)
\]
\[
u_i = C_i(\delta, \epsilon) \chi_i + D_i(\delta, \epsilon) \tilde{p}, \quad i \in Y', \quad (20b)
\]
where \( x_i \in \mathbb{R}^n \) is the controller state for agent \( i \), such that the Problem 2 is solvable for any given dwell time \( \tau_d > 0 \). That is, for any given \( \tau_d \), there exists an \( \rho^* \in (0,1] \) such that, for each \( \delta \in (0, \rho^*] \), there exists an \( \epsilon^* \) \( (\delta, \tau_d) \in (0,1] \) such that, for all \( \epsilon \in (0, \epsilon^* (\delta, \tau_d)] \), \( \lim_{t \to -\infty} (y_i(t) - y_\alpha(t)) = 0 \) for all \( i \in Y' \).

**Proof**: The proof follows from Lemmas 3 and 4 developed in the next two subsections. In Lemma 3, we shall show that state synchronization is achieved for a special agent dynamics specified later. In Lemma 4, we shall show that the state synchronization problem for the general agent dynamics can be convert to this special case by first augmenting the agents with dynamic pre-compensators.

A. Special Case

In this subsection, we consider special case where the agent dynamics (4) satisfies the following assumption

**Assumption 7**: (i) the pair \( (A_i, C_i) \) contains \( (S, R) \), (ii) the triple \( (A_i, B_i, C_i) \) is minimum-phase, invertible, and uniform rank with its order of the infinite zeros equal to \( p \), and (iii) the invariant zeros of the triple \( (A_i, B_i, C_i) \) are output-decoupling zeros.

We will show that Problem 2 is solvable for this special case by explicitly constructing the controllers. Our controller design is essentially based on the controller design in [19] for the fixed directed topology case. In what follows, we will present the design with some appropriate modification to the switching topology case.

Since Assumption 7 is satisfied, without loss of generality, we can assume that each agent dynamics (4) is given in the SCB form, and thus \( x_i \) can be partitioned as \( x_i = [x_{ia}; x_{id}] \), where \( x_{ia} \in \mathbb{R}^{n_p - p p} \) and \( x_{id} \in \mathbb{R}^{pp} \), and where
\[
\dot{x}_{ia} = A_{ia} x_{ia} + L_{iad} y_i, \quad (21a)
\]
\[
\dot{x}_{id} = A_{id} x_{id} + B_{id} (u_i + E_{id} x_{id}), \quad (21b)
\]
\[
y_i = C_{id} x_{id}, \quad (21c)
\]

The minimum degree of a matrix \( A \) is the minimum integer \( \ell \) such that \( A^\ell \) can be written as a linear combination of \( A^i \), for \( i = 0, 1, \ldots, \ell - 1 \).

for some appropriate matrices \( L_{iad} \) and \( E_{idd} \). The matrices \( A_d, B_d \) and \( C_d \) have the special form given in (12). Furthermore, the eigenvalues of \( A_{ia} \) are the invariant zeros of the triple \( (A_i, B_i, C_i) \).

Let \( K_{E} \) and \( F_{be} \) be designed as in Section III-A, however, in (13), \( \tau > 0 \) is the lower bound on the real parts of the eigenvalues of all the matrices \( L_0 + \text{diag} \{ \tau_1; \ldots; \tau_N \}, i \in \mathcal{P} \).

We then design the controller for each agent \( i \in Y' \):
\[
\dot{x}_{id} = A_{id} x_{id} + K_{E} (\eta_i - C_{id} x_{id}), \quad (22a)
\]
\[
u_i = F_{be} \dot{x}_{id}. \quad (22b)
\]

**Lemma 3**: Consider the network of non-identical linear agents (21), the exosystem (5) and the dynamic controller (22). Under Assumptions 5 and 7, Problem 2 is solvable for any given dwell time \( \tau_d > 0 \). That is, there exists an \( \delta_r \in (0,1] \) such that, for each \( \delta \in (0, \delta_r] \), there exists an \( \epsilon^* (\delta, \tau_d) \in (0,1] \) such that, for all \( \epsilon \in (0, \epsilon^* (\delta, \tau_d)] \), \( \lim_{t \to -\infty} (y_i(t) - y_\alpha(t)) = 0 \) for all \( i \in Y' \).

**Proof**: The proof is based on the proof of [19, Theorem 5] in conjunction with the multiple Lyapunov idea [25] as used in the proof of Theorem 1. The detailed proof can be found in [27].

B. General Case

In this section, we show how to recover the general case to the special case by designing pre-compensators. Due to the space limitation, we have omitted the details which can be found in [27].

**Pre-Compensator 1**: Consider the dynamic pre-compensator of the form
\[
\dot{z}_i = A_{ip_1} z_i + B_{ip_1} u_i^1, \quad (23a)
\]
\[
u_i = C_{ip_1} z_i, \quad (23b)
\]
where \( u_i^1 \in \mathbb{R}^p \) is a new input. The purpose of this pre-compensator is to add modes from the exosystem to agent \( i \in Y' \), so that the augmented dynamics contains the exosystem.

**Pre-Compensator 2**: Consider the pre-compensator given by the form (24)
\[
\dot{z}_i = A_{ip_2} z_i + B_{ip_2} u_i^2, \quad (24a)
\]
\[
u_i = C_{ip_2} z_i + D_{ip_2} u_i^2, \quad (24b)
\]
where \( u_i^2 \in \mathbb{R}^p \) is a new input. The purpose of this pre-compensator is to ensure that the augmented system is stabilizable, detectable, invertible, uniform rank, minimum-phase, and furthermore the invariant zeros become output-decoupling zeros.

**Pre-Compensator 3**: Each augmented agent dynamics of (4), (23) and (24) is uniform rank, however, the order of the infinite zeros for different agents are not the same. Let \( p_i \) be the order of the infinite zeros of the augmented agent dynamics for agent \( i \). The purpose of this step is to extend the length of the infinite zero chains of the augmented agent dynamics, such that the order of the infinite zeros of all augmented systems equal to \( \rho = \max_{i \in Y'} p_i \).

1We say that a matrix pair \( (A, C) \) contains the matrix pair \( (S, R) \) if there exists a matrix \( \Pi \) such that \( \Pi S = A \Pi \) and \( C \Pi = R \).
Define the following dynamic pre-compensator

\[
\dot{z}_{i3} = A_{ip3}z_{i3} + B_{ip3}u^i_3, \quad (25a)
\]
\[
u^i_2 = C_{ip3}z_{i3}, \quad (25b)
\]
where \(u^i_2 \in \mathbb{R}^P\) is a new input and where

\[
A_{ip3} = \begin{bmatrix} 0 & I_p \rho & -1 \end{bmatrix}, \quad B_{ip3} = \begin{bmatrix} 0 \ I_p \ 0 \end{bmatrix}, \quad C_{ip3} = \begin{bmatrix} I_p \ 0 \end{bmatrix}.
\]

By stacking the state of the original agent dynamics and the state of the three pre-compensators as \(\chi_i = [x_i; z_i1; z_i2; z_i3]\), we obtain the following augmented agent dynamics with the input \(u^i_3\):

\[
\dot{\chi}_i = A\chi_i + B\chi_i, \quad (26a)
\]
\[
y_i = C\chi_i, \quad (26b)
\]

where

\[
A = \begin{bmatrix}
0 & B_{i1}C_{ip1} & 0 & 0 \\
A_{i1} & B_{i1} & 0 & 0 \\
0 & A_{i2} & B_{i2} & 0 \\
0 & 0 & A_{i3} & B_{i3}
\end{bmatrix},
B = \begin{bmatrix}
0 \\
0 \\
0 \\
B_{i3}
\end{bmatrix},
C = [C_i, 0, 0].
\]

We have the following result whose proof can be found in [27], which shows that we can recover the general case to the special case by designing the above three pre-compensators, so that the augmented system satisfies Assumption 7.

**Lemma 4:** The augmented dynamics (26) satisfies Assumption 7. That is, (i) the triple \((\chi_i, \alpha_i, \beta_i)\) is minimum-phase, uniform rank with order of the infinite zeros \(\rho\), (ii) the invariant zeros of the triple \((\chi_i, \alpha_i, \beta_i)\) are output-decoupling zeros, and (iii) the pair \((\alpha_i, \beta_i)\) contains \((S, R)\).

**V. CONCLUSION**

This paper studied two synchronization problems under time-varying topologies, namely, state synchronization for networks of identical linear agents and regulated output synchronization for networks of non-identical linear agents. The agent are MIMO, minimum-phase, right invertible and are unable to exchange controller states. The time varying topology is assumed to switch among a finite set of digraphs which contains a directed spanning tree with an arbitrarily non-vanishing dwell time.

**REFERENCES**