Risk Adjusted Receding Horizon Control of Constrained Linear Parameter Varying Systems

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Summary. In the past few years, control of Linear Parameter Varying Systems (LPV) has been the object of considerable attention, as a way of formalizing the intuitively appealing idea of gain scheduling control for nonlinear systems. However, currently available LPV techniques are both computationally demanding and (potentially) very conservative. In this chapter we propose to address these difficulties by combining Receding Horizon and Control Lyapunov Functions techniques in a risk-adjusted framework. The resulting controllers are guaranteed to stabilize the plant and have computational complexity that increases polynomially, rather than exponentially, with the prediction horizon.

Key words: Model Predictive Control, LPV Systems, Risk Adjusted Control.

1 Introduction

A widely used practice to handle nonlinear dynamics is to linearize the plant around several operating points and then use gain-scheduled linear controllers. However, while intuitively appealing, this idea has several pitfalls [Shamma and Athans (1992)]. Motivated by these shortcomings, during the past few years considerably attention has been devoted to the problem of synthesizing controllers for Linear Parameter Varying Systems, where the state-space matrices of the plant depend on time-varying parameters whose values

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are not known a priori, but can be measured by the controller. This research has resulted in controller synthesis methods guaranteeing worst case performance bounds (for instance in an $\mathcal{H}_2$ or $\mathcal{H}_\infty$ sense, see e.g. [Balas *et. al.*(1997)] and references therein). While successful in many situations, these techniques are potentially very conservative in others, since they are based on sufficient conditions,. In addition, these methods are computationally very demanding, requiring the solution of a set of functional matrix inequalities. Obtaining computationally tractable problems requires using both finite expansion approximations as well as a gridding of the parameter space, leading to further conservatism.

The present chapter seeks to reduce both the computational complexity and conservatism entailed in currently available LPV synthesis methods by combining risk-adjusted (see eg. [Calafiore *et. al.* (2000), Chen and Zhou (1997), Ray and Stengel (1993)]) and Receding Horizon (see [Mayne *et. al.* (2000)] and references therein) ideas. Our main result shows that, by searching over a set of *closed-loop* strategies the problem can be reduced to finding a solution to a finite set of Linear Matrix Inequalities (LMIs). Finding the exact solution to this problem has computational complexity that grows exponentially with the horizon length. To circumvent this difficulty, we propose to use a stochastic approximation algorithm that is guaranteed to converge to the solution with probability one, and whose computational complexity grows only polynomially with the horizon.

The chapter draws inspiration, in addition to [Sznaier (1999)] and [Jadbabaie *et. al.* (1999)], from [Polyak and Tempo (2000)], [Fujisaki *et. al.* (2001)] and [Batina *et. al.* (2001)]. The main difference with [Polyak and Tempo (2000)] and [Fujisaki *et. al.* (2001)] is the use of Receding Horizon techniques and parameter dependent Lyapunov functions. Compared with [Batina *et. al.* (2001)], we consider the case of LPV dynamics and we obtain a controller that minimizes the worst case performance, rather than its expected value, over all trajectories compatible with the current parameter value. Finally, the use of closed-loop strategies, based on the solution of a set of LMIs, results in substantial reduction of computational complexity. In this sense the work presented here is related to the earlier work in [Tadmor (1992)] advocating the use of Riccati based receding horizon controllers for $\mathcal{H}_\infty$ control of Linear Time Varying systems, and to the work in [Kothare *et. al.* (1996)], proposing an LMI-based optimization of *closed-loop* control strategies.

2 Preliminaries

2.1 The LPV Quadratic Regulator Problem

In this chapter we consider the following class of LPV systems:

$$\begin{aligned} x(t+1) &= A[\rho(t)]x(t) + B_2[\rho(t)]u(t) \\ z(t) &= C_1[\rho(t)]x(t) + D_{12}[\rho(t)]u(t) \end{aligned} \tag{1}$$

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where $x \in R^{n_x}$, $u \in S_u \subseteq R^{n_u}$, and $z \in R^{m_z}$ represent the state, control, and regulated variables respectively, S_u is a convex set containing the origin in its interior, ρ denotes a vector of time-varying parameters that can be measured in real time, and where all matrices involved are continuous functions of ρ . Further, we will assume that the set of parameter trajectories is of the form:

$$\mathcal{F}_\Theta = \{\rho: \rho(t+1) \in \Theta[\rho(t)], t = 0, 1, \dots\} \quad (2)$$

where $\mathcal{P} \subset R^{n_\rho}$ is a compact set and $\Theta: \mathcal{P} \rightarrow \mathcal{P}$ is a given set valued map. Our goal is, given an initial condition x_o , and an initial value of the parameter ρ_o , to find an admissible parameter dependent state-feedback control law $u[x(t), \rho(t)] \in S_u$ that minimizes the index:

$$J(x_o, \rho_o, u) = \sup_{\rho \in \mathcal{F}_\Theta, \rho(0) = \rho_o} \sum_{k=0}^{\infty} z^T(k)z(k), x(0) = x_o \quad (3)$$

In the sequel, for simplicity, we make the following standard assumptions: $D_{12}^T D_{12} = I$, $C_1^T D_{12} = 0$. In addition, the explicit dependence of matrices on ρ will be omitted, when it is clear from the context.

Definition 1 A function $\Psi: R^{n_x} \times \mathcal{P} \rightarrow R_+$ that satisfies the following condition:

$$\max_{\rho \in \mathcal{P}} \left\{ \min_{u \in S_u} \left\{ \max_{\theta \in \Theta(\rho)} \Psi[A(\rho)x + B_2(\rho)u, \theta] - \Psi(x, \rho) \right\} \right\} < 0 \quad (4)$$

for all $x \neq 0$ is said to be a parameter dependent Control Lyapunov Function (CLF) for system (1).

The following result characterizes a family of CLFs in terms of the solution to a (functional) Affine Matrix Inequality.

Lemma 1 Assume that the pair $\{A(\cdot), C(\cdot)\}$ is uniformly observable for all parameter trajectories. If there exist a continuous matrix function $Y(\rho) > 0$ such that:

$$M(\rho) \doteq \begin{bmatrix} -Y(\rho) + B_2(\rho)B_2^T(\rho) & Y(\rho)A^T(\rho) - B_2(\rho)B_2^T(\rho) & Y(\rho)C_1^T(\rho) \\ A(\rho)Y(\rho) - B_2(\rho)B_2^T(\rho) & -Y(\theta) & 0 \\ C_1(\rho)Y(\rho) & 0 & -I \end{bmatrix} \leq 0 \quad (5)$$

for all $\rho \in \mathcal{P}$ and $\theta \in \Theta(\rho)$, then $V(x, \rho) = x^T Y^{-1}(\rho)x$ is a parameter dependent CLF for system (1), with associated control action given by

$$u(x, \rho) = -B_2^T(\rho)Y^{-1}(\rho)x \quad (6)$$

Moreover, the corresponding trajectory satisfies:

$$\sup_{\rho \in \mathcal{F}_\Theta, \rho(0) = \rho_o} \sum_{t=0}^{\infty} z^T(t)z(t) \leq x_o^T Y^{-1}(\rho_o)x_o \quad (7)$$

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Proof. Given any admissible parameter trajectory $\rho(\cdot) \in \mathcal{F}_\Theta$, let

$$v^T = [x(t)^T Y^{-1}[\rho(t)] \ x(t+1)^T Y^{-1}[\rho(t+1)] \ x(t)^T C_1^T[\rho(t)]] \quad (8)$$

Pre/postmultiplying $M[\rho(t)]$ by v^T and v yields:

$$x^T(t+1)Y^{-1}[\rho(t+1)]x(t+1) - x^T(t)Y^{-1}[\rho(t)]x(t) \leq -z^T(t)z(t) \quad (9)$$

Summing this inequality along the trajectories of the closed-loop system yields $\sum_0^\infty z^T(t)z(t) \leq x_o^T Y^{-1}[\rho(0)]x_o$. Since $Y > 0$ is a continuous function of $\rho(t) \in \mathcal{P}$ compact, it follows that there exist some constant c such that $Y^{-1}(\rho) < cI$. Thus $\sum_0^\infty z^T(k)z(k)$ is bounded, which implies that $z^T(t)z(t) \rightarrow 0$ as $t \rightarrow \infty$. Finally, it is easy to show that uniform observability of $\{A, C\}$, combined with continuity of all the matrices involved and the fact that $\rho \in \mathcal{P}$ compact, implies uniform observability of the closed loop system $\{A - B_2 B_2^T Y^{-1}, C_1 - D_{12} B_2^T Y^{-1}\}$. Thus $z(t) \rightarrow 0$ implies that $x(t) \rightarrow 0$.

2.2 Motivation: A Conceptual Algorithm for Receding Horizon Control of Constrained LPV Systems

In this section, motivated by the work in [Sznaier (1999)] we introduce a conceptual receding horizon control law for constrained LPV systems. This law provides the motivation for the risk-adjusted approach pursued latter in the chapter.

Let $\Psi: R^{n_x} \times \mathcal{P} \rightarrow R_+$ be a CLF for system (1) such that it satisfies the additional condition:

$$\max_{\rho \in \mathcal{P}} \left\{ \min_{u \in S_u} \left\{ \max_{\theta \in \Theta(\rho)} \Psi[A(\rho)x + B_2(\rho)u, \theta] + z^T z - \Psi(x, \rho) \right\} \right\} \leq 0 \quad (10)$$

and, given a horizon N , define (recursively) the following function $J(x, \rho, n, N)$:

$$\begin{aligned} J(x, \rho, N, N) &= \Psi(x, \rho) \\ J(x, \rho, i, N) &= \min_{u \in S_u} \left\{ z^T(i)z(i) + \max_{\theta \in \Theta(\rho)} J[A(\rho)x + B(\rho)u, \theta, i+1, N] \right\}, \quad i < N \end{aligned} \quad (11)$$

Finally, let $x(t), \rho(t)$ denote the present state and parameter values, and consider the following Receding Horizon control law:

$$u_{RH}[x(t), \rho(t)] = \underset{u \in S_u}{\operatorname{argmin}} J[x(t), \rho(t), t, t+N] \quad (12)$$

Theorem 1 *Assume that the pair $\{A(\cdot), C(\cdot)\}$ is uniformly observable for all admissible parameter trajectories $\rho(t) \in \mathcal{F}_\Theta$. Then (i) the control law u_{RH} renders the origin an asymptotically stable equilibrium point of (1), and (ii), as $N \rightarrow \infty$, its performance approaches optimality monotonically.*

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Proof: Begin by noting that (10) implies that

$$\begin{aligned}
& J(x, \rho, t + N, t + N + 1) \\
&= \min_{u \in S_u} \left\{ z^T(t + N)z(t + N) + \max_{\theta \in \Theta(\rho)} J(Ax + Bu, \theta, t + N + 1, t + N + 1) \right\} \\
&= \min_{u \in S_u} \left\{ z^T(t + N)z(t + N) + \max_{\theta \in \Theta(\rho)} \Psi(Ax + Bu, \theta) \right\} \\
&\leq \Psi(x, \rho) = J(x, \rho, t + N, t + N)
\end{aligned}$$

It follows that

$$\begin{aligned}
& J(x, \rho, t + N - 1, t + N + 1) \\
&= \min_{u \in S_u} \left\{ z^T(t + N - 1)z(t + N - 1) + \max_{\theta \in \Theta(\rho)} J(Ax + Bu, \theta, t + N, t + N + 1) \right\} \\
&\leq \min_{u \in S_u} \left\{ z^T(t + N - 1)z(t + N - 1) + \max_{\theta \in \Theta(\rho)} J(Ax + Bu, \theta, t + N, t + N) \right\} \\
&= J(x, \rho, t + N - 1, t + N)
\end{aligned}$$

A simple induction argument shows now that

$$J(x, \rho, t, t + N + 1) \leq J(x, \rho, t, t + N) \quad (13)$$

For a given initial condition and parameter trajectory, let

$$V(t) = J[x_{cl}(t), \rho(t), t, t + N]$$

where $x_{cl}(t)$ denotes the closed-loop trajectory corresponding to the control law u_{RH} . From the definition of J it follows that

$$\begin{aligned}
V(t) &= z^T(t)z(t) + \max_{\theta \in \Theta[\rho(t)]} J[x_{cl}(t + 1), \theta, t + 1, t + N] \\
&\geq z^T(t)z(t) + J[x_{cl}(t + 1), \rho(t + 1), t + 1, t + N] \\
&\geq z^T(t)z(t) + V(t + 1)
\end{aligned} \quad (14)$$

where the last inequality follows from (13). As before, asymptotic stability of the origin follows from a combination of standard Lyapunov and observability arguments. Property (ii) follows from (13) combined with Bellman's optimality principle [Bellman and Dreyfus (1962)].

Remark 1 In the unconstrained case (e.g. $S_u = \mathbb{R}^{n_u}$) a CLF is readily available from Lemma 1 since it can be easily shown that $\Psi = x^T Y^{-1}(x, \rho)x$, where Y satisfies the AMIs (5), satisfies the inequality (10). In addition, by summing (14) along the trajectories and using (13) it follows that this CLF choice yields the following worst-case performance bound:

$$\sup_{\rho \in \mathcal{F}_\Theta} \sum_{k=0}^{\infty} z^T(k)z(k) \leq V(0) \leq x_o^T Y^{-1}(\rho_o)x_o$$

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3 Risk Adjusted Receding Horizon

As shown in Theorem 1, the receding horizon control law u_{RH} is guaranteed to stabilize the system while optimizing a (monotonically improving with N) approximation to the performance index. However, solving the min-max optimization problem (11) is far from trivial, even when making several approximations such as assuming memoryless, arbitrarily fast time-varying parameters⁵. To avoid this difficulty, motivated by the work in [Tadmor (1992)], in the sequel we will search over *closed-loop strategies*, rather than control actions. For simplicity, we consider first the unconstrained control case and defer the treatment of constraints until section 3.3

3.1 The unconstrained control case

In the sequel, we will denote by $\mathbf{X}_N : \Theta \times \{n, n+1, \dots, n+N\} \rightarrow P$ the set of all bounded matrix functions that map N -length admissible parameter trajectories to P , the class of all symmetric positive definite matrices. With this definition, consider now the following receding horizon type control law:

Algorithm 1 0.- Data: A horizon N , a CLF $Y(\rho)$ that satisfies (5).

1.- Let $\rho(n)$ denote the measured value of the parameter at time n and solve the following LMI optimization problem in $\mathbf{X} \in \mathbf{X}_N$:

$$\begin{aligned} \min_{\mathbf{X} \in \mathbf{X}_N} \gamma \quad & \text{subject to:} \\ \begin{bmatrix} -\gamma & x^T(n) \\ x(n) & -X(n) \end{bmatrix} & \leq 0 \\ M(\rho) \doteq \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12}^T & -X(n+i+1) & 0 \\ A_{13}^T & 0 & -I \end{bmatrix} & \leq 0 \end{aligned} \quad (15)$$

$$\begin{aligned} A_{11} &= -X(n+i) + B_2[\rho(n+i)]B_2^T[\rho(n+i)] \\ A_{12} &= X(n+i)A^T[\rho(n+i)] - B_2[\rho(n+i)]B_2^T[\rho(n+i)] \\ A_{13} &= X(n+i)C_1^T[\rho(n+i)] \\ &\text{for all } \rho(n+i+1) \in \Theta[\rho(n+i)], i = 0, 1, \dots, N-1 \end{aligned}$$

with boundary condition $X(n+N) = Y[\rho(n+N)]$, and where, with a slight notational abuse, we denote $X[\rho(t), t]$ simply as $X(t)$.

2.- At time n use as control action

$$u[\rho(n), x(n), N] = -B_2^T[\rho(n)]X^{-1}[\rho(n), n]x(n) \quad (16)$$

3.- Set $n = n+1$ and go to step 1.

⁵ In this case the problem becomes essentially equivalent to synthesizing a controller for a system subject to parametric uncertainty, and it is well known that these problems are generically NP hard.

Theorem 2 Assume that the pair $\{A(\rho), C(\rho)\}$ is uniformly observable for all admissible parameter trajectories. Then, the control law (16) renders the origin an asymptotically stable point of the closed loop system for all $\rho(\cdot) \in \mathcal{F}_\Theta$. Moreover, performance improves monotonically with the horizon length N .

Proof To establish stability, let $x(n)$, $\rho(n)$ and $X(n+i)$, $i = 1, \dots, N$, denote the present state and parameter values and the corresponding solution to (15). Since

$$\{X(n+1), X(n+2), \dots, Y[\rho(n+N)], Y[\rho(n+N+1)]\}$$

is a feasible solution for (15) starting from any $\rho(n+1) \in \theta[\rho(n)]$, we have that

$$\begin{aligned} x^T(n+1)X^{-1}[\rho(n+1)]x(n+1) &\leq \\ x^T(n)X^{-1}[\rho(n)]x(n) - z^T(n)z(n) \end{aligned} \quad (17)$$

From an observability argument it can be easily shown that this, coupled with the fact that $X > 0$, implies that $x^T(n)X^{-1}(n)x(n) \rightarrow 0$ as $n \rightarrow \infty$. Combined with the fact that $X^{-1}(n)$ is bounded away from zero (since X is bounded), this implies that $x(n) \rightarrow 0$. Finally, the second property follows from the fact that if $\{X(n), X(n+1), \dots, Y[\rho(n+N)]\}$ is a feasible solution for (15) with horizon N , then

$$\{X(n), X(n+1), \dots, Y[\rho(n+N)], Y[\rho(n+N+1)]\}$$

is feasible for $N+1$. Thus $\gamma(N+1) \leq \gamma(N)$.

Remark 2 Note that, for any N , the sequence:

$$\{Y[\rho(n)], Y[\rho(n+1)], \dots, Y[\rho(n+N)]\} \quad (18)$$

is a feasible solution for (15). Thus, if there exist a sequence of matrices $X(\rho, i)$ that yields a lower value of the bound $x^T(n)X^{-1}[\rho(n), n]x(n)$ the optimization (15) will find it. Hence the proposed controller will outperforms the standard LPV control law (6). However, its computational complexity is comparable (or worse), since it requires finding feasible solutions to both (5) and (15). However, since the solution to (5) is used only as a boundary condition for (15) at $t = n+N$, even if a conservative solution is used here⁶, performance degradation is considerably less severe than the one incurred by using this conservative solution to compute the control action, for instance via (6). Further, as we show in the sequel, the computational complexity of solving (15) can be substantially reduced using a risk-adjusted approach.

⁶ Such a solution can be found using risk-adjusted methods following the approach used in [Fujisaki *et. al.* (2001)].

3.2 Stochastic Approximation

In principle, infinite-dimensional optimization problems of the form (15) can be (approximately) converted to a finite dimensional optimization by using a finite expansion $X(\rho, t) = \sum_{i=1}^m X_i(t)h_i(\rho)$, where $h_i(\cdot)$ are known continuous functions ([Balas *et. al.*(1997)]). However, the computational complexity of the resulting problem grows exponentially with the horizon length N .

In this chapter, we propose to avoid this difficulty by pursuing a stochastic approximation approach, whose complexity grows polynomially with N . To this effect, assume that $\hat{\rho} = [\rho(n+1)\rho(n+2)\cdots\rho(n+N)]$ has a non-zero probability density for all $\rho \in \mathcal{F}_\theta$.

Note that, to compute $u(n)$, one only needs $X[\rho(n)]$. This observation allows for reformulating the optimization problem (15) to eliminate the need to explicitly compute $X[\rho(n+1)]\dots X[\rho(n+N)]$ as follows: Given a fixed instant n , $\rho(n)$ and $\hat{\rho} \in \mathcal{F}_\theta$, define

$$\overline{M}(n) \doteq \begin{bmatrix} \begin{bmatrix} -\gamma & x^T(n) \\ x(n) & -X(n) \end{bmatrix} & & & \\ & M[\rho(n)] & & \\ & & \ddots & \\ & & & M[\rho(n+N-1)] \end{bmatrix}.$$

In terms of $\overline{M}$, the constraints in the optimization problem (15) can be expressed as:

$$\underline{\lambda}(\gamma, X(n), \rho(n), \hat{\rho}) \doteq \min_{X(n+1), \dots, X(n+N)} \lambda_{\max}[\overline{M}(n)] < 0.$$

Thus, in this context, one does not need to compute the explicit value of $X(i)$, $i = n+1, \dots, n+N$, as long as the minimum above can be computed. Note in passing, that this reformulation preserves convexity, since the function $\underline{\lambda}(\cdot)$ is a convex function of its arguments. To complete the approximation of the original optimization problem by a stochastic one, given $\zeta > 0$, define

$$\tilde{g}(x) = \frac{e^{\zeta x} - 1}{\zeta}$$

and collect all the optimization variables (e.g., γ and the entries of $X(n)$) in a vector $\mathbf{x}_n$. Define the following functions

$$f_0(\mathbf{x}_n, \rho(n), \hat{\rho}) \doteq \gamma;$$

$$f_1(\mathbf{x}_n, \rho(n), \hat{\rho}) \doteq \tilde{g}[\underline{\lambda}(\gamma, X(n), \rho(n), \hat{\rho})].$$

Consider now the following convex problem

$$\begin{aligned} \min \mathbf{E}_{\hat{\rho}} [f_0(\mathbf{x}_n, \rho(n), \hat{\rho})] &= \min \gamma \\ \text{s.t. } \mathbf{E}_{\hat{\rho}} [f_1(\mathbf{x}_n, \rho(n), \hat{\rho})] &\leq 0; \end{aligned} \tag{19}$$

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where $\mathbf{E}_{\hat{\rho}}[\cdot]$ denotes the expected value with respect to the random variable $\hat{\rho}$. It can be easily shown that the solution to this problem tends to the solution of the problem (15) as $\zeta \rightarrow \infty$. We are now ready to provide the main result of this section: an algorithm for solving problem (19) in polynomial time. For technical reasons, in the sequel we will assume that the solution to this problem is known to belong to a given compact convex set $\mathcal{X}$ (where the matrix $X(n)$ has bounded entries and is positive definite).

Let $\pi(\cdot)$ denote the projection onto $\mathcal{X}$; i.e.,

$$\pi(\mathbf{x}) = \arg \min_{\tilde{\mathbf{x}} \in \mathcal{X}} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2.$$

and consider the following algorithm:

Algorithm 2

1. Generate a feasible solution $Y(\rho)$ to (5), using for instance the procedure proposed in [Fujisaki et. al. (2001)].
2. Initialization: Determine $\mathbf{x}_n^0, y_i^0; i = 0, 1$ and z_1^0 . Let $k = 0$.
3. Generate a sample $\hat{\rho}^k = [\rho^k(n+1), \dots, \rho^k(n+N-1)]$
4. Let If $z_1^k \leq -\gamma_0/k^\gamma$ then

$$\mathbf{x}_n^{k+1} = \pi [\mathbf{x}_n^k - b_k y_0^k].$$

Otherwise, $\mathbf{x}_n^{k+1} = \pi [\mathbf{x}_n^k - b_k y_1^k] ..$

5. Let

$$y_i^{k+1} = y_i^k + a_k \left(\left. \frac{\partial f_i(\mathbf{x}_n, \rho(n), \hat{\rho}^k)}{\partial \mathbf{x}_n} \right|_{\mathbf{x}_n = \mathbf{x}_n^k} - y_i^k \right)$$

$i = 0, 1$ and

$$z_1^{k+1} = z_1^k + a_k (f_1(\mathbf{x}_n, \rho(n), \hat{\rho}^k) - z_1^k);$$

6. If $z_1^l < 0$ and $|\gamma_{k-1} - \gamma_k| < \epsilon$ for $l = k - N_{good} + 1, \dots, k + 1$ stop. Otherwise, let $k = k + 1$ and go to step 3.

Theorem 3 Let $a_k = \frac{\alpha_0}{k^\alpha}; b_k = \frac{\beta_0}{k^\beta}$, where $\alpha_0, \alpha, \beta_0$ and β are positive constants. Furthermore, assume that γ_0 and γ are also positive. Then, if

$$\sum_{k=0}^{\infty} a_k = \sum_{k=0}^{\infty} b_k = \infty; \quad \sum_{k=0}^{\infty} a_k^2 < \infty; \quad \lim_{k \rightarrow \infty} \frac{b_k}{a_k} = 0$$

and $2\beta - \alpha - 2\gamma > 1$ the sequence $\mathbf{x}_n^k$ converges with probability one to the solution of the problem (19).

Proof Direct application of Theorem 1 in [Gupal (1974)].

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3.3 Adding Control Constraints

In this section we briefly indicate how to modify the algorithm presented above to handle constraints in the control action. A difficulty here is that, while it is relatively easy to incorporate these constraints when searching for a control action in the minimization of (3), this is not the case when dealing instead with *closed-loop* control strategies of the form (16). Following [Sznaier and Suarez (2000)], we propose to accomplish this indirectly, by suitably scaling some of the terms in (15). Specifically, assume that the control constraint set is of the form $S_u \{u: \|u\|_\infty \leq 1\}$ and consider the following control law:

Algorithm 3

0.- *Data:* A horizon N , an initial condition $x(0) = x_o$

0.1- *Find a CLF* $Y(\rho, \tau, x)$ *that satisfies the following (functional) LMI, parametric in* ρ , x *and the scalar* τ :

$$\max_{\substack{\rho \in \mathcal{P} \\ \theta \in \Theta(\rho)}} \begin{bmatrix} -Y(\rho, \tau, x) & A_{12} & A_{13} & B_2(\rho) \\ A_{12}^T & -Y(\theta, \tau, x) & 0 & 0 \\ A_{13}^T & 0 & -\tau \cdot I & 0 \\ B_2^T(\rho) & 0 & 0 & -\tau \cdot I \end{bmatrix} \leq 0 \quad (20)$$

$$A_{12} = Y(\rho, \tau, x)A^T(\rho) - B_2(\rho)B_2^T(\rho)$$

$$A_{13} = Y(\rho, \tau, x)C_1^T(\rho)$$

$$\begin{bmatrix} 1 & x^T \\ x & Y(\rho, \tau, x) \end{bmatrix} \geq 0, \quad \begin{bmatrix} 1 & b_i^T \\ b_i & Y(\rho, \tau, x) \end{bmatrix} \geq 0, \quad i = 1, n_u \quad (21)$$

where b_i denotes the i^{th} column of $B_2(\rho)$.

0.2- Let $\tau_{\min}(\rho, x_o) = \inf \{\tau: (20)-(21) \text{ are feasible}\}$ with $x = x_o$ and define $\tau_o = \max_{\rho \in \mathcal{P}} \tau_{\min}(\rho, x_o)$.

1.- Let $\rho(n)$ denote the measured value of the parameter at time n and solve the following LMI optimization problem in $\mathcal{X} \in \mathbf{X}_N$ and the scalar γ :

$\min_{\mathcal{X} \in \mathbf{X}_N} \gamma$, *subject to:*

$$\begin{bmatrix} -\gamma & x^T(n) \\ x(n) & -X(n) \end{bmatrix} \leq 0 \quad (22)$$

$$\tau(n+i) \leq \tau_o(x_o)$$

$$\begin{bmatrix} 1 & x^T(n) \\ x(n) & X(n) \end{bmatrix} \geq 0, \quad \begin{bmatrix} 1 & b_j^T(n+i) \\ b_j(n+i) & X(n+i) \end{bmatrix} \geq 0; \quad (23)$$

$j = 1, \dots, n_u, \quad i = 0, 1, \dots, N-1$

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$$\begin{bmatrix} -X(n+i) & A_{12} & A_{13} & B_2(n+i) \\ A_{12}^T & -X(n+i+1) & 0 & 0 \\ A_{13}^T & 0 & -\tau(n+i) \cdot I & 0 \\ B_2^T(n+i) & 0 & 0 & -\tau(n+i) \cdot I \end{bmatrix} \leq 0 \quad (24)$$

for all $\rho(n+i+1) \in \Theta[\rho(n+i)]$, with boundary condition $X(n+N) = Y[\rho(n+N), \tau(n+N), x(n+N)]$, where

$$\begin{aligned} A_{12} &= X(n+i)A^T(n+i) - B_2(n+i)B_2^T(n+i) \\ A_{13} &= X(n+i)C_1^T(n+i) \text{ and} \end{aligned}$$

$$i = 1, 2, \dots, N$$

and where $Y(\cdot)$ satisfies the AMI (20).

2.- At time n use as control action

$$u(n, N) = -B_2^T[\rho(n)]X^{-1}[\rho(n), n]x(n) \quad (25)$$

3.- Set $n = n + 1$ and go to step 1.

Theorem 4 Assume that the origin is an exponentially stable equilibrium point of $A[\rho(\cdot)]$ for all $\rho \in \mathcal{F}_\Theta$ and the pair $\{A(\rho), C(\rho)\}$ is uniformly observable for all admissible parameter trajectories. Then, the control law (25) (i) is admissible, in the sense that it satisfies the control constraints, and (ii) it renders the origin a globally exponentially stable point of the closed loop system for all admissible parameter trajectories.

Proof. See the Appendix.

Remark 3 If the open loop system is not exponentially stable but the pairs (A, B_2) and (A, C_1) are uniformly controllable and observable respectively, the algorithm above will locally stabilize the system in some neighborhood of the origin, which can be estimated by computing the set $\mathcal{S} = \bigcap_{\rho \in \mathcal{P}} \mathcal{S}(\rho)$, where

$$\mathcal{S}(\rho) = \left\{ x : \lim_{\tau \rightarrow \infty} x^T X^{-1}(\rho, \tau) x < \infty \right\}$$

Remark 4 As before, the computational complexity of the optimization problem above grows exponentially with the horizon, even when approximating X by a finite expansion. However, the stochastic approach used in Algorithm 2 can also be used here, with minimal modifications, to obtain polynomial growth approximations.

4 Illustrative Examples

In this section we provide several examples illustrating the proposed approach.

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4.1 A simple academic example

Our first example is an academic one, to illustrate some of the issues involved in applying the algorithm, in a simple setup. Consider the following discrete time LPV system:

$$A(\rho(t)) = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 - 0.1\rho(k) \end{bmatrix}, \quad B_2(\rho(t)) = [0 \ 0.0787]'$$

$$C_1(\rho(t)) = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \quad D_{12}(\rho(t)) = \frac{1}{\sqrt{2}} [1 \ 1]'$$

with admissible parameter set

$$\mathcal{F}_\Theta = \{\rho(t) : \rho(t+1) \in [-1, 1], \ t = 0, 1, \dots\}. \quad (26)$$

It can be verified that the matrix function:

$$Y(\rho_k) = Y_0 + Y_1\rho_k + Y_2\rho_k^2$$

$$Y_0 = \begin{bmatrix} 0.0296 & -0.0196 \\ -0.0196 & 0.0290 \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} 0.0003 & -0.0016 \\ -0.0016 & 0.0044 \end{bmatrix}$$

$$Y_2 = 10^{-3} \begin{bmatrix} -0.0691 & 0.0302 \\ 0.0302 & 0.2542 \end{bmatrix}$$

satisfies (5). Thus $V(x, \rho) = x^T Y^{-1}(\rho)x$ is a parameter dependent CLF for the plant.

Figure 1 shows a typical parameter trajectory and the corresponding cost evolution. In this case the initial condition was $x_0 = [0.1 \ 0.1]'$, and the following values were used for the risk adjusted RH controller: $N = 10$, $\zeta = 30$, $\alpha = 0.6$, $\alpha_0 = 1$, $\beta = 1$, $\beta_0 = 10^{-3}$, $\gamma = 0.15$ and $\gamma_0 = 10^{-6}$. At each time instant, an approximate solution to (15) was obtained using 100 iterations of the stochastic approximation. On a Intel Pentium III 1.6GHz, the total CPU time to perform this iteration is 81.88 seconds. As reference, solving the parametric LMI (5) takes 474.44 seconds, although this needs to be done only once, off-line. The overall cost of the trajectory is $J = 1.0886$. For comparison, the standard LPV controller (6) results in a cost $J = 1.2805$. Thus, in this case the risk adjusted controller yields roughly a 20% performance improvement. Similar results were obtained for other initial conditions and parameter trajectories.

4.2 Additional Examples

In the final version of the chapter, this section will include two additional examples. One is the Stirred Tank Reactor proposed by the editors, the second is a simplified version of the dynamics of an autonomous aerial vehicle (a blimp) controlled using computer vision in the feedback loop.

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Fig. 1. Top: Parameter Trajectory. Bottom: Cost Evolution

5 Conclusions

Practical tools for synthesizing controllers for LPV systems have emerged relatively recently and are still far from complete. Among others, issues not completely solved yet include non-conservative handling of performance specifications and overall computational complexity.

In this chapter we take some steps towards removing these limitations by combining Receding Horizon and Control Lyapunov Function ideas in a risk-adjusted framework. Motivated by some earlier results on regulation of LTV and LPV systems ([Tadmor (1992), Sznaier (1999)]) the main idea of

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the chapter is to recast the infinite horizon regulation problem into a (approximately) equivalent finite horizon form, by searching over a suitable set of closed-loop strategies. As shown here, this leads to a globally stabilizing control law that is guaranteed to outperform techniques currently used to deal with LPV systems. However, in principle this is achieved at the expense of computational complexity, since this law requires the on-line solution of a set of functional LMIs. We propose to address this difficulty by using a risk-adjusted approach, where in exchange for a slight probability of constraint violation, one obtains a substantial reduction in computational complexity. Moreover, this approach scales polynomially, rather than exponentially, with system size ([Khargonekar and Tikku (1996), Tempo *et. al.* (1996)]).

These results were illustrated with a simple example where a risk-adjusted receding horizon controller was used to control a second order LPV plant. As shown there the proposed risk-adjusted receding horizon controller improves performance vis-a-vis a conventional LPV controller, while substantially reducing the computational effort required by a comparable Receding Horizon controller.

Research is currently under way seeking to extend these results to the output feedback case and to address the issue of model uncertainty, both parametric and dynamic.

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References

- [Balas *et. al.*(1997)] G. BALAS *et. al.* (1997). Lecture Notes, Workshop on Theory and Application of Linear Parameter Varying Techniques, *1997 ACC*.
- [Barmish and Lagoa (1997)] B. R. Barmish and C. M. Lagoa (1997). "The Uniform Distribution: A Rigorous Justification for its Use in Robustness Analysis", *Mathematics of Control, Signals and Systems*,10, 203–222.
- [Batina *et. al.* (2001)] I. Batina, A. A. Stoorvogel and S. Weiland (2001). "Stochastic Disturbance Rejection in Model Predictive Control by Randomized Algorithms," *Proc. 2001 ACC*, Arlington, Va, pp. 732–737.
- [Bellman and Dreyfus (1962)] R. E. Bellman and S. E. Dreyfus, (1962). **Applied Dynamic Programming**, Princeton University Press.
- [Calafiore *et. al.* (2000)] G. Calafiore, F. Dabbene and R. Tempo (2000). "Randomized Algorithms for Probabilistic Robustness with Real and Complex Structured Uncertainty," *IEEE Trans. on Autom. Contr.*, 45,12, 2218–2235.
- [Chen and Zhou (1997)] X. Chen and K. Zhou (1997). "A Probabilistic Approach to Robust Control", *Proc. 36 IEEE CDC*, pp. 4894–4895.

This text differs from actual publication

- [Fujisaki *et. al.* (2001)] Y. Fujisaki, F. Dabbene and R. Tempo (2001) “Probabilistic Robust Design of LPV Control Systems,” *Proc. 2001 IEEE CDC*, Orlando, FL, pp. 2019–2024.
- [Gupal (1974)] A. M. Gupal (1974). “A Certain Stochastic Programming Problem with Constraints of a Probabilistic Nature,” *Kibernetika (Kiev)*, n. 6, pp. 94–100.
- [Jadbabaie *et. al.* (1999)] A. Jadbabaie, J. Yu and J. Hauser (1999). “Receding horizon control of the Caltech ducted fan: a control Lyapunov function approach,” *Proc. 1999 IEEE CCA*, pp. 51–56,.
- [Khargonekar and Tikku (1996)] P. Khargonekar and A. Tikku (1996). “Randomized Algorithms for Robust Control Analysis and Synthesis Have Polynomial Complexity”, *Proceedings of the 35th IEEE CDC*, pp. 3470 - 3475.
- [Kothare *et. al.* (1996)] M. Kothare, V. Balakrishnan and M. Morari (1996). “Robust Constrained Model Predictive Control Using Linear Matrix Inequalities,” *Automatica*, 32, 10, pp. 1361–1379.
- [Mayne *et. al.* (2000)] D. Q. Mayne, J. B. Rawlings, C.V. Rao and P. O. M. Scokaert (2000). “Constrained Model Predictive Control: Stability and Optimality,” *Automatica*, 36, 6, pp. 789–814.
- [Ray and Stengel (1993)] L. Ray and R. Stengel (1993). “A Monte Carlo Approach to the Analysis of Control System Robustness”, *Automatica*, 29, 1, 229–236.
- [Polyak and Tempo (2000)] B. T. Polyak and R. Tempo (2000). “Probabilistic Robust Design with Linear Quadratic Regulators,” *Proc. 39 IEEE CDC*, pp. 1037–1042.
- [Shamma and Athans (1992)] J. S. Shamma and M. Athans (1992). “Gain Scheduling: Potential Hazards and Possible Remedies,” *IEEE. Contr. Sys. Mag.*, 12, 1, pp. 101–17.
- [Sznaier (1999)] M. Sznaier (1999). “Receding Horizon: An easy way to improve performance in LPV systems,” *Proc. 1999 ACC*, pp. 2257–2261.
- [Sznaier and Suarez (2000)] M. Sznaier and R. Suarez (2000). “Control of Constrained Systems using Receding Horizon Control Lyapunov Functions,” *Workshop on Systems with Time-Domain Constraints*, Eindhoven University of Technology, Eindhoven, The Netherlands.
- [Tadmor (1992)] G. Tadmor (1992). “Receding horizon revisited: and easy way to robustly stabilize an LTV system,” *Systems and Control Letters*, 18, pp. 285–294.
- [Tempo *et. al.* (1996)] R. Tempo, E. W. Bai, and F. Dabbene, F. (1996). “Probabilistic Robustness Analysis: Explicit Bounds for the Minimum Number of Samples,” *Proc. 35th IEEE CDC*, pp. 3424 - 3428.

A Computing Subgradient of $\lambda_{\max}(M(\mathbf{x}))$

In this section we describe how to compute a subgradient of $\lambda_{\max}(M(\mathbf{x}))$. To compute such a subgradient first note that, given $\tilde{\mathbf{x}}$,

$$\lambda_{\max}(M(\tilde{\mathbf{x}})) = \max_{\|y\|_2=1} y^T M(\tilde{\mathbf{x}}) y = y^{*T} M(\tilde{\mathbf{x}}) y^*$$

where y^* is an eigenvector of euclidean norm one associated with maximum eigenvalue of $M(\tilde{\mathbf{x}})$. Given that the maximum above is achieved by y^* then

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$$\left. \frac{\partial \lambda_{\max}(M(\mathbf{x}))}{\partial \mathbf{x}} \right|_{\mathbf{x}=\tilde{\mathbf{x}}} = \left. \frac{\partial y^{*T} M(\mathbf{x}) y^*}{\partial \mathbf{x}} \right|_{\mathbf{x}=\tilde{\mathbf{x}}}.$$

This can be easily computed since, in the case addressed in this chapter, $M(\mathbf{x})$ is an affine function of $\mathbf{x}$.

B Proof of Theorem 4

In order to prove this theorem we need the following preliminary result:

Lemma 2 *Assume that the origin is an exponentially stable equilibrium point of the open loop system $A[\rho(t)]$ for all admissible parameter trajectories and the pair $\{A, C\}$ is uniformly observable. Let $Y[\rho(t), \tau]$ denote a feasible solution to (20) corresponding to a given τ . Then, given $\tau_1 > \tau_2$, (20) admits a solution $Y[\rho(t), \tau_1] > Y[\rho(t), \tau_2]$.*

Proof: Let $\delta = \tau_2 - \tau_1$. From the hypothesis it follows that for all admissible parameter trajectories $\rho \in \mathcal{F}_\theta$ there exists some $\Delta[\rho(t)] > 0$ that satisfies the inequality:

$$\begin{bmatrix} -\Delta[\rho(t)] & \Delta[\rho(t)] A^T[\rho(t)] & \Delta[\rho(t)] C_1^T[\rho(t)] & 0 \\ A[\rho(t)] \Delta[\rho(t)] & -\Delta[\rho(t+1)] & 0 & 0 \\ C_1[\rho(t)] \Delta[\rho(t)] & 0 & -\delta \cdot I & 0 \\ 0 & 0 & 0 & -\delta \cdot I \end{bmatrix} < 0 \quad (27)$$

Consider now any solution of (20) $Y[\rho(t), \tau_2]$ and let $Y[\rho(t), \tau_1] \doteq Y[\rho(t), \tau_2] + \Delta[\rho(t)] > Y[\rho(t), \tau_2]$. Adding (27) and (20) we have that $Y[\rho(t), \tau_1]$ satisfies (20) for the value $\tau = \tau_1$.

Proof of Theorem 4

Controllability of the pair $\{A(\cdot), B(\cdot)\}$ implies that for some $\tau > 0$ large enough, (20) has a solution $Y > 0$. From Lemma 2 it follows then that as $\tau \rightarrow \infty$, the LMI (20) always admits a solution $Y \rightarrow \infty$, which in turn implies that, for any initial condition x_o , there always exist some finite τ such that (21) is feasible. Since $\mathcal{P}$ is compact, τ_o in step [0.2] is well defined.

Consider now a fixed value of $\tau \leq \tau_o$ and let

$$v = \left[x(t)^T Y^{-1}[\rho(t)] \quad x(t+1)^T Y^{-1}[\rho(t+1)] \quad \frac{x(t)^T C_1^T[\rho(t)]}{\tau} \quad \frac{B_2^T[\rho(t)] Y^{-1}[\rho(t)] x(t)}{\tau} \right] \quad (28)$$

Pre/postmultiplying (20) by v and v^T yields, after some algebra:

$$x^T(t+1) Y^{-1}[\rho(t+1), \tau] x(t+1) - x^T(t) Y^{-1}[\rho(t)] x(t) \leq -\frac{\|z(t)\|^2}{\tau} \quad (29)$$

From the definition of τ_o it follows that $Y(\rho, \tau_o)$ satisfies (20) for all admissible parameter trajectories $\rho \in \mathcal{F}_\theta$. This fact, together with (29) implies that the

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LMIs(22)–(24) are feasible for all parameter trajectories, since $X = Y$ is a feasible solution. A compactness argument shows that along the trajectories of the closed loop system $c_1 I > X(t) > c_2 I$ for some constants c_1, c_2 . Finally, pre/postmultiplying (24) by

$$v = \left[x(t)^T X^{-1}(t) \ x(t+1)^T X^{-1}(t+1) \ \frac{x(t)^T C_1^T(t)}{\tau} \ \frac{B_2^T(t) X^{-1}(t) x(t)}{\tau} \right]$$

and v^T yields:

$$x^T(t+1)X^{-1}(t+1)x(t+1) - x^T(t)X^{-1}(t)x(t) \leq -\frac{\|z(t)\|^2}{\tau(t)} \quad (30)$$

From this point on, the proof of asymptotic stability proceeds along the lines of the proof of Theorem 4 using (30) instead of (17).

To show that the control law (25) satisfies the constraints $|u_i(\cdot)| \leq 1$, note that

$$|u_i(n)| = |-b_i^T(n)X^{-1}(n)x(n)| \leq \|b_i^T(n)X^{-\frac{1}{2}}(n)\|_2 \|X^{-\frac{1}{2}}(n)x(n)\|_2 \leq 1$$

where the last inequality follows from the LMIs (23), coupled with the fact that, along the closed loop trajectories

$$x^T(n+i)X^{-1}x(n+i) \leq x^T(n)X^{-1}x(n) \leq 1$$

.

Note in passing that from the proof above it follows that

$$\frac{1}{\tau_o} \sum_{t=0}^{\infty} z^T(t)z(t) \leq \sum_t \frac{z^T z}{\tau(t)} \leq x_o^T X^{-1}(x_o, \rho_o, \tau_o)x_o$$

This gives an intuitive explanation of τ : it is a parameter used to scale the performance index so that the control constraints are satisfied.

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