Synchronization in networks of weakly-non-minimum-phase, non-introspective agents without exchange of controller states

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Abstract. Synchronization in networks has been an active research area in recent years. In this paper we consider networks where the only measurements available to an agent are relative output information from their neighbors. No direct information of its own internal agents is available nor is there an exchange of internal controller states with neighboring agents. We consider the homogeneous case where all agents are identical SISO linear systems of arbitrary order. The prime extension compared to earlier work is that we only assume the systems to be weakly-non-minimum-phase.

I. INTRODUCTION

Synchronization in networks has been an active research area in recent years although decentralized control has a long history (see for instance [2], [11]). Its revival started with works such as [7], [6], [8] and is based on crucial work [12], [13]. In recent years lots of extensions have been considered. Initially systems were often single or double integrators but recently also general linear systems have been considered. In the literature we often see all agents identical (the homogeneous case) but also the case of different agents (heterogeneous case) has been considered. Initially papers assumed that relative state measurements were available. An important extension is to only consider relative output measurements which was studied in [5]. However in that paper it was assumed that there is also an exchange of controller states between neighboring agents. This exchange is in itself not very natural so an obvious quest became to investigate whether this exchange of controller states can be avoided. A first paper that avoided this exchange was the paper [10] using a low-gain controller design. That paper imposed a limitations on the poles of the agents which needed to be in the closed left-half plane. In the paper [3] a low-high gain design was derived to deal with minimum-phase systems. The prime objective of this paper is to extend this latter paper to the case of weakly-non-minimum-phase systems. This requires substantial modifications in the design since we really need to include an observer for the zero dynamics since the unstable zero dynamics directly affects our synchronization. This paper is an initial step where assume in contrast to [3] that we are dealing with an undirected network of identical agents. In other words, we are looking at an undirected network of homogeneous agents.

II. PROBLEM FORMULATION

We consider a network of $N$ SISO identical agents:

\[
\begin{align*}
\dot{x}_i &= A x_i + B u_i \\
y_i &= C x_i
\end{align*}
\]

(1)

The information each agent has available comes from the network. To be precise agent $i$ has access to the following quantity:

\[
\zeta_i = \sum_{j=1}^{N} a_{ij} (y_i - y_j)
\]

with $a_{ij} = a_{ji} > 0$. This can be rewritten as:

\[
\zeta_i = \sum_{j=1}^{N} g_{ij} y_j
\]

(2)

where $G = [g_{ij}]$ is such that its row sums are zero and all off-diagonal elements are nonpositive while its diagonal elements are nonnegative. The Laplacian matrix $G$ is associated to an undirected graph $\mathcal{G}$ and we assume that this graph is connected which implies that the matrix $G$ has a single eigenvalue in 0 while all other eigenvalues are real and strictly positive.

In practice, the network is often not precisely known. For this reason our design will only rely on a lower bound $\tau$ for the non-zero eigenvalues of $G$ and will not depend on other characteristics of the network.

Our goal is to achieve state synchronization among the agents, i.e. our objective is to guarantee:

\[
\lim_{t \to \infty} (x_i - x_j) = 0
\]

for all $i, j \in \{1, \ldots, N\}$ and for all initial conditions of the individual agents and their controllers.

For our design it will be convenient to bring the agents into an appropriate form through a state space transformation, see [9]

\[
\begin{align*}
\dot{x}_a &= A_a x_a + L_{ad} y_i \\
\dot{x}_d &= A_d x_d + B_d (u_i + E_{da} x_a + E_{dd} x_d) \\
y_i &= C_d x_d
\end{align*}
\]

(3)
where
\[
A_d = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 1 \\
0 & \cdots & \cdots & 1 & 0
\end{pmatrix}, \quad B_d = \begin{pmatrix}
0 \\
\vdots \\
\vdots \\
\vdots \\
0
\end{pmatrix}
\]
and
\[
C_d = \begin{pmatrix}
1 & 0 & \cdots & 0
\end{pmatrix}.
\]

We consider a controller for agent \(i\) of the form:
\[
\dot{x}_{ia} = A_d \dot{x}_{ia} + L_{ad} C_d \dot{x}_{id} + K_1 (\zeta_i - C_d \dot{x}_{id}) \\
\dot{x}_{id} = A_d \dot{x}_{id} + B_d (u_i + E_{da} \dot{x}_{ia} + E_{dd} \dot{x}_{id}) \\
+ K_2 (\zeta_i - C_d \dot{x}_{id})
\]
(4)

\[
u_i = F_1 \dot{x}_{ia} + F_2 \dot{x}_{id}
\]
where \(\zeta_i\) is defined by (2). We should realize that the states \(\dot{x}_{ia}\) and \(\dot{x}_{id}\) are actually an estimate of
\[
\sum_{i=1}^{N} g_{ij} x_{ja}, \quad \text{and} \quad \sum_{i=1}^{N} g_{ij} x_{jd},
\]
respectively. In other words, we estimate the linear combination of the neighbor’s relative states as described by the network instead of estimating the state of the agent directly. The main difference with the minimum-phase case studied in [3] is \(\dot{x}_{ia}\) which is not needed in the minimum-phase case since the zero dynamics are asymptotically stable and hence do not affect synchronization.

We choose
\[
F_1 = \varepsilon^{-\rho} \bar{F}_1, \\
F_2 = \varepsilon^{-\rho} \bar{F}_2 S_e, \\
K_1 = \varepsilon^{-\rho} \bar{K}_1, \\
K_2 = \varepsilon^{-1} S_e^{-1} \bar{K}_2
\]
where \(\varepsilon \in (0, 1]\) is a high-gain parameter which speeds up the estimation of the \(x_{id}\)-dynamics. Here, the matrix \(S_e\) is defined by:
\[
S_e = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & \varepsilon & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 0 \\
0 & \cdots & \cdots & \varepsilon^{\rho-1}
\end{pmatrix}
\]
where \(\rho\) is the dimension of \(x_{id}\) which is equal to the relative degree of the agents. Moreover, we choose \(\bar{K}_2\) and \(\bar{F}_2\) such that \(A_d - K_2 B_d\) is asymptotically stable while \(\bar{F}_2 = -B_d P_d\) where \(P_d\) is the solution of the algebraic Riccati equation:
\[
P_d A_d + A_d^T P_d - \tau P_d B_d B_d^T P_d + \delta I = 0
\]
(5)
with \(\tau\) a lower bound for the smallest non-zero eigenvalue of the Laplacian \(G\) while \(\delta \in (0, 1]\) is a low-gain parameter which needs to be chosen sufficiently small.

**Theorem II.1** Consider the network with agents described by (3) with the measurements given by (2). Let the network be described by an undirected, connected graph whose Laplacian matrix \(G\) is symmetric and its smallest non-zero eigenvalue is larger than \(\tau > 0\).

In that case there exist a dynamic controller described by (4), which only depends on the dynamics of the agents (3), and the lower bound \(\tau\) such that
\[
\lim_{t \to \infty} \left[ y_i(t) - y_j(t) \right] = 0
\]
(6)
for all \(i, j \in 1, \ldots, N\) provided \(\delta\) and \(\varepsilon\) are chosen sufficiently small where the choice of \(\varepsilon\) depends on our choice of \(\delta\).

Before we prove the above theorem, we first establish the following lemma which will be needed in the proof.

**Lemma II.2** Assume a SISO system \((A, B, C)\) which is controllable and observable is given. Moreover, let \(M > 0\) be given. Then there exists matrices \(K\) and \(F\) such that
\[
\begin{pmatrix}
A + KC & \alpha BF \\
-KC & A + BF
\end{pmatrix}
\]
(7)
is asymptotically stable for all \(\alpha \in (-\frac{1}{M}, M]\).

**Proof:** Let \(P\) and \(Q\) be such that
\[
A'P + PA - PBB'P + I = 0 \\
AQ + QA' - QC'CQ + I = 0
\]
and choose \(F = -B'P\) and \(K = -QC'\).

The proof will be established by contradiction. Assume (7) is unstable for some \(\alpha \in (-\frac{1}{M}, M]\). Since the matrix is asymptotically stable for \(\alpha = 0\), this implies there exists some \(\alpha_0 \in (-\frac{1}{M}, M]\) such that (7) has an imaginary axis eigenvalue \(s\). This implies that there exists \(x_1, x_2\) which are not both equal to zero such that:
\[
(A + KC)x_1 + \alpha_0 BF x_2 = sx_1 \\
-KCx_1 + (A + BF)x_2 = sx_2
\]

It is then easily verified that \(y = (sI - A - BF)x_2\) satisfies:
\[
[I + \alpha KC(sI - A - KC)^{-1} BF(sI - A - BF)^{-1}]y = 0
\]
and \(y \neq 0\). Here we use that \(A + KC\) and \(A + BF\) are asymptotically stable. Since
\[
[I + \alpha KC(sI - A - KC)^{-1} BF(sI - A - BF)^{-1}]
\]
is therefore singular, it is easily seen that
\[
1 + \alpha g(s) h(s) = 0
\]
(8)
where
\[
g(s) = F(sI - A - BF)^{-1} B \\
h(s) = C(sI - A - KC)^{-1} K
\]
From classical LQ theory (see [1]) we know that
\[
|1 + g(s)| \leq 1, |1 + h(s)| \leq 1,
\]
(9)
(recall that \( s \) is on the imaginary axis). This implies \(|g(s)| \leq 2\) and \(|h(s)| \leq 2\) which contradicts (8) if \(|\alpha| < \frac{1}{4}\).

Remains to consider the case where \( \alpha > \frac{1}{4} \). If (8) is satisfied with \( \alpha \) a positive real number, then \( g(s)h(s) \) must be real and hence there exists real numbers \( a, b \) and \( r < 0 \) such that:

\[
g(s) = a + bi \tag{10}
\]

\[
h(s) = ra - rb \tag{11}
\]

On the other hand (9) implies:

\[(1 + a)^2 + b^2 \leq 1 + (ra)^2 + (rb)^2 < 1\]

The first equation implies \( a \in [-2,0]\) while the second equation implies \( ra \in [-2,0]\). This yields a contradiction with \( r < 0 \) and therefore (8) cannot be satisfied. We conclude that (7) cannot have eigenvalues in the closed right half plane which implies that (7) is asymptotically stable.

Using the above lemma, we can now present the proof of Theorem II.1.

**Proof of Theorem II.1**: For each \( i \in 1, \ldots, N - 1 \), define

\[e_i = \gamma_N - y_i, \quad \tilde{g}_{ij} = g_{ij} - g_{Nj},\] and let

\[\hat{x}_i = \begin{pmatrix} \tilde{x}_{ia} \\ \tilde{x}_{id} \end{pmatrix} = \begin{pmatrix} x_{Na} - x_{ia} \\ x_{Nd} - x_{id} \end{pmatrix} , \quad \hat{\xi}_i = \begin{pmatrix} \hat{x}_{ia} \\ \hat{x}_{id} \end{pmatrix} = \begin{pmatrix} \tilde{x}_{Na} - \tilde{x}_{ia} \\ \tilde{x}_{Nd} - \tilde{x}_{id} \end{pmatrix}\]

Using this notation, we can write,

\[\hat{x}_{ia} = A_a \tilde{x}_{ia} + L_{ad} C_d \tilde{x}_{id} \]

\[\hat{x}_{id} = A_d \tilde{x}_{id} + B_d (F_1 \hat{x}_{ia} + F_2 \hat{x}_{id} + E_{da} \tilde{x}_{ia} + E_{dd} \tilde{x}_{id})\]

\[e_i = C_d \tilde{x}_{id}\]

Moreover,

\[\hat{x}_{ia} = A_a \tilde{x}_{ia} + L_{ad} C_d \tilde{x}_{id} + \sum_{j=1}^{N-1} \tilde{g}_{i,j} K_1 C_d \tilde{x}_{jd} - K_1 C_d \hat{\xi}_{id}\]

\[\hat{x}_{id} = A_d \tilde{x}_{id} + B_d (F_1 \hat{x}_{ia} + F_2 \hat{x}_{id} + E_{da} \tilde{x}_{ia} + E_{dd} \tilde{x}_{id}) + \sum_{j=1}^{N-1} \tilde{g}_{i,j} K_2 C_d \tilde{x}_{jd} - K_2 C_d \hat{\xi}_{id}\]

Next, define

\[\hat{\xi}_{ia} = \tilde{x}_{ia}, \quad \hat{\xi}_{id} = S_e \tilde{x}_{id}\]

Then,

\[\hat{x}_{ia} = A_a \hat{\xi}_{ia} + V_{ad} \hat{\xi}_{id}\]

\[\hat{x}_{id} = A_d \hat{\xi}_{id} + B_d \bar{F}_{1} \hat{\xi}_{ia} + B_d \bar{F}_{2} \hat{\xi}_{id} + V_{dd} \hat{\xi}_{ia} + V_{ad} \hat{\xi}_{id}\]

\[e_{ia} = A_a \hat{x}_{ia} + B_d \bar{F}_{1} \tilde{x}_{ia} + E_{da} \tilde{x}_{ia} + E_{dd} \tilde{x}_{id}\]

\[e_{id} = A_d \hat{x}_{id} + B_d \bar{F}_{1} \tilde{x}_{id} + B_d \bar{F}_{2} \hat{\xi}_{id} + E_{da} \tilde{x}_{ia} + E_{dd} \tilde{x}_{id}\]

\[\hat{\xi}_{ia} = A_a \hat{\xi}_{ia} + V_{ad} \hat{\xi}_{id}\]

\[\hat{\xi}_{id} = A_d \hat{\xi}_{id} + B_d \bar{F}_{1} \hat{\xi}_{ia} + B_d \bar{F}_{2} \hat{\xi}_{id} + V_{dd} \hat{\xi}_{ia} + V_{ad} \hat{\xi}_{id}\]

where

\[V_{ad} = L_{ad} C_d, \quad V_{da} = e^p B_d E_{da} \quad \text{and} \quad V_{dd} = e^p B_d E_{dd} S_e^{-1}\]

Define \( \bar{G} = [\tilde{g}_{ij}] \) for \( i, j \in 1, \ldots, N - 1 \) and let

\[\hat{\xi}_a = \begin{pmatrix} \hat{\xi}_{ia} \\ \hat{\xi}_{id} \\ \hat{\xi}_{(N-1)a} \end{pmatrix}, \quad \hat{\xi}_d = \begin{pmatrix} \hat{\xi}_{id} \\ \hat{\xi}_{(N-1)d} \end{pmatrix}\]

\[\hat{\xi}_a = \begin{pmatrix} \hat{\xi}_{ia} \\ \hat{\xi}_{id} \\ \hat{\xi}_{(N-1)a} \end{pmatrix}, \quad \hat{\xi}_d = \begin{pmatrix} \hat{\xi}_{id} \\ \hat{\xi}_{(N-1)d} \end{pmatrix}\]

Then we have,

\[\hat{\xi}_a = (I_{N-1} \otimes A_a) \hat{\xi}_a + V_{ad} \hat{\xi}_d + (\bar{G} \otimes \bar{K}_1 C_d) \hat{\xi}_d - (I_{N-1} \otimes \bar{K}_1 C_d) \hat{\xi}_d\]

\[\hat{\xi}_d = (I_{N-1} \otimes A_d) \hat{\xi}_d + V_{dd} \hat{\xi}_a + (I_{N-1} \otimes B_d \bar{F}_1) \hat{\xi}_a + (I_{N-1} \otimes B_d \bar{F}_2) \hat{\xi}_d\]

Define \( \bar{G} = U J U^{-1} \), where \( J \) is the Jordan form of \( \bar{G} \), and let

\[v_a = (J U^{-1} \otimes I_{n-p}) \hat{\xi}_a, \quad \tilde{v}_a = v_a - (J U^{-1} \otimes I_{n-p}) \hat{\xi}_a, \quad v_d = (J U^{-1} \otimes I_p) \hat{\xi}_d, \quad \tilde{v}_d = v_d - (J U^{-1} \otimes I_p) \hat{\xi}_d\]

Then,

\[\tilde{v}_a = (I_{N-1} \otimes A_a) \tilde{v}_a + W_{ad} v_d\]

\[\tilde{v}_d = (I_{N-1} \otimes A_d) \tilde{v}_d + W_{dd} v_d\]

\[W_{ad} = (J U^{-1} \otimes I_{n-p}) V_{ad} (U J^{-1} \otimes I_p) = V_{ad}, \quad W_{dd} = (J U^{-1} \otimes I_p) V_{dd} (U J^{-1} \otimes I_{n-p}) = V_{dd}\]

Define \( \bar{G} = [\tilde{g}_{ij}] \) for \( i, j \in 1, \ldots, N - 1 \) and let

\[\hat{\xi}_a = \begin{pmatrix} \hat{\xi}_{ia} \\ \hat{\xi}_{ia} \\ \hat{\xi}_{(N-1)a} \end{pmatrix}, \quad \hat{\xi}_d = \begin{pmatrix} \hat{\xi}_{id} \\ \hat{\xi}_{id} \\ \hat{\xi}_{(N-1)d} \end{pmatrix}\]

\[\hat{\xi}_a = \begin{pmatrix} \hat{\xi}_{ia} \\ \hat{\xi}_{ia} \\ \hat{\xi}_{(N-1)a} \end{pmatrix}, \quad \hat{\xi}_d = \begin{pmatrix} \hat{\xi}_{id} \\ \hat{\xi}_{id} \\ \hat{\xi}_{(N-1)d} \end{pmatrix}\]

\[W_{ad} = (J U^{-1} \otimes I_{n-p}) V_{ad} (U J^{-1} \otimes I_p) = V_{ad}, \quad W_{dd} = (J U^{-1} \otimes I_p) V_{dd} (U J^{-1} \otimes I_{n-p}) = V_{dd}\]

We consider the following decomposition:

\[v_a = \begin{pmatrix} v_{ia} \\ v_{ia} \\ v_{(N-1)a} \end{pmatrix}, v_d = \begin{pmatrix} v_{id} \\ v_{id} \\ v_{(N-1)d} \end{pmatrix}\]
\( \hat{v}_a = \left( \begin{array}{c} \hat{v}_{1a} \\ \vdots \\ \hat{v}_{(N-1)a} \end{array} \right) \), \( \hat{v}_d = \left( \begin{array}{c} \hat{v}_{1d} \\ \vdots \\ \hat{v}_{(N-1)d} \end{array} \right) \).

and we define:

\[ \eta_{ia} = \left( \begin{array}{c} v_{ia} \\ \hat{v}_{ia} \end{array} \right) \quad \eta_{id} = \left( \begin{array}{c} v_{id} \\ \hat{v}_{id} \end{array} \right) \]

for \( i = 1, \ldots, N-1 \) and we obtain the following dynamics:

\[
\begin{aligned}
\dot{\eta}_{ia} &= \bar{A}_{iaa} \eta_{ia} + \bar{A}_{iad} \eta_{id} \\
\dot{\eta}_{id} &= \bar{A}_{idd} \eta_{ia} + \bar{A}_{idd} \eta_{id} + \bar{W}_{ida} \eta_{ia} + \epsilon \bar{W}_{ida} \eta_{id}
\end{aligned}
\]  

(12)

for \( i = 1, \ldots, N-1 \) with

\[
\begin{aligned}
\bar{A}_{iaa} &= \left( \begin{array}{cc} A_L & 0 \\ 0 & A_R \end{array} \right) \\
\bar{A}_{iad} &= \left( \begin{array}{cc} L_{ad} C_d & 0 \\ (1 - \lambda_1) L_{ad} C_d & \lambda_1 L_{ad} C_d - e^{-\varphi} \lambda_1 \bar{K}_1 C_d \end{array} \right) \\
\bar{A}_{ida} &= \left( \begin{array}{cc} B_d \tilde{F}_1 & -B_d \bar{F}_1 \\ (1 - \lambda_1^{-1}) B_d \tilde{F}_1 & -(1 - \lambda_1^{-1}) B_d \bar{F}_1 \end{array} \right) \\
\bar{A}_{idd} &= \left( \begin{array}{cc} A_d & 0 \\ 0 & A_d - \bar{K}_2 C_d \end{array} \right) \\
+ & \left( \begin{array}{cc} \lambda_1 B_d \tilde{F}_2 & -\lambda_1 B_d \bar{F}_2 \\ (\lambda_1 - 1) B_d \tilde{F}_2 & -(\lambda_1 - 1) B_d \bar{F}_2 \end{array} \right) \\
\bar{W}_{ida} &= e^{\varphi} \left( \begin{array}{cc} B_d E_{da} & 0 \\ (1 - \lambda_1^{-1}) B_d E_{da} & \lambda_1^{-1} B_d E_{da} \end{array} \right) \\
\bar{W}_{idd} &= e^{\varphi-1} \left( \begin{array}{cc} B_d E_{dd} S_{\epsilon^{-1}} & 0 \\ 0 & B_d E_{dd} S_{\epsilon^{-1}} \end{array} \right)
\end{aligned}
\]

Note that the dynamics expressed in (12) for different \( i \) have no coupling between them so it remains the check the stability of the individual systems. We will use a singular perturbation approach to establish stability for sufficiently small \( \varepsilon \) (see e.g. [4]). We note that the stability of the fast dynamics is determined by the stability of the matrix \( \bar{A}_{idd} \).

This stability was already established before in [3] (to be precise this is equivalent to the argument to prove that \( \bar{A}_S \) in equation (8) is asymptotically stable).

Remains to prove that the slow dynamics is asymptotically stable. We get:

\[ \dot{\eta}_{ia} = \bar{A}_{iaa} \eta_{ia} + \bar{A}_{iad} \eta_{id} \]

where

\[
0 = \bar{A}_{ida} \eta_{ia} + \bar{A}_{idd} \eta_{id} + \bar{W}_{ida} \eta_{ia} + \epsilon \bar{W}_{ida} \eta_{id}
\]  

(13)

Let

\[ \eta_{d1} = [\eta_{d11}, \eta_{d12}, \ldots, \eta_{d1N}, \eta_{d1\rho}, \eta_{d11}, \eta_{d12}, \ldots, \eta_{d1\rho}]^T \]

By examining (13) and using the structure of \( A_d \) and \( B_d \) we get:

\[ \eta_{d12} = \eta_{d13} = \ldots = \eta_{d1\rho} = 0 \]

and

\[ -\bar{K}_{21} \eta_{d11} = \left( \begin{array}{c} \eta_{d12} \\ \eta_{d13} \\ \vdots \\ \eta_{d1\rho} \end{array} \right) \]

where

\[ \bar{K}_2 = \left( \begin{array}{cc} K_1 & 0 \\ 0 & K_2 \end{array} \right) \quad \bar{F}_2 = \left( \begin{array}{cc} F_21 & F_22 \\ F_21 & F_22 \end{array} \right) \]

with \( \bar{K}_2 \) and \( \bar{F}_2 \) both scalars. We define

\[ \bar{K}_1 = \bar{K}_2 \bar{K}_1, \quad \bar{F}_1 = \frac{1}{F_21} \bar{F}_1. \]

It can then we shown that for \( \varepsilon \) small enough the dynamics of \( \eta_{ia} \) is approximately described by:

\[
\dot{\eta}_{ia} = \left( \begin{array}{cc} A_d + \bar{K}_1 E_{da} & \frac{\lambda_i - 1}{\lambda_i} L_{ad} \bar{F}_1 \\ -\bar{K}_1 E_{da} & A_d - L_{ad} \bar{F}_1 \end{array} \right) \eta_{ia}
\]  

(14)

where

\[ \eta_{ia} = \left( \begin{array}{cc} 0 & I \\ I & -I \end{array} \right) \eta_{ia} \]

According to Lemma II.2 there exists \( \bar{F}_1 \) and \( \bar{K}_1 \) such that this system is asymptotically stable provided

\[ \frac{\lambda_i - 1}{\lambda_i} \in [-M, \frac{1}{4}) \]

Since our network is undirected, we know that all \( \lambda_i \) are real and, by assumption, we have \( \lambda_i \in [\tau, m] \). By a simple scaling of our measurement, we can assume, without loss of generality, that \( 1 - \frac{1}{m} < \frac{1}{4} \) and hence using Lemma II.2 we can find \( \bar{K}_1 \) and \( \bar{F}_1 \) such that the system (14) is asymptotically stable for all possible \( \lambda_i \). Singular perturbation theory then guarantees that our system (12) is asymptotically stable for sufficiently small \( \varepsilon \) for all \( i = 1, \ldots, N - 1 \). The latter implies that \( \bar{x}_i \) converges to zero and hence we achieve state synchronization which obviously implies (6).

\[ \Box \]

III. Conclusion

In the paper [3], it was shown that we can achieve synchronization while only using relative output information of neighboring agents. This was achieved under the assumption that all agents were minimum-phase. In this paper we established that in the case of identical SISO systems and an undirected graph, we can weaken the assumption of minimum-phase to weakly non-minimum-phase and still achieve synchronization. Obvious extensions of the work in this paper are to the heterogeneous case (where agents can be different) as well as the MIMO case as well as directed graphs. The latter is the subject of our current research.
REFERENCES