Chapter 7

Decomposition and Aggregation in Queueing Networks

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Abstract This chapter considers the decomposition and aggregation of multiclass queueing networks with state-dependent routing. Combining state-dependent generalisations of quasi-reversibility and biased local balance, sufficient conditions are obtained under which the stationary distribution of the network is of product-form. This product-form factorises into one part that describes the nodes of the network in isolation, and one part that describes the routing and the global network state. It is shown that a decomposition holds for general nodes if the input-output behaviour of these nodes is suitably compensated by the state-dependent routing function. When only a subset of the nodes is of interest, it is shown that the other nodes may be aggregated into nodes that only capture their global behaviour. The results both unify and extend existing classes of product-form networks, as is illustrated by several cases and an example of an assembly network.

7.1 Introduction

In the analysis of queueing networks, two at first sight different techniques have been used to derive product form results: quasi-reversibility and local balance. Quasi-reversibility is a property of the nodes of the network, roughly stating that they should preserve input and output flows when they are considered in isolation and fed by a Poisson process. If such nodes are coupled into a network by Markov

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routing, the stationary distribution factorises over the nodes, i.e., is of product form (see [18, 25]). When using local balance, however, the nodes are not analysed in isolation first. Instead, the local balance equations for the entire network are considered and shown to hold in more detailed form (usually per node) under the assumed product form stationary distribution (see [5, 14]). This technique has the advantage that state-dependent routing can be analysed too.

Recently, both techniques have been combined. Boucherie [3] considers a network of quasi-reversible nodes linked by state-dependent routing. If the process associated with the routing (called the global process) satisfies local balance, the stationary distribution of the network is shown to factorise into the stationary distributions of the nodes in isolation, and the stationary distribution of the global process. Chao and Miyazawa [11] extend the definition of quasi-reversibility, allowing input and output rates of customers to differ from each other. When nodes satisfying this extended form of quasi-reversibility are coupled into a network by Markov routing, the network is shown to have a product form stationary distribution. In [12] it is demonstrated that this product form result can be proved using *biased local balance*. This is an extension of local balance allowing unbalance in the local balance equations to be compensated by a constant bias term. When the nodes are quasi-reversible with equal input and output rates, the bias terms are zero, and biased local balance reduces to ordinary local balance.

This chapter combines and extends the results of [3] and [12] to networks with more general nodes, and more general state-dependent routing. As in [3], we introduce local processes describing the nodes in isolation, and a global process describing the routing process. For the global process the definition of biased local balance of [12] is extended, allowing state-dependent bias terms. For the local processes, quasi-reversibility is further generalised to include state-dependent input rates, and a state-dependent difference between input and output rates. This difference can be interpreted as the bias of the local process with respect to the outside of a node, similar to the bias of the global process. If the bias of the nodes with respect to their outside is suitably compensated by the bias of the global process, the network allows a decomposition into the global process and the local processes. Thus, this chapter combines state-dependent generalisations of the quasi-reversibility results in [3] and of the biased local balance results in [12].

Decomposition

The first part of this chapter is concerned with the *decomposition* of queueing networks. A queueing network can be decomposed if its stationary distribution factorises into the stationary distributions of the nodes of which the network is comprised; the network is then of product form. Apart from the theoretical interest, decomposition results are also of substantial practical importance: finding the stationary distribution of an entire queueing network usually requires an enormous computational effort, whereas the stationary distribution of a single node can be found relatively easily.

The first, and perhaps most famous, decomposition results for queueing networks have been reported by Jackson [17], who considered a single class queueing net-

work of queues with exponential service times, where customers move between the queues according to fixed routing probabilities, and arrive at the network according to a Poisson process with rate equal to the throughputs that can be obtained from the routing probabilities via the so-called traffic equations. Extensions of this result include closed queueing networks, specific service disciplines for non-exponential service times, and multiclass queueing networks, where classes differ in routing and - again under certain service disciplines - in service times, see, for example, the BCMP networks [2].

It was shown that these results were a consequence of local balance [26, 27], and later that these results were also a consequence of a special input/output property of the queues in the network, called quasi-reversibility (see, for example [18]): when a queue is considered in isolation with Poisson arrivals, the time-reversed Process describing this queue also has Poisson arrivals with the same rates as the original (time-forward) process. The two worlds of local balance and quasi-reversibility have since then moved on parallel tracks. Some product-form results, such as those for networks with blocking [5] were developed by local balance conditions, and are believed not to be available via quasi-reversibility. Other results, such as for networks with negative customers [15] were rapidly shown to be due to an extension of local balance [7]. Later, also the concept of quasi-reversibility was extended by allowing that customer classes depart from the nodes at a different rate from which they entered, which allows customers to change class in the queue, and includes negative customers, see [12]. Networks of quasi-reversible queues linked via state-dependent routing were considered in [3]. Due to the state-dependent nature of the routing, it is not possible to determine the throughput from the traffic equations. Instead, the traffic equations are replaced by a stochastic process, called the global process, that describes the number of customers in each node of the network. A decomposition of the network into the stationary distributions of the nodes and the stationary distribution of the global process is obtained under the condition that all nodes are quasi-reversible with arrival rate one, and the global process - describing the number of customers in each node, as if each node emits customers with constant rate one - satisfies local balance. Via these results, the worlds of local balance and quasi-reversibility seem to re-join the same track. This chapter provides a unified framework for quasi-reversibility and local balance.

Aggregation

The second part of this chapter is concerned with *aggregation* of queueing networks. A stochastic process is the aggregation of a queueing network with respect to an aggregation function on the state of the network, if this process describes - in probability, as well as in probability flow - the evolution of the aggregate state in the network, see [9] for a general definition.

Aggregation results are commonly referred to as Norton's theorem. Norton's theorem for queueing networks states that under certain conditions on the structure of the queueing network it is possible to replace a subset of the queueing network by a single station such that for the feature of interest (e.g. equilibrium distribution, throughput, average number of customers) the behaviour of the rest of the network

remains unchanged. Norton's theorem for queueing networks was originally introduced by Chandy *et al.* [10] as an efficient aggregation method for queueing networks similar to Norton's theorem from electrical circuit theory. They prove the aggregation method to be correct for queueing networks of the BCMP-type [2] consisting of two subnetworks of which the subnetwork of interest is a single station. The results of [10] can easily be generalised to subnetworks consisting of several stations such that customers enter the subnetwork through a single input node and leave the subnetwork through a single output node. Balsamo and Iazeolla [1], Kritzing *et al.* [19], and Vantilborgh [23] extend Norton's theorem to BCMP-networks consisting of two arbitrary subnetworks. A further extension is given by Towsley [22], where elementary state-dependent routing is incorporated. An additional extension is presented in Hsiao and Lazar [16], where it is shown that Norton's equivalent can be seen as a conditional expectation.

The relation between quasi-reversibility and Norton's theorem is introduced in Walrand [24]. Walrand considers a queueing network containing two quasi-reversible components, and shows that a quasi-reversible component may be replaced by an equivalent server. In Brandt [8] this result is extended to queueing networks of multiple quasi-reversible components linked by Markov routing, that is by state-independent routing. Pellaumail [21] shows that components of a closed network with state-dependent routing can be replaced by equivalent servers under a type of quasi-reversibility condition. Both the method and the construction of the equivalent servers require the network to be a closed network. Boucherie and van Dijk [6] discuss Norton's theorem for queueing networks consisting of product form components linked by state-dependent routing. All components can be aggregated into equivalent servers independently, and for the detailed behaviour of components it is allowed to analyse the behaviour of components as open networks in isolation (not part of the queueing network). Additional results for networks consisting of multiple components linked by state-dependent routing are reported in Van Dijk [13], where product form results for networks in which the routing probabilities depend only on the total number of customers present in the components are derived. Boucherie [3] combines the results of Boucherie and van Dijk [6] and Brandt [8]. This gives an extension of Norton's theorem to queueing networks comprised of quasi-reversible components linked by state-dependent routing. This is an extension of the results of [6] since the components in isolation are now assumed to be quasi-reversible and of [8] since the routing process is allowed to be state-dependent, such as most notably including blocking and alternative routing. A key difference with other methods is that subnetworks are analysed as open networks in isolation and not by shortcircuiting of the components. This substantially simplifies the construction of the equivalent servers.

In this chapter we extend the aggregation result of [3] to our model: we show that the global process is the aggregation of the network with respect to the global state. Moreover, we show that under some additional restrictions on the arrival rates, the local processes are also aggregations of the network with respect to the detailed state of the nodes. To obtain the necessary arrival rates for this aggregation, an iterative algorithm can be used. This algorithm appears to be similar in spirit to Marie's

method [20] to compute approximations for the steady-state distribution in queueing networks with non-quasi-reversible nodes and fixed routing, and thus allows development of new approximation methods, allowing global processes that do not satisfy local balance, allowing state-dependent routing, and general global states.

Examples and outline

To make the relation with the models and assumptions of [3] and [12] more explicit, we consider them as a special case. Somewhat surprisingly, it appears that our results reduce to those of [3] if there is only one customer class, and the global state represents the number of customers in a node: the state-dependent arrival and departure rates do not lead to further extensions. This, however, only holds for single class networks. By defining a trivial global state, our model and results reduce to those of [12]. This is, in fact, almost immediate, since in this way all state-dependence is reduced. We then proceed with pull networks, in which a transition is initiated by the arrival of a customer to a queue, and subsequently a customer is removed from the originating queue [4]. Finally, we consider decomposition for assembly network.

The chapter is organised as follows. In section 7.2 the network model is described and the definitions of the global and the local processes are given. Section 7.3 presents our decomposition results, and section 7.4 our aggregation results. Examples are included in section 7.5.

7.2 Model

Consider a network comprised of N interacting nodes, labelled $n = 1, 2, \dots, N$, and an outside node, labelled node 0, in which customers of classes $\bigcup_{n=0}^N \{\mathcal{A}_n \cup \mathcal{D}_n\}$ route among the nodes, where $\mathcal{A}_n$ resp. $\mathcal{D}_n$ is the set of customer classes that may arrive to resp. depart from node n , $n = 0, \dots, N$. Interaction among the nodes is due to customers routing among the nodes as well as due to the state of nodes influencing the behaviour of other nodes. This interaction is specified below. First, we will describe the nodes. Then, the interaction between the nodes is characterised.

7.2.1 The nodes

Consider the state-space $\mathcal{S}_n$, with states x_n . Define the mapping $G_n : \mathcal{S}_n \rightarrow G_n(\mathcal{S}_n)$, and $X_n = G_n(x_n)$. We will refer to X_n as global state corresponding to the detailed state x_n . The global state may be seen as an aggregate state (thus containing aggregate information of the node that is of interest for its performance, such as the number of customers), but will also play a more technical role in describing the interaction between the nodes (i.e. arrival and departure processes, and the routing between the nodes). The set $G_n(\mathcal{S}_n)$ will be referred to as the global state-space of node n .

We distinguish three types of state changes: due to an arrival, due to a departure, and due to an internal change, only. The behaviour of node n in isolation is characterised as follows, see [28] for a similar characterisation.

Definition 7.1 (Local process). Consider node n . $\mathcal{A}_n$ resp. $\mathcal{D}_n$ is the set of customer classes that may arrive resp. depart from node n . For each $c \in \mathcal{A}_n \cup \mathcal{D}_n$, let $A_n^c : G_n(\mathcal{S}_n) \rightarrow G_n(\mathcal{S}_n)$, and $D_n^c : G_n(\mathcal{S}_n) \rightarrow G_n(\mathcal{S}_n)$ 1 – 1 mappings such that D_n^c is the inverse of A_n^c .

- In an arrival transition, upon arrival of a class $c \in \mathcal{A}_n$ customer at node n , the detailed state changes from $x_n \in \mathcal{S}_n$ to $x'_n \in \mathcal{S}_n$ with probability $a_n^c(x_n, x'_n)$, and the global state changes from $X_n = G_n(x_n)$ to $A_n^c(X_n)$, where $a_n^c(x_n, x'_n)$ is an honest probability function:

$$\sum_{x'_n \in \mathcal{S}_n} a_n^c(x_n, x'_n) = 1, \quad x_n \in \mathcal{S}_n, c \in \mathcal{A}_n. \quad (7.1)$$

- In a departure transition in detailed state x_n a state change to state x'_n causing a departure of a class $c \in \mathcal{D}_n$ customer occurs at rate $d_n^c(x_n, x'_n)$. This detailed state change results in a global state change from $X_n = G_n(x_n)$ to $D_n^c(X_n)$.
- Node n initiates internal transitions from state x_n to state x'_n with rate $i_n(x_n, x'_n)$. Internal transitions do not cause a departure or arrival and do not change the global state, i.e., $G_n(x_n) = G_n(x'_n)$.
- Consider the set of functions $\lambda_n = (\lambda_n^c : G_n(\mathcal{S}_n) \rightarrow \mathbb{R}_0^+; c \in \mathcal{A}_n)$. The local process $\mathcal{L}_n(\lambda_n)$ is the Markov chain with state-space $\mathcal{S}_n$ and transition rates $q_n(x_n, x'_n; \lambda_n)$ from state $x_n \in \mathcal{S}_n$ to state $x'_n \in \mathcal{S}_n$ defined by

$$q_n(x_n, x'_n; \lambda_n) = \sum_{c \in \mathcal{A}_n} \lambda_n^c(G_n(x_n)) a_n^c(x_n, x'_n) + \sum_{c \in \mathcal{D}_n} d_n^c(x_n, x'_n) + i_n(x_n, x'_n). \quad (7.2)$$

Observe that, upon arrival of a class c customer in state x_n , the global state changes from $X_n = G_n(x_n)$ to $X'_n = A_n^c(X_n)$, and the detailed state may change to all $x'_n \in \{x : G_n(x) = A_n^c(X_n)\}$, which also implies that $a_n^c(x_n, x'_n) = 0$ if $G_n(x'_n) \neq A_n^c(G_n(x_n))$. The detailed state may represent the detailed content of a queue, and the global state the number of customers in this queue: upon arrival of a single customer, the global state then always changes from X_n to $X_n + 1$, where the detailed state change then may reflect the position of the customer in the queue, see e.g. the (ϕ, γ, δ) protocol introduced in [18], chapter 3, to represent queue disciplines such as FIFO, LIFO and PS. A class c customer may also represent a batch of customers by defining $A_n^c(X_n) = X_n + b_n^c$, where b_n^c denotes the class c batch size arriving at node n . Moreover, b_n^c may be set to a negative value: the number of customers is then decreased upon arrival of a class c customer. Such a customer may reflect a signal in a computer network, that removes tasks at a server. In literature, such customers have also been referred to as negative customers, see e.g. [15]. Departure transitions satisfy similar conditions as arrival transitions. Upon a departure, the global state change is unique, determined solely by the current global state and the class of the departing customer, whereas the detailed state may change from x_n to all $x'_n \in \{x : G_n(x) = D_n^c(x_n)\}$,

which also implies that $d_n^c(x_n, x'_n) = 0$ if $G_n(x'_n) \neq D_n^c(G_n(x_n))$. Internal transitions may correspond e.g. to completion of service phases, and - in nodes representing a subnetwork of queues - movements of customers between the queues in the subnetwork. As internal transitions do not change the global state, it must be that $i_n(x_n, x'_n) = 0$ if $G_n(x'_n) \neq G_n(x_n)$.

Remark 7.1. The class of arriving customers $\mathcal{A}_n$ is not required to coincide with the class of departing customers $\mathcal{D}_n$. As a consequence, the inverse A_n^c of D_n^c needs not be a function that corresponds to the global state change of an arriving transition, i.e., it may be that class c customers arrive to node n , but do not depart from node n . $\square$

We assume that the local process $\mathcal{L}_n(\lambda_n)$ is ergodic. Let $\pi_n(x_n; \lambda_n)$ denote the stationary probability that $\mathcal{L}_n(\lambda_n)$ is in state x_n , i.e., for all $x_n \in \mathcal{S}_n$,

$$\sum_{x'_n \in \mathcal{S}_n} \{ \pi_n(x_n; \lambda_n) q(x_n, x'_n; \lambda_n) - \pi_n(x'_n; \lambda_n) q(x'_n, x_n; \lambda_n) \} = 0,$$

and let

$$p_n(X_n; \lambda_n) = \sum_{\{x_n: G_n(x_n)=X_n\}} \pi_n(x_n; \lambda_n), \quad (7.3)$$

denote the stationary probability that $\mathcal{L}_n(\lambda_n)$ is in global state X_n .

Observe that the transition rates (7.2) characterise the arrival rate of customers to node n via the state-dependent arrival rate functions λ_n . The arrival processes at node n can be described by a state-dependent Poisson process, whose rate $\lambda_n^c(G_n(x_n))$ is assumed to depend on the global state $X_n = G_n(x_n)$ of this node, only. For the departure process, which - in correspondence with [18, 12] - will be described by the arrival rate in the time-reversed process, a similar assumption is made.

Assumption 7.2.1 *For the local process $\mathcal{L}_n(\lambda_n)$, $c \in \mathcal{D}_n$, we assume that the arrival rate of class c customers in state x_n of the stationary time-reversed process of $\mathcal{L}_n(\lambda_n)$ depends on x_n through the global state $X_n = G_n(x_n)$, only. We will denote this rate by $\mu_n^c(X_n; \lambda_n)$:*

$$\mu_n^c(X_n; \lambda_n) = \sum_{x'_n \in \mathcal{S}_n} \frac{\pi_n(x'_n; \lambda_n)}{\pi_n(x_n; \lambda_n)} d_n^c(x'_n, x_n), \quad X_n \in G_n(\mathcal{S}_n). \quad (7.4)$$

Quasi-reversibility plays a key-role in the theory of product form networks. Kelly [18] calls a node quasi-reversible, if, for a constant arrival rate function, the arrival rate of the time-reversed local process is constant, and equal to the arrival rate in the original (time-forward) process. This, in particular, implies that both the arrival and departure processes are Poisson processes with equal intensity, and independent of the state of a node. Chao and Miyazawa [12] have extended this definition by allowing arrival and departure rates to differ from each other: in their definition a node is quasi-reversible, if, for constant arrival rate functions, the departure process is a Poisson process that is independent of the state of a node. To distinguish these two

definitions, we will call the latter form generalised quasi-reversible. We summarise the above in the following definition.

Definition 7.2 ((Generalised) quasi-reversibility). Let $\hat{\lambda}_n = (\hat{\lambda}_n^c : G_n(\mathcal{S}_n) \rightarrow \mathbb{R}_0^+; c \in \mathcal{A}_n)$ be a set of constant functions. If $\mathcal{A}_n = \mathcal{D}_n$, and, for $c \in \mathcal{D}_n$, $\mu_n^c(X_n; \hat{\lambda}_n)$ is constant in X_n and equal to $\hat{\lambda}_n^c$, then the local process $\mathcal{L}_n(\hat{\lambda}_n)$ is said to be quasi-reversible. If, for $c \in \mathcal{D}_n$, $\mu_n^c(X_n; \hat{\lambda}_n)$ is constant in X_n , then the local process is said to be generalised quasi-reversible.

In the analysis below, we do not require generalised quasi-reversibility. Instead, we use the more general form of Assumption 7.2.1, and invoke a more general form of partial balance.

7.2.2 Interaction between the nodes

Nodes are coupled via a global process. Let $X = (X_1, \dots, X_N)$ denote the global state of the network, with X_n the global state of node n . The global state-space of the network, $\mathcal{S}_g \subseteq G_1(\mathcal{S}_1) \times \dots \times G_N(\mathcal{S}_N)$, is the set of all possible global states in the network. The global state of the network affects the interaction in three ways. Routing of customers between the nodes may depend on the global state of the network, arrivals to and departure from the network may depend on the global state, and the global state of a node may cause nodes to speed up or slow down. We use the following notation. For $X \in \mathcal{S}_g$, $T_{nn'}^{cc'}(X)$ denotes the vector obtained from X , by replacing the n -th component by $D_n^c(X_n)$, and the n' -th component by $A_{n'}^{c'}(X_{n'})$, $n, n' = 0, \dots, N$, where $n = 0$, or $n' = 0$ does not result in a change of state of that component.

Definition 7.3 (Global process). Let $\mathcal{A}_0$ resp. $\mathcal{D}_0$ denote the set of customer classes that may leave resp. enter the network. Consider state $X \in \mathcal{S}_g$.

- A class $c \in \mathcal{D}_0$ customer enters the network at rate $M_0^c(X)$, and arrives at node n' , $n' = 1, \dots, N$, as a class $c' \in \mathcal{A}_{n'}$ customer with probability $R_{0n'}^{cc'}(X)$. The global state changes from X to $T_{0n'}^{cc'}(X)$.
- A class $c \in \mathcal{D}_n$ customer departing from node n leaves the network as a class $c' \in \mathcal{A}_0$ customer with probability $R_{n0}^{cc'}(X)$. The global state changes from X to $T_{n0}^{cc'}(X)$.
- A class $c \in \mathcal{D}_n$ customer departing from node n , $n = 1, \dots, N$, routes to node n' , $n' = 1, \dots, N$, $n' \neq n$, as a class $c' \in \mathcal{A}_{n'}$ customer with probability $R_{nn'}^{cc'}(X)$. The global state changes from X to $T_{nn'}^{cc'}(X)$.
- The rate of change of node n , $n = 1, \dots, N$, for internal and departure transitions is $N_n(X)$.
- The routing probabilities $R_{nn'}^{cc'}(X)$ are honest:

$$\sum_{n'=0, n' \neq n}^N \sum_{c' \in \mathcal{A}_{n'}} R_{nn'}^{cc'}(X) = 1, \quad X \in \mathcal{S}_g, c \in \mathcal{D}_n, n = 0, \dots, N. \quad (7.5)$$

- Consider the set of functions $M = (M_n^c : G_n(\mathcal{S}_n) \rightarrow \mathbb{R}_0^+; c \in \mathcal{D}_n, n = 1, \dots, N)$. The global process $\mathcal{G}(M)$ is the Markov chain with state-space $\mathcal{S}_g$ and transition rates $Q(X, X'; M)$ from state $X \in \mathcal{S}_g$ to state $X' \in \mathcal{S}_g$ defined by

$$Q(X, T_{nn'}^{cc'}(X); M) = \begin{cases} M_0^c(X) R_{0n'}^{cc'}(X) & n = 0, \\ M_n^c(X_n) N_n(X) R_{nn'}^{cc'}(X) & n = 1, \dots, N, \end{cases}$$

for $n' = 0, \dots, N, n' \neq n, c \in \mathcal{D}_n$ and $c' \in \mathcal{A}_{n'}$.

The global process describes the global state of the network, as if node n in isolation (i.e. without the multiplication factor $N_n(X)$) emits customers at rate $M_n^c(X_n)$. We will call $M_n^c(X_n)$ the nominal departure rate of class c customers from node n . The global and local processes are closely intertwined, as will become clear later. In the formulation of the global process, the nominal departure rates $M_n^c(X_n)$ depend on the local process. Furthermore, the arrival rates $\lambda_n^c(G_n(x_n))$ in the local processes depend on the global process. These relations will be made explicit when we define our network in Definition 7.4.

We assume that the global process $\mathcal{G}(M)$ is ergodic. Let $\Pi(X; M)$ denote the stationary probability that $\mathcal{G}(M)$ is in state X , i.e., for all $X \in \mathcal{G}(M), c \in \mathcal{D}_n, n = 0, \dots, N$,

$$\sum_{n, n'=0, n' \neq n}^N \sum_{c \in \mathcal{A}_n, c' \in \mathcal{A}_{n'}} \{ \Pi(X; M) Q(X, T_{nn'}^{cc'}(X); M) - \Pi(T_{nn'}^{cc'}(X); M) Q(T_{nn'}^{cc'}(X), X; M) \} = 0. \quad (7.6)$$

Let

$$P_n(X_n; M) = \sum_{\{X' : X'_n = X_n\}} \Pi(X'; M),$$

denote the marginal stationary probability that the global state of node n is X_n .

Our results are formulated via the nominal departure rates $M_n^c(X_n)$, and the departure rates of the time-reversed process that will be used to characterise the arrival processes at the nodes. Let $\Lambda_0^c(X; M)$ denote the class $c \in \mathcal{D}_0$ departure rate in the time-reversed process of $\mathcal{G}(M)$. Then

$$\Lambda_0^c(X; M) = \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} \frac{\Pi(T_{0n'}^{cc'}(X); M)}{\Pi(X; M)} M_{n'}^{c'}(A_{n'}^{c'}(X_{n'})) N_{n'}^{c'}(T_{0n'}^{cc'}(X)) R_{n'0}^{c'c}(T_{0n'}^{cc'}(X)). \quad (7.7)$$

The nominal departure rates $M_n^c(X_n)$ of node n depend only on the global state of node $n, n = 1, \dots, N$. We assume that this is also the case for the nominal departure rates in the time-reversed process.

Assumption 7.2.2 For the global process $\mathcal{G}(M)$, $c \in \mathcal{A}_n$, $n = 1, \dots, N$, and $X \in \mathcal{S}_g$, we assume that the nominal departure rate of class c customers from node n in state X of the stationary time-reversed process of $\mathcal{G}(M)$ depends on the global state X_n only. We will denote this nominal departure rate by $\Lambda_n^c(X_n; M)$:

$$\begin{aligned} \Lambda_n^c(X_n; M)N_n(X) &= \sum_{c' \in \mathcal{D}_0} \frac{\Pi(T_{n0}^{cc'}(X); M)}{\Pi(X; M)} M_0^{c'}(T_{n0}^{cc'}(X)) R_{0n}^{c'c}(T_{n0}^{cc'}(X)) \\ &+ \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} \frac{\Pi(T_{nn'}^{cc'}(X); M)}{\Pi(X; M)} M_{n'}^{c'}(A_{n'}^{c'}(X_{n'})) N_{n'}^{c'}(T_{nn'}^{cc'}(X)) R_{n'n}^{c'c}(T_{nn'}^{cc'}(X)). \end{aligned} \quad (7.8)$$

In general, the time-reversed departure rate (7.8) will depend on the global state X of the network. The assumption that this rate is equal to $\Lambda_n^c(X_n; M)N_n(X)$, where $\Lambda_n^c(X_n; M)$ depends on X through the global state X_n of node n , only, seems to be rather restrictive. This is not the case. Assumption 7.2.2 includes local balance, a common assumption for queueing networks with state-dependent routing. To this end, note that if $\mathcal{A}_n = \mathcal{D}_n$, and $\Lambda_n^c(X_n; M) = M_n^c(X_n)$, $X_n \in G_n(\mathcal{S}_n)$, $c \in \mathcal{A}_n$, $n = 1, \dots, N$, then, from (7.8),

$$\begin{aligned} M_n^c(X_n)N_n(X)\Pi(X; M) &= \sum_{c' \in \mathcal{D}_0} \Pi(T_{n0}^{cc'}(X); M) M_0^{c'}(T_{n0}^{cc'}(X)) R_{0n}^{c'c}(T_{n0}^{cc'}(X)) \\ &+ \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} \Pi(T_{nn'}^{cc'}(X); M) M_{n'}^{c'}(A_{n'}^{c'}(X_{n'})) N_{n'}^{c'}(T_{nn'}^{cc'}(X)) R_{n'n}^{c'c}(T_{nn'}^{cc'}(X)), \end{aligned}$$

and thus the global process satisfies local balance

$$\sum_{n'=0}^N \sum_{c' \in \mathcal{A}_{n'}} \{ \Pi(X; M) Q(X, T_{nn'}^{cc'}(X); M) - \Pi(T_{nn'}^{cc'}(X); M) Q(T_{nn'}^{cc'}(X), X; M) \} = 0.$$

7.2.3 The network

Combining the descriptions of the nodes and their interaction, we obtain a queueing network of nodes in which the detailed behaviour of the node is specified in Definition 7.1, and the interaction among the nodes is specified in Definition 7.3. This network allows a Markovian description with state $x = (x_1, \dots, x_N)$. Denote $G(x) = (G_1(x_1), \dots, G_N(x_N))$.

Definition 7.4 (Network). The network $\mathcal{N}$ is the Markov-chain with state-space $\mathcal{S} \subseteq \{x = (x_1, \dots, x_N) : x_n \in \mathcal{S}_n, G(x) \in \mathcal{S}_g\}$, and transition rates $q(x, x')$ from state $x = (x_1, \dots, x_N)$ to state $x' = (x'_1, \dots, x'_N)$ given by

$$q(x, x') = \sum_{c \in \mathcal{D}_n, c' \in \mathcal{A}_{n'}} d_n^c(x_n, x'_n) N_n(G(x)) R_{nn'}^{cc'}(G(x)) a_{n'}^{c'}(x'_n, x'_n),$$

if $x'_n \neq x_n$, $x'_{n'} \neq x_{n'}$, and $x'_k = x_k$, for $k \neq n, n'$,

$$\begin{aligned} q(x, x') = & i_n(x_n, x'_n) N_n(G(x)) + \sum_{c \in \mathcal{D}_0} M_0^c(G(x)) \sum_{c' \in \mathcal{A}_n} R_{0n}^{cc'}(G(x)) a_n^{c'}(x_n, x'_n) \\ & + \sum_{c \in \mathcal{D}_n} d_n^c(x_n, x'_n) N_n(G(x)) \sum_{c' \in \mathcal{A}_0} R_{n0}^{cc'}(G(x)), \end{aligned}$$

if $x'_n \neq x_n$, and $x'_k = x_k$ for $k \neq n$.

We assume that the network $\mathcal{N}$ is ergodic, and define $\pi(x)$ as the stationary probability that the network is in state x .

Arrivals and departures in the global process have been characterised via assumptions on the nominal departure rates, $M_n^c(X_n)$, and their time-reversed counterparts, $\Lambda_n^c(X_n; M)$, that are restricted to depend on the global state X_n , only. In contrast, arrivals and departures in the local processes have been characterised via assumptions on the arrival rates $\lambda_n^c(X_n)$, and their time-reversed counterparts $\mu_n^c(X_n; \lambda_n)$. This may seem somewhat inconvenient at first glance. However, arrivals to a node at local level are determined by departures from nodes at global level and subsequent routing of customers at global level. In our analysis below, we will make this relation explicit, thus characterising the relation between λ and M . Further, note that characterisation of local processes via arrival rates in the forward and time-reversed process provides a direct link with quasi-reversibility, whereas characterisation of the global process via departure rates in the forward and time-reversed processes provides a link with local balance. We may thus view our network as a network of further generalised quasi-reversible nodes linked via a process that satisfies a generalised form of local balance.

The aim of this chapter is twofold. First, we want to establish sufficient conditions on the arrival rate functions $\lambda_n^c(X_n)$, $\mu_n^c(X_n; \lambda_n)$, and the nominal departure rate functions $M_n^c(X_n)$, $\Lambda_n^c(X_n; M)$ under which the network can be decomposed, i.e. the stationary distribution $\pi(x)$ of the network can be factorised into the stationary distributions $\pi_n(x_n; \lambda_n)$ of the local processes, and the stationary distribution $\Pi(X; M)$ of the global process. Second, our aim is to investigate when the global process and the local processes are aggregations of the network, i.e., the distribution and the rates of the global process describe the evolution of the global state of the network, and the distribution and the rates of the local processes describe the evolution of the detailed state of a node in the network. Roughly said, these aggregations require that not only the stationary distribution of the network $\mathcal{N}$ can be decomposed into the stationary distributions of the local and global processes, but also the process $\mathcal{N}$ itself can be decomposed into the processes $\mathcal{L}_n(\lambda_n)$ and $\mathcal{G}(M)$.

7.3 Decomposition

This section considers the decomposition of the stationary distribution $\pi(x)$ of the network $\mathcal{N}$ into the stationary distributions of the global process and the local processes. We show that such a decomposition holds if the nominal departure rates $M_n^c(X)$ and the nominal time-reversed departure rates $\Lambda_n^c(X; M)$ of the global process equal the corresponding rates in the local processes, to be specified below. As an illustration, in Section 7.5 we consider the two models that are studied in [3] and [12]. These models fall into our class of queueing networks via specific assumptions on the form of the global state. We will show that for both models the conditions of our general result are satisfied if and only if the assumptions that are made in [3] and [12] are satisfied. In addition, we will describe pull networks [4], and derive some new decomposition results for so-called assembly networks.

The conditional probability of x_n given X_n for local process $\mathcal{L}_n(\lambda_n)$ equals $\pi_n(x_n; \lambda_n)/p_n(X_n; \lambda_n)$. Let $\tilde{M}_n^c(X_n; \lambda_n)$ denote the conditional expected class $c \in \mathcal{D}_n$ departure rate given state X_n of the local process $\mathcal{L}_n(\lambda_n)$. Then

$$\tilde{M}_n^c(X_n; \lambda_n) = \sum_{\{x_n: G_n(x_n)=X_n\}} \frac{\pi_n(x_n; \lambda_n)}{p_n(X_n; \lambda_n)} \sum_{x'_n \in \mathcal{S}_n} d_n^c(x_n, x'_n) \quad (7.9)$$

$$\begin{aligned} &= \frac{1}{p_n(X_n; \lambda_n)} \sum_{\{x'_n: G_n(x'_n)=D_n^c(X_n)\}} \pi_n(x'_n; \lambda_n) \mu_n^c(D_n^c(X_n); \lambda_n) \\ &= \frac{p_n(D_n^c(X_n); \lambda_n)}{p_n(X_n; \lambda_n)} \mu_n^c(D_n^c(X_n); \lambda_n), \end{aligned} \quad (7.10)$$

where the second equality is obtained from the definition of μ_n^c given in (7.4). Similarly, let $\tilde{\Lambda}_n^c(X_n; \lambda_n)$ denote the conditional expected class $c \in \mathcal{D}_n$ arrival rate given state X_n of the local process $\mathcal{L}_n(\lambda_n)$. Then

$$\begin{aligned} \tilde{\Lambda}_n^c(X_n; \lambda_n) &= \sum_{\{x_n \in \mathcal{S}_n: G_n(x_n)=X_n\}} \frac{\pi_n(x_n; \lambda_n)}{p_n(X_n; \lambda_n)} \sum_{x'_n \in \mathcal{S}_n} \frac{\pi_n(x'_n; \lambda_n)}{\pi_n(x_n; \lambda_n)} \lambda_n^c(G_n(x'_n)) a_n^c(x'_n, x_n) \\ &= \frac{p_n(D_n^c(X_n); \lambda_n)}{p_n(X_n; \lambda_n)} \lambda_n^c(D_n^c(X_n)), \end{aligned} \quad (7.11)$$

where the last equality is due to the restrictions on x'_n due to $a_n^c(x'_n, x_n)$, i.e., $x'_n \in \{x: G_n(x) = D_n^c(X_n)\}$, and due to $a_n^c(x'_n, x_n)$ being honest.

It is interesting to observe that under Assumption 7.2.1 resp. Assumption 7.2.2 we obtain flow balance under time-reversal as specified below for the local processes, resp. the global process. These observations start from the global balance equations for the local processes, for $\pi_n(x_n; \lambda_n)$ the stationary distribution of local process $\mathcal{L}_n(\lambda_n)$,

$$\pi_n(x_n; \lambda_n) \sum_{x'_n \in \mathcal{S}_n} \left(\sum_{c \in \mathcal{A}_n} \lambda_n^c(G_n(x_n)) a_n^c(x_n, x'_n) + \sum_{c \in \mathcal{D}_n} d_n^c(x_n, x'_n) + i_n(x_n, x'_n) \right)$$

$$\begin{aligned}
&= \sum_{x'_n \in \mathcal{S}_n} \pi_n(x'_n; \lambda_n) \left(\sum_{c \in \mathcal{A}_n} \lambda_n^c(G_n(x'_n)) a_n^c(x'_n, x_n) \right. \\
&\quad \left. + \sum_{c \in \mathcal{D}_n} d_n^c(x'_n, x_n) + i_n(x_n, x'_n) \right), \tag{7.12}
\end{aligned}$$

and the global balance equations for the global process, for $\Pi(X; M)$ the stationary distribution of the global process $\mathcal{G}(M)$,

$$\begin{aligned}
&\Pi(X; M) \sum_{n=0}^N \sum_{c \in \mathcal{D}_n} M_n^c(X) N_n(X) \\
&= \sum_{n=0}^N \sum_{c \in \mathcal{A}_n} \sum_{n'=0}^N \sum_{c' \in \mathcal{D}_{n'}} \Pi(T_{nn'}^{cc'}(X); M) M_{n'}^{c'}(T_{nn'}^{cc'}(X)) N_{n'}^{c'}(T_{nn'}^{cc'}(X)) R_{n'n}^{c'c}(T_{nn'}^{cc'}(X)). \tag{7.13}
\end{aligned}$$

Summing the global balance equations (7.12) for fixed X_n over all x_n with $G_n(x_n) = X_n$, the internal transitions cancel out. The definition of $\mu_n^c(X_n; \lambda_n)$ in Assumption 7.2.1 then yields, noting that $G_n(x_n) = X_n$,

$$\begin{aligned}
&\sum_{\{x_n: G_n(x_n)=X_n\}} \pi_n(x_n; \lambda_n) \sum_{c \in \mathcal{A}_n} \lambda_n^c(G_n(x_n)) + \sum_{c \in \mathcal{D}_n} \sum_{\{x'_n: G_n(x'_n)=D_n^c(X_n)\}} \pi_n(x'_n; \lambda_n) \\
&\quad \times \mu_n^c(D_n^c(G_n(x_n)); \lambda_n) = \sum_{c \in \mathcal{A}_n} \sum_{\{x'_n: G_n(x'_n)=D_n^c(X_n)\}} \pi_n(x'_n; \lambda_n) \\
&\quad \times \lambda_n^c(D_n^c(G_n(x_n))) + \sum_{\{x_n: G_n(x_n)=X_n\}} \pi_n(x_n; \lambda_n) \mu_n^c(G_n(x_n); \lambda_n).
\end{aligned}$$

The definition of $p_n(X_n; \lambda_n)$ now implies that for the local process $\mathcal{L}_n(\lambda_n)$ the sum of the total arrival rates and the total mean departure rates in each global state X_n does not change under time reversal:

$$\begin{aligned}
&\sum_{c \in \mathcal{A}_n} \lambda_n^c(X_n) + \sum_{c \in \mathcal{D}_n} \frac{p_n(D_n^c(X_n); \lambda_n)}{p_n(X_n; \lambda_n)} \mu_n^c(D_n^c(X_n); \lambda_n) \\
&= \sum_{c \in \mathcal{D}_n} \mu_n^c(X_n; \lambda_n) + \sum_{c \in \mathcal{A}_n} \frac{p_n(D_n^c(X_n); \lambda_n)}{p_n(X_n; \lambda_n)} \lambda_n^c(D_n^c(X_n)) \tag{7.14}
\end{aligned}$$

To obtain our decomposition result, we will assume that for the global process the arrival rate to node n equals the departure rate to node n , as characterized via the time-reversed process:

$$M_n^c(X_n) = \tilde{M}_n^c(X_n; \lambda_n) \tag{7.15}$$

$$\Lambda_n^c(X_n; M) = \tilde{\Lambda}_n^c(X_n; \lambda_n). \tag{7.16}$$

Invoking (7.10), (7.15), (7.11), and (7.16) we obtain

$$\begin{aligned}
& \sum_{c \in \mathcal{A}_n} \lambda_n^c(G_n(x_n)) - \sum_{c \in \mathcal{A}_n} \Lambda_n^c(G_n(x_n); M) \\
&= \sum_{c \in \mathcal{D}_n} \mu_n^c(G_n(x_n); \lambda_n) - \sum_{c \in \mathcal{D}_n} M_n^c(G_n(x_n)), \tag{7.17}
\end{aligned}$$

i.e., the net input due to the local and global processes equals the net output due to the local and global processes.

A further consequence of (7.14) is that the $p_n(X_n; \lambda_n)$ can be computed recursively:

$$\begin{aligned}
p_n(X_n; \lambda_n) = & \frac{\sum_{c \in \mathcal{A}_n} p_n(D_n^c(X_n); \lambda_n) \lambda_n^c(D_n^c(X_n)) - \sum_{c \in \mathcal{D}_n} p_n(D_n^c(X_n); \lambda_n) \mu_n^c(D_n^c(X_n); \lambda_n)}{\sum_{c \in \mathcal{A}_n} \lambda_n^c(X_n) - \sum_{c \in \mathcal{D}_n} \mu_n^c(X_n; \lambda_n)}
\end{aligned}$$

for $\sum_{c \in \mathcal{A}_n} \lambda_n^c(X_n) \neq \sum_{c \in \mathcal{D}_n} \mu_n^c(X_n; \lambda_n)$. For $\sum_{c \in \mathcal{A}_n} \lambda_n^c(X_n) = \sum_{c \in \mathcal{D}_n} \mu_n^c(X_n; \lambda_n)$, we find

$$p_n(X_n; \lambda_n) = \frac{\sum_{c \in \mathcal{A}_n} p_n(D_n^c(X_n); \lambda_n) \lambda_n^c(D_n^c(X_n))}{\sum_{c \in \mathcal{D}_n} \tilde{M}_n^c(X_n; \lambda_n)},$$

which, for example, is the case for quasi-reversible nodes.

Invoking Assumption 7.2.2 on the nominal departure rates $\Lambda_n^c(X; M)$ in the right-hand side of the global balance equations (7.13) implies that for the global process $\mathcal{G}(M)$, the total departure rate in each state X does not change under time-reversal:

$$\begin{aligned}
& \sum_{c \in \mathcal{D}_0} M_0^c(X) + \sum_{n=1}^N \sum_{c \in \mathcal{D}_n} M_n(X) N_n(X) = \\
&= \sum_{c \in \mathcal{A}_0} \Lambda_0^c(X; M) + \sum_{n=1}^N \sum_{c \in \mathcal{A}_n} \Lambda_n^c(X; M) N_n(X). \tag{7.18}
\end{aligned}$$

We are now ready to state the main theorem of this section.

Theorem 7.3.1 Assume that, for $n = 1, \dots, N$, $X_n \in G_n(\mathcal{S}_n)$,

$$\begin{aligned}
M_n^c(X_n) &= \tilde{M}_n^c(X_n; \lambda_n) \\
\Lambda_n^c(X_n; M) &= \tilde{\Lambda}_n^c(X_n; \lambda_n).
\end{aligned}$$

Then the stationary distribution of the network $\mathcal{N}$ is

$$\pi(x) = \Pi(G(x); M) \prod_{n=1}^N \frac{\pi_n(x_n; \lambda_n)}{p_n(G_n(x_n); \lambda_n)}, \quad x \in \mathcal{S}. \tag{7.19}$$

Observe that (7.15) and (7.16) place severe restrictions on the departure rates from a node in the local processes and the global process, and thus relate the sets of functions $M = (M_n^c : G_n(\mathcal{S}_n) \rightarrow \mathbb{R}_0^+; c \in \mathcal{D}_n, n = 1, \dots, N)$ to the sets of functions $\lambda_n = (\lambda_n^c : G_n(\mathcal{S}_n) \rightarrow \mathbb{R}_0^+; c \in \mathcal{A}_n), n = 1, \dots, N$.

Proof of Theorem 7.3.1. It is sufficient to show that $\pi(x)$ solves the balance equations for the network, that read when inserting the proposed form (7.19), and dividing by $\pi(x)$:

$$\begin{aligned}
& \sum_{c \in \mathcal{D}_0} M_0^c(G(x)) + \sum_{n=1}^N N_n(G(x)) \sum_{x'_n} \left(\sum_{c \in \mathcal{D}_n} d_n^c(x_n, x'_n) + i_n(x_n, x'_n) \right) \\
&= \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} \sum_{c \in \mathcal{A}_0} \frac{\Pi(T_{0n'}^{cc'}(G(x)); M)}{\Pi(G(x); M)} N_{n'}^{c'}(T_{0n'}^{cc'}(G(x))) R_{n'0}^{c'c}(T_{0n'}^{cc'}(G(x))) \\
&\quad \times \frac{p_{n'}(G_{n'}(x_{n'}); \lambda_{n'})}{p_{n'}(A_{n'}^{c'}(G_{n'}(x_{n'})); \lambda_{n'})} \left(\sum_{x'_{n'}} \frac{\pi_{n'}(x'_{n'}; \lambda_{n'})}{\pi_{n'}(x_{n'}; \lambda_{n'})} d_{n'}^{c'}(x'_{n'}, x_{n'}) \right) \\
&+ \sum_{n=1}^N \sum_{c \in \mathcal{A}_n} \sum_{x'_n} \frac{\pi_n(x'_n; \lambda_n)}{\pi_n(x_n; \lambda_n)} a_n^c(x'_n, x_n) \frac{p_n(G_n(x_n); \lambda_n)}{p_n(D_n^c(G_n(x_n)); \lambda_n)} \left(\sum_{c' \in \mathcal{D}_0} \frac{\Pi(T_{n0}^{cc'}(G(x)))}{\Pi(G(x))} \right. \\
&\times M_0^{c'}(T_{n0}^{cc'}(G(x))) R_{n0}^{c'c}(T_{n0}^{cc'}(G(x))) + \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} \frac{\Pi(T_{nn'}^{cc'}(G(x)); M)}{\Pi(G(x); M)} N_{n'}^{c'}(T_{nn'}^{cc'}(G(x))) \\
&\quad \times R_{n'n}^{c'c}(T_{nn'}^{cc'}(G(x))) \frac{p_{n'}(G_{n'}(x_{n'}); \lambda_{n'})}{p_{n'}(A_{n'}^{c'}(G_{n'}(x_{n'})); \lambda_{n'})} \left(\sum_{x'_{n'}} \frac{\pi_{n'}(x'_{n'}; \lambda_{n'})}{\pi_{n'}(x_{n'}; \lambda_{n'})} d_{n'}^{c'}(x'_{n'}, x_{n'}) \right) \Bigg) \\
&\quad + \sum_{n=1}^N \sum_{x'_n} \frac{\pi_n(x'_n)}{\pi_n(x_n)} N_n(G(x)) i_n(x'_n, x_n).
\end{aligned}$$

Invoking (7.4), (7.10), and (7.15), and (7.7), the first term on the right hand side equals $\sum_{c \in \mathcal{A}_0} \Lambda_0^c(G(x); M)$. Invoking (7.4), (7.10), (7.15), (7.8), (7.16) and (7.11), the second and third term in the right hand side equal:

$$\sum_{n=1}^N \sum_{c \in \mathcal{A}_n} \sum_{x'_n} \frac{\pi_n(x'_n; \lambda_n)}{\pi_n(x_n; \lambda_n)} a_n^c(x'_n, x_n) \lambda_n^c(D_n^c(x_n); M) N_n(x).$$

Inserting these expressions in the right hand side, and invoking global balance for the nodes (7.12), implies that it is sufficient to show that

$$\begin{aligned}
& \sum_{c \in \mathcal{D}_0} M_0^c(G(x)) + \sum_{n=1}^N \sum_{c \in \mathcal{D}_n} \sum_{x'_n} \frac{\pi_n(x'_n)}{\pi_n(x_n)} d_n^c(x'_n, x_n) N_n(G_n(x_n)) \\
&= \sum_{n=1}^N N_n(G(x)) \sum_{c \in \mathcal{A}_n} \lambda_n^c(G_n(x_n)) + \sum_{c \in \mathcal{A}_0} \Lambda_0^c(G(x); M) \quad (7.20)
\end{aligned}$$

Inserting (7.17) into (7.18) yields (7.20), which completes the proof. $\square$

The decomposition of Theorem 7.3.1 does not establish a complete decomposition of the nodes and the global process, in the sense that the state of the nodes and the global state of the network are independent. Equation (7.19) states that the detailed states of the nodes are independent, conditioned on the global state of the nodes:

$$\frac{\pi(x)}{\Pi(X;M)} = \prod_{n=1}^N \frac{\pi_n(x_n; \lambda_n)}{p_n(X_n; \lambda_n)}.$$

The proof of Theorem 7.3.1 relies heavily on (7.14) but does not require additional properties of $p_n(X_n; \lambda_n)$. An immediate generalisation of Theorem 7.3.1 is obtained replacing $p_n(X_n; \lambda_n)$ by any function satisfying (7.14).

Theorem 7.3.2 *Let $f_n : \mathcal{S}_n \rightarrow \mathbb{R}_0^+$ be a function satisfying*

$$\begin{aligned} & \sum_{c \in \mathcal{A}_n} \lambda_n^c(X_n) + \sum_{c \in \mathcal{D}_n} \frac{f_n(D_n^c(X_n); \lambda_n)}{f_n(X_n; \lambda_n)} \mu_n^c(D_n^c(X_n); \lambda_n) \\ &= \sum_{c \in \mathcal{D}_n} \mu_n^c(X_n; \lambda_n) + \sum_{c \in \mathcal{A}_n} \frac{f_n(D_n^c(X_n); \lambda_n)}{f_n(X_n; \lambda_n)} \lambda_n^c(D_n^c(X_n)), \end{aligned} \quad (7.21)$$

and assume that the following conditions are satisfied:

$$M_n^c(X_n) = \frac{f_n(D_n^c(X_n); \lambda_n)}{f_n(X_n; \lambda_n)} \mu_n^c(D_n^c(X_n); \lambda_n), \quad (7.22)$$

$$\Lambda_n^c(X_n; M) = \frac{f_n(D_n^c(X_n); \lambda_n)}{f_n(X_n; \lambda_n)} \lambda_n^c(D_n^c(X_n)). \quad (7.23)$$

Then the stationary distribution $\pi(x)$ of the network $\mathcal{N}$ is

$$\pi(x) = C^{-1} \Pi(G(x); M) \prod_{n=1}^N \frac{\pi_n(x_n; \lambda_n)}{f_n(G_n(x_n); \lambda_n)}, \quad x \in \mathcal{S}, \quad (7.24)$$

with

$$C = \sum_{x \in \mathcal{S}} \Pi(G(x); M) \prod_{n=1}^N \frac{\pi_n(x_n; \lambda_n)}{f_n(G_n(x_n); \lambda_n)}.$$

For generalised quasi-reversible nodes conditions (7.22), (7.23) are satisfied with $f_n(X_n; \lambda_n) = 1$. In this case, a complete decomposition can be obtained from Theorem 7.3.2.

As a consequence of Theorem 7.3.2, we can simplify the formula for the stationary distribution in case the local processes are extended quasi-reversible: then $f_n(X_n) = 1$ satisfies condition (7.21), and the following Corollary follows immediately from Theorem 7.3.2.

Corollary 7.3.3 *Assume that the local processes $\mathcal{L}_n(\lambda_n)$ are generalised quasi-reversible, say with arrival rates $\lambda_n^c(X_n) = \hat{\lambda}_n^c$ and time-reversed arrival rates*

$\mu_n^c(X_n; \lambda_n) = \hat{\mu}_n^c$. Let $M_n^c(X_n)$ be given by

$$M_n^c(X_n) = \begin{cases} \hat{\mu}_n^c & \text{if } D_n^c(X_n) \in G_n(\mathcal{S}_n), \\ 0 & \text{otherwise.} \end{cases}$$

If for all X_n with $D_n^c(X_n) \in G_n(\mathcal{S}_n)$, $\Lambda_n^c(X_n; M) = \lambda_n^c$, then the stationary distribution of the network $\mathcal{N}$ is given by

$$\pi(x) = C^{-1} \Pi(G(x)) \prod_{n=1}^N \pi_n(x_n; \lambda_n), \quad x \in \mathcal{S},$$

with

$$C = \sum_{x \in \mathcal{S}} \Pi(G(x)) \prod_{n=1}^N \pi_n(x_n; \lambda_n).$$

Theorems 7.3.1, 7.3.2 and Corollary 7.3.3 require the values for the arrival rates $\lambda_n = (\lambda_n^c : G_n(\mathcal{S}_n) \rightarrow \mathbb{R}_0^+; c \in \mathcal{A}_0)$ that relate the local processes and the global process. These arrival rates are the solution of the fixed point problem consisting of the equations.

Corollary 7.3.4 (Fixed point equations for arrival rates: decomposition) *The arrival rates $\lambda_n = (\lambda_n^c : G_n(\mathcal{S}_n) \rightarrow \mathbb{R}_0^+; c \in \mathcal{A}_0)$ are a solution of the fixed point equations:*

$$\begin{aligned} (7.12) \quad \pi_n &= \pi_n(\lambda_n) \\ (7.4) \quad \mu_n^c &= \mu_n^c(\pi_n, \lambda_n) \\ (7.10) \quad \tilde{M}_n^c &= \tilde{M}_n^c(\pi_n, \mu_n, \lambda_n) \\ (7.15) \quad M &= \tilde{M} \\ (7.13) \quad \Pi &= \Pi(M) \\ (7.7) \quad \Lambda_n^c &= \Lambda_n^c(M, \Pi) \\ (7.16) \quad \tilde{\Lambda} &= \Lambda \\ (7.11) \quad \lambda &= \lambda(\tilde{\Lambda}) \end{aligned}$$

These equations may be solved using the following algorithm:

- Step i: For $n = 1, \dots, N$ initialize with a starting value for $\hat{\lambda}_n$ for λ_n .
- Step ii: Use (7.12), (7.4), (7.10) to obtain $\tilde{M}_n^c$.
- Step iii: Use (7.15), (7.13), (7.7) to obtain Λ_n^c .
- Step iv: If $\tilde{\Lambda}_n^c = \Lambda_n^c$ for $c \in \mathcal{D}_n$, $n = 1, \dots, N$ then stop, and λ_n is obtained, else use (7.11) to let

$$\lambda_n^c(X_n) = \frac{p_n(A_n^c(X_n); \lambda_n)}{p_n(X_n; \lambda_n)} \Lambda_n^c(A_n^c(X_n); M).$$

and go to Step ii.

Notice that existence of a fixed point is an implicit assumption that we made for the results of Theorems 7.3.1, 7.3.2 to be valid.

The arrival rates and time-reversed arrival rates of our network depend on the state x through the global state $G(x)$ only. For the network to satisfy Assumption 7.2.1 additional assumptions on the global state are required. Section 7.5 provides examples of networks that have this structure.

7.4 Aggregation

This section considers aggregation of the nodes in our network. We first show that under the conditions of Theorem 7.3.1, the global process is the aggregation of the network with respect to the global state, that is, for the analysis of the global network the detailed behaviour of the nodes is not required. We then investigate under which conditions the local processes are the aggregation of the network with respect to the detailed state of a single node, that is, for the analysis of the detailed behaviour of a single node the detailed behaviour of the *other* nodes is not required. It appears that this requires some extra restrictions on the arrival rates: the local arrival rates should equal the global arrival rates. Our generalisation results in an aggregation algorithm that generalises the method developed by Marie in [20].

The following definition is adapted from Brandwajn [9].

Definition 7.5 (Aggregation). Consider two Markov chains $\mathcal{M}_1$ and $\mathcal{M}_2$ with state spaces S_1 and S_2 , transition rates $q_1(y_1, y'_1)$, $y_1, y'_1 \in S_1$, and $q_2(y_2, y'_2)$, $y_2, y'_2 \in S_2$, and stationary distributions $\pi_1(y_1)$, $y_1 \in S_1$, and $\pi_2(y_2)$, $y_2 \in S_2$. The Markov chain $\mathcal{M}_2$ is said to be the aggregation of $\mathcal{M}_1$ with respect to a function $h : S_1 \rightarrow S_2$ if the following two conditions are satisfied:

$$\pi_2(y_2) = \sum_{\{y_1 \in S_1 : h(y_1) = y_2\}} \pi_1(y_1), \quad y_2 \in S_2, \quad (7.25)$$

$$\pi_2(y_2)q_2(y_2, y'_2) = \sum_{\{y_1, y'_1 \in S_1 : h(y_1) = y_2, h(y'_1) = y'_2\}} \pi_1(y_1)q_1(y_1, y'_1), \quad y_2, y'_2 \in S_2. \quad (7.26)$$

The definition of aggregation requires both the equilibrium distribution and the probability flows to match. Boucherie [3] refers to this form of aggregation as first order equivalence. The intuition for Theorem 7.4.1 is encapsulated in (7.15), (7.16) of Theorem 7.3.1: $M_n^c(X_n) = \tilde{M}_n^c(X_n; \lambda_n)$, $\Lambda_n^c(X_n; M) = \tilde{\Lambda}_n^c(X_n; \lambda_n)$. These equations state that for the global process the arrival rate to node n equals the departure rate to node n , as characterized via the time-reversed process, which expresses conservation of probability flow.

Theorem 7.4.1 (Aggregation with respect to the global state function) *Assume that, for $n = 1, \dots, N$, $X_n \in G_n(\mathbb{S}_n)$,*

$$\begin{aligned} M_n^c(X_n) &= \tilde{M}_n^c(X_n; \lambda_n) \\ \Lambda_n^c(X_n; M) &= \tilde{\Lambda}_n^c(X_n; \lambda_n). \end{aligned}$$

Then the global process $\mathcal{G}(M)$ is the aggregation of the network $\mathcal{N}$ with respect to the global state function $G: \mathcal{S} \rightarrow \mathcal{S}_g$.

Proof. Condition (7.25) is almost immediate:

$$\sum_{\{x: G(x)=X\}} \pi(x) = \Pi(X) \sum_{\{x: G(x)=X\}} \prod_{n=1}^N \frac{\pi_n(x_n; \lambda_n)}{p_n(X_n; \lambda_n)} = \Pi(X), \quad X \in \mathcal{S}_g.$$

For condition (7.26), we first consider a transition from global state X to $T_{nn'}^{cc'}(X)$ with $n, n' \neq 0$, $c \in \mathcal{D}_n$, and $c' \in \mathcal{A}_{n'}$. The aggregate probability flow for this transition is

$$\begin{aligned} & \sum_{\substack{\{x, x' : G(x) = X, \\ G(x') = T_{nn'}^{cc'}(G(x))\}}} \Pi(X; M) \prod_{i=1}^N \frac{\pi_i(x_i; \lambda_i)}{p_i(X_i; \lambda_i)} d_n^c(x_n, x'_n) N_n(X) R_{nn'}^{cc'}(X) a_{n'}^{c'}(x_{n'}, x'_{n'}) \\ &= \Pi(X; M) N_n(X) R_{nn'}^{cc'}(X) \sum_{\substack{\{x_n, x'_n : G_n(x_n) = X_n, \\ G_n(x'_n) = D_n^c(X_n)\}}} \frac{\pi_n(x_n; \lambda_n)}{p_n(X_n; \lambda_n)} d_n^c(x_n, x'_n) \Pi(X) N_n(X) R_{nn'}^{cc'}(X) M_n^c(X_n), \end{aligned}$$

which is the corresponding probability flow in the global process $\mathcal{G}(M)$. For transitions from state X to state $T_{0n'}^{cc'}(X)$ and state $T_{n0}^{cc'}(X)$, condition (7.26) is proved analogously. $\square$

Let us now study conditions for the local processes to be the aggregation of the network with respect to the detailed state of a node. The multiplication factor $N_n(X)$ in the transition rates for the network is not incorporated in the local processes, so that we must set $N_n(X) = N_n(X_n)$. We will restrict the network to

$$N_n(X) = 1, \quad n = 1, \dots, N, \quad X \in \mathcal{S}_g.$$

For aggregation with respect to the nodes, we need additional conditions. To this end, observe that Theorem 7.3.1 has been obtained under the condition that the departure and time-reversed departure rates of the local processes equal the corresponding rates in the global processes. Intuitively, for the local processes to be the aggregation of the network with respect to the nodes, it is also required that the local arrival rates equal the corresponding rates in the global process. Let us first specify the arrival rates in the global process that will be used in the formulation of our aggregation result.

Let $\hat{\lambda}_n^c(X_n)$ denote the mean class $c \in \mathcal{A}_n$ arrival rate at node n in state X_n , $n = 1, \dots, N$, of the global process $\mathcal{G}(M)$. Then

$$\tilde{\lambda}_n^c(X_n) = \sum_{Y:Y_n=X_n} \frac{\Pi(Y;M)}{P_n(X_n;M)} \left(\sum_{c' \in \mathcal{D}_0} M_0^{c'}(X) R_{0n}^{c'c}(X) + \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} M_{n'}^{c'}(X_{n'}) R_{n'n}^{c'c}(X) \right) \quad (7.27)$$

$$\begin{aligned} &= \frac{1}{P_n(X_n;M)} \sum_{Y:Y_n=A_n^c(X_n)} \left(\sum_{c' \in \mathcal{D}_0} \Pi(T_{n0}^{cc'}(Y)) M_0^{c'}(T_{n0}^{cc'}(Y)) R_{0n}^{c'c}(T_{n0}^{cc'}(Y)) \right. \\ &\quad \left. + \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} \Pi(T_{nn'}^{cc'}(Y)) M_{n'}^{c'}(A_{n'}^{c'}(Y_{n'})) R_{n'n}^{c'c}(T_{nn'}^{cc'}(Y)) \right) \\ &= \frac{P_n(A_n^c(X_n);M)}{P_n(X_n;M)} \Lambda_n^c(A_n^c(X_n);M), \end{aligned} \quad (7.28)$$

where the term

$$\sum_{c' \in \mathcal{D}_0} M_0^{c'}(X) R_{0n}^{c'c}(X) + \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} M_{n'}^{c'}(X_{n'}) R_{n'n}^{c'c}(X)$$

in the first line (7.27) is the class c arrival rate at node n in state X of the global process, and the last equality follows from the definitions of $\Lambda_n^c(X_n;M)$ and $P_n(X_n;M)$.

Under the conditions (7.15), (7.16) of Theorem 7.3.1 the local class $c \in \mathcal{A}_n$ arrival rate $\lambda_n^c(X_n)$ is related to $\Lambda_n^c(X_n;M)$ by

$$\lambda_n^c(X_n) = \frac{p_n(A_n^c(X_n); \lambda_n)}{p_n(X_n; \lambda_n)} \Lambda_n^c(A_n^c(X_n);M).$$

The following theorem shows that if this rate equals the corresponding rate $\tilde{\lambda}_n^c(X_n)$ as specified in (7.28) for the global process, the local processes are the aggregation of the network with respect to the nodes. Note that this further implies that the aggregate probability $p_n(X_n; \lambda_n)$ that the local process is in state X_n equals the corresponding probability $P_n(X_n;M)$ for the global process.

Theorem 7.4.2 (Aggregation with respect to the detailed state of the nodes) *Assume that $N_n(X) = 1$ for all n and X , and that, for $n = 1, \dots, N$, $X_n \in G_n(\mathcal{S}_n)$,*

$$\begin{aligned} M_n^c(X_n) &= \tilde{M}_n^c(X_n; \lambda_n) \\ \Lambda_n^c(X_n;M) &= \tilde{\Lambda}_n^c(X_n; \lambda_n). \end{aligned}$$

Further assume that for $n = 1, \dots, N$, $X_n \in G_n(\mathcal{S}_n)$,

$$\tilde{\lambda}_n^c(X_n) = \lambda_n^c(X_n). \quad (7.29)$$

Then, for $n = 1, \dots, N$, $X_n \in G_n(\mathcal{S}_n)$,

$$P_n(X_n;M) = p_n(X_n; \lambda_n) \quad (7.30)$$

and the local process $\mathcal{L}_n$ is the aggregation of $\mathcal{N}$ with respect to the aggregation function $h(x) = x_n$.

Proof. First observe that condition (7.29) implies that

$$\frac{p_n(A_n^c(X_n); \lambda_n)}{p_n(X_n; \lambda_n)} = \frac{P_n(A_n^c(X_n); M)}{P_n(X_n; M)}.$$

Since both $p_n(\cdot)$ and $P_n(\cdot)$ are probabilities over $G_n(\mathcal{S}_n)$, it must be that (7.30) is satisfied.

The aggregate probability that the state of node n in the network equals x_n is given by

$$\begin{aligned} \sum_{\{y: y_n = x_n\}} \pi(y) &= \sum_{\{y: y_n = x_n\}} \Pi(G(y); M) \prod_{i=1}^N \frac{\pi_i(y_i; \lambda_i)}{p_i(G_i(y_i); \lambda_i)} \\ &= \sum_{\{Y: Y_n = G_n(x_n)\}} \Pi(Y; M) \frac{\pi_n(x_n; \lambda_n)}{p_n(G_n(x_n); \lambda_n)} \\ &= P_n(G_n(x_n); M) \frac{\pi_n(x_n; \lambda_n)}{p_n(G_n(x_n); \lambda_n)} = \pi_n(x_n; \lambda_n), \end{aligned}$$

the corresponding probability in the local process. Hence, condition (7.25) is satisfied.

It remains to prove that condition (7.26) is satisfied. For internal transitions, note that the probability flow of an internal transition of node n from state x_n to x'_n in the network is given by

$$\sum_{\{y: y_n = x_n\}} \sum_{\{y': y'_n = x'_n\}} \pi(y) i_n(y_n, y'_n) = \pi_n(x_n; \lambda_n) i_n(x_n, x'_n).$$

For departure transitions, condition (7.26) is proved similarly. Let us now consider a class $c \in \mathcal{A}_n$ arrival transition from state x_n to state x'_n . The probability flow of this transition in the network is given by

$$\begin{aligned} &\sum_{\{y: y_n = x_n\}} \left(\sum_{c' \in \mathcal{D}_0} \sum_{\substack{\{y': y'_n = x'_n\} \\ G(y') = T_{0n}^{c'}(G(y))}} \pi(y) M_0^{c'}(G(y)) R_{0n}^{c'}(G(y)) a_n^c(y_n, y'_n) \right. \\ &\quad \left. + \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} \sum_{\substack{\{y': y'_n = x'_n\} \\ G(y') = T_{n'n}^{c'}(G(y))}} \pi(y) d_{n'}^{c'}(y_{n'}, y'_n) R_{n'n}^{c'}(G(y)) a_n^c(y_n, y'_n) \right) \\ &= \pi_n(x_n) a_n^c(x_n, x'_n) \sum_{\{Y: G_n(Y_n) = G_n(x_n)\}} \Pi(Y; M) \frac{1}{p_n(Y_n; \lambda_n)} \end{aligned}$$

$$\begin{aligned} & \left(\sum_{c' \in \mathcal{D}_0} M_0^{c'}(Y) R_{0n}^{c'c}(Y) + \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} M_{n'}^{c'}(Y_{n'}) R_{n'n}^{c'c}(Y) \right) \\ &= \pi_n(x_n; \lambda_n) \lambda_n(G_n(x_n)) a_n^c(x_n, x'_n), \end{aligned}$$

which is the corresponding rate in the local process. Note that the last equality is obtained using (7.27) and (7.30). $\square$

Note that the conditions of Theorem 7.4.2 include those of Theorem 7.4.1. Thus under the conditions of Theorem 7.4.2 both aggregations hold. Note also that the decomposition (7.19) still holds: the stationary distribution of the network thus may be factorised such that the local processes are aggregations of the network with respect to the nodes, and the global process is the aggregation of the network with respect to the global state.

Under the conditions of Theorem 7.4.2, the arrival rates $\lambda_n = (\lambda_n^c : G_n(\mathcal{S}_n) \rightarrow \mathbb{R}_0^+; c \in \mathcal{A}_0)$ are a solution of a set of fixed point equations that comprises those of Corollary 7.3.4 and in addition (7.29) and (7.30). To simplify this set of equations, note that (7.30) implies that both (7.29): $\lambda = \tilde{\lambda}$, and (7.16): $\tilde{\Lambda} = \Lambda$ are satisfied. We have the following result.

Corollary 7.4.3 (Fixed point equations for arrival rates: aggregation) *Under the conditions of Theorem 7.4.2, the arrival rates $\lambda_n = (\lambda_n^c : G_n(\mathcal{S}_n) \rightarrow \mathbb{R}_0^+; c \in \mathcal{A}_0)$ are a solution of the fixed point equations:*

$$\begin{aligned} (7.12) \quad & \pi_n = \pi_n(\lambda_n) \\ (7.4) \quad & \mu_n^c = \mu_n^c(\pi_n, \lambda_n) \\ (7.10) \quad & \tilde{M}_n^c = \tilde{M}_n^c(\pi_n, \mu_n, \lambda_n) \\ (7.15) \quad & M = \tilde{M} \\ (7.13) \quad & \Pi = \Pi(M) \\ (7.28) \quad & \tilde{\lambda}_n^c = \tilde{\lambda}_n^c(\Lambda_n^c) \\ (7.30) \quad & P_n = p_n \\ & \lambda_n^c(X_n) = \frac{P_n(A_n^c(X_n); M)}{P_n(X_n; M)} \Lambda_n^c(A_n^c(X_n); M). \end{aligned}$$

These equations may be solved using the following algorithm:

- Step i: For $n = 1, \dots, N$ initialize with a starting value for $\hat{\lambda}_n$ for λ_n .
- Step ii: Use (7.12), (7.4), (7.10) to obtain $\tilde{M}_n^c$.
- Step iii: Use (7.15), (7.13), (7.28) to obtain $\tilde{\lambda}_n^c$.
- Step iv: If $P_n(X_n) = p_n(X_n)$ for $c \in \mathcal{D}_n$, $X_n \in G_n(\mathcal{S}_n)$, $n = 1, \dots, N$ then stop, and λ_n is obtained, else let

$$\lambda_n^c(X_n) = \frac{P_n(A_n^c(X_n); M)}{P_n(X_n; M)} \Lambda_n^c(A_n^c(X_n); M)$$

and go to Step ii.

Remark 7.2 (Marie's decomposition and aggregation method). The algorithm of Corollary 7.4.3 requires Assumptions 7.2.1 and 7.2.2. Observe, however, that the

algorithm can also be evaluated if these assumptions do not hold by replacing formula (7.10) for the mean local departure rate by (7.9), and formula (7.28) for the mean global arrival rates by (7.27). This gives an approximation algorithm that extends Marie's method [20] to include state-dependent routing, general global states and to global processes that do not satisfy local balance. $\square$

7.5 Examples

This section provides some examples to illustrate the results of Sections 7.3 and 7.4. The first three examples relate our results to known cases from the literature that have motivated the results of this paper. Section 7.5.1 describes a network of quasi-reversible nodes linked via state-dependent routing as studied in [3]. Section 7.5.2 describes biased local balance and a network with negative customers and signals as studied in [12]. The third example in Section 7.5.3 is concerned with pull networks as studied in [4] for which the partial balance equations are different from the standard equations for Jackson type networks. Finally, Section 7.5.4 provides a novel example of assembly networks. We obtain novel product form results and novel decomposition results.

7.5.1 Quasi-reversible nodes linked via state-dependent routing

Consider a network of N interacting nodes containing customers of a single class, say $\mathcal{A}_n = \mathcal{D}_n = \{1\}$ for all $n = 0, \dots, N$. Let the global state X_n of node $n = 1, \dots, N$ represent the total number of customers in node n . Let $A_n^1(X_n) = X_n + 1$, $D_n^1(X_n) = X_n - 1$, i.e. an arriving customer increases the number of customers by one, and a departing customer decreases the number of customers by one. For simplicity, we also assume that $G_n(\mathcal{S}_n) = \{0, \dots, M\}$, where M may represent infinity. This assumption, however, is not essential for the results below.

In (7.14) we have shown that for the local process $\mathcal{L}_n(\lambda_n)$ the sum of the total arrival rates and the total mean departure rates in each global state X_n does not change under time reversal. This implies for a network containing only a single class of customers that the local time-reversed arrival rates equal the time-forward arrival rates:

$$\mu_n^1(X_n; \lambda_n) = \lambda_n^1(X_n) \quad n = 1, \dots, N. \quad (7.31)$$

To see this, first note that for $X_n = 0$, the result follows since $p_n(-1) = 0$. Now suppose $\mu_n^1(X_n; \lambda_n) = \lambda_n^1(X_n)$ for $X_n < M$. Then, again by (7.14), $\mu_n^1(X_n + 1; \lambda_n) = \lambda_n^1(X_n + 1)$, since $p_n(X_n + 1) > 0$ by the ergodicity of the local processes.

Equation (7.31) states that the outside of the nodes in the local process should satisfy local balance (with possibly state-dependent arrival rates). In the following

lemma we show that this property is equivalent to quasi-reversibility (i.e., with constant arrival rates).

Lemma 7.5.1 *Assume that $\mathcal{A}_n = \mathcal{D}_n = \{1\}$ for all $n = 0, \dots, N$. Let the global state X_n of node $n = 1, \dots, N$ represent the total number of customers in node n . Let $A_n^1(X_n) = X_n + 1$, $D_n^1(X_n) = X_n - 1$. Then $\mu_n^1(X_n; \lambda_n) = \lambda_n^1(X_n)$ if and only if node n is quasi-reversible when the arrival rate equals one.*

Proof. Suppose node n is quasi-reversible with arrival rate one, and $\pi_n(x_n; 1)$ is its stationary distribution. By substitution in the balance equations, we obtain that

$$\pi_n(x_n; 1) \prod_{y=0}^{G_n(x_n)-1} \lambda_n^1(y) \quad (7.32)$$

is the stationary distribution of node n with arrival rate $\lambda_n^1(X_n)$, and that $\mu_n(X_n; \lambda_n) = \lambda_n(X_n)$. Similarly, if $\mu_n(X_n; \lambda_n) = \lambda_n(X_n)$ and node n has stationary distribution $\pi_n(x_n; \lambda_n)$, then

$$\pi_n(x_n; \lambda_n) \left(\prod_{y=0}^{G_n(x_n)-1} \lambda_n^1(y) \right)^{-1}$$

is the stationary distribution of node n with arrival rate 1, and $\mu_n(X_n; 1) = 1$. $\square$

Let us now consider the implications of a single class with global state representing the number of customers in the global process. By Lemma 7.5.1 we immediately see that

$$\Lambda_n^c(X_n; M) = M_n^c(X_n),$$

since for (generalised) quasi-reversible nodes we may invoke Theorem 7.3.2 with $f_n(X_n; \lambda_n) = 1$, or Corollary 7.3.3. The global process thus must satisfy local balance. The following lemma shows that the choice of the departure rates $M_n^c(X_n)$ does not effect the local balance of the global process: local balance of the global process is a property that is only determined by the coupling of the nodes, and not by the nodes themselves.

Lemma 7.5.2 *Assume that $\mathcal{A}_n = \mathcal{D}_n = \{1\}$ for all $n = 0, \dots, N$. Let the global state X_n of node $n = 1, \dots, N$ represent the total number of customers in node n . Let $A_n^1(X_n) = X_n + 1$, $D_n^1(X_n) = X_n - 1$. Then $M_n^1(X_n) = \Lambda_n^1(X_n; M)$ if and only if the global process satisfies local balance when $M_n(X_n) = 1$.*

Proof. Suppose the global process satisfies local balance with $M_n(X_n) = 1$, and let $\Pi(X; \mathbf{1})$ denote the stationary distribution when $M_n(X_n) = 1$. Then it is readily verified by substitution in the balance equations for the global process that

$$\Pi(X; \mathbf{1}) \prod_{n=1}^N \left(\prod_{y=1}^{X_n} M_n(y) \right)^{-1} \quad (7.33)$$

is the stationary distribution for the global process with departure rates $M_n(X_n)$, and that $\Lambda_n(X_n; M) = M_n(X_n)$. Similarly, if $\Pi(X; M)$ is the stationary distribution of the global process with departure rates $M_n(X_n)$, and $\Lambda_n(X_n; M) = M_n(X_n)$, then

$$\Pi(X; M) \prod_{n=1}^N \left(\prod_{y=1}^{X_n} M_n(y) \right)$$

is the stationary distribution for the global process with departure rates equal to one, and satisfies local balance for the global process. $\square$

We summarize the above results in the following theorem, that states the conditions on the nodes and the local processes of [3]. We want to stress that the results presented above need all conditions stated here. Theorem 7.5.3 generally will not hold for multiclass queueing networks, networks with batch movements, or networks with negative customers.

Theorem 7.5.3 *Assume that $\mathcal{A}_n = \mathcal{D}_n = \{1\}$ for all $n = 0, \dots, N$. Let the global state X_n of node $n = 1, \dots, N$ represent the total number of customers in node n . Let $A_n^1(X_n) = X_n + 1$, $D_n^1(X_n) = X_n - 1$. The conditions of Theorems 7.3.1 and 7.3.2, and of Corollary 7.3.3 are satisfied if and only if the nodes are quasi-reversible with arrival rate one, and the global process satisfies local balance with departure rates one.*

If $\lambda_n^1(X_n) = \mu_n^1(X_n; \lambda_n)$, then any function f_n satisfies (7.21). Hence, Theorem 7.3.2 allows the global process to be analysed by arbitrary departure rate functions. From (7.32) and (7.33) we find that the stationary distribution in Theorem 7.3.2 takes the form

$$\begin{aligned} C \Pi(G(x); \mathbf{1}) \prod_{n=1}^N \left(\prod_{y=1}^{G_n(x_n)} \lambda_n^1(y-1) \frac{f_n(y-1)}{f_n(y)} \right)^{-1} & \frac{\pi_n(x_n; 1)}{f_n(G_n(x_n))} \prod_{y=0}^{G_n(x_n)-1} \lambda_n^1(y) \\ & = C \prod_{n=1}^N f_n(0) \Pi(G(x); \mathbf{1}) \prod_{n=1}^N \prod_{m=1}^N \pi_n(x_n; 1), \end{aligned}$$

in correspondence with Corollary 7.3.3.

7.5.2 Biased local balance

For the global process, we have assumed in Assumption 7.2.2 that the nominal departure rate of class c customers from node n in state X of the stationary time-reversed process of $\mathcal{G}(M)$ depends on the global state X_n only, i.e.,

$$\Lambda_n^c(X_n; M) N_n(X) = \sum_{c' \in \mathcal{D}_0} \frac{\Pi(T_{n0}^{cc'}(X); M)}{\Pi(X; M)} M_0^{c'}(T_{n0}^{cc'}(X)) R_{0n}^{c'c}(T_{n0}^{cc'}(X))$$

$$+ \sum_{n'=1}^N \sum_{c' \in \mathcal{D}_{n'}} \frac{\Pi(T_{nn'}^{cc'}(X); M)}{\Pi(X; M)} M_{n'}^{c'}(A_{n'}^{c'}(X_{n'})) N_{n'}^{c'}(T_{nn'}^{cc'}(X)) R_{n'n}^{c'c}(T_{nn'}^{cc'}(X)).$$

We have shown that, if $\mathcal{A}_n = \mathcal{D}_n$, and $\Lambda_n^c(X_n; M) = M_n^c(X_n)$, $X_n \in G_n(\mathcal{S}_n)$, $c \in \mathcal{A}_n$, $n = 1, \dots, N$, then this assumption implies that the global process satisfies local balance (where, for notational convenience, $N_0^c(X) = 1$ for all c, X)

$$\begin{aligned} & M_n^c(X_n) N_n(X) \Pi(X; M) \\ &= \sum_{n'=0}^N \sum_{c' \in \mathcal{D}_{n'}} \Pi(T_{nn'}^{cc'}(X); M) M_{n'}^{c'}(A_{n'}^{c'}(X_{n'})) N_{n'}^{c'}(T_{nn'}^{cc'}(X)) R_{n'n}^{c'c}(T_{nn'}^{cc'}(X)), \end{aligned}$$

otherwise we do not have equality. Following Chao and Miyazawa [12] we introduce *biased local balance*, and say that $\Pi(X; M)$ satisfies biased local balance with bias $\gamma_n^c(X; M)$ if

$$\begin{aligned} & (M_n^c(X_n) N_n(X) + \Gamma_n^c(X; M)) \Pi(X; M) \\ &= \sum_{n'=0}^N \sum_{c' \in \mathcal{D}_{n'}} \Pi(T_{nn'}^{cc'}(X); M) M_{n'}^{c'}(A_{n'}^{c'}(X_{n'})) N_{n'}^{c'}(T_{nn'}^{cc'}(X)) R_{n'n}^{c'c}(T_{nn'}^{cc'}(X)), \end{aligned} \quad (7.34)$$

Our definition of biased local balance is closely related to the concept of biased local balance, introduced by Chao and Miyazawa [12]. However, in [12] the bias is required to be constant, and thus the existence of the bias imposes conditions on the global process. By allowing the bias to be state-dependent, the bias can be defined for every global process.

Note that global balance implies that

$$\sum_{n=0}^N \sum_{c \in \mathcal{A}_n \cup \mathcal{D}_n} \Gamma_n^c(X; M) = 0, \quad X \in \mathcal{S}_g. \quad (7.35)$$

Further note that Assumption 7.2.2 implies that

$$\Gamma_n^c(X; M) = (\Lambda_n^c(X_n; M) - M_n^c(X_n)) N_n(X), \quad (7.36)$$

i.e., we have a strict condition on the state dependence of the bias.

We now define the bias of the local process as the difference in arrival and departure rates to a node. For the local processes $\mathcal{L}_n(\lambda_n)$, we call $\gamma_n^c(x_n; \lambda_n)$ the *bias of node n with respect to the outside and c* , if for all $x_n \in \mathcal{S}_n$

$$\pi_n(x_n; \lambda_n) (\lambda_n^c(G_n(x_n)) + \gamma_n^c(x_n; \lambda_n)) = \sum_{x_n'} \pi_n(x_n'; \lambda_n) d_n^c(x_n', x_n). \quad (7.37)$$

Similar to the bias of the global process, the bias indicates the unbalance in local balance equations: if $\gamma_n^c(x_n; \lambda_n) = 0$, equation (7.37) corresponds to node n being locally balanced with respect to its outside and type u , and thus, if in addition λ_n^c is a constant function, (7.37) corresponds to node n being quasi-reversible in the

definition of [18]. When $\gamma_n^c(x_n; \lambda_n)$ is constant, but not necessarily zero, and λ_n^c is a constant function, (7.37) states that node n is quasi-reversible according to the generalised definition of [11]. Again, allowing the bias to be state-dependent, it can be defined for every node n , without requiring conditions on this node. Assumption 7.2.1 implies that

$$\gamma_n^c(x_n; \lambda_n) = \mu_n^c(G_n(x_n); \lambda_n) - \lambda_n^c(G_n(x_n)) \quad (7.38)$$

From our assumptions, invoking (7.10), (7.15), (7.11), and (7.16) we have obtained (7.17), that may be rewritten as

$$N_n(X) \gamma_n^c(X_n; \lambda_n) = -\Gamma_n^c(X; M)$$

i.e., the bias of the local process equals the bias of the global process. Our results of Section 7.3 thus show that if the bias of the nodes is suitably compensated by the bias of the global process, the network allows a decomposition of the stationary distribution.

Chao and Miyazawa [12] introduced the concept of biased local balance to extend the definition of quasi-reversibility allowing the input and output rate of customers at the nodes to differ from each other. The model of [12] has no global state for the nodes, say $G_n = 0$. Routing then is necessarily state-independent, and the multiplication factors $N_n(X)$ may be omitted, i.e., we may set $N_n(X) = 1$. Removing the global state also implies removing the state-dependence of the arrival and departure rates. The following theorem summarizes the product form result of [12].

Theorem 7.5.4 *Assume that $X_n = 0$ for all n . Then the conditions of Theorems 7.3.1, 7.3.2 and Corollary 7.3.3 are satisfied if and only if each node is generalised quasi-reversible, say with $\hat{\lambda}_n^c$ and $\hat{\mu}_n^c$, and the following traffic equations hold:*

$$\lambda_n^c = \sum_{n'=0}^N \sum_{c' \in \mathcal{D}'_n} \mu_n^c R_{n'n}^{c'c} \quad (7.39)$$

7.5.3 A pull network

In a Jackson network a transition is initiated by the service of a customer at a node, and subsequently this customer is routed to its destination. This behaviour is sometimes referred to as push network: a customer is pushed from one queue to the next queue. We now consider a pull network in which a transition is initiated by the destination node that pulls a customer from another node.

Consider a network of N interacting nodes containing customers of a single class, say $\mathcal{A}_n = \mathcal{D}_n = \{1\}$ for all $n = 0, \dots, N$. Let the global state X_n of node $n = 1, \dots, N$ represent the total number of customers in node n . Let $A_n^1(X_n) = X_n - 1$, $D_n^1(X_n) = X_n + 1$. A departure from node n increases the number of customers in node n by one, and with probability $R_{nn'}^{11}(X)$ decreases the number of customers in node n' by one:

node n thus pulls a customer with probability $R_{nn'}^{11}(X)$ from node n' . For simplicity, we also assume that $G_n(\mathcal{S}_n) = \{0, \dots, M\}$, where M may represent infinity. The following results are easily proved in the same way as in Section 7.5.1.

First, we may show that $\mu_n^1(X_n; \lambda_n) = \lambda_n^1(X_n)$ for all $n = 1, \dots, N$, and $\mu_n^1(X_n; \lambda_n) = \lambda_n^1(X_n)$ if and only if node n is quasi-reversible when the arrival rate equals one. Furthermore, we have that $M_n^1(X_n) = \Lambda_n^1(X_n; M)$ if and only if the global process satisfies local balance when $M_n(X_n) = 1$. Summarizing, we have the following result.

Theorem 7.5.5 *Assume that $\mathcal{A}_n = \mathcal{D}_n = \{1\}$ for all $n = 0, \dots, N$. Let the global state X_n of node $n = 1, \dots, N$ represent the total number of customers in node n . Let $A_n^1(X_n) = X_n - 1$, $D_n^1(X_n) = X_n + 1$. The conditions of Theorem 7.3.1 are satisfied if and only if the nodes are quasi-reversible with arrival rate one, and the global process satisfies local balance with departure rates one.*

Thus, the seemingly distinct formulations of the local balance equations for push and pull networks that are described in [4] are a consequence of the same notion of local balance.

7.5.4 An assembly network

Consider a simple assembly network consisting of three nodes. Node 1 and node 2 each represent a subnetwork, on which we make no other assumption than that they produce units at nominal rate one. The units produced by node 1 are referred to as class 1 units; the units that are produced by node 2 as class 2 units. Both nodes send their units to node 3, where a class one and a class two unit are assembled into a class 3 unit. Assembly takes an exponentially distributed time with mean $\beta^{-1} < 1$, and clearly requires that both a class 1 and a class 2 unit are present at node 3.

We assume the following control mechanism in the network. If there are no class 1 units in node 3, node 2 is slowed down by a factor $\phi < 1$. Similarly, if no class 2 units are present in node 3, node 1 is slowed down by the same factor ϕ . This control mechanism thus tries to save production costs by producing less units when these units do not directly lead to output. We will show that for a specific choice of ϕ the network has a product form solution, and the time-reversed class 3 arrival rate is constant.

Let us first consider the local processes. For node 1 and node 2 we need no arrival transitions. We will omit the λ_n , $n = 1, 2$, from the notation. The stationary distributions π_1 and π_2 of the local processes for node $n = 1, 2$ thus are the unique distributions satisfying

$$\pi_n(x_n) \sum_{x'_n \in \mathcal{S}_n} (i_n(x_n, x'_n) + d_n^n(x_n, x'_n)) = \sum_{x'_n \in \mathcal{S}_n} \pi_n(x'_n) (i_n(x'_n, x_n) + d_n^n(x'_n, x_n)).$$

By the assumption that nodes 1 and 2 produce units at nominal rate one, we have

$$\sum_{x'_n \in \mathcal{S}_n} \frac{\pi_n(x'_n)}{\pi_n(x_n)} d_n^n(x'_n, x_n) = 1.$$

Therefore, no global state for node 1 and 2 is required (note that the routing is fixed, and the control mechanism is only influenced by node 3). As the global state for nodes 1, 2 is not required, we may set $X_1 = X_2 = 0$, and, hence, $p_1(0) = 1$, $p_2(0) = 1$.

The state of node 3 is described by $x_3 = (u_1, u_2)$, with u_n denoting the number of class n units in node 3. Since upon arrival of a class $n = 1, 2$ unit, the number of class n units is increased by one, arrival transitions are given by

$$a_3^n(x_n, x_n + e_n) = 1,$$

with e_n denoting the n -th unit vector of dimension 2. Departure transitions take place at rate β , as long as there are both type 1 and a type 2 units present in node 3. As a class 3 departure reduces the number of class 1 and class 2 units by one, the departure transitions are thus given by

$$d_3^3((u_1, u_2), (u_1 - 1, u_2 - 1)) = \beta, \quad u_1, u_2 > 0.$$

Internal transitions do not occur, as the service times of node 3 are exponential. To model the desired control mechanism, we define the global state of node 3 equal to the detailed state. Note that this is allowed by the exponential service times, and the unique changes of the state at arrival and service transitions. Then $p_3 = \pi_3$ and the functions $A_3^c(X_3)$ and $D_3^c(X_3)$ for $c = 1, 2, 3$ are given by

$$A_3^c(X_3) = \begin{cases} X_3 + e_c & \text{for } c = 1, 2, \\ X_3 + e_1 + e_2 & \text{for } c = 3, \end{cases}$$

$$D_3^c(X_3) = \begin{cases} X_3 - e_c & \text{for } c = 1, 2, \\ X_3 - e_1 - e_2 & \text{for } c = 3. \end{cases}$$

To define a local process for node 3, we need an initial guess for the arrival rates of class 1 and class 2 units. An obvious choice is the following.

$$\lambda_3^1((u_1, u_2)) = \begin{cases} \phi & \text{for } u_2 = 0 \\ 1 & \text{otherwise} \end{cases} \quad (7.40)$$

$$\lambda_3^2((u_1, u_2)) = \begin{cases} \phi & \text{for } u_1 = 0 \\ 1 & \text{otherwise} \end{cases} \quad (7.41)$$

The stationary distribution of the resulting local process $\mathcal{L}_3(\lambda_3)$ is provided in the following lemma for a specific choice of ϕ .

Lemma 7.5.6 *Let $\lambda_3 = (\lambda_3^1, \lambda_3^2)$ be given by (7.40) and (7.41). For $\phi = \frac{1}{2}\beta\alpha^2$, with*

$$\alpha = -\frac{1}{2} + \frac{1}{2\beta}\sqrt{\beta^2 + 8\beta}, \quad (7.42)$$

the stationary distribution π_3 of the local process $\mathcal{L}_3(\lambda_3)$ is given by

$$\pi_3((u_1, u_2); \lambda_3) = (1 - \alpha)^2 \alpha^{u_1 + u_2}. \quad (7.43)$$

Under these conditions, the time-reversed class 3 arrival rate $\mu_3^3((u_1, u_2); \lambda_3)$ is constant and equal to $\beta \alpha^2$.

Proof. As (7.43) sums to one, it is sufficient to prove that (7.43) satisfies the balance equations. For $u_1, u_2 > 0$, these equations are given by

$$\begin{aligned} & \pi((u_1, u_2); \lambda_3)(2 + \beta) \\ &= \pi((u_1 - 1, u_2); \lambda_3) + \pi((u_1, u_2 - 1); \lambda_3) + \beta \pi((u_1 + 1, u_2 + 1); \lambda_3). \end{aligned}$$

Substitution of (7.43) and dividing by $(1 - \alpha^2) \alpha^{u_1 + u_2 - 1}$ results in

$$\alpha(2 + \beta) = 2 + \beta \alpha^3.$$

This implies that either $\alpha = 1$, or

$$\beta \alpha^2 + \beta \alpha - 2 = 0. \quad (7.44)$$

As α , as given by (7.42) solves this equation, the proposed form for π_3 satisfies the balance equations for $u_1, u_2 > 0$. For $u_1 = u_2 = 0$, the balance equations are easily seen to be satisfied for $\phi = \frac{1}{2} \beta \alpha^2$. For $u_2 = 0$ and $u_1 > 0$, the balance equations are given by

$$\pi_3((u_1, 0); \lambda_3)(\phi + 1) = \pi((u_1 + 1, 1); \lambda_3) \beta + \pi((u_1 - 1, 0); \lambda_3) \phi.$$

Substituting (7.43) and dividing by $(1 - \alpha)^2 \alpha^{u_1 - 1}$, we have

$$\alpha(\phi + 1) = \alpha^3 \beta + \phi.$$

As $\phi = \frac{1}{2} \beta \alpha^2$, this equation is equivalent to (7.44) and thus satisfied by the form of α . As the model is symmetric in u_1 and u_2 , the first statement is proved.

By definition, the time-reversed arrival rate is given by

$$\mu_n((u_1, u_2); \lambda_3) = \frac{\pi_3((u_1 + 1, u_2 + 1); \lambda_3)}{\pi_3((u_1, u_2); \lambda_3)} \beta.$$

The second statement of the Theorem now follows from (7.43). $\square$

Let us now consider the network and the global process. The routing functions are obviously given by $R_{13}^{11} = 1$, $R_{23}^{22} = 1$ and $R_{30}^{33} = 1$. Furthermore, the control mechanism is incorporated in the model by

$$N_1((u_1, u_2)) = \begin{cases} \phi & \text{for } u_2 = 0 \\ 1 & \text{otherwise} \end{cases}$$

$$N_2((u_1, u_2)) = \begin{cases} \phi & \text{for } u_1 = 0 \\ 1 & \text{otherwise,} \end{cases}$$

and $N_3((u_1, u_2)) = 1$. Note that we have defined no global state for node 1 and 2, and thus the global state of the network is given by the global state of node 3. According to Theorem 7.3.1, the departure rate function are given by $M_1^1(0) = 1$, $M_2^2(0) = 1$, and using Lemma 7.5.6 we find

$$M_3^3((u_1, u_2)) = \frac{\pi_3((u_1 - 1, u_2 - 1); \lambda_3)}{\pi_3((u_1, u_2); \lambda_3)} \alpha^2 = \begin{cases} \beta & \text{for } u_1, u_2 > 0 \\ 0 & \text{otherwise} \end{cases}$$

Constructing the rates of the global process by Definition 7.3, and using the definition of the time-reversed departure rates of the global process, we obtain the following lemma.

Lemma 7.5.7 *The global process equals the local process for node 3 and satisfies Assumption 7.2.2 with*

$$\Lambda_3^1((u_1, u_2); M) = \begin{cases} \phi \alpha^{-1} & \text{for } u_2 = 0, \\ \alpha^{-1} & \text{otherwise,} \end{cases}$$

$$\Lambda_3^2((u_1, u_2); M) = \begin{cases} \phi \alpha^{-1} & \text{for } u_1 = 0, \\ \alpha^{-1} & \text{otherwise.} \end{cases}$$

According to Theorem 7.3.1, the class 1 arrival rate of the local process for node 3 corresponding with $\Lambda_3^1((u_1, u_2); M)$ should be equal to

$$\frac{p_3((u_1 + 1, u_2); \lambda_3)}{p_3((u_1, u_2); \lambda_3)} \Lambda_3^1((u_1, u_2); M) = \begin{cases} \phi & \text{for } u_2 = 0, \\ 1 & \text{otherwise.} \end{cases}$$

Similarly, the local class 2 arrival rate should be equal to ϕ for $u_1 = 0$ and equal to 1 otherwise. Hence, our initial guess for these local arrival rates was correct, and by Theorem 7.3.1 and Lemma 7.5.6, we have the following result.

Theorem 7.5.8 *The stationary distribution of the assembly network is of product-form: $\pi_1(x_1)\pi_2(x_2)\pi_3(x_3)$. The time-reversed class 3 arrival rate of the network is constant and equals $\beta\alpha^2$. $\square$*

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